Transient Analysis of Single Machine Production Line Dynamics

Farhood Rismanchian

Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the Degree of

Master of Science
in
Industrial Engineering

Eastern Mediterranean University
January 2013
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

Prof. Dr. Elvan Yılmaz
Director

I certify that this thesis satisfies the requirements as a thesis for the degree of Master of Science in Industrial Engineering.

Asst. Prof. Dr. Gökhan İzbırak
Chair, Department of Industrial Engineering

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Industrial Engineering.

Prof. Dr. Alagar Rangan
Supervisor

Examining Committee

1. Prof. Dr. Alagar Rangan
2. Prof. Dr. Bela Vizvari
3. Assoc. Prof. Dr. Adham Mackieh

ABSTRACT

In this thesis a single machine production line is modeled by an alternating renewal process. We derive efficient approximations for the first and second order transient performance measures of a production line which can be in one of the two states up(working) or down(failed), modeling the production line by an alternating renewal process. A due date performance measure is derived and discussed. The thesis also discusses two optimizations problems in the production line. Numerical examples are provided to illustrate the procedure.

Keywords: production line, alternating renewal process, renewal function, availability function, due date performance.

ÖZ

Bu tezde almaşık yenileme süreci ile tek makinalı bir üretim hattı modellenmiştir. Durumu yukarıda(çalışan) veya aşağıda(başarısız) olabilen bir üretim hattı'nın birinci ve ikinci derece geçici başarım ölçütlerine verimli yaklaşıklama türetiyoruz. Bu üretim hattı almaşik yenileme süreci ile modellenmiştir. Ayrıca bir vade tarihi başarim ölçütü türetilmiş ve tartışılmıştır. Bu tez ayni zamanda üretim hattinda iki eniyileme problemi irdelemektedir. Prosedürü göstemek için sayısal örnekler sağlanmiştir.

Anahtar kelimeler: üretim hatti, almaşik yenileme süreci, yenileme fonksiyonu, vade tarihi başarımı.

To My Parents

ACKNOWLEDGMENTS

When I think about the people who have directly helped me with my thesis, my parents would be on top of the list. They have always helped me in every single step of my life, and their love and support has been a huge help for me to achieve this goal. Their value to me only grows with age.

I would like to express my deepest appreciation to my supervisor Prof. Alagar. Rangan. No matter how many times I have disturbed this great man regarding this thesis, he has continually showed great support and spirit towards me and my research from the beginning until the final revision of this research. Without his guidance, reaching this goal would not have been possible for me. His early insights launched the greater part of this dissertation.

My gratitude and appreciation to Asst. Prof. Dr. Gokhan Izbirak, head of the Industrial Engineering department who has showed eternal support not only for me, but for all the students of this department. I would like to thank him for providing such a great, friendly, and healthy environment for the students of the Industrial Engineering department.

Also, it is with a sense of appreciation that I acknowledge the contributions and support of my faculty members who have given a great deal of time, valuable advices, and continuous guidance so that I could reach to this point. And my special thanks to Asst. Prof. Dr. Emine Atasoylu for the Turkish translation of my thesis abstract.

Above all, I sincerely thank my close friends who have been unstintingly supportive throughout this research, and have guided me with their great knowledge and good humor.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

Chapter 1

INTRODUCTION

In the modern days of mass production systems, production lines play a key role. Mathematical and stochastic models which address to the manufacturing system design and performance measures for the production line control have been the subject of intense investigation of researchers. The major focus of such investigations has been to apply queuing networks models. These studies however have been restricted to steady state analysis. These analysis provide in a compact form of certain measures like the production rate, throughput, buffer etc. For a good review of production models one can refer to Papadopoulos and Heavy (1996) and Dallery and Gershwin (1992). One cannot ignore the importance of the closed form solutions necessary for the computer implementation provided by the steady state analysis. However, in many production lines with a limited planning horizon the steady state measures may not be in synchronization with the actual situation and what really needed may be the transient measures. In the modern days, customer orders are to be met with minimum lead time. Further production systems are governed by JIT deliveries. With such changing environments, the planning periods are decidedly reduced.

Production system operators are generally interested in optimizing the first order performance measures such as maximizing the expected throughput or minimizing the expected buffer. Such an approach has been practiced because the first order measures are amenable for a compact and close form. While it is important that the production system delivers on the average a pre specified number of items, it is equally important that the variation in the output remains under control. For instance, between two production lines whose average output in a given time period is the same, production managers will prefer the production line which exhibit lesser variation. Gershwin (1993) observes that the production line output has high variability often lying in the interval mean $\pm 10\%$ of standard deviation. Tan, (1999) has given the data on the number of units of certain appliance manufactured each day to underscore the importance of variability in a production line. We reproduce below the figure given by the same author.

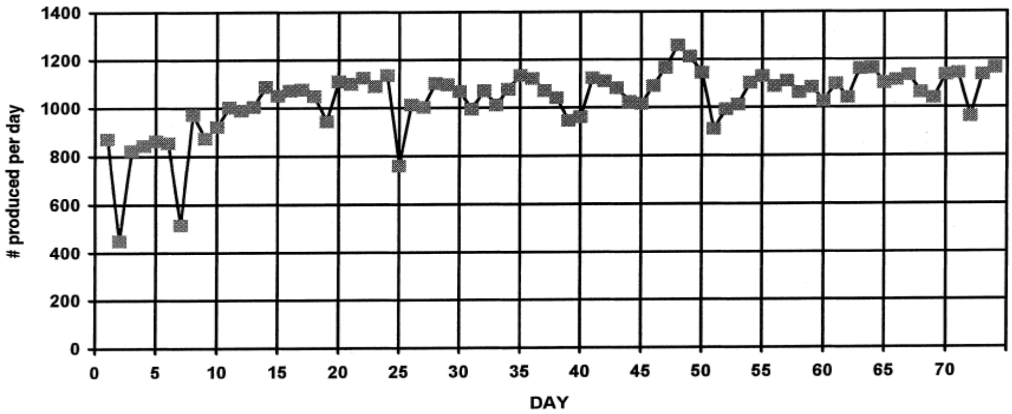

Figure 1: Plot of the number of appliances produced per day

The figure shows that while the average production per day was 1043.67 the standard deviation of the output was 112.91 which is 10.2% of the mean. It is to be observed that the requirement of the company during the said period was 1100 units each day.

In a dynamic and fast changing environment, we are of the considered view that the steady state as well as the first order measures alone is not sufficient to give a correct picture of the production dynamics. We believe that a transient analysis incorporating the first and second order performance measures will provide a powerful decision support tool which alone can bring out the nuances in the production dynamics. Thus the contribution of the present thesis is twofold:

(i). To derive certain first and second order transient performance measures in single machine production line which can be in up(working) or down(failed) state

(ii). To analyze certain optimization problems in production lines using these performance measures.

1.1 Brief Literature Review

In a production line, the output of a manufacturing subsystem is usually the input to one or more downstream subsystems in the production process. More generally the output process of the production line becomes the arrival process to the next subsystem in the line. This aspect has encouraged researchers to use queuing networks to model production lines. For early surveys on the results of the application of queuing networks one can referred to Papadopoulos and Heavey, (1996).

Hendricks (1992) provided certain results for the output process of serial production lines of exponential machines with finite buffers. There are several studies using the first order performance measures of production lines. Yeralan and Muth (1987) considered two station production lines using several assumptions. Papadopoulos (1994) discussed multi state production lines with no buffers in between. The mean performance of multi station production lines with and without interstation buffers under several operational assumptions was determined exactly by generating the state based model and solving the resulting system of equations. Such results were determined approximately by decomposition method (Tan & Yeralan (1997)), Dallery and Frein (1993)).The above mentioned studies try to find and use average measures of the characteristics of interest. However the available literatures on the variability of the output in manufacturing systems are scanty. Miltenburg (1987) was perhaps the first to present a method which determines the asymptotic variance of the output per unit time. He used the Markov chain theory to determine the asymptotic mean and variance of the time spent in each of states. Hendricks (1992) developed an analytical approach which unfortunately was computer intensive and thus was not useful for large number of machines.

Tan (1997a) modeled production lines with finite buffer using Markov reward systems and computed the asymptotic variance rate of production. He has also considered production lines with no interstation buffers (Tan (1997b)). It is also interesting to note that (Tan (1999)) has dealt with discrete flow production line with cycle dependent failures. Gershwin (1993) presented a crucial result which enables one to extend the basic results for a single machine to N station production lines. He first calculated the production variance for a single machine exactly. Then he developed decomposition techniques for larger production lines.

All the literatures cited so far have the Markov property built into the model. More specifically, the residence times are assumed to be exponentially distributed whose lack of memory property gives the modeler lots of flexibility leading to explicit analytical results. However in real life situations one is confronted with arbitrary distributions which render the analysis intractable. Thus it is not surprising that researchers resort to Markov modeling. This thesis attempts to obtain the first and second order characteristics of the throughput of a single station production line with arbitrary up and down times. We do hope Gershwin`s decomposition technique will help us to obtain the corresponding characteristics of a N station production line. Our modeling could also be viewed as a generalization of the Markov reward model for a discrete material flow production line of Tan (1999). The single machine that we consider could be operational (up state) or failed (down state), so that we model the system using an alternating renewal process. The characteristics of interest require computation of performance measures such as the mean and variance of the number of visits to the up state as well as the availability function. However no analytical solutions are available for these measures excepting when the up and down times are exponentially distributed which corresponds to the Markov model. A notable contribution of this thesis lies in developing useful approximations to determine (i) the expected number of visits to the up state known as the renewal function (ii) variance of the number of visits to the up stare and (iii) the availability function which gives the probability that the system is found in the up state at any arbitrary time. In order to understand the theory and concepts of renewal function and availability function, we present in following sections a brief write up on them. The materials presented in the write up are readily available in any standard text book on stochastic processes.

1.2 Renewal Processes

Let $\{X_n, n = 1,2, \dots\}$ be a sequence of continuous, non-negative independently and identically distributed random variables with a common distribution function F. Let

$$E[X_n] = \int_0^{\infty} x dF(x) = \mu$$

Let us assume

$$s_0 = 0, s_n = X_1 + X_2 + X_3 + \cdots + X_n \ ,$$

Let us denote the distribution function of $S_n, n \geq 1$ by $F_n(x) = Pr\{S_n \leq x\}$.

$$F_0(x) = \begin{cases} 1 & if \;\; x \geq 0 \\ 0 & if \;\; x < 0 \end{cases}$$

We further define a new random variable $N(t) = sup\{n; S_n \leq t\}$. The integer valued stochastic process $\{N(t), t \geq 0\}$ is referred to as a renewal process whose distribution is F. The expectation of the random variable $N(t)$ denoted by the function $M(t) = E\{N(t)\}$ plays a crucial role in the theory of renewal processes. This function is referred to as the renewal function of the process. Using elementary probability arguments one can show that $M(t)$ satisfy the following integral equation.

$$M(t) = F(t) + \int_0^t M(t-x)dF(x) \tag{1.1}$$

Applying the Laplace transformation to the left and right hand side of the above equation we obtain

$$M^*(s) = \frac{f^*(s)}{s[1-f^*(s)]} \tag{1.2}$$

The derivative $m(t)$of $M(t)$ is called the renewal density. We have

$$m(t) = \lim_{\Delta t \to \infty} \frac{Pr\{one\ or\ more\ renewals\ in\ (t, t+\Delta t)\}}{\Delta t} =$$

$$\sum_{n=1}^{\infty} \lim_{\Delta t \to \infty} \frac{Pr\{n^{th}\ renewal\ occurs\ in\ (t, t+\Delta t)\}}{\Delta t} =$$

$$\sum_{n=1}^{\infty} \lim_{\Delta t \to \infty} \frac{f_n(t)\Delta t + o(\Delta t)}{\Delta t} = \sum_{n=1}^{\infty} f_n(t) = \sum_{n=1}^{\infty} F_n'(t) = M'(t)$$

The function $m(t)$ gives the average number of renewals which are to be expected in a small tine interval $[t, t + dt]$. We wish to observe that $m(t)$ is not a PDF. The Laplace transforms of the renewal density can easily seem to be:

$$m^*(s) = \frac{f^*(s)}{1 - f^*(s)} \tag{1.3}$$

The renewal equation given in (1.1) can be identifying to be a Volterra integral equation. This equation cannot be explicitly solved to get a closed form solution which is possible only if the distribution function $F(x)$ is either exponential or gamma distribution. Since the renewal function plays a crucial role in several real life applications, several authors have proposed approximations to the renewal function. These approximations can be broadly classified as below. We give only the references which have contributed to these methods. Interested reader can refer to them.

- Method of substitution:

In this method some of the terms in the integrand of the renewal equation are substituted. Some of the notable contributions came from Bartholomew (1993), Deligonul (1985), Smeitink and Dekker (1990), Politis and Pitts (1998).Kambo (2012).

- Methods based on Riemann-Stieltjes integral:

In this method researchers approximate the integral on the right hand side of (1.1) by an infinite sum. The major contributions came from Xie (1989), Ayhan (1999), Xie (2003).

- Bounds:

These methods analyses the asymptotic nature and bounds to the solution of the renewal equations. Some important references using this method are Marshall (1973), Deley (1976), Li and Luo (2005) and Ran (2006).

- Method of moment matching:

In these methods the distribution function F(x) is approximated by mostly phase type distributions such that first few moments of the two distributions match. Notable literature using this method include Marie (1980), Whitt (1982), Altiok (1985), Lindsay (2000), Cui and Xie (2003) and Bux and Herzog (1997).

1.3 Alternating (or two stage) Renewal Process

In an ordinary renewal process, the system is identified with only one state, for instance the working state of a system. It is tacitly assumed that the detection of failure and replacement are instantaneous so that a renewal occurs at the termination of the working state. Let us consider now that the detection and replacement are not instantaneous but takes a non-negligible amount of time. The system now has two states, the working state (hereafter referred to as up state) and failed state (down state). If the working states and the failed states are specified by two sequences of independently and identically distributed random variables then the system is said to be governed by an alternating renewal process.

Consider an alternating renewal process. Denote the up and down states by U and D. Let the duration of the two states be specified by the sequence of i.i.d random variables $X_n$with distribution function $F_U(.)$ and Y_n with distribution $F_D(.)$. Denote by $N_i(t)$, $i = U, D$, the number of renewals of state i in $[0, t]$. Then $\{N_D(t), t \geq 0\}$ is an ordinary renewal process generated by the sequence of random variable $\{X_i + Y_i\}$ having distribution $H = F_U * F_D$ (i.e. H is given by convolution of F_U and F_D) and $\{N_U(t), t \geq 0\}$ is a modified renewal process with initial distribution F_U (i.e. initial inter arrival time X_1) and subsequent distribution $H = F_U * F_D$ (i.e. subsequent inter arrival times $X_{i+1} + Y_i, i = 1,2, ...$).

