ABSTRACT

The concept of thin-shell wormholes has been considered in Einstein's theory of gravity coupled with different matter sources like Maxwell, Yang-Mills, Born-Infeld-Hoffman fields, generalized Chaplygin gas and dilaton. Our consideration is in higher dimensions where the bulk spacetime is introduced there. We mainly concentrate on the stability of possible thin shell wormholes and the amount of normal or exotic matter needed to support such kind of wormholes. In addition to the Einstein Gravity (i.e General Relativity) we also consider the Gausse-Bonnet gravity which is an extension of Einstein's gravity. This extended version of general relativity enabled us to construct thin-shell wormholes which are supported by normal matter. Most of our calculations are numeric together with some plots. The results given in this thesis are published during the recent years.

Keywords: Black hole, Wormholes, Thin-shell, Normal matter, Exotic matter, Stability.

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Stability of thin-shell wormholes

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ÖZET

Maxwell, Yang - Mills, Born - Infeld - Hoffman, genellestirilmis Chaplygin gazi ve dilaton alan katkili Einstein yerekim kurami ierisindeki ince - kabuklu uzay solucan delikleri incelenmistir. Bu kuramlarda yksek boyutlar ierisinde esas ilgi alanimiz ince - kabuklu solucan deliklerinin kararliligi, normal ve normal - disi (ekzotik) madde miktarinin varligi olmustur. Einstein ekim kuraminin Gauss - Bonnet ainimi alismamizda zel bir yer teskil etmekte , zira burada normal madde ile solucan delikleri kararli olabilmektedir. Yapilan islemlerin byk ogunlugunu sekillerle desteklenmis sayisal hesaplar olusturuyor. Elde edilen sonular son birka yil ierisinde yayinlanmistir.

Anahtar Kelimeler: kara delik, solucan delikleri, ince - kabuklu, normal madde, ekzotik madde, stability.

Dedicated to my father

and

to my husband Habib Mazhari

CHAPTER 1

INTRODUCTION

Although the idea of wormholes entered into physics literature in 1930s under the name of Einstein-Rosen bridge, its popularization came with the seminal paper of Morris and Thorne [1]. Therein, Morris and Thorne defined the concept of Traversable Wormholes and the conditions under which such objects can turn into reality in the real life. Expectedly, the conditions were insurmountable: the energy conditions satisfied by such objects were not physical. Then the question arises, is it possible to consider alternative theories of gravity which admits different energy conditions in order to make wormholes viable? This question has partly been answered in the literature by many researchers during the recent decade. Within the Einstein gravity, however, a satisfactory theory of wormholes with reasonable energy conditions and stability requirement has never been met. In other words, with the Einstein-Hilbert Lagrangian which is described by the Lagrangian $\sqrt{-g}R,$ where g refers to the spacetime metric determinant and R is the Ricci scalar, there is no physical energy momentum $T_{\mu\nu}$ that renders construction of wormholes possible. If we imagine that such an object has been constructed momentarily it will be collapsed into the singularity of space time. What is needed therefore is an energy-momentum that acts against attractive gravity to balance such a wormhole. The absence of such an energy-momentum in Einstein's gravity enforces us to consider extended theories among which the Gauss-Bonnet (GB) theory is rather popular. In the latter beside the Ricci scalar, which appears linearly in the Einstein-Hilbert Lagrangian we consider quadratic invariants in a particular combination that the field equations are still second order so that no ghosts, or non-physical degrees of freedom arise. In this thesis we shall show that by employing Gauss-Bonnet invariant as supplementary to the Einstein-Hilbert term desired wormholes can be obtained.

Beside the geometrical Gauss-Bonnet term we consider also extended physical sources in the theory. The first and simplest source to be considered is naturally a Maxwell field of linear electrodynamics. Due to divergence problem for the electric field (i.e. the Coulomb problem) Born-Infeld (BI) introduced in 1930s a non-linear electromagnetic theory that may also have a curing property for singularities in general relativity [41]. Further, the Hoffmann modification of the Born-Infeld theory, under the name of Born-Infeld-Hoffmann (BIH), has also been considered in this thesis. Another source is naturally a massless scalar field with exponential coupling to the electromagnetic field, such a scalar field is called the dilaton which has been proved much useful in various theories. The dilaton plays the role of 'cement' to glue gravity with the electromagnetic field.

In addition to the linear and non-linear electromagnetic sources we consider also the magnetically charged Yang-Mills (YM) field through the Wu-Yang ansatz which is effective in the higher dimensions $d > 4$, [7]. For $d = 4$ the YM field reduces to the Abelian Maxwell field in a particular gauge. Being an isotropic gauge the Wu-Yang ansatz is applicable to the spherically symmetric spacetime metrics. During the recent decade another interesting physical source that attracted attention is the Chaplygin [63] source which is characterized by the equation of state $p = -\frac{A}{\rho}$, for the constant $A > 0$, [63]. For the generalized Chaplygin gas we have $p = -\left|\frac{A}{\rho}\right|^{\nu}$ where the parameter $\nu > 0$. The main reason that the Chaplygin gas attracted interest is that it yields negative pressure is negative which is required for a repulsive force. In the cosmological problems due to the accelerated expansion of the universe a negative pressure may play the dominant role to provide the expansion. In analogy, since in the wormholes also a repulsive force is required to overcome the ever attractive gravity the Chaplygin gas may play such a role.

As stated in the beginning, traversable wormholes were introduced and discussed thoroughly by Morris and Thorne [1]. The idea in such a construction is to connect distant points of the same spacetime or two asymptotically flat spacetimes by a short cut route. By this procedure distances of billions light years can be reduced to a very short time and distance through the passage from a wormhole. Physical construction of wormholes in real life, as stated above amounts to finding appropriate energy-momentum that will enable such objects. To reduce the problems further, instead of general wormholes we prefer to consider in this thesis a particular class of wormholes, known as thin-shell wormholes (TSWs). The advantage of the latter is that energy is confined on an infinitely thin layer so that it can be adjusted in an easier manner. The junction, or the boundary conditions in this particular construction becomes the important matter to be overcome. These junction conditions for thin-layers were developed long ago by Isreal [14] which were modified later on in accordance with the theory involved. By this formalism the surface energy-momentum on the thin-shell is determined by the surrounding energy-momenta such that the Einstein field equations are satisfied everywhere including the thin-shell. The resulting expressions are rather tedious so that we can only treat them numerically. The perturbation equation of the thin-shell has been reduced to the particle equation with a potential. The analysis of the potential becomes decisive to determine whether our thin-shell is stable or not. For certain range of parameters our thin-shells turn out to be stable against linear radial perturbations. Physically this implies that such objects can be constructed and may exist in certain parts of our universe as remnants from the big bang. Given the conditions it is also possible to produce such wormholes in the high-energy collision experiments that undergo at CERN.

Organization of the thesis is as follows. In Chapter 2 we discuss the higher dimensional TSWs in Einstein-Yang-Mills-Gauss-Bonnet gravity. Chapter 3 investigates the stability of TSWs supported by normal matter in Einstein-Maxwell-Gauss-Bonnet gravity. TSWs in Einstein-Yang-Mills-Dilaton gravity is considered in Chapter 4. Black holes and TSWs in Hoffmann modified Born-Infeld theory is studied in Chapter 5. Chapter 6 takes into account the generalized Chaplygin gas in Einstein-Maxwell-Gauss-Bonnet theory. We complete the thesis with Conclusion which appears in Chapter 7.

CHAPTER 2

THIN SHELL WORMHOLE IN EINSTEIN-YANG-MILLS-GAUSS-BONNET GRAVITY

2.1. Overview

Constraction of a traversable wormholes, using the curvature of spacetime and physical energy-momenta is one of the long standing problem in general relativity [1, 2]. Most of the sources to support wormholes to date, unfortunately consists of exotic matter which violates the energy conditions [3, 4]. However, there are examples of TSWs that resist against collapse when sourced entirely by physical (normal) matter satisfying the energy conditions [5, 6, 7]. From this token, it has been observed that pure Einstein's gravity consisting of Einstein-Hillbert (EH) action with familiar sources alone doesn't suffice to satisfy the criteria required for normal matter. This leads automatically to taking into account the higher curvature corrections known as the Lovelock hierarchy [8]. Most prominent term among such higher order corrections is the Gauss-Bonnet (GB) term to modify the EH Lagrangian. There is already a growing literature on Einstein-Gauss-Bonnet (EGB) gravity and wormhole constructions in such a theory.