Denote the renewal function

$$M_i(t) = E\{N_i(t)\}, \qquad i = U, D$$

Let f_U and f_D be the p.d.f of X_i and Y_i respectively and let $M_i^*(s)$ and $f_i^*(s)$ be the Laplace transforms of the renewal functions and probability density functions.

The renewal functions $M_D(t)$ and $M_U(t)$ satisfying the renewal equations

$$M_D(t) = F_{U+D}(t) + \int_0^t M_D(t-x)dF_{U+D}(x) \qquad (1.4)$$

$$M_U(t) = F_U(t) + \int_0^t M_U(t-x)dF_{U+D}(x) \qquad (1.5)$$

The corresponding Laplace transforms of the above equations can be seen to be

$$M_D^*(s) = \frac{f_U^*(s).f_D^*(s)}{s[1-f_U^*(s).f_D^*(s)]} \qquad (1.6)$$

$$M_U^*(s) = \frac{f_U^*(s)}{s[1-f_U^*(s).f_D^*(s)]} \qquad (1.7)$$

Results*:*

For the system described by an alternating renewal process (starting with state U at $t = 0$), the probability that the system will be in states U and D respectively at time t are given by

$$P_U(t) = M_D(t) - M_U(t) + 1$$

$$P_D(t) = M_U(t) - M_D(t)$$

The probability $P_U(t)$ is generally known in the literature as the availability function and denoted by $A(t)$. It is interesting to note that in the limiting case as $t \to \infty$ the limiting availability is given by

$$A = \lim_{t\to\infty} A(t) = \frac{E(X)}{E(X)+E(Y)} \tag{1.8}$$

Chapter 2

THE MATHEMATICAL MODEL

For our model a discrete material flow production process which consists of a single station with station break downs is considered. We assume that the station is neither starved nor blocked. The station works for a random amount of time (up time) before it fails. The station is sent for repair which takes a random amount of time (down time) when it becomes operational again. The sum of the up and down time will be referred to as a cycle. The up and down times in the nth cycle are denoted by random variables X_n and Y_n. The sequence of random variables $\{X_n; n \geq 1\}$ and $\{Y_n; n \geq 1\}$ are assumed to be independent. The single station then can be described by an alternating renewal process. In a cycle, when the production line operates, it is assumed that an item is produced. Thus the number of items produced in an arbitrary time interval$[0, \mathrm{t})$ equals the number of times the up state has been visited by the process in the said interval. We wish to observe that the production of one unit in an up state is only a convenient assumption for the sake of clarity. However one can assume that a fixed number of items are produced during an up time in a cycle or assign a production rate during the up time with very minor changes to the model. One can even assume a reward for each of the visits to the up state as done in Tan (1999).We also wish to observe that Tan constructed a Markov reward model where the single station material flow production line has been modeled as a discrete time Markov chain. The present model is clearly a generalization of his model to continuous time processes. Also the restrictive assumptions of exponential up and down times are relaxed to accommodate general distributions.

With the above model assumptions, the single station can be represented by an alternating renewal process with two states {U, D}. We assume that the station has become just operational so that it is in state U initially. This is not a restrictive assumption as the model could be easily be worked out starting with the down state as well. Thus $M_U(t)$ and $M_D(t)$ respectively denote the expected number of times the up and down states have been visited in $[0, t)$. With the assumption of the production of one unit in each of the states, $M_D(t)$ also gives the number of units produced in the same interval. As mentioned in chapter 1, $M_U(t)$ is given by the renewal equation (1.4) and its Laplace transform by (1.6). There is no explicit solution for this renewal equation excepting in the case of alternating renewal processes driven by exponential up and down times. While there have been approximations available for the renewal function $M_D(t)$, we are not aware of any approximation to the function $M_U(t)$ perhaps because of its structure. We present below a theorem which gives an efficient approximation procedure to compute the renewal function $M_U(t)$.

Theorem**:**

Assume that the first three raw moments of the random variables U and U+D exist and are known. Then the following results hold for renewal function $M_U(t)$.

$$M_U(t) = A.t - \frac{B.\left(1-e^{s_0'.t}\right)}{s_0} \tag{2.1}$$

where

$$A = \frac{1}{\mu'_{1(U+D)}} \tag{2.2}$$

$$B = -S_0{}' \frac{\mu'_{2(U+D)} - 2\mu'_{1(U)}\mu'_{1(U+D)}}{2\mu'^{2}_{1(U+D)}} \tag{2.3}$$

and

$$S_0{}' = -\frac{6\left(\mu'_{2(U+D)} - 2\mu'_{1(U)}\mu'_{1(U+D)}\right)\mu'_{1(U+D)}}{2\mu'_{3(U+D)}\mu'_{1(U+D)} - 3\mu'^{2}_{2(U+D)} + 6\mu'_{2(U+D)}\mu'_{1(U+D)}\mu'_{1(U)} - 6\mu'_{2(U)}\mu'_{1(U+D)}} \tag{2.4}$$

Proof:

We know that the Laplace transform of the renewal density $m_U(t)$ is given by

$$m_U^*(s) = \frac{f_U^*(s)}{1 - f_U^*(s).f_D^*(s)} \tag{2.5}$$

We note that there is a singularity at the origin for the function $m_U^*(s)$. Thus the function

$m_U^*(s)$ is approximated with the help of ration function as below.

$$m_U^*(s) = \frac{A}{s} + \frac{B}{s - s_0'} \tag{2.6}$$

Inverting the above equation result in (2.1). Now the constants A, B and s_0' are obtained as follows:

We express $f^*(s)$, the Laplace transform of the pdf as a power series as below.

$$f^*(s) = \sum_{n=0}^{\infty} \frac{(-1)^n . s^n}{n!} \mu'_n = 1 - \frac{s^1 \mu'_1}{1!} + \frac{s^2 \mu'_2}{2!} - \frac{s^3 \mu'_3}{3!} + \cdots$$

Using (2.5) and (2.6) we obtain

$$\frac{A}{s} + \frac{B}{s - s_0{}'} = \frac{1 - \frac{S^1 . \mu'_{1(U)}}{1!} + \frac{S^2 . \mu'_{2(U)}}{2!} - \frac{S^3 . \mu'_{3(U)}}{3!} + \cdots}{1 - \left[1 - \frac{S^1 . \mu'_{1(U+D)}}{1!} + \frac{S^2 . \mu'_{2(U+D)}}{2!} - \frac{S^3 . \mu'_{3(U+D)}}{3!} + \cdots\right]} \tag{2.7}$$

Comparing the coefficients of S, S^2, S^3 on both the sides of (2.7) and after some algebra we obtain the constants A, B and S'_0 as given in (2.2), (2.3) and (2.4) respectively. Thus finally we obtain

$$M_U(t) = \frac{t}{\mu'_{1(U+D)}} + \frac{\left(\mu'_{2(U+D)} - 2\mu'_{1(U)}\mu'_{1(U+D)}\right).\left(1 - e^{S_0{}'.t}\right)}{2\mu'^2_{1(U+D)}} \tag{2.8}$$

where

$$S_0{}' = -\frac{6\left(\mu'_{2(U+D)} - 2\mu'_{1(U)}\mu'_{1(U+D)}\right)\mu'_{1(U+D)}}{2\mu'_{3(U+D)}\mu'_{1(U+D)} - 3\mu'^2_{2(U+D)} + 6\mu'_{2(U+D)}\mu'_{1(U+D)}\mu'_{1(U)} - 6\mu'_{2(U)\mu'_{1(U+D)}}}$$

Using a similar analysis as in the previous theorem we can obtain the renewal function $M_D(t)$ as

$$M_D(t) = \frac{t}{\mu'_{1(U+D)}} + \frac{\left(\mu'_{2(U+D)} - 2\mu'^2_{1(U+D)}\right).\left(1 - e^{s_0.t}\right)}{2\mu'^2_{1(U+D)}} \tag{2.9}$$

where

$$S_0 = -\frac{6(\mu'_{2(U+D)} - 2\mu'^2_{1(U+D)})\mu'_{1(U+D)}}{2\mu'_{3(U+D)}\mu'_{1(U+D)} - 3\mu'^2_{2(U+D)}}$$

It should be noted that for the approximations (2.8) and (2.9) to be valid the constants S_0 and $S_0{}'$ must be less than zero. The restriction that S_0 and $S_0{}'$ are less than zero is not very restrictive because we have seen that this condition is satisfied by gamma, mixture of exponential, lognormal, phase type distributions and Weibull which are commonly used in production and reliability analysis. The condition is also met for distribution like Truncated Normal and Inverse Gaussian but under certain conditions.

The availability function $A(t)$ which gives the probability that the system is in up state at an arbitrary time t is given by

$$A(t) = M_D(t) - M_U(t) + 1 \tag{2.10}$$

Using (2.8) and (2.9) in the above equation and after some algebra we obtain

$$A(t) = -\frac{\mu'_{2(U+D)}(e^{S_0.t} - e^{S_0{}'.t}) - 2\mu'^2_{1(U+D)}e^{S_0.t} - 2\mu'_{1(U)}\mu'_{1(U+D)}\left(1 - e^{S_0{}'.t}\right)}{2\mu'^2_{1(U+D)}} \tag{2.11}$$

Finally, our interests lie not only on the first order characteristics of the number distributions but on the second order as well. In order to calculate the variance of $N_D(t)$ which is needed in our analysis, we use the following relation which is given in any standard text book on stochastic processes.

$$var\{N_D(t)\} = M_D(t) + 2\int_0^t M(t-x)dM(x) - [M(t)]^2 \tag{2.12}$$

Use of the above equation with $M_D(t)$ as specified in (2.9) yields

$$var[N_D(t)] = \frac{\sigma^2}{\left[\mu'_{1(U+D)}\right]^3} t + \left\{\frac{2\sigma^2}{\left[\mu'_{1(U+D)}\right]^2} + \frac{3}{4} + \frac{5\sigma^4}{4\left[\mu'_{1(U+D)}\right]^4} - \frac{2\left[\mu'_{3(U+D)}\right]}{3\left[\mu'_{1(U+D)}\right]^3}\right\} -$$

$$\left\{\frac{5\sigma^2}{2\left[\mu'_{1(U+D)}\right]^2} + \frac{1}{2} + \frac{\sigma^4}{\left[\mu'_{1(U+D)}\right]^4} - \frac{2\mu'_{3(U+D)}}{3\left[\mu'_{1(U+D)}\right]^3}\right\} e^{s_0 t} + 2tv\left(\frac{1}{\mu} + vs\right) e^{s_0 t} -$$

$$v^2 e^{2s_0 t} \tag{2.13}$$

where $v = \frac{\mu'_{2(U+D)} - 2\mu'^2_{1(U+D)}}{2\mu'^2_{1(U+D)}}$ and $\sigma^2 = \mu'_{2(U+D)} - \mu'^2_{1(U+D)}$

2.1 Special Cases

Having obtained the approximations for the first and second order of characteristics of the production process, in the following sub sections we proceed to obtain these characteristics for certain special cases.

2.1.1 Case 1

The first case we consider assumes exponential up times and deterministic down times. Such cases arise when the type of failure and the repair times are known in advance. We assume

$$f_U(x) = \lambda e^{-\lambda x}$$

$$f_D\,(x) = constant = c$$

so that

$\mu'_{1(U)} = \frac{1}{\lambda}$ $\qquad \mu'_{2(U)} = \frac{2}{\lambda^2}$ $\qquad \mu'_{3(U)} = \frac{6}{\lambda^3}$

$\mu'_{1(D)} = c$ $\qquad \mu'_{2(D)} = c^2$ $\qquad \mu'_{3(D)} = c^3$

$$\mu'_{1(U+D)} = \mu'_{1(U)} + \mu'_{1(D)} = \frac{1}{\lambda} + c$$

$$\mu'_{2(U+D)} = \mu'_{2(U)} + \mu'_{2(D)} + 2\mu'_{1(U)}\mu'_{1(D)} = \frac{2}{\lambda^2} + c^2 + \frac{2c}{\lambda}$$

$$\mu'_{3(U+D)} = \mu'_{3(U)} + \mu'_{3(D)} + 3\mu'_{1(U)}\mu'_{2(D)} + 3\mu'_{2(U)}\mu'_{1(D)} = \frac{6}{\lambda^3} + \frac{6c}{\lambda^2} + \frac{3c^2}{\lambda} + c^3$$

The approximations give

$$E[N_D(t)] = M_D(t) = \frac{t}{\frac{1}{\lambda} + c} - \frac{c\lambda(2 + c\lambda)(1 - e^{S_0 t})}{2(1 + c\lambda)^2}$$

$$\begin{aligned} var\{[N_D(t)]\} = {} & \frac{\lambda t}{(1 + c\lambda)^3} + \frac{2}{(1 + c\lambda)^2} + \frac{3}{4} + \frac{5}{4(1 + c\lambda)^4} \\ & - \frac{2}{3}\left(\frac{6 + 6c\lambda + 3c^2\lambda^2 + c^3\lambda^3}{(1 + c\lambda)^3}\right) - \frac{5e^{S_0 t}}{2(1 + c\lambda)^2} - \frac{e^{S_0 t}}{2} - \frac{e^{S_0 t}}{(1 + c\lambda)^4} \\ & + \frac{2(6 + 6c\lambda + 3c^2\lambda^2 + c^3\lambda^3)e^{S_0 t}}{3(1 + c\lambda)^3} \\ & + \frac{tc(c\lambda + 2)(-2 - 2c\lambda + c^2 S_0\lambda + 2cS_0)\lambda^2 e^{S_0 t}}{2(1 + c\lambda)^4} \\ & - \frac{c^2\lambda^2(c\lambda + 2)^2 e^{2S_0 t}}{4(1 + c\lambda)^4} \end{aligned}$$

$$S_0 = \frac{-6(1 + c\lambda)(c\lambda + 2)}{c(6 + c^2\lambda^2 + 4c\lambda)}$$

$$M_U(t) = \frac{\lambda[2t + 2ct\lambda + c^2\lambda - c^2\lambda e^{S'_0 t}]}{2(1 + c\lambda)^2}$$

where $S'_0 = \frac{6(1+c\lambda)}{c(c\lambda-2)}$; $c < \frac{2}{\lambda}$

2.1.2 Case 2

We assume in this case, the up times and down times to be exponentially distributed so that the single station production line can be described by a two state Markov process. We choose

$$f_U(x) = \lambda e^{-\mu x}$$

$$f_D(x) = \mu e^{-\lambda x}$$

so that

$\mu'_{1(U)} = \frac{1}{\lambda}$ $\mu'_{2(U)} = \frac{2}{\lambda}$ $\mu'_{3(U)} = \frac{6}{\lambda^3}$

$\mu'_{1(D)} = \frac{1}{\mu}$ $\mu'_{2(D)} = \frac{2}{\mu^2}$ $\mu'_{3(D)} = \frac{6}{\mu^3}$

$$\mu'_{1(U+D)} = \frac{\mu + \lambda}{\lambda\mu}$$

$$\mu'_{2(U+D)} = \frac{2(\mu^2 + \lambda^2 + \lambda\mu)}{\mu^2\lambda^2}$$

$$\mu'_{3(U+D)} = \frac{6(\mu^3 + \lambda^2\mu + \lambda\mu^2 + \lambda^3)}{\mu^3\lambda^3}$$

The proposed approximations yields

$$M_D(t) = \frac{\mu\lambda[t\mu + t\lambda - 1 + e^{S_0 t}]}{(\mu + \lambda)^2}$$

$$M_U(t) = \frac{\mu[t\mu\lambda + t\lambda^2 + \mu - \mu e^{S'_0 t}]}{(\mu + \lambda)^2}$$

$$var\{[N_D(t)]\} = \frac{(\mu^2 + \lambda^2)\lambda\mu t}{(\mu + \lambda)^3} - \frac{2(\mu^2 + \lambda^2)}{(\mu + \lambda)^2} + \frac{3}{4} + \frac{5}{4}\frac{(\mu^2 + \lambda^2)^2}{(\mu^2 + \lambda^2)^4} + \frac{3}{2}\frac{(\mu^2 + \lambda^2)e^{S_0 t}}{(\mu + \lambda)^2} - \frac{(\mu^2 + \lambda^2)^2 e^{S_0 t}}{(\mu + \lambda)^4} - \frac{4t\mu^2\lambda^2 e^{S_0 t}}{(\mu + \lambda)^3} - \frac{\mu^2\lambda^2 e^{2S_0 t}}{(\mu + \lambda)^4} - \frac{e^{S_0 t}}{2}$$

$$A(t) = \frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} e^{S_0 t}$$

where $S_0 = S'_0 = -(\mu + \lambda)$

The availability function $A(t)$ for a two state Markov process is well known in the literature. [See page 242 of Ross (1996)]. It is interesting that our approximation gives the same expression for the availability function.

2.1.3 Case 3

In this case, we consider the up times to be gamma distributed with constant repair times. Thus

$$f_U(x) = \frac{1}{\Gamma(k)} \lambda^k x^{k-1} e^{-\lambda x}$$

$$f_D(x) = constant = c$$

We obtain

$\mu'_{1(U)} = \frac{k}{\lambda}$ $\quad\quad$ $\mu'_{2(U)} = \frac{k(k+1)}{\lambda^2}$ $\quad\quad$ $\mu'_{3(U)} = \frac{k(k+1)(k+2)}{\lambda^3}$

$\mu'_{1(D)} = c$ $\quad\quad$ $\mu'_{2(D)} = c^2$ $\quad\quad$ $\mu'_{3(D)} = c^3$

$$\mu'_{1(U+D)} = \frac{k}{\lambda} + c$$

$$\mu'_{2(U+D)} = \frac{k(k+1)}{\lambda^2} + c^2 + \frac{2ck}{\lambda}$$

$$\mu'_{3(U+D)} = \frac{(k+2)(k+1)}{\lambda^3} + \frac{3kc(k+1)}{\lambda^2} + \frac{3kc^2}{\lambda} + c^3$$

The characteristics of production process are given by

$$M_D(t) = \frac{t\lambda}{c\lambda + k} - \frac{(k^2 - k + c^2\lambda^2 + 2kc\lambda)(1 - e^{S_0 t})}{2(k + c\lambda)^2}$$

$$S_0 = -\frac{6(k + c\lambda)(k^2 - k + c^2\lambda^2 + 2kc\lambda)\lambda}{3k^4 + 4k^3 + 6k^2c^2\lambda^2 + 6k^3c\lambda - 3k^2 + 4k^2c\lambda + c^4\lambda^4 + 4c^3\lambda^3k - 4k - 6kc\lambda - 4c\lambda}$$

$$M_U(t) = \frac{t\lambda}{c\lambda + k} + \frac{(c^2\lambda^2 - k^2 + k)(1 - e^{S_0' t})}{2(k + c\lambda)^2}$$

$$S_0' = \frac{6k(k+2c\lambda-1)(k+c\lambda)\lambda}{-4k^3+2k^2c\lambda+3k^2+6kc\lambda+4k+4c\lambda+6k^2c^2\lambda^2+8kc^3\lambda^3+2c^4\lambda^4-3k^4}$$

$$A(t) = -\frac{(k^2-k+c^2\lambda^2+2kc\lambda)(1-e^{S_0t})+(c^2\lambda^2-k^2+k)(1-e^{S_0't})}{2(k+c\lambda)^2}+1$$

2.1.4 Case 4

This case assumes exponential up times and gamma distributed down times so that

$$f_U(x) = \lambda e^{-\lambda x}$$

$$f_D(x) = \frac{1}{\Gamma(k)}\mu^k x^{k-1}e^{-\mu x}$$

$$\mu'_{1(U)} = \frac{1}{\lambda} \qquad \mu'_{2(U)} = \frac{2}{\lambda^2} \qquad \mu'_{3(U)} = \frac{6}{\lambda^3}$$

$$\mu'_{1(D)} = \frac{k}{\mu} \qquad \mu'_{2(D)} = \frac{k(k+1)}{\mu^2} \qquad \mu'_{3(D)} = \frac{k(k+1)(k+2)}{\mu^3}$$

$$\mu'_{1(U+D)} = \frac{k\lambda+\mu}{\lambda\mu}$$

$$\mu'_{2(U+D)} = \frac{\lambda^2k^2+2\mu^2+k\lambda^2+2k\lambda\mu}{\mu^2\lambda^2}$$

$$\mu'_{3(U+D)} = \frac{2k\lambda^3+6\mu^3+3k^2\lambda^2\mu+3k\lambda^2\mu+6k\lambda\mu^2+k^3\lambda^3+3k^2\lambda^3}{\mu^3\lambda^3}$$

$$M_D(t) = \frac{t\lambda\mu}{\mu+k\lambda} - \frac{k\lambda(k\lambda-\lambda+2\mu)(1-e^{S_0t})}{2(\mu+k\lambda)^2}$$

$$S_0 = -\frac{6(k\lambda+\mu)(k\lambda-\lambda+2\mu)\mu}{6\mu^2k+6\mu^2+k^3\lambda^2+4k^2\mu\lambda-k\lambda^2-4\mu\lambda}$$

$$M_U(t) = \frac{t\lambda\mu}{\mu+k\lambda} + \frac{k\lambda^2(k+1)(1-e^{S_0't})}{2(\mu+k\lambda)^2}$$

$$S_0' = \frac{6\mu(\mu+k\lambda)}{\lambda k^2-2k\mu-k\lambda^2-\lambda k-4\mu}$$

$$A(t) = \frac{-k\lambda(k\lambda-\lambda+2\mu)(1-e^{S_0t})-k\lambda^2(k+1)(1-e^{S_0't})}{2(\mu+k\lambda)^2}+1$$

2.1.5 Case 5

In this section we assume the up and down times to be distributed according to gamma distributions. Such cases arise when the system failure can be identified with a sequence of stages with each stage being exponentially distributed. Further the repairs are carried out in stages with each stage being exponentially distributed. Specifically we assume

$$f_U(x) = \frac{1}{\Gamma(k)} \lambda^k x^{k-1} e^{-\lambda x}$$

$$f_D(x) = \frac{1}{\Gamma(k)} \lambda^k x^{k-1} e^{-\lambda x}$$

we obtain

$\mu'_{1(U)} = \mu'_{1(D)} = \frac{k}{\lambda}$ $\mu'_{2(U)} = \mu'_{2(D)} = \frac{k(k+1)}{\lambda^2}$ $\mu'_{3(U)} = \mu'_{3(D)} = \frac{k(k+1)(k+2)}{\lambda^3}$

$$\mu'_{1(U+D)} = \frac{2k}{\lambda}$$

$$\mu'_{2(U+D)} = \frac{2k(2k+1)}{\lambda^2}$$

$$\mu'_{3(U+D)} = \mu'_{3(U)} + \mu'_{3(D)} + 3\mu'_{1(U)}\mu'_{2(D)} + 3\mu'_{2(U)}\mu'_{1(D)} = \frac{4k(k+1)(2k+1)}{\lambda^3}$$

Using the approximations method, characteristics of the production process are given by

$$M_D(t) = \frac{t\lambda}{2k} - \frac{(2k-1)(1-e^{S_0 t})}{4k}$$

where $s_0 = \frac{-6\lambda}{2k+1}$

$$M_U(t) = \frac{t\lambda}{2k} + \frac{(1-e^{s'_0 t})}{4k}$$

where $S_0' = \frac{-6\lambda}{2k^2+1}$

$$A(t) = \frac{2k + e^{S_0 t}(2k-1) + e^{s_0' t}}{4k}$$

Lukas (2008) has given explicit formulae for the computation of $M_U(t)$ and $M_D(t)$ when the up and down time are gamma distributed by expressing an infinite series in terms of finite sum that involves complex numbers. His formulae can be expressed as

$$M_D(t) = \frac{\lambda t}{k} + \frac{1}{k}\sum_{r=1}^{k-1} \frac{\varepsilon^r}{1-\varepsilon^r}\left(1 - e^{-\lambda t(1-\varepsilon^r)}\right)$$

$$M_U(t) = \frac{\lambda t}{k} + \frac{1}{k}\sum_{r=1}^{k-1} \frac{\varepsilon^{r(m+1)}}{1-\varepsilon^r}\left(1 - e^{-\lambda t(1-\varepsilon^r)}\right)$$

$$A(t) = \frac{1}{k}\sum_{r=1}^{k-1} \frac{\varepsilon^r}{1-\varepsilon^r}\left(1 - e^{-\lambda t(1-\varepsilon^r)}\right)(1-(\varepsilon^r)^m) + 1$$

where $\varepsilon = \exp\left(\frac{2\pi i}{k}\right), \varepsilon^0 = 1, \varepsilon^r = \exp(\frac{2\pi r i}{k})$ and i is imaginary unit satisfying $i^2 = -1$

Again it is interesting to note that our approximation provides the exact results for the above mentioned characteristics coinciding with the formulae given by Lukas.

The special cases given above are only illustrative in nature and can be worked out for any given distribution of up times and down times provided the corresponding S_0 and S_0' are negative.

2.2 Optimization Problems

An operations research manager`s interests lie in maximizing the output. However recent advances in manufacturing systems with management techniques have identified the variability in production as an important tool in the design criterion. These two facts together take care of the dependability in terms of the output and predictability by controlling the variation of production systems. Thus there has been a greater demand on the part of the management to include variability of the output in analytical models. Another important criterion that the management looks into is the availability of the system. Thus in this thesis we study two important optimization problems commonly faced by the operations manager in any production line which are given below:

Problem 1:

$$\max \mathrm{Z} = M_D(t)$$

$$subject\ to:$$

$$var\{N_D(t)\} \leq \alpha$$

$$A(t) \geq \beta$$

Problem 2:

$$\min \mathrm{Z} = var\{N_D(t)\}$$

$$subject\ to:$$

$$M_D(t) \geq \gamma$$

$$A(t) \geq \beta$$

In chapter 3 we present the optimality results for various special cases of up and down time distributions.

2.3 Due Date Performance Measure

One of the main jobs of an operations manager in a production line is in fulfilling the orders on time without recourse to back log or lost sales. Thus a good due date performance measure to know whether the output matches the demand can be defined to be the probability that the customer`s demands are fulfilled on time.

Let Q be the ordered quantity and T_Q the due date of the same order. If the quantity produced in $(0, T_Q)$ exceeds Q, then the production line is able to meet the customer`s order on time. Thus a due date performance measure can be defined as

$$D = \Pr[N(T_Q) \geq Q]$$

To compute this measure one should be know the probability distribution of N_t. However if T_Q is sufficiently large, central limit theorem can be invoked to establish that the random variable $N(t)$ is asymptotically normal. This gives us

$$D = \Pr\left[N\left(T_Q\right) \geq Q\right] = 1 - \frac{1}{\sqrt{2\pi}.\sigma_{N_{T_Q}}} \int_{-\infty}^{Q} e^{-\frac{\left[x - E\left[N_{T_Q}\right]\right]^2}{2\sigma_{N_{T_Q}}^2}} dx = 1 - \Phi(Q)$$

where Φ is cumulative normal probability.

Chapter 3

NUMERICAL RESULTS AND DISCUSSIONS

This chapter presents certain numerical results for the optimization problems and due date performance problem mentioned in sections 2.2 and 2.3 by assuming various distributions for the up and down times. These results will be supplemented by discussions and observations. The numerical results are intended to show the variations in the optimal decision variables owing to the selected distributions for up and down times as well as the variations in the parametric value of the same distribution.

3.1 Case 1: Exponential Up Times and Constant Down Times

In this subsection we continue with some numerical result for the first case which was discussed in section 2.1.1.

Figure 2 plots the renewal function for various failure rates. Firstly we note that the renewal function is a monotonically non-decreasing function. Also for a constant repair time, as the failure rate λ increases so that the mean working time decreases, the number of units produces increases. Although this may look counterintuitive, with the assumption of one unit produced in each working interval, this is to be expected. However if we make the assumption of a repair rate so that the number of units produced varies with the length of the working times such a result cannot be expected.

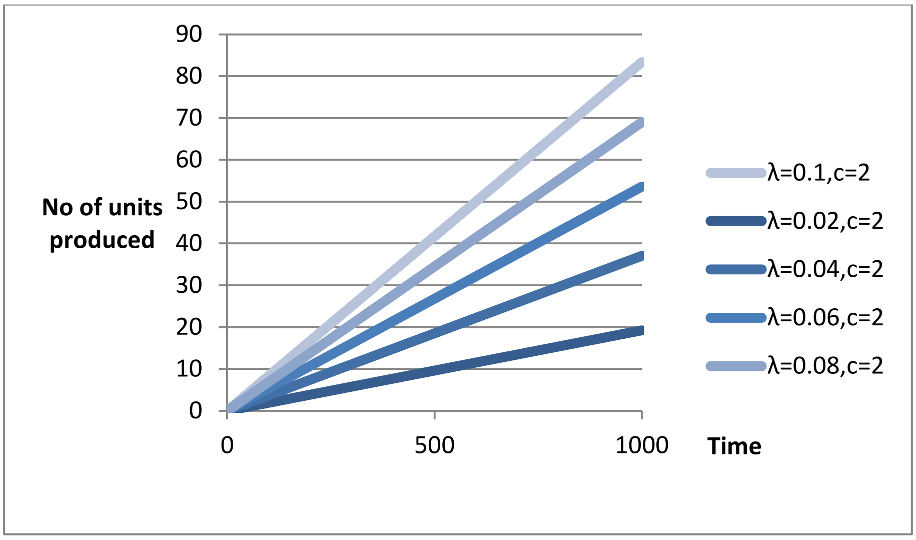

Figure 2: The expected number of units produced in a time interval T (case 1)

In figure 3 we plot the availability function for various values of repair rates λ and a constant repair time $c = 2$. We note that as the mean working time increases the probability of the system being found in a working state increases. We also observe that the availability function reaches the steady state availability A given in (1.8) rather quickly.