In this Chapter we intend to fill a gap in this line of thought which concerns Einstein-Yang-Mills (EYM) theory amended with the GB term. More specifically, we construct TSWs that are supported by normal (i.e. non-exotic) matter. By employing the higher dimensional Wu-Yang ansatz which has been described in Ref.s [9, 10] we first construct higher dimensional ($d \geq 5$) exact black hole solutions in EYMGB theory. In this regard EYM solution becomes simpler in comparison with the Einstein-Maxwell (EM) solutions. This motivates us to seek for TSWs by cutting / pasting method in EYM theory. Another point of utmost importance is the GB parameter (α), whose sign plays a crucial role in the positivity of energy of the system. Although in string theory this parameter is chosen positive for some valid reasons, when it comes to the subject of wormholes our choice favors the negative values ($\alpha < 0$), for the GB parameter. One more item that we consider in detail in this study is to investigate the stability of such wormholes against linear perturbations when the pressure and energy density are linearly related.

2.2. Einstein-Yang-Mills-Gauss-Bonnet Black Hole

The exact solution to EYMGB gravity that we shall introduce were found by Mazharimousavi and Halilsoy [9, 10]. The d-dimensional, static and spherically symmetric line element is given by [10]

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2, \tag{2.1}$$

where $f(r)$ is an unknown function to be found and

$$d\Omega_{d-2}^2 = d\theta_1^2 + \sum_{i=2}^{d-2} \prod_{j=1}^{i-1} \sin^2\theta_j \; d\theta_i^2, \tag{2.2}$$

in which

$$0 \leq \theta_{d-2} \leq 2\pi, 0 \leq \theta_i \leq \pi, \quad 1 \leq i \leq d-3.$$

The Wu-Yang ansatz in higher dimension follows by introducing the YM potentials as

$$\mathbf{A}^{(a)} = \frac{Q}{r^2} C^{(a)}_{(i)(j)} \, x^i dx^j, \quad r^2 = \sum_{i=1}^{d-1} x_i^2, \tag{2.3}$$

$$2 \leq j+1 \leq i \leq d-1, \text{ and } 1 \leq a \leq (d-2)(d-1)/2,$$

$$x_1 = r\cos\theta_{d-3}\sin\theta_{d-4}...\sin\theta_1, \; x_2 = r\sin\theta_{d-3}\sin\theta_{d-4}...\sin\theta_1,$$

$$x_3 = r\cos\theta_{d-4}\sin\theta_{d-5}...\sin\theta_1, \; x_4 = r\sin\theta_{d-4}\sin\theta_{d-5}...\sin\theta_1,$$

$$...$$

$$x_{d-2} = r\cos\theta_1.$$

Herein $C^{(a)}_{(b)(c)}$ is the structure constants [11] and Q is the YM magnetic charge. Next we find the YM invariant $\mathcal{F}$ which is given by

$$\mathcal{F} = \mathbf{Tr}(F^{(a)}_{\lambda\sigma} F^{(a)\lambda\sigma}) = \frac{(d-3)}{r^4} Q^2, \tag{2.4}$$

and the energy momentum tensor reads

$$T^{\nu}_{\mu} = -\frac{1}{2}\mathcal{F}\text{diag}\left[1, 1, \kappa, \kappa, .., \kappa\right], \text{ and } \kappa = \frac{d-6}{d-2}. \tag{2.5}$$

The Einstein-Yang-Mills-Gauss-Bonnet field equations also are given by

$$G^{E}_{\mu\nu} + \alpha G^{GB}_{\mu\nu} = T_{\mu\nu}, \tag{2.6}$$

in which

$$G_{\mu\nu}^{GB}=2\left(-R_{\mu\sigma\kappa\tau}R^{\kappa\tau\sigma}{}_{\nu}-2R_{\mu\rho\nu\sigma}R^{\rho\sigma}-2R_{\mu\sigma}R^{\sigma}{}_{\nu}+RR_{\mu\nu}\right)-\frac{1}{2}\mathcal{L}_{GB}g_{\mu\nu}\ , \tag{2.7}$$

α is the GB parameter and GB Lagrangian $\mathcal{L}_{GB}$ is given by

$$\mathcal{L}_{GB}=R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta}-4R_{\mu\nu}R^{\mu\nu}+R^2. \tag{2.8}$$

The exact solutions which we shall use are found in [9, 10]

$$f_{\pm}\left(r\right)=\begin{cases}1+\frac{r^2}{4\alpha}\left(1\pm\sqrt{1+\frac{32\alpha M_{ADM}}{3r^4}+\frac{16\alpha Q^2\ln r}{r^4}}\right), & d=5\\ 1+\frac{r^2}{2\tilde{\alpha}}\left(1\pm\sqrt{1+\frac{16\tilde{\alpha}M_{ADM}}{r^{d-1}(d-2)}+\frac{4(d-3)\tilde{\alpha}Q^2}{(d-5)r^4}}\right), & d\geq 6\end{cases}\ , \tag{2.9}$$

in which $\tilde{\alpha}=\left(d-3\right)\left(d-4\right)\alpha$, with the GB parameter α. Here M_{ADM} stands for the usual ADM mass of the black hole and Q is the YM charge. When compared with Ref.s [9] (for $d=5$) and [10] (for $d>5$) the meaning of M_{ADM} implies that $M_{ADM}=\frac{3}{2}\left(m+2\alpha\right)$ and $M_{ADM}=\frac{1}{4}m\left(d-2\right)$, respectively. Let us also add that in Ref. [10] we set $Q=1$ through scaling. The crucial point in our solution is that the YM term under the square root has a fixed power $\frac{1}{r^4}$ for all $d\geq 6$. As it can be checked, the negative branch gives the correct limit of higher dimensional black hole solution in EYM theory of gravity if $\alpha\rightarrow 0$, and therefore in the sequel we only consider this specific case.

Here, in order to explore the physical properties of the above solutions we investigate some essential thermodynamic quantities. Since $d=5$ case has been studied elsewhere [12] we shall concentrate on $d\geq 6$.

Radius of the event horizon (i.e., r_h) of the negative branch black hole $f_{-}\left(r\right)$, with positive α is the maximum root of $f_{-}(r_h) = 0$. It is not difficult to show that in terms of event horizon radius one can write

$$M_{ADM} = \frac{(d-2)}{4}\left[\left(\tilde{\alpha} + r_h^2\right) - \frac{(d-3)}{(d-5)}Q^2\right] r_h^{d-5}. \tag{2.10}$$

Also we find the Hawking temperature T_H in terms of r_h, i.e.,

$$T_H = \frac{1}{4\pi} f'(r_h) = \frac{(d-3)\left(r_h^2 - Q^2\right) + \tilde{\alpha}(d-5)}{4\pi r_h \left(2\tilde{\alpha} + r_h^2\right)}. \tag{2.11}$$

To complete our thermodynamical quantities we use the standard definition of the specific heat capacity with the constant charge

$$C_Q = T_H \left(\frac{\partial S}{\partial T_H}\right)_Q, \tag{2.12}$$

in which S is the standard entropy defined as

$$S = \frac{A}{4} = \frac{(d-1)\,\pi^{\frac{d-1}{2}}}{4\Gamma\left(\frac{d+1}{2}\right)} r_h^{d-2}, \tag{2.13}$$

to show the possible thermodynamical phase transition. After some manipulation we find

$$C_Q = \frac{(d-2)(d-1)\left(2\tilde{\alpha}+r_h^2\right)\pi^{\frac{d-1}{2}} r_h^{d-2}\left[(d-5)\tilde{\alpha}+(d-3)\left(r_h^2-Q^2\right)\right]}{4\Gamma\left(\frac{d+1}{2}\right)\left\{2\tilde{\alpha}[Q^2(d-3)-\tilde{\alpha}(d-5)]+[3Q^2(d-3)-\tilde{\alpha}(d-9)]r_h^2-(d-3)r_h^4\right\}}. \tag{2.14}$$