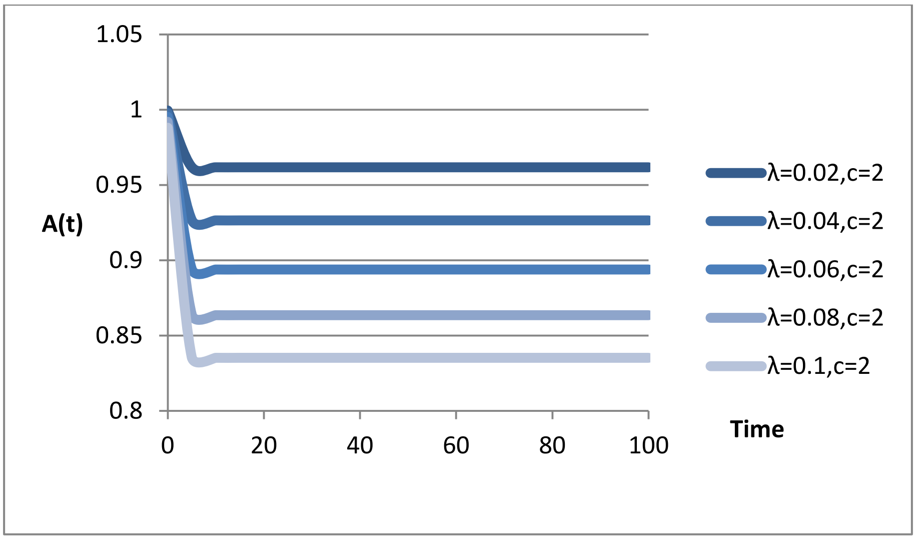

Figure 3: Availability function A(t) (case 1)

In figure 4 we present the due date performance measure which is specified by the probability that a given order size Q is fulfilled within the due date given for certain values of Q. The values of λ and c where fixed to be 0.04 and 2 respectively. It is seen immediately that as the due date t increases for a given order size Q, the probability of fulfilling that order is an increasing function of t and tends to unity as t tends to infinity. Also for a given t such a probability is a decreasing function of Q. The due date curve exhibits more shoulder for smaller values of Q and is steeper for larger values of Q.

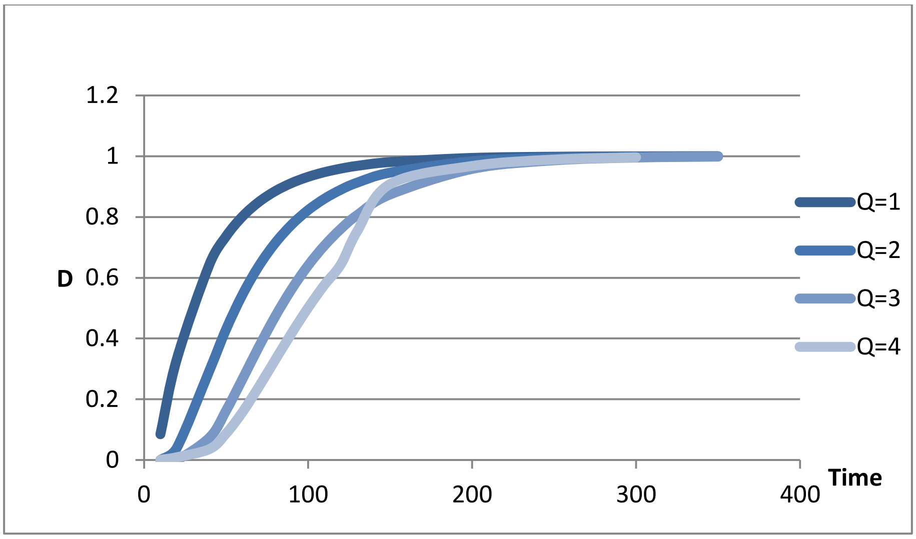

Figure 4: The due date performance measure Pr $[N(T_Q) \geq Q]$ (case 1)

In table 1 below we give the optimal number of units to be produced and the corresponding time for the optimality problem 1 for various values of failure rates λ and repair times c. we observe that the number of units produced is relatively insensitive to the failure rate λ whereas the time needed to produce the same is quite sensitive

Table 1: Optimal Z* for problem 1 and the corresponding t* for different repair times (case 1)

$c = 0.5$			$c = 1$		$c = 2$		$c = 2.5$	
λ	z*	t*	z*	t*	z*	t*	z*	t*
0.01	3.03	305.02	3.06	310.08	3.18	220.46	3.15	192.53
0.02	3.06	155.04	3.12	160.15	3.24	170.61	3.31	152.88
0.03	3.09	105.06	3.18	110.23	3.37	120.93	3.46	123.22
0.04	3.12	80.08	3.24	85.31	3.49	96.25	3.62	101.98
0.05	3.15	65.10	3.31	70.39	3.62	81.58	3.78	87.50
0.06	3.18	55.11	3.37	60.46	3.75	71.92	3.95	78.04
0.07	3.21	47.99	3.43	53.40	3.88	65.11	4.12	71.44
0.08	3.24	42.65	3.49	48.13	4.02	60.10	4.29	66.65
0.09	3.27	38.51	3.56	44.04	4.15	56.29	4.47	63.05
0.10	3.31	35.19	3.62	40.79	4.29	53.32	4.64	60.30

Figures 5 and 6 plot the results of table 2

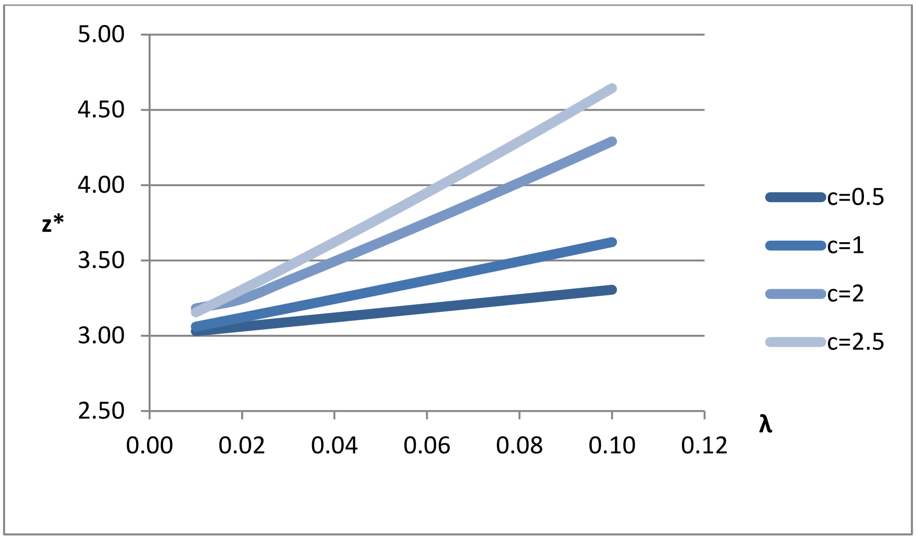

Figure 5: Optimal Z* for various repair times (constant) for problem 1(case 1)

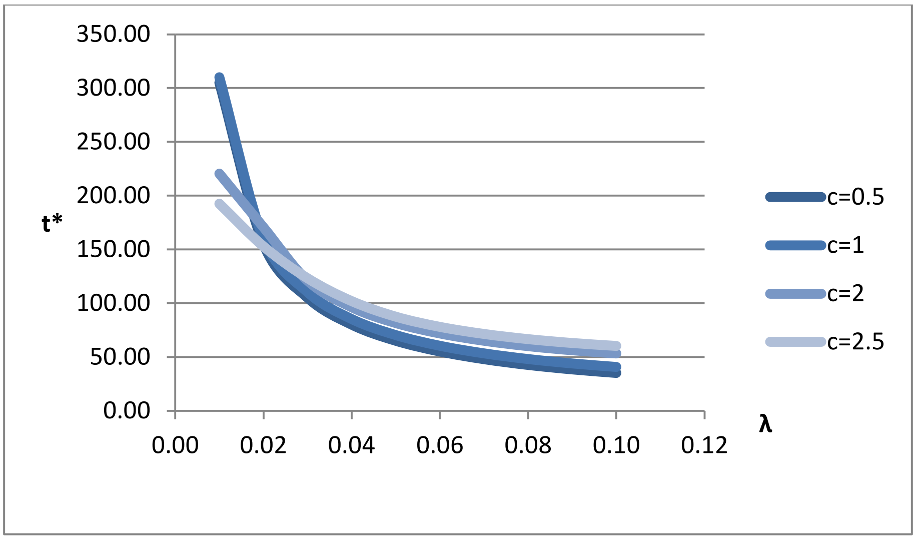

Figure 6: Optimal t* for various repair times (constant) for problem 1 (case 1)

In table 2 below we give the optimal number of units to be produced and the corresponding time for the optimality problem 2 for various values of failure rates λ and repair times c. We wish to remind that problem 2 was of the minimization of the variability in production. From table 2 we observe that in this case, the decision variable Z* which is the variance of the number of units produced and the time needed to produce the same are both sensitive to the failure rate λ.

Table 2: Optimal Z* for problem 2 and the corresponding t* for different repair times (case1)

	c=0.5		c=1		c=2		c=2.5		c=3	
λ	z*	t*	z*	t*	z*	t*	z*	t*	z*	t*
0.01	19.76	1677.17	19.61	2021.00	19.22	2041.98	19.04	2052.47	18.96	2056.66
0.02	19.61	1010.50	19.22	1020.99	18.49	1041.96	18.14	1052.44	18.01	1056.63
0.03	19.41	677.16	18.85	687.65	17.80	708.61	17.31	719.08	17.12	723.27
0.04	19.22	510.50	18.49	520.98	17.15	541.93	16.54	552.39	16.30	556.57
0.05	19.04	410.49	18.14	420.98	16.54	441.91	15.81	452.36	15.54	456.54
0.06	18.85	343.83	17.80	354.31	15.95	375.23	15.14	385.67	14.83	389.85
0.07	18.67	296.21	17.47	306.68	15.40	327.59	14.50	338.03	14.17	342.20
0.08	18.49	260.49	17.15	270.96	14.88	291.86	13.91	302.29	13.55	306.46
0.09	18.32	232.71	16.84	243.18	14.38	264.07	13.35	274.49	12.97	278.66
0.10	18.14	210.49	16.54	220.95	13.91	241.83	12.83	252.25	12.43	256.41

Figures 7 and 8 plot the results of table 2

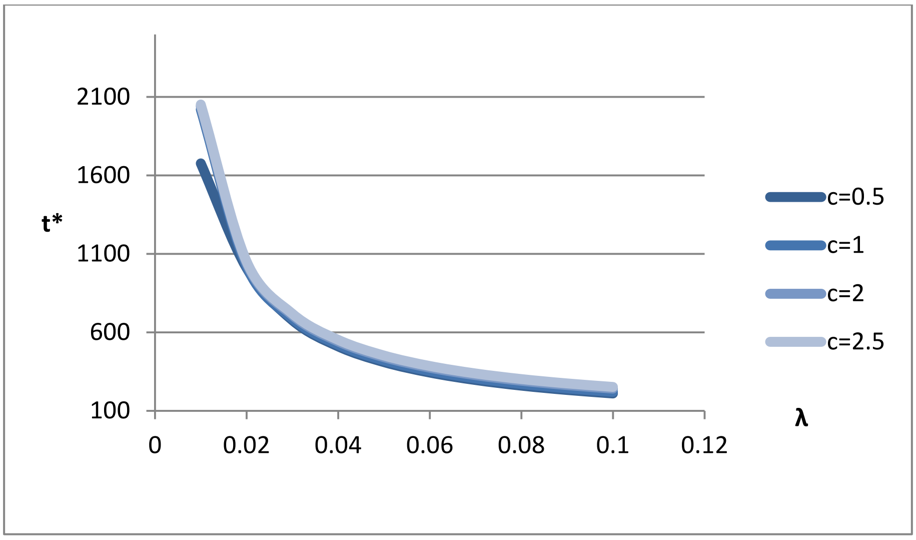

Figure 7: Optimal t* for various repair times (constant) for problem 2 (case 1)

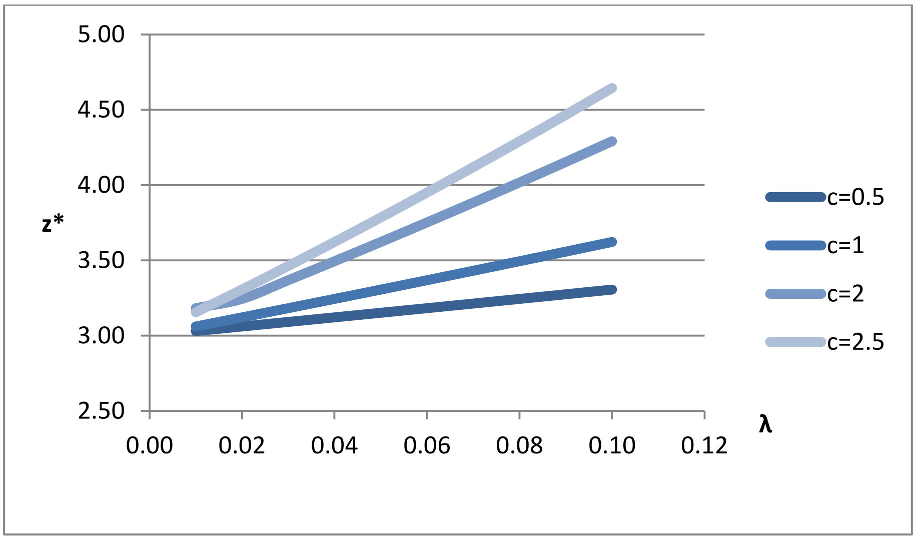

Figure 8: Optimal Z* for various repair times (constant) for problem 2 (case 1)

In the next three subsections, we will make a similar analysis for the three other cases that were introduced in subsections 2.1.2, 2.1.3 and 2.1.4. Since the analysis and conclusions run similar to this section, we confine ourselves only to presenting the tables and figures without discussions.

3.2 Case 2: Exponential Up and Down Times

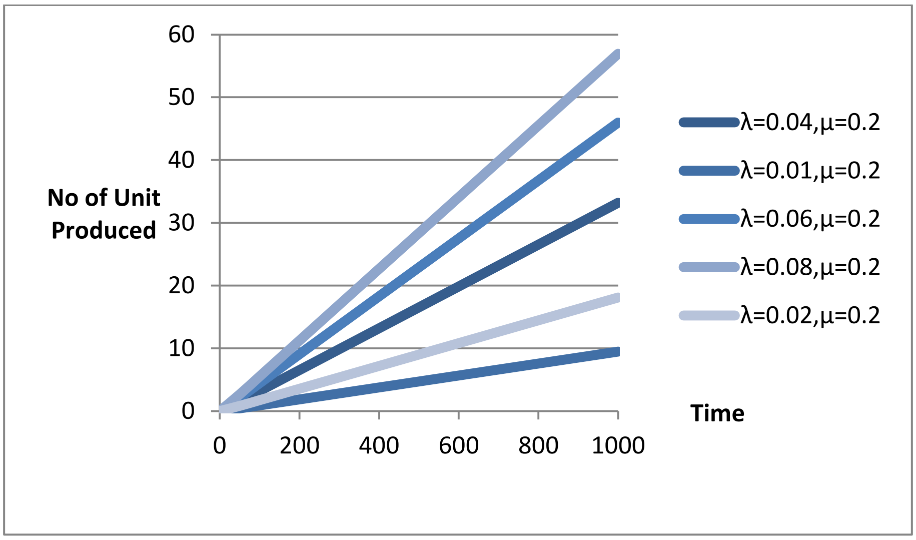

Figure 9: The expected number of units produced in a time interval T (case 2)

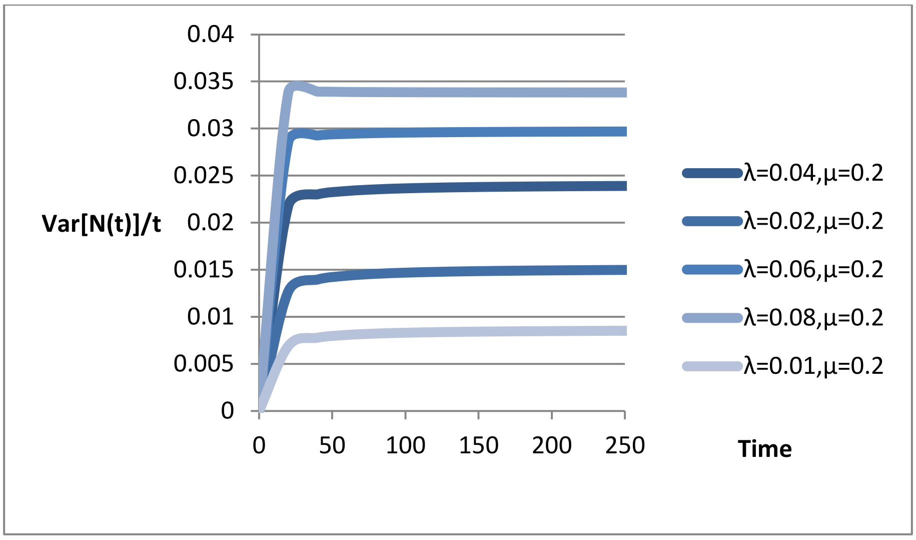

Figure 10: Var [N (t)]/t as a function of time (case 2)

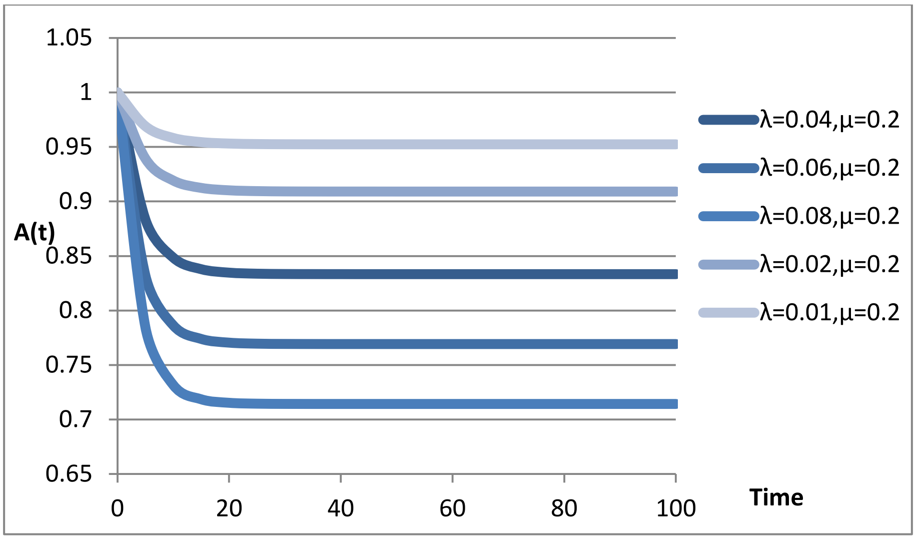

Figure 11: Availability function A(t) (case 2)

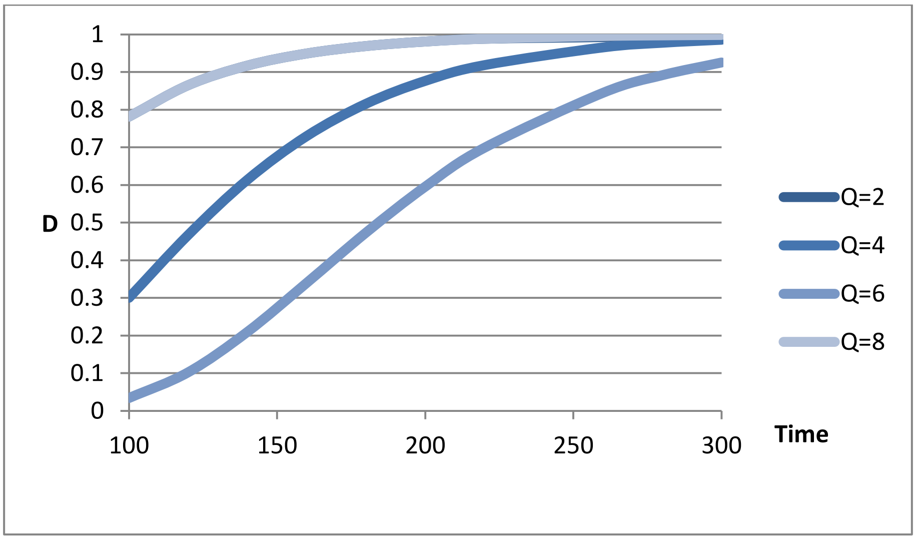

Figure 12: The due date performance measure Pr $[N(T_Q) \geq Q]$ (case 2)

Table 3: Optimal Z* for problem 1 and the corresponding t* for different repair times (case 2)

$\mu = 0.2$			$\mu = 0.6$			$\mu = 1$		
λ	z*	t*	λ	z*	t*	λ	z*	t*
0.010	3.148	325.122	0.010	3.099	316.721	0.010	3.060	310.020
0.020	3.292	175.235	0.020	3.197	166.774	0.020	3.119	160.039
0.040	4.074	126.378	0.040	3.387	91.871	0.040	3.235	85.076
0.060	4.505	101.450	0.060	3.570	66.956	0.060	3.349	60.111
0.080	4.858	88.584	0.080	3.745	54.528	0.080	3.461	47.644
0.090	4.189	59.053	0.090	3.830	50.392	0.090	3.515	43.493
$\mu = 1.4$			$\mu = 1.8$			$\mu = 2.2$		
λ	z*	t*	λ	z*	t*	λ	z*	t*
0.010	3.043	307.153	0.010	3.033	305.562	0.010	3.027	304.550
0.020	3.085	157.163	0.020	3.066	155.568	0.020	3.054	154.554
0.040	3.169	82.182	0.040	3.132	80.580	0.040	3.108	79.562
0.060	3.252	57.201	0.060	3.197	55.591	0.060	3.161	54.570
0.080	3.333	44.719	0.080	3.261	43.102	0.080	3.214	42.077
0.090	3.373	40.561	0.090	3.292	38.941	0.090	3.240	37.914

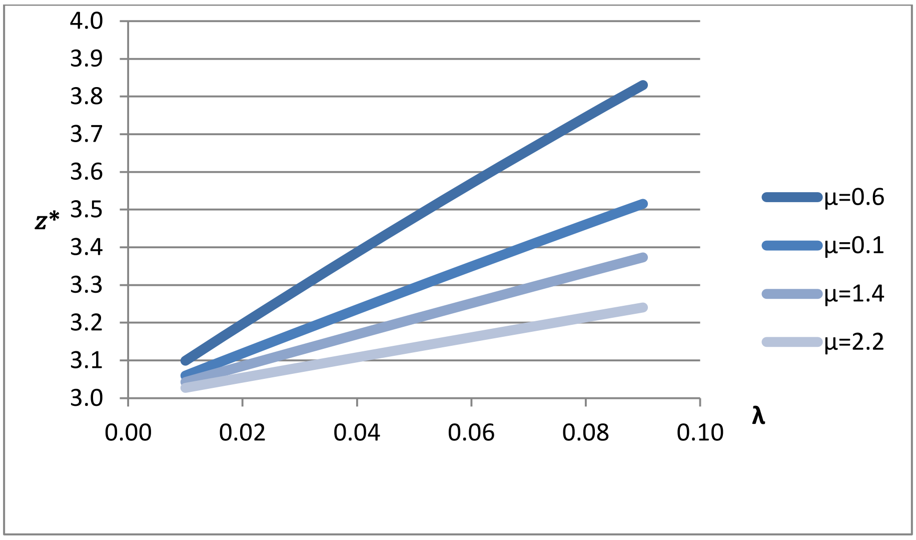

Figure 13: Optimal Z* for various repair times (constant) for problem 1 (case 2)

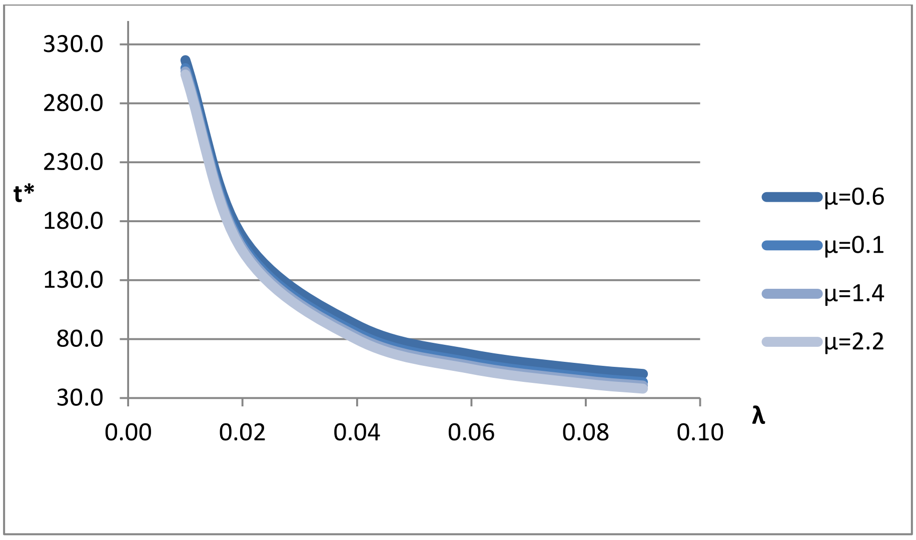

Figure 14: Optimal t* for various repair times (constant) for problem 1 (case 2)

Table 4: Optimal Z* for problem 2 and the corresponding t* for different repair times (case2)

$\mu = 0.2$			$\mu = 0.6$			$\mu = 1$		
λ	z*	t*	λ	z*	t*	λ	z*	t*
0.01	9.099	1054.762	0.01	9.678	1018.306	0.01	9.804	1010.990
0.02	8.368	554.545	0.02	9.379	518.280	0.02	9.617	510.980
0.04	7.280	304.167	0.04	8.838	268.229	0.04	9.264	260.962
0.06	6.544	220.513	0.06	8.368	184.848	0.06	8.941	177.610
0.08	6.043	178.571	0.08	7.956	143.137	0.08	8.642	135.926
0.09	5.933	170.186	0.09	7.770	129.227	0.09	8.502	122.029
$\mu = 1.4$			$\mu = 1.8$			$\mu = 2.2$		
λ	z*	t*	λ	z*	t*	λ	z*	t*
0.01	9.859	1007.852	0.01	9.890	1006.108	0.01	9.910	1004.998
0.02	9.723	507.847	0.02	9.783	506.105	0.02	9.822	504.996
0.04	9.462	257.837	0.04	9.576	256.099	0.04	9.650	254.992
0.06	9.217	174.494	0.06	9.379	172.760	0.06	9.485	171.655
0.08	8.985	132.819	0.08	9.190	131.087	0.08	9.326	129.984
0.09	8.875	118.925	0.09	9.099	117.196	0.09	9.249	116.093

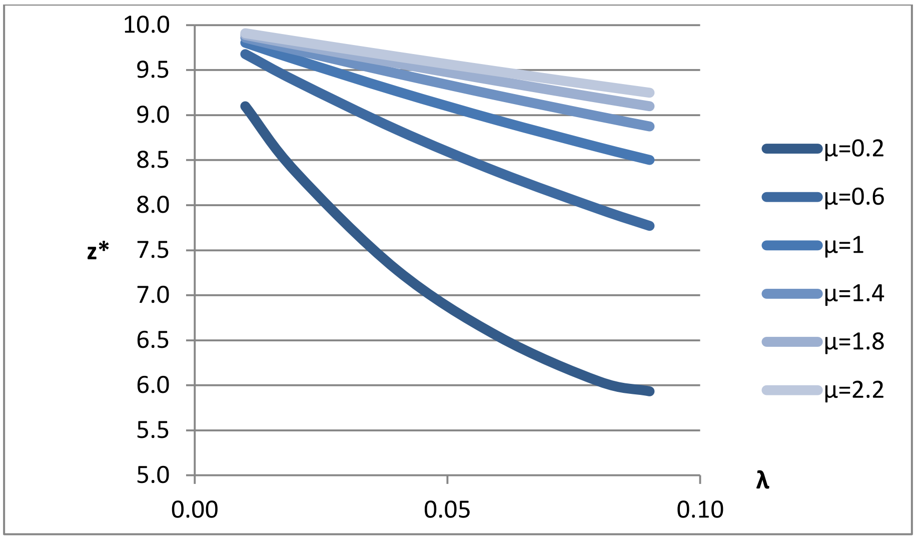

Figure 15: Optimal z* for various repair times (constant) for problem 2 (case 2)