The phase transition is taking place at the real and positive root(s) of the denominator, i.e.,

$$2\tilde{\alpha}\left[Q^2(d-3)-\tilde{\alpha}(d-5)\right]+\left[3Q^2(d-3)-\tilde{\alpha}(d-9)\right]r_h^2-(d-3)\,r_h^4=0. \tag{2.15}$$

One can show that under the condition

$$\frac{Q^2}{\tilde{\alpha}}<\frac{7d-39}{9(d-3)} \tag{2.16}$$

there is no phase transition, while if

$$\frac{7d-39}{9(d-3)}<\frac{Q^2}{\tilde{\alpha}}<\frac{d-5}{d-3} \tag{2.17}$$

we will observe two phase transitions. Finally upon choosing

$$\frac{d-5}{d-3}\leq\frac{Q^2}{\tilde{\alpha}} \tag{2.18}$$

there exists only one phase transition. Also, if $\frac{Q^2}{\tilde{\alpha}}=\frac{7d-39}{9(d-3)}$ one phase transition occurs at $r_h=\sqrt{\frac{6(d-3)}{7d-39}Q^2}$. These results show that the dimensionality of spacetime plays a crucial role in the thermodynamical behavior of the EYMGB system.

For negative α in the negative branch we write $\tilde{\alpha}=-|\tilde{\alpha}|$ and therefore the horizon radius r_h is given by solving

$$1-\frac{2|\tilde{\alpha}|}{r_h^2}=\sqrt{1-\frac{16|\tilde{\alpha}|M_{ADM}}{r_h^{d-1}(d-2)}-\frac{4(d-3)|\tilde{\alpha}|Q^2}{(d-5)\,r_h^4}}. \tag{2.19}$$

The method of establishing the TSW, based on the black hole solutions given in (2.9), follows the standard procedure which has been employed in many recent works [5, 6, 7].

2.3. Dynamic Thin Shell Wormholes in $d-$Dimensions

The method of establishing a TSW in the foregoing geometry goes as follows. We cut two copies of the EYMGB spacetime

$$M^{\pm} = \{r^{\pm} \geq a, \;\; a > r_h\} \tag{2.20}$$

and paste them at the boundary hypersurface $\Sigma^{\pm} = \{r^{\pm} = a, \; a > r_h\}$. These surfaces are identified on $r = a$ with a surface energy-momentum of a thin-shell whose radius coincides also with the throat radius such that geodesic completeness holds for $M = M^{+} \cup M^{-}$. Following the Darmois-Israel formalism [13, 14, 15, 16] in terms of the original coordinates $x^{\gamma} = (t, r, \theta_1, \theta_2, ...)$ (i.e. in M) the induced metric $\xi^i = (\tau, \theta_1, \theta_2, ...)$, on Σ is given by (Latin indices run over the induced coordinates i.e., $\{1, 2, 3, .., d-1\}$ and Greek indices run over the original manifold's coordinates i.e., $\{1, 2, 3, .., d\}$)

$$g_{ij} = \frac{\partial x^{\alpha}}{\partial \xi^i} \frac{\partial x^{\beta}}{\partial \xi^j} g_{\alpha\beta}. \tag{2.21}$$

Here τ is the proper time and

$$g_{ij} = \text{diag}\left(-1, a^2, a^2 \sin^2\theta, a^2 \sin^2\theta \sin^2\phi, ...\right), \tag{2.22}$$

while the extrinsic curvature is defined by

$$K_{ij}^{\pm} = -n_{\gamma}^{\pm} \left(\frac{\partial^2 x^{\gamma}}{\partial \xi^i \partial \xi^j} + \Gamma_{\alpha\beta}^{\gamma} \frac{\partial x^{\alpha}}{\partial \xi^i} \frac{\partial x^{\beta}}{\partial \xi^j} \right)_{r=a}. \tag{2.23}$$

It is assumed that Σ is non-null, whose unit $d-$normal in $M^{\pm}$ is given by

$$n_{\gamma}=\left(\pm\left|g^{\alpha\beta}\frac{\partial F}{\partial x^{\alpha}}\frac{\partial F}{\partial x^{\beta}}\right|^{-1/2}\frac{\partial F}{\partial x^{\gamma}}\right)_{r=a}, \tag{2.24}$$

in which F is the equation of the hypersurface Σ, i.e.

$$\Sigma:F\left(r\right)=r-a\left(\tau\right)=0. \tag{2.25}$$

The generalized Darmois-Israel conditions on Σ determines the surface energy-momentum tensor S_{ab} which is expressed by [17]

$$S_{i}^{j}=-\frac{1}{8\pi}\left(\left\langle K_{i}^{j}\right\rangle-K\delta_{i}^{j}\right)-\frac{\alpha}{16\pi}\left\langle 3J_{i}^{j}-J\delta_{i}^{j}+2P_{imn}^{\ \ \ \ j}K^{mn}\right\rangle. \tag{2.26}$$

Here a bracket implies a jump across Σ. The divergence-free part of the Riemann tensor P_{abcd} and the tensor J_{ab} (with trace $J=J_{a}^{a}$) are given by

$$P_{imnj}=R_{imnj}+\left(R_{mn}g_{ij}-R_{mj}g_{in}\right)-\left(R_{in}g_{mj}-R_{ij}g_{mn}\right)+\frac{1}{2}R\left(g_{in}g_{mj}-g_{ij}g_{mn}\right), \tag{2.27}$$

$$J_{ij}=\frac{1}{3}\left[2KK_{im}K_{j}^{m}+K_{mn}K^{mn}K_{ij}-2K_{im}K^{mn}K_{nj}-K^{2}K_{ij}\right]. \tag{2.28}$$

By employing these expressions through (2.26) we find the energy density and surface pressures for a generic metric function $f\left(r\right)$, with $r=a\left(\tau\right)$. The results are given by

$$\sigma=-S_{\tau}^{\tau}=-\frac{\Delta\left(d-2\right)}{8\pi}\left[\frac{2}{a}-\frac{4\tilde{\alpha}}{3a^{3}}\left(\Delta^{2}-3\left(1+\dot{a}^{2}\right)\right)\right], \tag{2.29}$$

$$S_{\theta_i}^{\theta_i} = p = \frac{1}{8\pi}\left\{\frac{2\left(d-3\right)\Delta}{a} + \frac{2\ell}{\Delta} - \frac{4\tilde{\alpha}}{3a^2}\mathfrak{S}\right\} \tag{2.30}$$

in which

$$\mathfrak{S} = \left[3\ell\Delta - \frac{3\ell}{\Delta}\left(1+\dot{a}^2\right) + \frac{\Delta^3}{a}\left(d-5\right) - \frac{6\Delta}{a}\left(a\ddot{a} + \frac{d-5}{2}\left(1+\dot{a}^2\right)\right)\right].$$

Herein $\ell = \ddot{a} + f'\left(a\right)/2$ and $\Delta = \sqrt{f\left(a\right) + \dot{a}^2}$ in which

$$f\left(a\right) = \left. f_-\left(r\right)\right|_{r=a}. \tag{2.31}$$

We note that in our notation a 'dot' denotes derivative with respect to the proper time τ and a 'prime' with respect to the argument of the function. It can be checked by direct substitution from (2.29) and (2.30) that the conservation equation

$$\nabla_i S^{ij} = \frac{d}{d\tau}\left(\sigma a^{(d-2)}\right) + p\frac{d}{d\tau}\left(a^{(d-2)}\right) = 0. \tag{2.32}$$

holds true.