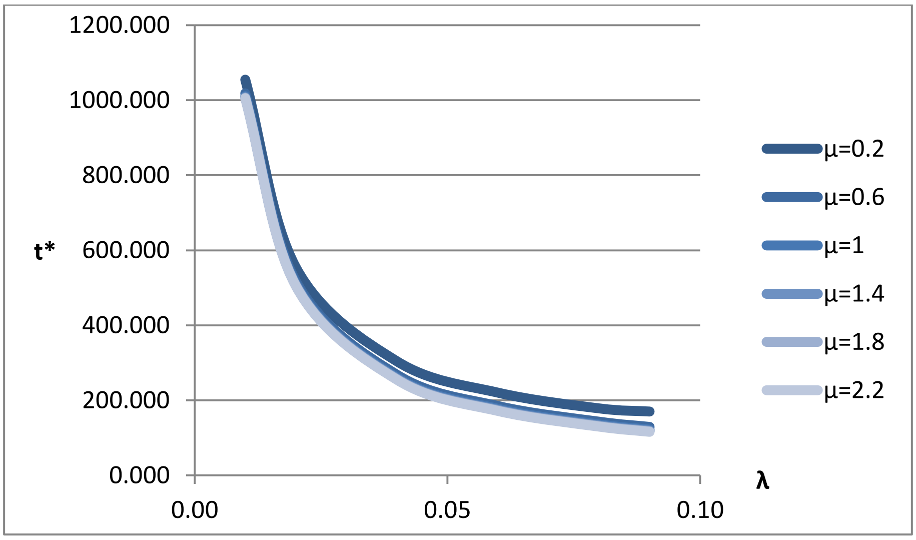

Figure 16: Optimal t* for various repair times (constant) for problem 2 (case 2)

3.3 Case 3: Gamma Up Times and Constant Down Times

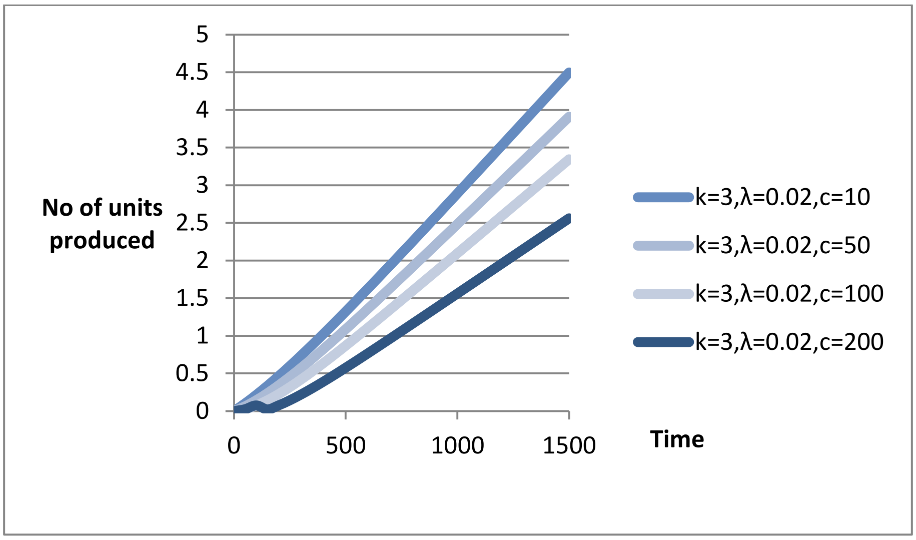

Figure 17: The expected number of units produced in a time interval T (case 3)

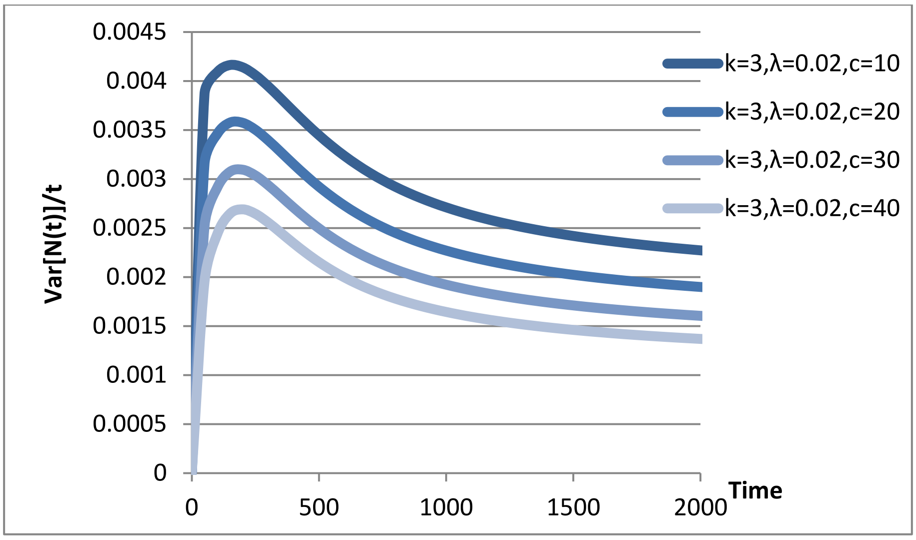

Figure 18: Var [N (t)]/t as a function of t (case 3)

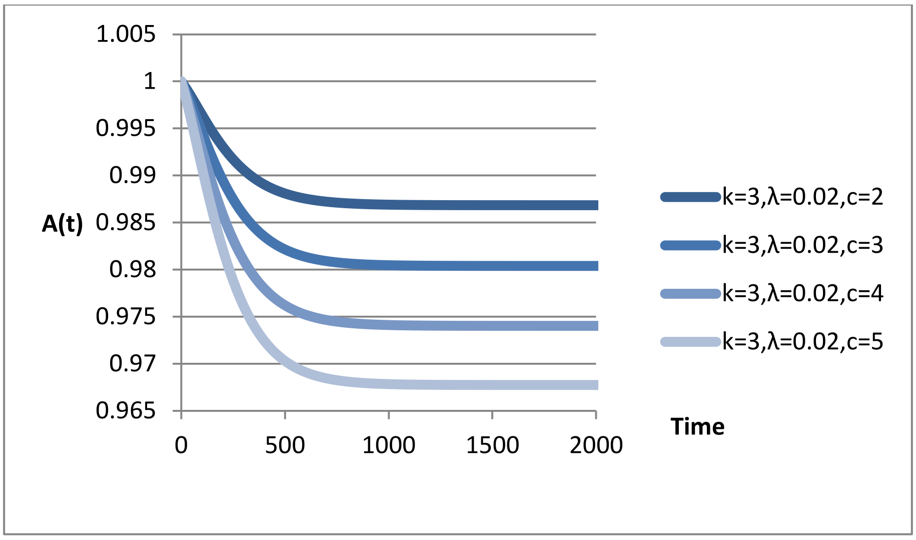

Figure 19: Figure 20: Availability function A(t) (case 3)

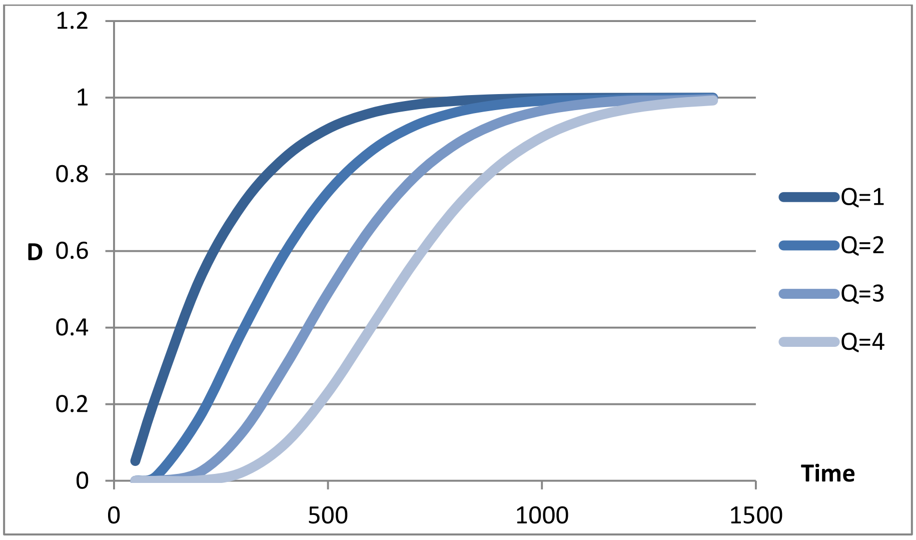

Figure 21: The due date performance measure Pr $[N(T_Q) \geq Q]$ (case 3)

Table 5: Optimal Z* for problem 1 and the corresponding t* for different repair times (case 3)

k=3,λ=0.02			k=3,λ=0.04			k=3,λ=0.06		
c	z	t	c	z	t	c	z	t
1	5.652784	904.1479	1	5.78295	465.135	1	5.915012	318.9787
2	5.78295	930.2701	2	6.04876	6.04876	2	6.320577	346.6502
3	5.915012	956.9361	3	6.320577	519.9753	3	6.736918	375.6924
4	6.04876	984.1202	4	6.597148	548.802	4	7.16119	405.9873
5	6.184005	1011.799	5	6.877559	578.4831	5	7.591853	437.4759
6	6.320577	1039.951	6	7.16119	608.9809	6	8.765	470.988
k=4,λ=0.02			k=4,λ=0.04			k=4,λ=0.06		
c	z	t	c	z	t	c	z	t
1	7.700444	1623.41	1	7.83613	829.5714	1	7.972327	565.0832
2	7.83613	1659.143	2	8.10901	865.8633	2	8.383772	601.9281
3	7.972327	1695.25	3	8.383772	902.8922	3	8.799246	639.8729
4	8.10901	1731.727	4	8.66032	940.6539	4	9.256	680.765
5	8.246163	1768.572	5	8.9635	980.987	5	9.8873	720.87

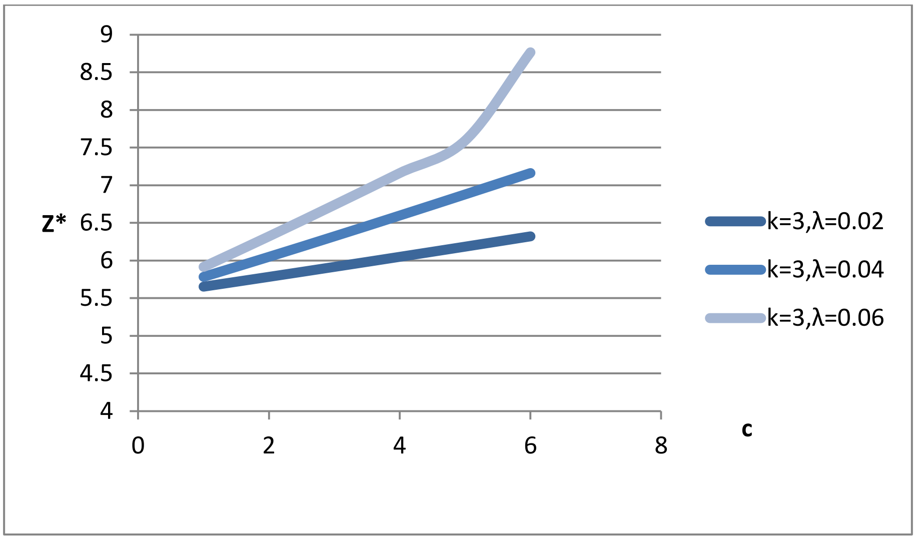

Figure 22: Optimal Z* for various repair times (constant) for problem 1(case 3)

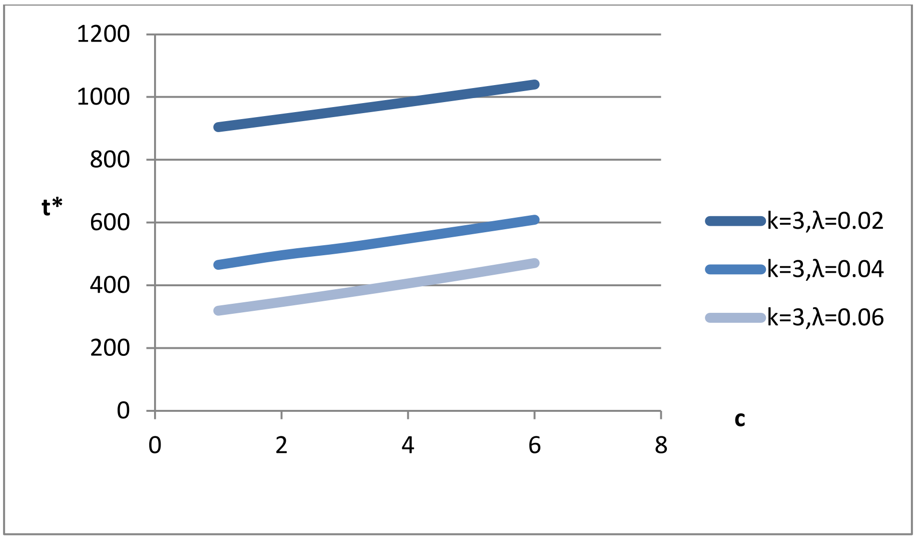

Figure 23: Optimal t* for various repair times (constant) for problem 1 (case 3)

Table 6: Optimal Z* for problem 2 and the corresponding t* for different repair times (case3)

k=3,λ=0.02			k=3,λ=0.04			k=3,λ=0.06		
c	z	t	c	z	t	c	z	t
1	7.730544	3070.666	1	7.62386	1545.664	1	7.519393	1037.33
2	7.62386	3091.329	2	7.41708	1566.325	2	7.218686	1057.987
3	7.519393	3111.99	3	7.19929	1589.046	3	6.935849	1078.638
4	7.41708	3132.649	4	7.028231	1607.633	4	6.669495	1099.284
5	7.316863	3153.306	5	6.845297	1628.281	5	6.418367	1119.924
k=4,λ=0.02			k=4,λ=0.04			k=4,λ=0.06		
c	z	t	c	z	t	c	z	t
1	6.044895	4095.624	1	5.981442	2058.124	1	5.934	1380.467
2	5.981442	4116.248	2	5.857529	2078.745	2	5.737469	1399.576
3	5.918994	4136.869	3	5.737469	2099.364	3	5.56426	1420.192
4	5.857529	4157.49	4	5.621104	2119.981	4	5.398869	1440.805
5	5.797027	4178.11	5	5.508285	2140.595	5	5.240833	1461.415

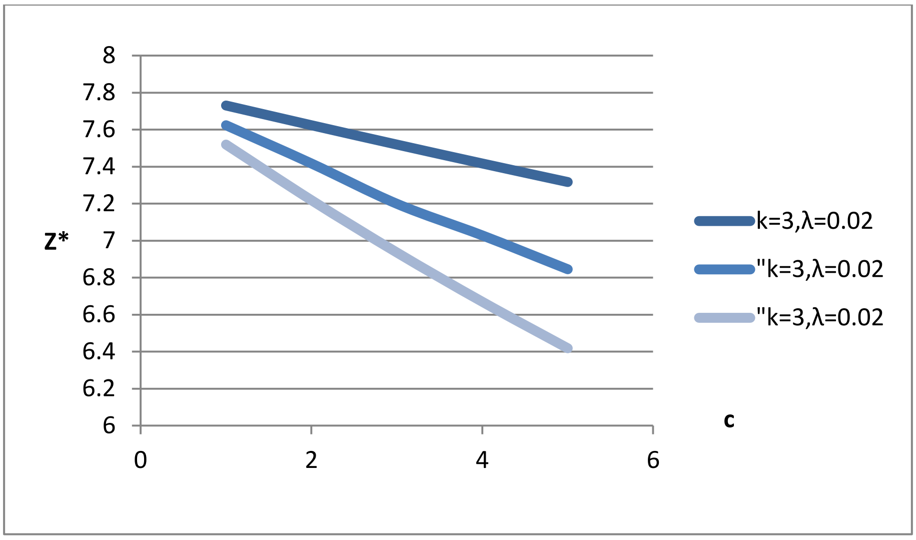

Figure 24: Optimal z* for various repair times (constant) for problem 2 (case 3)

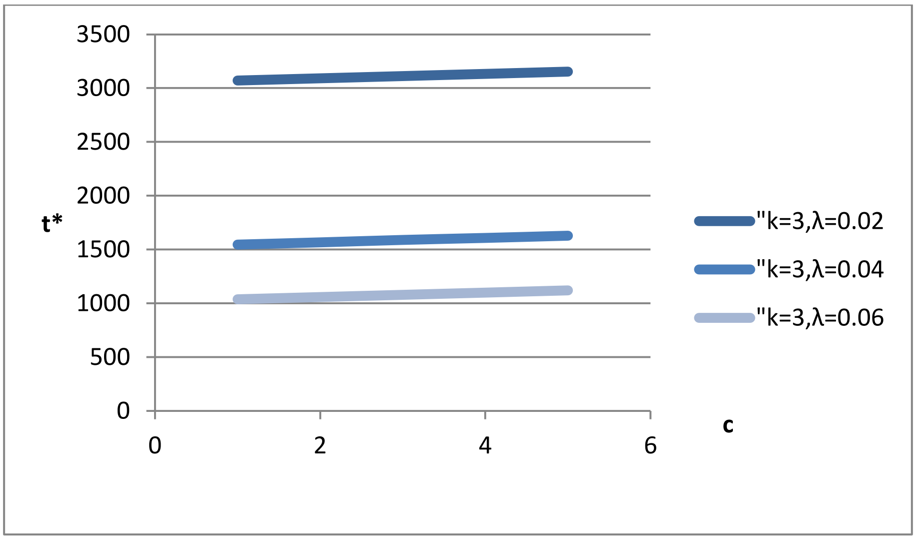

Figure 25: Optimal t* for various repair times (constant) for problem 2 (case 3)

3.4 Case 4: Exponential Up Times and Gamma Distributed Down Times

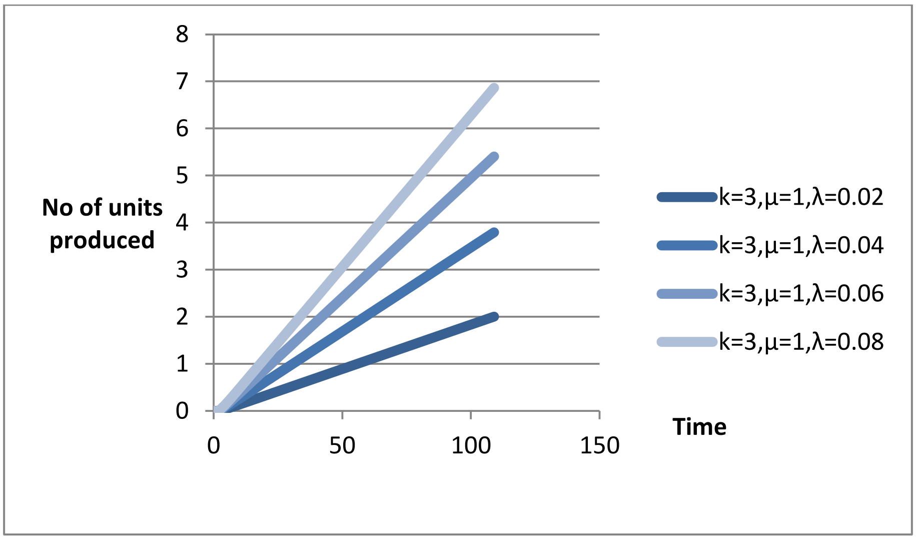

Figure 26: The expected number of units produced in a time interval T (case 4)

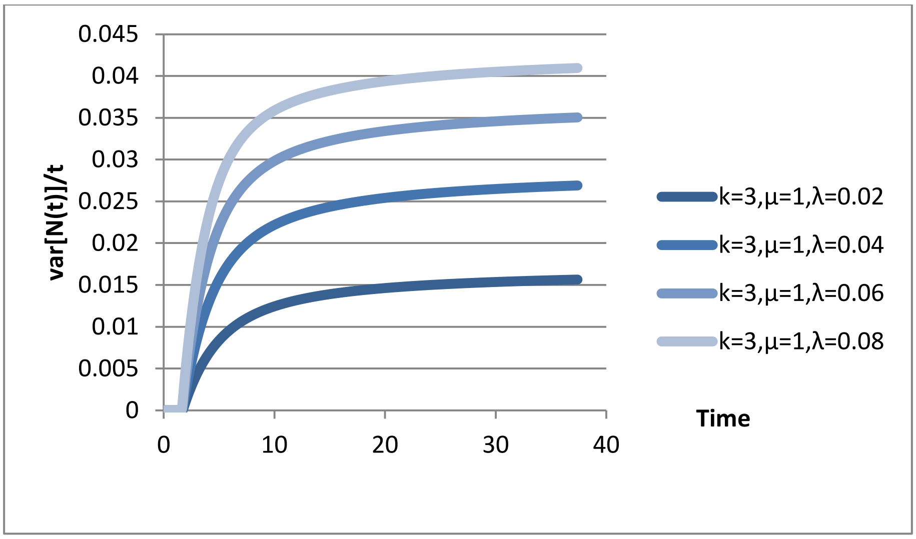

Figure 27: Var [N (t)]/t as a function of t (case 4)

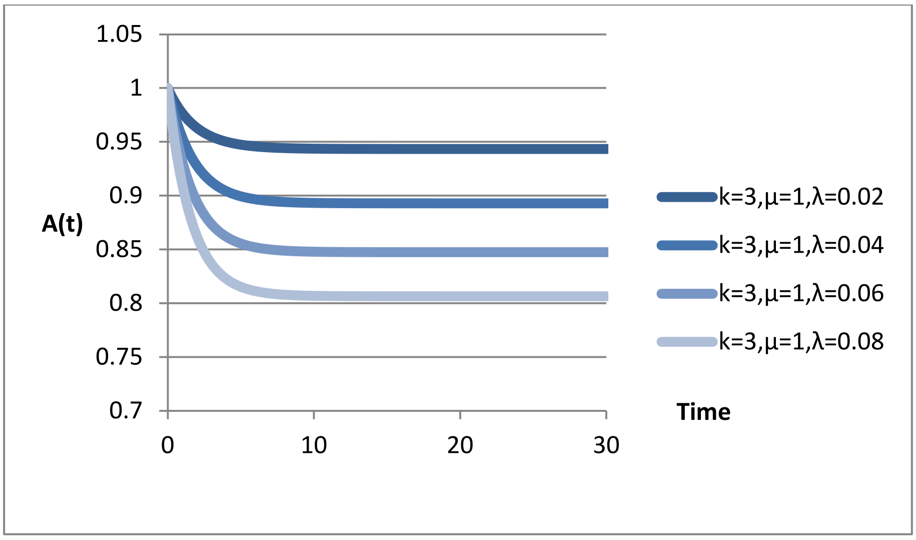

Figure 28: Figure 29: Availability function A(t) (case 4)

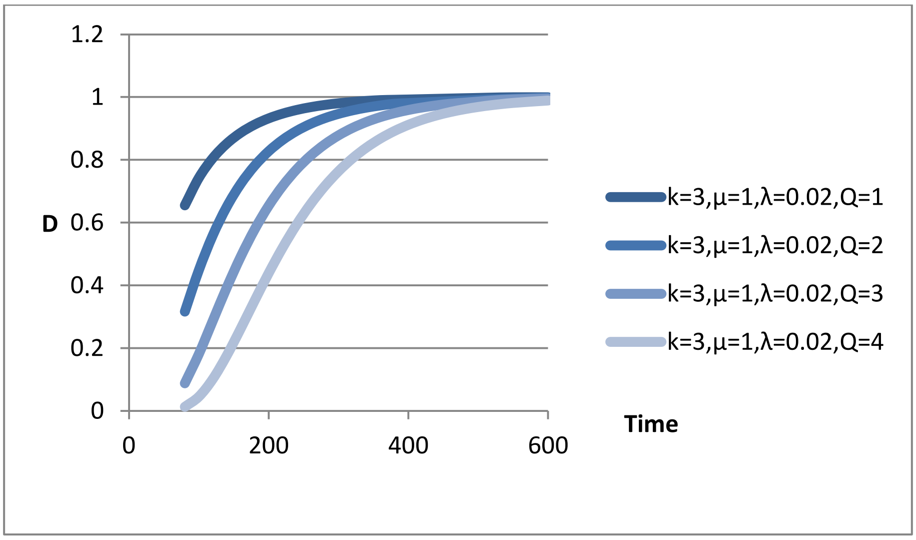

Figure 30: The due date performance measure Pr $[N(T_Q) \geq Q]$ (case 4)

Table 7: Optimal Z* for problem 1 and the corresponding t* for different repair times (case 4)

k=3						k=4					
μ=1			μ=1.5			μ=1			μ=1.5		
λ	z*	t*	λ	z*	t*	λ	z*	t*	λ	z*	t*
0.01	3.06	310.06	0.01	3.12	320.23	0.01	3.21	335.07	0.01	3.16	327.11
0.02	3.36	181.04	0.02	3.24	170.46	0.02	3.42	185.81	0.02	3.32	177.55
0.04	3.73	107.10	0.04	3.48	95.93	0.04	3.84	112.35	0.04	3.65	103.47
0.06	4.09	83.17	0.06	3.73	71.40	0.06	4.27	88.92	0.06	3.98	79.41
0.08	4.46	71.74	0.08	3.97	59.37	0.08	4.71	78.00	0.08	4.32	67.86
0.09	4.64	68.10	0.09	4.09	55.45	0.09	4.93	74.63	0.09	4.49	64.17
μ=2			μ=2.5			μ=2			μ=2.5		
λ	z*	t*	λ	z*	t*	λ	z*	t*	λ	z*	t*
0.01	3.09	315.13	0.01	3.07	312.08	0.01	3.12	320.25	0.01	3.10	316.16
0.02	3.18	165.26	0.02	3.14	162.16	0.02	3.24	170.50	0.02	3.19	166.32
0.04	3.36	90.52	0.04	3.29	87.33	0.04	3.49	96.01	0.04	3.39	91.64
0.06	3.54	65.78	0.06	3.43	62.50	0.06	3.73	71.53	0.06	3.58	66.97
0.08	3.73	53.55	0.08	3.58	50.17	0.08	3.98	59.55	0.08	3.78	54.81
0.09	3.82	49.52	0.09	3.65	46.09	0.09	4.11	55.65	0.09	3.88	50.81
μ=3			μ=3.5			μ=3			μ=3.5		
λ	z*	t*	λ	z*	t*	λ	z*	t*	λ	z*	t*
0.01	3.06	310.06	0.01	3.05	308.61	0.01	3.08	313.44	0.01	3.07	311.51
0.02	3.12	160.11	0.02	3.10	158.66	0.02	3.16	163.55	0.02	3.14	161.59
0.04	3.24	85.23	0.04	3.21	83.74	0.04	3.32	88.78	0.04	3.28	86.75
0.06	3.36	60.35	0.06	3.31	58.82	0.06	3.49	64.00	0.06	3.42	61.92
0.08	3.48	47.96	0.08	3.41	46.41	0.08	3.65	51.74	0.08	3.56	49.59
0.09	3.54	43.86	0.09	3.47	42.29	0.09	3.73	47.68	0.09	3.63	45.51

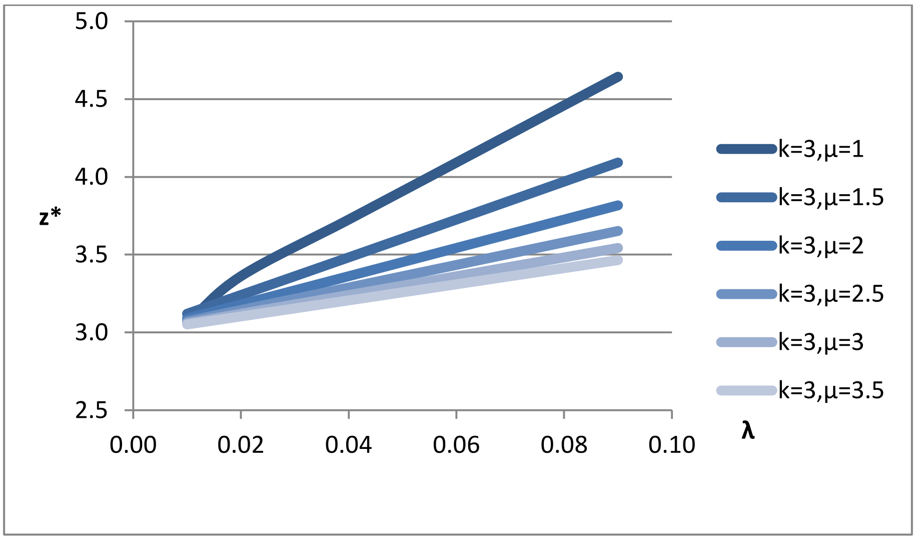

Figure 31: Optimal Z* for various repair times (constant) for problem 1(case 4)