Once we know precisely the energy density and surface pressures, we can study the energy conditions and the amount of exotic / normal matter that is to support the above TSW. Let us start with the weak energy condition (WEC) which implies for any timelike vector V_μ we must have $T_{\mu\nu}V^\mu V^\nu \geq 0$. Also by continuity, WEC implies the null energy condition (NEC), which states that for any null vector U_μ, $T_{\mu\nu}U^\mu U^\nu \geq 0$ [2]. It is not difficult to show that in an orthonormal basis these conditions read as

$$\begin{aligned} WEC: &\quad \rho \geq 0,\ \rho + p_i \geq 0, \\ NEC &\quad \rho + p_i \geq 0, \end{aligned} \tag{2.33}$$

in which $i \in \{2, 3, ..., d-1\}$. Here in the spherical TSWs, the radial pressure p_r is zero and $\rho = \delta(r-a)\sigma$ which imply WEC and NEC coincide as $\sigma \geq 0$. Note that $\delta(r-a)$ stands for the Dirac delta-function. By looking at σ given in (2.30) one may conclude that these conditions reduce to

$$\frac{3}{2}a^2 \leq \tilde{\alpha}\left(f(a) - 2\dot{a}^2 - 3\right). \tag{2.34}$$

For the static configuration with $\dot{a} = 0, \ddot{a} = 0$ and $a = a_0$ it is not difficult to see that for $\tilde{\alpha} \geq 0$ the latter condition is not satisfied. In other words, both WEC and NEC are violated. This is simply from the fact that the metric function is asymptotically flat and $f(a) < 1$ for $a \geq r_h$. Unlike $\tilde{\alpha} \geq 0$, for the case of $\tilde{\alpha} < 0$ this condition in arbitrary dimensions is satisfied. Direct consequence of these results can be seen in the total matter in supporting the TSW. The standard integral definition of the total matter is given by

$$\Omega = \int (\rho + p_r)\sqrt{-g}d^{d-1}x \tag{2.35}$$

which gives

$$\Omega = \frac{2\pi^{\frac{d-1}{2}} a_0^{d-2}}{\Gamma\left(\frac{d-1}{2}\right)}\sigma_0 \tag{2.36}$$

in which

$$\sigma_0 = -\frac{\sqrt{f(a_0)}(d-2)}{8\pi}\left[\frac{2}{a_0} - \frac{4\tilde{\alpha}}{3a_0^3}(f(a_0) - 3)\right]. \tag{2.37}$$

We note that the sign of σ_0 indicates the nature of the matter which supports the wormhole. It is obvious from Ω that similar to σ_0, in static configuration the total matter which supports the TSW is exotic if $\tilde{\alpha} \geq 0$ and normal if $\tilde{\alpha} < 0$. This result is independent of dimensions and other parameters.

2.4. Stability of the Thin Shell Wormholes for $d \geq 5$

To study the stability of the TSW, constructed above, we consider a radial perturbation of the radius of the throat a. After the linear perturbation we may consider a linear relation between the energy density and radial pressure, namely [18]

$$p = p_0 + \beta^2 (\sigma - \sigma_0) . \tag{2.38}$$

Here the constant σ_0 is given by (2.37) and p_0 reads as

$$p_0 = \frac{\sqrt{f(a_0)}}{8\pi} \left\{ \frac{2(d-3)}{a_0} + \frac{f'(a_0)}{f(a_0)} - \frac{4\tilde{\alpha}}{a_0^2} \mathfrak{A} \right\} . \tag{2.39}$$

where

$$\mathfrak{A} = \left[\frac{f'(a_0)}{2} - \frac{f'(a_0)}{2f(a_0)} + \frac{f(a_0)(d-5)}{3a_0} - \frac{d-5}{a_0} \right] .$$

The constant parameter β^2 for the wormhole supported by normal matter is related to the speed of sound. By considering (2.38) in (2.32), one finds

$$\sigma(a) = \left(\frac{\sigma_0 - p_0}{\beta^2 + 1} \right) \left(\frac{a_0}{a} \right)^{(d-2)\left(\beta^2+1\right)} + \frac{\beta^2 \sigma_0 - p_0}{\beta^2 + 1} \tag{2.40}$$

in which a_0 is the radius of the throat in static equilibrium wormhole and $\sigma_0(p_0)$ is the static energy density (pressure) on the thin-shell. By equating the latter expression and the one found by using Einstein equation on the shell (2.29), we find the equation of motion of the wormhole which reads

$$\dot{a}^2 + V(a) = 0, \tag{2.41}$$

where

$$V(a) = f(a) - \left(\left[\sqrt{\mathbb{A}^2 + \mathbb{B}^3} - \mathbb{A} \right]^{1/3} - \frac{\mathbb{B}}{\left[\sqrt{\mathbb{A}^2 + \mathbb{B}^3} - \mathbb{A} \right]^{1/3}} \right)^2 \tag{2.42}$$

and

$$\mathbb{A} = \frac{3\pi a^3}{2(d-2)\tilde{\alpha}} \left[\left(\frac{\sigma_0 + p_0}{\beta^2 + 1} \right) \left(\frac{a_0}{a} \right)^{(d-2)(\beta^2+1)} + \frac{\beta^2 \sigma_0 - p_0}{\beta^2 + 1} \right], \tag{2.43}$$

$$\mathbb{B} = \frac{a^2}{4\tilde{\alpha}} + \frac{1 - f(a)}{2}. \tag{2.44}$$

Here $V(a)$ is called the potential of the wormhole's motion and it helps us to figure out the regions of stability for the wormhole under our linear perturbation. According to the standard method of stability of TSWs, we expand $V(a)$ as a series of $(a - a_0)$. One can show that both $V(a_0)$ and $V'(a_0)$ vanish and the first non-zero term in this expansion is $\frac{1}{2}V''(a_0)(a - a_0)^2$. Now, in a small neighborhood of the equilibrium point a_0 we have

$$\dot{a}^2 + \frac{1}{2}V''(a_0)(a - a_0)^2 = 0, \tag{2.45}$$

which implies that with $V''(a_0) > 0$, $a(\tau)$ will oscillate about a_0 and make the wormhole stable. At this point it will be in order also to clarify the status of parameter β since ultimately the three-dimensional (i.e. $V''(a_0) > 0$, β, a_0) stability plots will make use of it. First of all although in principle $\beta < 0$ is possible we shall restrict ourselves only to the case $\beta > 0$. Unfortunately β can only be expressed implicitly as a function of a_0, through (2.38) and expressions for p, σ, p_0 and σ_0. It turns out that the usual expression for stability, namely $V''(a_0) > 0$, can be plotted as a projection onto the plane formed by β and a_0. This must not give the impression, however, that the relation $\beta = \beta(a_0)$ is known explicitly.

2.4.1. $d = 5$

Let us first eliminate α from the equations, by using the solution given in (2.9). To do so we introduce new variables and parameters as

$$\begin{aligned} \tilde{a} &= \frac{a}{\sqrt{|\alpha|}}, \tilde{\tau} = \frac{\tau}{\sqrt{|\alpha|}}, \tilde{Q}^2 = \frac{Q^2}{|\alpha|}, \\ \tilde{m} &= \frac{2M_{ADM}}{3|\alpha|} + \frac{Q^2}{2|\alpha|} \ln|\alpha|. \end{aligned} \tag{2.46}$$

Upon these changes of variables, the other quantities change according to

$$\begin{aligned} f(a) &= f(\tilde{a}), \sigma(a) = \frac{\sigma(\tilde{a})}{\sqrt{|\alpha|}}, p(a) = \frac{p(\tilde{a})}{\sqrt{|\alpha|}}, \\ \mathbb{A}(a) &= \mathbb{A}(\tilde{a}), \mathbb{B}(a) = \mathbb{B}(\tilde{a}), V(a) = V(\tilde{a}). \end{aligned} \tag{2.47}$$

Finally the wormhole equation reads

$$\left(\frac{d\tilde{a}}{d\tilde{\tau}}\right)^2 + \tilde{V}(\tilde{a}) = 0. \tag{2.48}$$

Now, we consider two distinct cases, for $\alpha > 0$ and $\alpha < 0$, separately.

2.4.1.1. with $\alpha > 0$. In this section we consider $\alpha > 0$, such that the negative branch of the EYM black hole solution reads

$$f(\tilde{a}) = 1 + \frac{\tilde{a}^2}{4}\left(1 - \sqrt{1 + \frac{16\tilde{m}}{\tilde{a}^4} + \frac{16\tilde{Q}^2 \ln \tilde{a}}{\tilde{a}^4}}\right), \tag{2.49}$$

in which the condition

$$1 + \frac{16\tilde{m}}{\tilde{a}^4} + \frac{16\tilde{Q}^2 \ln \tilde{a}}{\tilde{a}^4}\bigg|_{\tilde{a}=\tilde{a}_0} \geq 0, \tag{2.50}$$

and

$$\mathbb{A}^2 + \mathbb{B}^3\big|_{\tilde{a}=\tilde{a}_0} \geq 0 \tag{2.51}$$

must hold. The latter equation automatically is valid and the final relation between the parameters reduces to (2.50). Based on this solution we find $\tilde{V}''(\tilde{a}_0)$ in terms of the other parameters. Fig. 2.1 shows the stability regions and also $f(\tilde{a})$ and $\sigma(\tilde{a}_0)$.

2.4.1.2. with $\alpha < 0$. Next, we concentrate on the case $\alpha < 0$. With this choice negative branch of the EYM black hole solution reads

$$f(\tilde{a}) = 1 - \frac{\tilde{a}^2}{4}\left(1 - \sqrt{1 - \frac{16\tilde{m}}{\tilde{a}^4} - \frac{16\tilde{Q}^2 \ln \tilde{a}}{\tilde{a}^4}}\right). \tag{2.52}$$

Based on this solution we study $\tilde{V}''(\tilde{a}_0)$ in terms of the other parameters.

In this case also we have some constraint on the parameters in order to get $f(\tilde{a}_0) \geq 0$, $\sigma(\tilde{a}_0) \geq 0$, and $\mathbb{A}^2 + \mathbb{B}^3\big|_{\tilde{a}=\tilde{a}_0} \geq 0$. It is not difficult to see that all

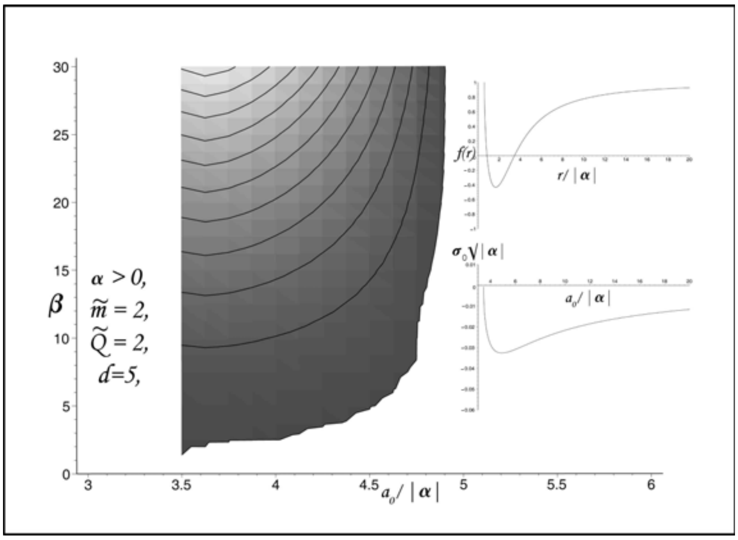

Figure 2.1. Region of stability (i.e. $V''(a_0) > 0$) for the thin-shell in $d = 5$ and for $\alpha > 0$. The $f(r)$ and σ_0 plots are also given. It can easily be seen that the energy density σ_0 is negative which implies exotic matter.

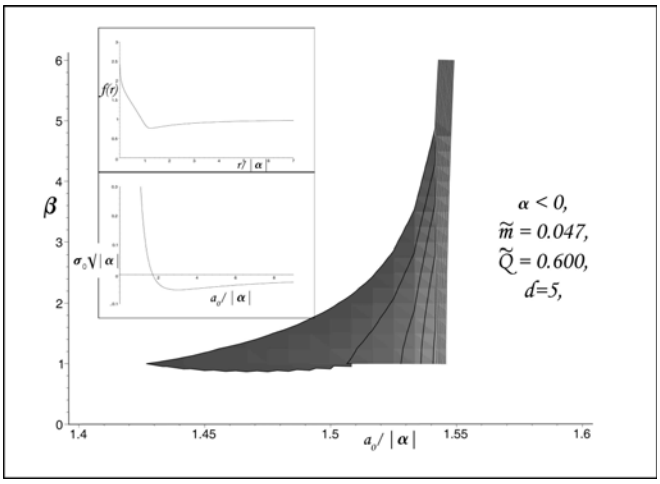

Figure 2.2. For $d = 5$ and $\alpha < 0$ case with the chosen parameters $f(r)$ has no zero but σ_0 has a small band of positivity with the presence of normal matter. We note also that $\beta < 1$ in a small band.

these conditions reduce to

$$0 \leq \frac{1}{4}\sqrt{1-\frac{16\tilde{m}}{\tilde{a}_0^4}-\frac{16\tilde{Q}^2\ln\tilde{a}_0}{\tilde{a}_0^4}} \leq \frac{4}{\tilde{a}_0^2}-1, \tag{2.53}$$

and

$$1-\frac{16\tilde{m}}{\tilde{a}_0^4}-\frac{16\tilde{Q}^2\ln\tilde{a}_0}{\tilde{a}_0^4} \geq 0. \tag{2.54}$$

After some manipulation, the parameters must satisfy the following constraint

$$\tilde{a}_0^4 \geq 16\left(\tilde{m}+\tilde{Q}^2\ln\tilde{a}_0^2\right) \tag{2.55}$$

where $0 \leq \tilde{a}_0^2 \leq 4$. The stability region for this case is given in Fig. 2.2.

2.4.2. $d \geq 6$

Here also we eliminate $\tilde{\alpha}$ from the equations. By introducing

$$\tilde{a}=\frac{a}{\sqrt{|\tilde{\alpha}|}}, \tilde{\tau}=\frac{\tau}{\sqrt{|\tilde{\alpha}|}}, \tilde{Q}^2=\frac{Q^2}{|\tilde{\alpha}|},\ \tilde{m}=\frac{M_{ADM}}{|\tilde{\alpha}|^{\frac{d-3}{2}}}. \tag{2.56}$$

the other quantities become

$$\begin{aligned} f(a) &= f(\tilde{a}), \sigma(a)=\frac{\sigma(\tilde{a})}{\sqrt{|\alpha|}}, p(a)=\frac{p(\tilde{a})}{\sqrt{|\alpha|}}, \\ \mathbb{A}(a) &= \mathbb{A}(\tilde{a}), \mathbb{B}(a)=\mathbb{B}(\tilde{a}), V(a)=V(\tilde{a}), \end{aligned} \tag{2.57}$$

and the wormhole equation is given by

$$\left(\frac{d\tilde{a}}{d\tilde{\tau}}\right)^2+\tilde{V}(\tilde{a})=0. \tag{2.58}$$

2.4.2.1. with $\alpha > 0$. In this section we consider $\alpha > 0$, such that the negative branch of the EYMGB black hole solution reads

$$f_{-}(\tilde{a}) = 1 + \frac{\tilde{a}^2}{2}\left(1 - \sqrt{1 + \frac{16\tilde{m}}{\tilde{a}^{d-1}(d-2)} + \frac{4(d-3)\tilde{Q}^2}{(d-5)\tilde{a}^4}}\right) \tag{2.59}$$

Here we comment that constraints always restrict our free parameters. In the case of $\alpha > 0$ the first constraint is given by

$$\mathbb{A}^2 + \mathbb{B}^3\big|_{\tilde{a}=\tilde{a}_0} \geq 0, \tag{2.60}$$

which upon substitution and manipulation automatically is satisfied for all value of parameters. Based on this solution we find $\tilde{V}''(\tilde{a}_0)$ in terms of the other parameters. Fig.s 2.3, 2.4 and 2.5 shows the stability regions and also $f(\tilde{a})$ and $\sigma(\tilde{a}_0)$ for dimensions $d = 6, 7$ and 8.