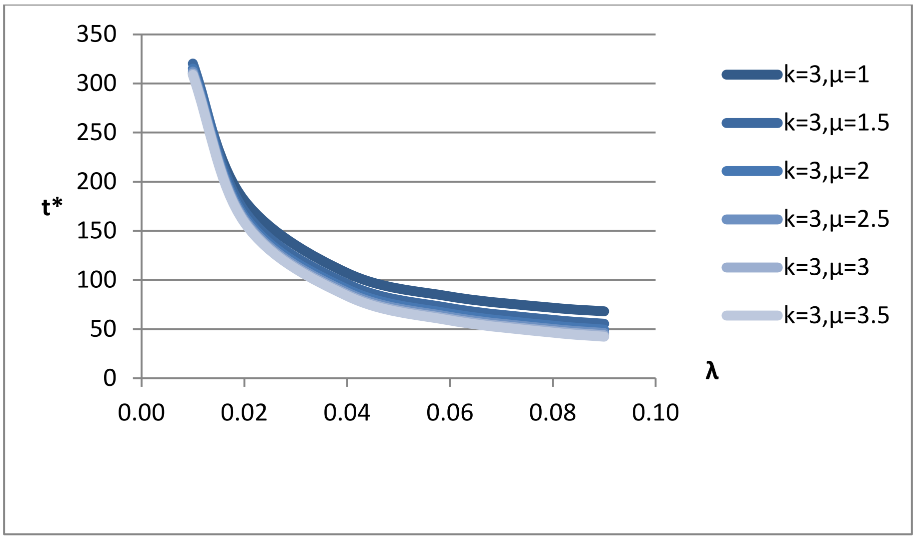

Figure 32: Optimal t* for various repair times (constant) for problem 1(case 4)

Table 8: Optimal Z* for problem 2 and the corresponding t* for different repair times (case 4)

k=3						k=4					
μ=1			μ=1.5			μ=1			μ=1.5		
λ	z*	t*	λ	z*	t*	λ	z*	t*	λ	z*	t*
0.01	47.15	5152.94	0.01	48.07	5101.97	0.01	46.75	5175.04	0.01	47.45	5135.96
0.02	44.56	2652.89	0.02	46.25	2601.95	0.02	43.84	2674.97	0.02	45.10	2635.92
0.04	40.07	1402.79	0.04	42.97	1351.90	0.04	38.85	1424.85	0.04	40.95	1385.84
0.06	36.33	986.03	0.06	40.07	935.19	0.06	34.77	1008.07	0.06	37.42	969.10
0.08	33.19	777.61	0.08	37.50	726.82	0.08	31.40	799.64	0.08	34.38	760.71
0.09	31.80	708.13	0.09	36.33	657.35	0.09	29.93	29.93	0.09	33.03	691.23
μ=2			μ=2.5			μ=2			μ=2.5		
λ	z*	t*	λ	z*	t*	λ	z*	t*	λ	z*	t*
0.01	48.54	5076.49	0.01	48.82	5061.19	0.01	48.06	5101.98	0.01	48.44	5081.58
0.02	47.15	2576.47	0.02	47.69	2561.18	0.02	46.25	2601.95	0.02	46.96	2581.57
0.04	44.56	1326.44	0.04	45.56	1311.16	0.04	42.94	1351.91	0.04	44.22	1331.54
0.06	42.21	909.75	0.06	43.59	894.48	0.06	40.02	935.20	0.06	41.73	914.85
0.08	40.07	701.39	0.08	41.76	686.13	0.08	37.42	726.83	0.08	39.47	706.49
0.09	39.07	631.94	0.09	40.90	616.68	0.09	36.23	657.36	0.09	38.42	637.03
μ=3			μ=3.5			μ=3			μ=3.5		
λ	z*	t*	λ	z*	t*	λ	z*	t*	λ	z*	t*
0.01	49.02	5050.99	0.01	49.16	5043.71	0.01	48.70	5067.99	0.01	48.88	5058.28
0.02	48.07	2550.99	0.02	48.33	2543.70	0.02	47.45	2567.98	0.02	47.80	2558.27
0.04	46.25	1300.97	0.04	46.76	1293.70	0.04	45.10	1317.96	0.04	45.75	1308.25
0.06	44.56	884.30	0.06	45.27	877.02	0.06	42.94	901.27	0.06	43.85	891.57
0.08	42.97	675.95	0.08	43.86	668.68	0.08	40.95	692.92	0.08	42.07	683.23
0.09	42.21	606.50	0.09	43.19	599.23	0.09	40.02	623.47	0.09	41.23	613.77

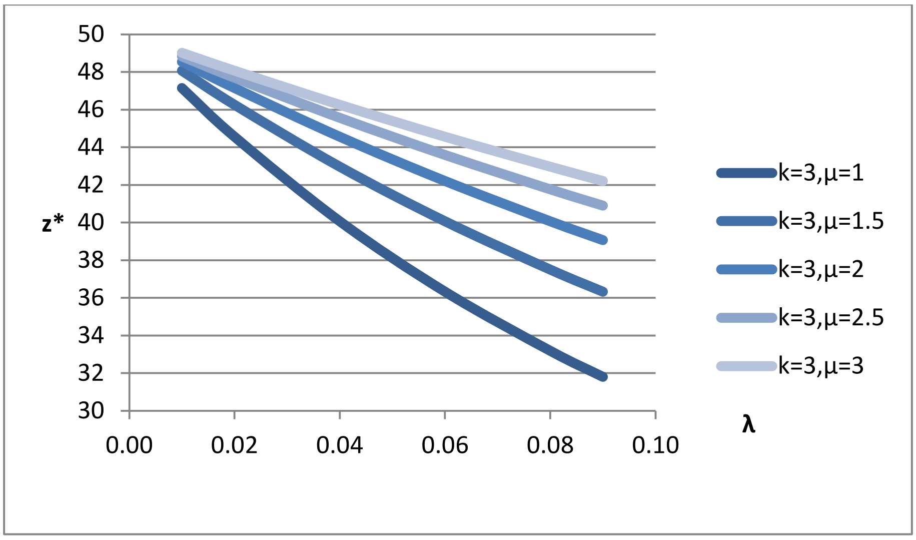

Figure 33: Optimal z* for various repair times (constant) for problem 2 (case 4)

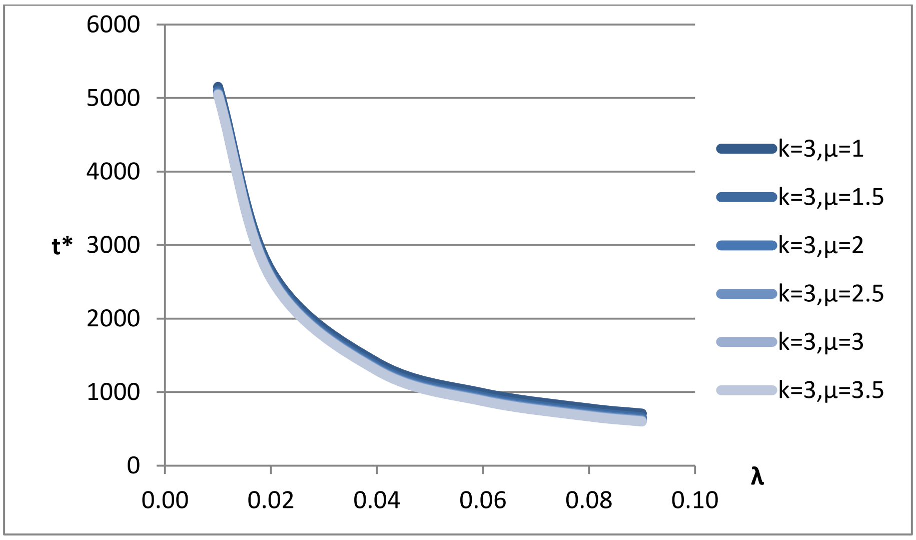

Figure 34: Optimal t* for various repair times (constant) for problem 2 (case 4)

Chapter 4

CONCLUDING REMARKS

This thesis presents a transient analysis of a single machine production line, modeling the system using an alternating renewal process. The existing models have the Markov property of the up and down time built in. Also these models make use of the steady state analysis of the system. Indeed the study of short time production variability has all along been considered to be a difficult problem in the literature for quite some time. (Tan 1999).

Thus we have moved forward using arbitrary probability distributions as well as using transient analysis in our study. The major contribution of our work is in deriving some useful approximations for the renewal functions as well as the availability function. It is not possible to obtain explicit solutions for these functions for arbitrary up and downtime distributions. These approximations are easy to implement and depend only on the first three moments of the underlying distributions. We also analyze some optimization problems in the production line using the average and variability of the throughput as well as availability of the system. Using the average and variability of the throughput we have suggested a due date performance measure for relatively large values of due date. This is because we invoke central limit theorem for N_t, the number of units produced in an arbitrary time t.

We conclude the thesis with some direction for future work. The generalization form the exponential to arbitrary distributions necessitated the use of approximations for the performance measures. However one could approximate the arbitrary distribution functions by phase type distributions. There are extensive literatures available for such approximations. The advantage of such approximations is that the performance measures for phase type distributions are available in explicit form. Secondly we have assumed the up times and down times to be mutually independent. It may be interesting to make a study if these variables are correlated. Such a case arises when there are different types of failures and the repair times depend on the type of the failure. Finally it is natural that this study is carried forward for N-station production lines and later on with buffer in between the stations.

REFRENCES

Altiok, T. (1985). On the phase-type approximations of general distributions. *IIE Transactions, 17*(2), 110-116.

Ayhan, H., Limon-Robles, J., & Wortman, A. M. (1999). An approach for computing tight numerical bounds on renewal functions. *IEEE Transactions on Reliabbility, 48*(2), 182-188.

Cui, L., & Xie, M. (2003). Some normal approximation for renewal function of large Weibull shape parameter. *Communication in Statistics, Simulation and Computation, 32*(1), 1-16.

Daley, J. D. (1976). Another upper bound for the renewal function. *The Annals of Probability*, *4*(1), 109-114.

Dallery, Y., & Frein, Y. (1993). On decomposition methods for tandem queuing networks with blocking. *Operations Research*, 386-399.

Dallery, Y., & Gershwin, S. (1992). Manufacturing flow line systems:A review of models and analytical results. *Queueing Systems Theory and Applications, Special Issue on Queueing Models of Manufacturing Systems, 12*, 3-94.

Deligonul, S. Z. (1985). An approximate solution of the integral equation of renewal theory. *Applied Probabbility*, *22*, 926–931.

He, X.-F., Wu, S., & Li, Q.-L. (2007). Production variability of production lines. *International Journal of Production Economics, 107*(1), 78–87.

Hendricks, K. B. (1992). The output processes of serial production lines of exponential machines with finite buffers. *Operation Research Society of America, 40*(6), 1139-1147.

Kambo, N. s., Rangan, A., & Hadji, E. M. (2012). Moments-Based Approximation to the Renewal. *Communications in Statistics - Theory and Methods, 41*, 851-868.

Li, G., & Luo, G. (2005). Upper and lower bounds for the solutions of Markov renewal equations. *Mathematival Methods and Operations Research, 26,* 243–253.

Lindsay, G. B., Pilla, S. R., & Basak, P. (2000). Moment-based approximations of distributions using mixtures:Theory and applications. *Annals of the Institute of Statistical Mathematics, 52*(2), 215-230.

LUkasz , B. (2008). *Method for Approximating the Availibilty Function, Master thesis, Delft University of Technology*. Delft.

Marshall, T. K. (1973). Linear bounds on the renewal function. *SIMA Journal of Applied Mathematics, 42*(2), 245-250.

Medhi, J. (1994). *Stochastic processes.* New Delhi: Wiley Eastern.

Miltenburg, G. J. (1987). Variance of the number of units produced on a transfer line with buffer inventories during a period of length T. *Naval Research Logistics, 34*(6), 811-822.

Papadopoulos, H., & Heavey, C. (1996). Queueing theory in manufacturing systems analysis and design:. *European Journal of Operational Research, 92*, 1-27.

Papadopoulos,, H. T. (1994). An analytic formula for the mean throughput of K-station production lines with no intermediate buffers. *European Journal of Operational Research, 91*, 481-494.

Politis, K., & Pitts, M. S. (1998). Approximations for solutions of renewal-type equations. *Stochastic Process Application, 78*, 195–216.

Ran, L., Cui, L., & Xie, M. (2006). Some analytical and numerical bounds on the renewal function. *Communications in Statistics- Theory and Methods, 35*, 1815–1827.

Sheldon, R. M. (1996). *Stochastic Processes.* New York: John Wiley.

Smeitink, E., & Dekker, R. (1990). A simple approximation to the renewal function. *IEEE Transactions on Reliability, 39*(1), 71–75.

Tan, B. (1997a). An efficient method for variance of the output in a production line with a finite buffer. *International Workshop on Performance Evaluation and Optimization of Production Lines,* (pp. 135-157). Samos,Greece.

Tan, B. (1997b). Variance of the throughput of an N-station production line with no intermediate buffers and time dependent failures. *1997*, 560-576.

Tan, B. (1998). An analytical formula for variance of output from a series-parallel production system with no interstation buffers and time-dependent failures. *Mathematical and Computer Modelling, 27*(6), 95-112.

Tan, B. (1999). Theory and Methodology, Variance of the output as a function of time,production line dynamics. *European Journal of Operational Research, 101*, 470-484.

Tan, B. (1999). Theory and Methodology,Variance of the output as a function of time, production line dybamics. *European Journal of Operational Research, 117,* 470-484.

Tan, B. (1999). Variance of the output as a function of time: Production line dynamics. *European Journal of Operational Research, 117*(3), 470-484.

Tan, B., & Yeralan, S. (1997). A decomposition model for continuous materials flow production systems. *International Journal of Production Research, 35*(10), 2759-2772.

Tan, B., & Yeralan, S. (1997). Analysis of Multistation Production Systems with Limited Buffer Capacity Part 2: The Decomposition Method. *25*, 109-123.

Whitt, W. (1982). Approximate a point process by a renewal process, I: two basic methods. *Journal of Operation Research, 30*(1), 125-147.

Xie, M. (1989). On the solution of renewal-type integral equations. *Statistical Computation and Simulation, 18*(1), 281–293.

Xie, M., Preuss, W., & Cui, L. (2003). Error analysis of some integration procedures for renewal equation and convolution integrals. *Statistical Computation and Simulation*, *73*(1), 59-70.

Yeralan, S., & Muth, E. J. (1987). A general model of a production line with intermediate buffer and station breakdown. *IIE Transactions*, *19*(2), 130-139.

Yeralan, S., & Tan, B. (1997). Analysis of Multistation Production Systems with Limited Buffer Capacity Part 1: The Subsystem Model. *Mathematical and Computer Modelling* , *25*(7), 109-122.