2.4.2.2. with $\alpha < 0$. Next, we concentrate on the case $\alpha < 0$. With this choice negative branch of the EYMGB black hole solution reads

$$f_{-}(\tilde{a}) = 1 - \frac{\tilde{a}^2}{2}\left(1 - \sqrt{1 - \frac{16\tilde{m}}{\tilde{a}^{d-1}(d-2)} - \frac{4(d-3)\tilde{Q}^2}{(d-5)\tilde{a}^4}}\right). \tag{2.61}$$

Based on this solution we study $\tilde{V}''(\tilde{a}_0)$ in terms of the other parameters. Fig. 2.6 show the stability regions and also $f(\tilde{a})$ and $\sigma(\tilde{a}_0)$. In order to set $f(\tilde{a}_0) \geq 0$, $\sigma(\tilde{a}_0) \geq 0$, and $\mathbb{A}^2 + \mathbb{B}^3|_{\tilde{a}=\tilde{a}_0} \geq 0$ it is enough to satisfy

$$0 < \sqrt{1 - \frac{16\tilde{m}}{\tilde{a}_0^{d-1}(d-2)} - \frac{4(d-3)\tilde{Q}^2}{(d-5)\tilde{a}_0^4}} < \frac{2}{\tilde{a}_0^2} - 1, \tag{2.62}$$

and

$$1-\frac{16\tilde{m}}{\tilde{a}^{d-1}\left(d-2\right)}-\frac{4\left(d-3\right)\tilde{Q}^{2}}{\left(d-5\right)\tilde{a}^{4}}>0, \tag{2.63}$$

where $0<\tilde{a}_0^2<2$.

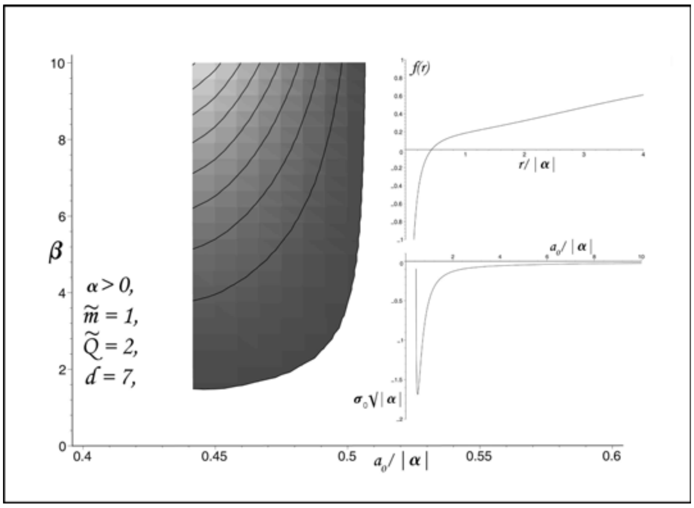

Figure 2.3. For $d = 7$ and $\alpha > 0$ the stability region is plotted which is seen to have exotic matter alone.

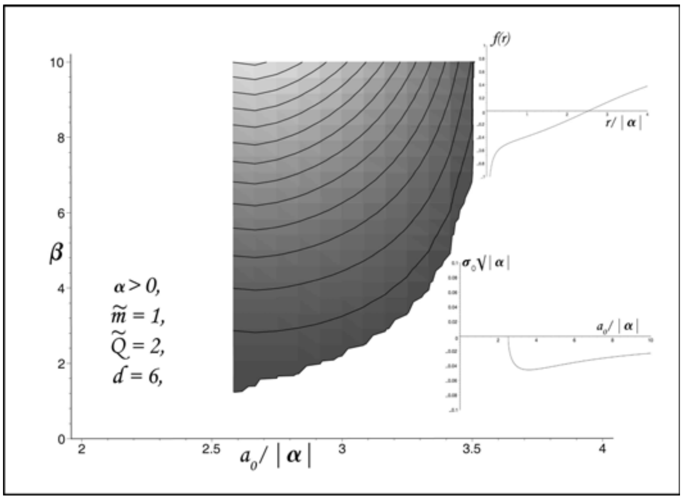

Figure 2.4. For $d = 6$ and $\alpha > 0$ also a region of stability is available but with $\sigma_0 < 0$. Note that $d = 6$ is special, since from Eq. (2.5) in the text we have $\kappa = 0$ and the energy-momentum takes a simple form.

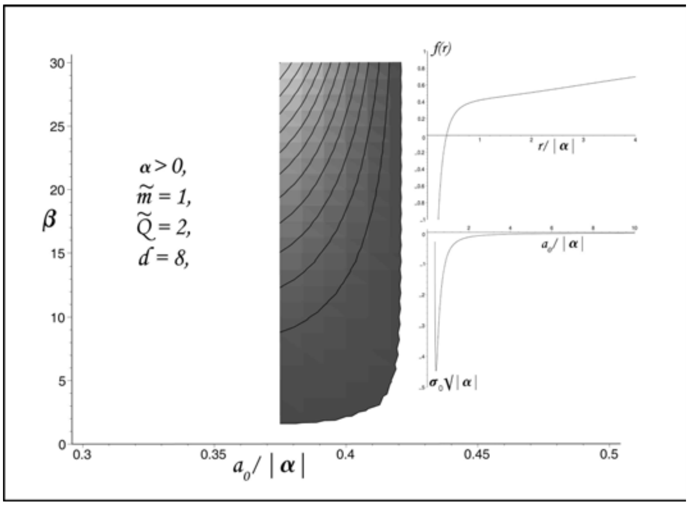

Figure 2.5. For $d = 8$ with $\alpha > 0$ exotic matter is seen to be indispensable.

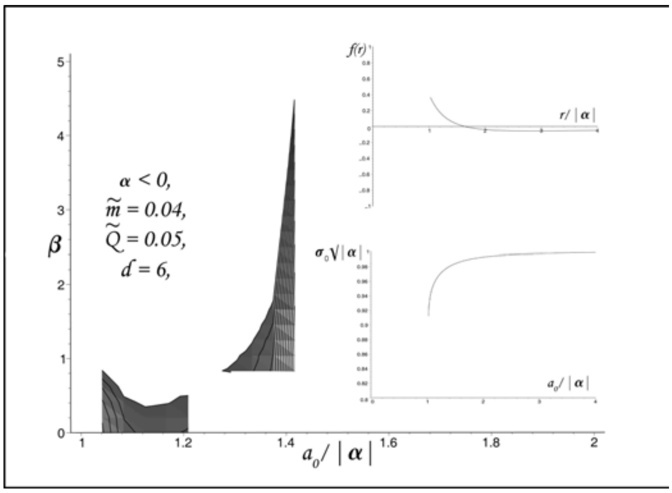

Figure 2.6. For $d = 6$ with $\alpha < 0$ there are two disjoint regions of stability for the thin-shell and in contrast to the $\alpha > 0$ case in Fig. 2.4, we have $\sigma_0 > 0$. We notice in this case also that $\beta < 1$ is possible.

CHAPTER 3

WORMHOLES SUPPORTED BY NORMAL MATTER IN EINSTEIN-MAXWELL-GAUSS-BONNET GRAVITY

3.1. Overview

Einstein-Gauss-Bonnet (EGB) gravity, with additional sources such as Maxwell, Yang-Mills, dilaton etc. has already been investigated extensively in the literature [19, 21, 22, 23, 24, 25, 9]. Not to mention, all these theories admit black hole, wormhole [26, 32, 28] and other physically interesting solutions. As it is the usual trend in theoretical physics, each new parameter invokes new hopes and from that token, the GB parameter α does the same. Although the case $\alpha > 0,$ has been exalted much more than the case $\alpha < 0$ in EGB gravity so far [29, 30] (and references cited therein), it turns out here in the stable, normal matter TSWs that the latter comes first time to the fore. Construction and maintenance of TSWs has been the subject of a large literature, so that we shall provide only a brief review here. Instead, one class [5, 6] that made use of non-exotic matter for its maintenance attracted our interest and we intend to analyze its stability in this chapter. This is the $5-$dimensional thin-shell solution of Einstein-Maxwell-Gauss-Bonnet (EMGB) gravity, whose radius is identified with the minimum radius of the wormhole. For this purpose we employ radial, linear perturbations to cast the motion into a potential-well problem in the background. In doing this, a reasonable assumption employed routinely, which is adopted here also, is to relate pressure and energy density by a linear expression [31, 14, 15]. For special choices of parameters we obtain islands of stability for such wormholes. To this end, we make use of numerical computation and plotting since the problem involves highly intricate functions for an analytical treatment.

3.2. A Brief Review of 5-Dimensional Einstein-Maxwell-Gauss-Bonnet Thin Shell Wormholes

The action of EMGB gravity in $5-$dimensions (without cosmological constant, i.e. $\Lambda = 0$) is

$$S = \kappa \int \sqrt{|g|} d^5x \left(R + \alpha \mathcal{L}_{GB} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \tag{3.1}$$

in which κ is related to the $5-$dimensional Newton constant and α is the GB parameter. Beside the Maxwell Lagrangian $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ the GB Lagrangian $\mathcal{L}_{GB}$ is given by (2.8) and hence the variational principle of S with respect to $g_{\mu\nu}$ yields EGB equation

$$G_{\mu\nu} + 2\alpha G_{\mu\nu}^{(GB)} = \kappa^2 T_{\mu\nu} \tag{3.2}$$

in which $G_{\mu\nu}$ is the Einstein tensor, the GB tensore $G_{\mu\nu}^{(GB)}$ is given by (2.7) and the energy momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = F_{\mu\alpha} F_{\nu}^{\alpha} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}. \tag{3.3}$$

The metric element (2.1) with $d = 5$ which explicitly becomes

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta\left(d\phi^2 + \sin^2\phi d\psi^2\right)\right), \tag{3.4}$$

in which $f(r)$ is the only unknown function to be found. A TSW is constructed in EMGB theory as it has been shown in Sec. 2.3 with $d = 5$.

The EMGB solution that will be employed as a thin-shell solution with a normal matter [5, 6] is given by (with $\Lambda = 0$)

$$f(r) = 1 + \frac{r^2}{4\alpha}\left(1 - \sqrt{1 + \frac{8\alpha}{r^4}\left(\frac{2M}{\pi} - \frac{Q^2}{3r^2}\right)}\right) \tag{3.5}$$

with constants, M =mass and Q =charge. For a black hole solution the inner (r_-) and event horizons ($r_+ = r_h$) are

$$r_\pm = \sqrt{\frac{M}{\pi} - \alpha \pm \left[\left(\frac{M}{\pi} - \alpha\right)^2 - \frac{Q^2}{3}\right]^{1/2}}. \tag{3.6}$$

By employing the solution (3.5) we determine the surface energy-momentum on the thin-shell, which will play the major role in the perturbation. We shall address this problem in the next section.

In order to study the radial perturbations of the wormhole we take the throat radius as a function of the proper time, i.e., $a = a(\tau)$. Based on the generalized Birkhoff theorem, for $r > a(\tau)$ the geometry will be given still by (3.4). For the metric function $f(r)$ given in (3.5) one finds the energy density and pressures as

[5, 6]

$$\sigma = -S_{\tau}^{\tau} = -\frac{\Delta}{4\pi}\left[\frac{3}{a} - \frac{4\alpha}{a^3}\left(\Delta^2 - 3\left(1+\dot{a}^2\right)\right)\right], \tag{3.7}$$

$$S_{\theta}^{\theta} = S_{\phi}^{\phi} = S_{\psi}^{\psi} = p = \frac{1}{4\pi}\left[\frac{2\Delta}{a} + \frac{\ell}{\Delta} - \frac{4\alpha}{a^2}\left(\ell\Delta - \frac{\ell}{\Delta}\left(1+\dot{a}^2\right) - 2\ddot{a}\Delta\right)\right], \tag{3.8}$$

where $\ell = \ddot{a} + f'(a)/2$ and $\Delta = \sqrt{f(a) + \dot{a}^2}$ in which

$$f(a) = 1 + \frac{a^2}{4\alpha}\left(1 - \sqrt{1 + \frac{8\alpha}{a^4}\left(\frac{2M}{\pi} - \frac{Q^2}{3a^2}\right)}\right). \tag{3.9}$$

We note that in our notation a 'dot' denotes derivative with respect to the proper time τ and a 'prime' implies differentiation with respect to the argument of the function. By a simple substitution one can show that, the conservation equation

$$\frac{d}{d\tau}\left(\sigma a^3\right) + p\frac{d}{d\tau}\left(a^3\right) = 0. \tag{3.10}$$

is satisfied. The static configuration of radius a_0 has the following density and pressures

$$\sigma_0 = -\frac{\sqrt{f(a_0)}}{4\pi}\left[\frac{3}{a_0} - \frac{4\alpha}{a_0^3}\left(f(a_0) - 3\right)\right], \tag{3.11}$$

$$p_0 = \frac{\sqrt{f(a_0)}}{4\pi}\left[\frac{2}{a_0} + \frac{f'(a_0)}{2f(a_0)} - \frac{2\alpha}{a_0^2}\frac{f'(a_0)}{f(a_0)}\left(f(a_0) - 1\right)\right]. \tag{3.12}$$

In what follows we shall study small radial perturbations around the radius of equilibrium a_0. To this end we adapt a linear relation between p and σ as in (2.38)

[31, 18]. The 5-dimensional form of (2.40) explicitly becomes

$$\sigma(a) = \left(\frac{\sigma_0 - p_0}{\beta^2 + 1}\right)\left(\frac{a_0}{a}\right)^{3\left(\beta^2+1\right)} + \frac{\beta^2 \sigma_{0-} p_0}{\beta^2 + 1}. \tag{3.13}$$

This, together with (3.7) lead us to the equation of motion for the radius of the throat, which reads

$$-\frac{\sqrt{f(a) + \dot{a}^2}}{4\pi}\left[\frac{3}{a} - \frac{4\alpha}{a^3}\left(f(a) - 3 - 2\dot{a}^2\right)\right] = \left(\frac{\sigma_{0+} p_0}{\beta^2 + 1}\right)\left(\frac{a_0}{a}\right)^{3\left(\beta^2+1\right)} + \frac{\beta^2 \sigma_{0-} p_0}{\beta^2 + 1}. \tag{3.14}$$

After some manipulation this can be cast into

$$\dot{a}^2 + V(a) = 0, \tag{3.15}$$

where $V(a)$ is given by (2.42)-(2.44) with $d = 5$ and $\tilde{\alpha} = 2\alpha$. We notice that $V(a)$, and more tediously $V'(a)$, both vanish at $a = a_0$. The stability requirement for equilibrium reduces therefore to the determination of $V''(a_0) > 0$, and it is needless to add that, $V(a)$ is complicated enough for an immediate analytical result. For this reason we shall proceed through numerical calculation to see whether stability regions/ islands develop or not. Since the hopes for obtaining TSWs with normal matter when $\alpha > 0$, have already been dashed [29, 30], we shall investigate here only the case for $\alpha < 0$.

In order to analyze the behavior of $V(a)$ (and its second derivative) we introduce new parameterization as follows

$$\tilde{a}^2 = -\frac{a^2}{\alpha},\ m = -\frac{16M}{\pi\alpha},\ q^2 = \frac{8Q^2}{3\alpha^2},\ \tilde{\sigma}_0 = \sqrt{-\alpha}\sigma_0,\ p_0 = \sqrt{-\alpha}p_0 \tag{3.16}$$

Accordingly, our new variables $f(\tilde{a})$, $\tilde{\sigma}_0$, $\tilde{p}_0$, A and B take the following forms

$$f(\tilde{a}) = 1 - \frac{\tilde{a}^2}{4} + \frac{\tilde{a}^2}{4}\sqrt{1 - \frac{m}{\tilde{a}^4} + \frac{q^2}{\tilde{a}^6}} \tag{3.17}$$

and

$$\tilde{\sigma}_0 = -\frac{\sqrt{f(\tilde{a}_0)}}{4\pi}\left[\frac{3}{\tilde{a}_0} + \frac{4}{\tilde{a}_0^3}\left(f(\tilde{a}_0) - 3\right)\right], \tag{3.18}$$

$$\tilde{p}_0 = \frac{\sqrt{f(\tilde{a}_0)}}{4\pi}\left[\frac{2}{\tilde{a}_0} + \frac{f'(\tilde{a}_0)}{2f(\tilde{a}_0)} + \frac{2}{\tilde{a}_0^2}\frac{f'(\tilde{a}_0)}{f(\tilde{a}_0)}\left(f(\tilde{a}_0) - 1\right)\right], \tag{3.19}$$

and $V(a)$ is the same as the Eq. (2.42) in which the functions A and B are

$$\mathbb{A} = -\frac{\pi\tilde{a}^3}{4}\left[\left(\frac{\tilde{\sigma}_{0+}\tilde{p}_0}{\beta^2+1}\right)\left(\frac{\tilde{a}_0}{\tilde{a}}\right)^{3\left(\beta^2+1\right)} + \frac{\beta^2\tilde{\sigma}_{0-}\tilde{p}_0}{\beta^2+1}\right], \tag{3.20}$$

$$\mathbb{B} = -\frac{\tilde{a}^2}{8} + \frac{1-f(\tilde{a})}{2}. \tag{3.21}$$

Following this parametrization our Eq. (3.15) takes the form

$$\left(\frac{d\tilde{a}}{d\tau}\right)^2 + \tilde{V}(\tilde{a}) = 0, \tag{3.22}$$

where

$$\tilde{V}(\tilde{a}) = -\frac{V(\tilde{a})}{\alpha}. \tag{3.23}$$

In the next section we explore all possible constraints on our parameters that must satisfy to materialize a stable, normal matter wormhole through the requirement $V''(\tilde{a}) > 0.$

3.3. Constraints Versus Finely-Tuned Parameters

i) Starting from the metric function we must have

$$1-\frac{m}{\tilde{a}_0^4}+\frac{q^2}{\tilde{a}_0^6}\geq 0. \tag{3.24}$$

ii) In the potential, the reality condition requires also that

$$\mathbb{A}^2+\mathbb{B}^3\geq 0. \tag{3.25}$$

At the location of the throat this amounts to

$$\left(-\frac{\pi\tilde{a}_0^3}{4}\tilde{\sigma}_0\right)^2+\left(-\frac{\tilde{a}_0^2}{8}+\frac{1-f\left(\tilde{a}_0\right)}{2}\right)^3\geq 0 \tag{3.26}$$

or after some manipulation it yields

$$f\left(\tilde{a}_0\right)-2+\frac{\tilde{a}_0^2}{2}\leq 0. \tag{3.27}$$

This is equivalent to

$$0\leq 1-\frac{m}{\tilde{a}_0^4}+\frac{q^2}{\tilde{a}_0^6}\leq\left(\frac{4}{\tilde{a}_0^2}-1\right)^2. \tag{3.28}$$

iii) Our last constraint condition concerns, the positivity of the energy density, which means that

$$\tilde{\sigma}_0>0. \tag{3.29}$$

This implies, from (3.18) that

$$\left[\frac{3}{\tilde{a}_0}+\frac{4}{\tilde{a}_0^3}\left(f\left(\tilde{a}_0\right)-3\right)\right]<0 \tag{3.30}$$

or equivalently

$$0\leq 1-\frac{m}{\tilde{a}_0^4}+\frac{q^2}{\tilde{a}_0^6}<4\left(\frac{4}{\tilde{a}_0^2}-1\right)^2. \tag{3.31}$$

It is remarkable to observe now that the foregoing constraints ($i-iii$) on our parameters can all be expressed as a single constraint condition, namely

$$0\leq 1-\frac{m}{\tilde{a}_0^4}+\frac{q^2}{\tilde{a}_0^6}\leq\left(\frac{4}{\tilde{a}_0^2}-1\right)^2. \tag{3.32}$$

We plot $\tilde{V}''\left(\tilde{a}\right)$ from (3.23) for various fixed values of mass and charge, as a projection into the plane with coordinates β and $\tilde{a}_0$. In other words, we search and identify the regions for which $\tilde{V}''\left(\tilde{a}\right)>0$, in $3-$dimensional figures considered as a projection in the $\left(\beta,\tilde{a}_0\right)$ plane. The metric function $f\left(r\right)$ and energy density $\tilde{\sigma}_0>0$, behavior also are given in Fig.s 3.1-3.4.

It is evident from Fig.s 1-4 that for increasing charge the stability regions shrink to smaller domains and tends ultimately to disappear completely. For smaller $\tilde{a}_0$ bounds we obtain fluctuations in $\tilde{V}''\left(\tilde{a}\right)$, which is smooth otherwise. In each plot it is observed that the maximum of $\tilde{V}''\left(\tilde{a}\right)$ occurs at the right-below corner (say, at $a_{\max}$) which decreases to the left (with $\tilde{a}_0$) and in the upward direction (with β). Beyond certain limit (say $a_{\min}$), the region of instability takes the start. The proper

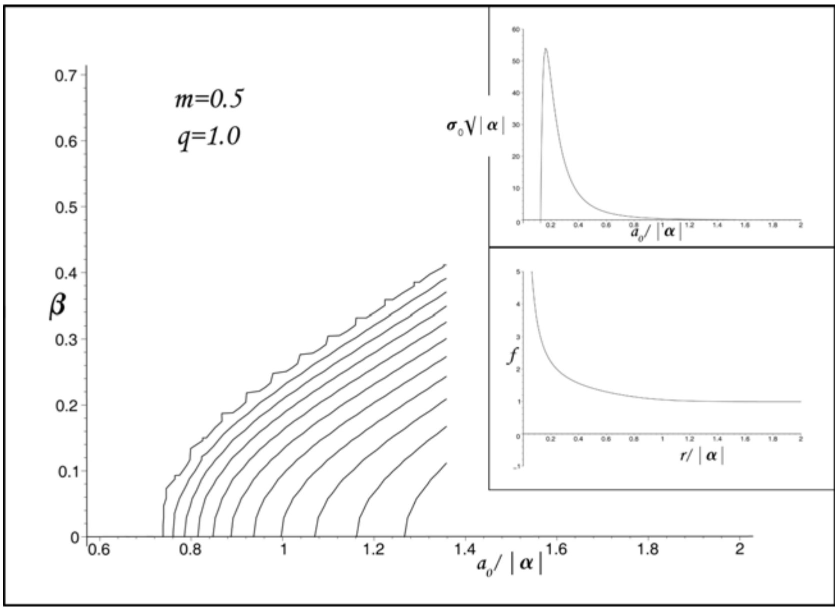

Figure 3.1. $\tilde{V}''(\tilde{a}) > 0$ region ($m = 0.5$, $q = 1.0$) for various ranges of β and $\tilde{a}_0$. The lower and upper limits of the parameters are evident in the figure. The metric function $f(\tilde{r})$ and $\tilde{\sigma}_0 > 0$, are also indicated in the smaller figures.

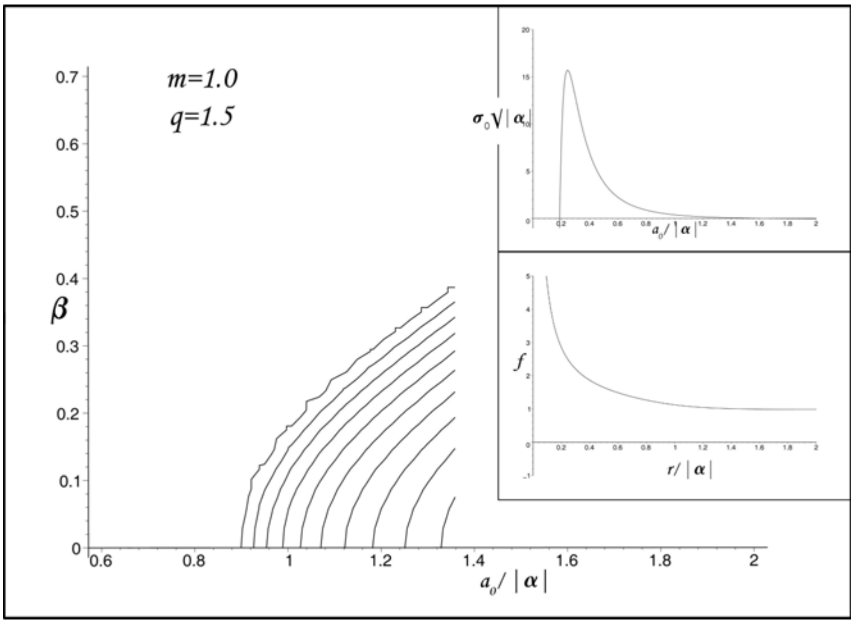

Figure 3.2. $\tilde{V}''(\tilde{a}) > 0$ plot for $m = 1.0$, $q = 1.5$. The stability region is seen clearly to shrink with the increasing charge. This effect reflects also to the $\tilde{\sigma}_0 > 0$, behavior.

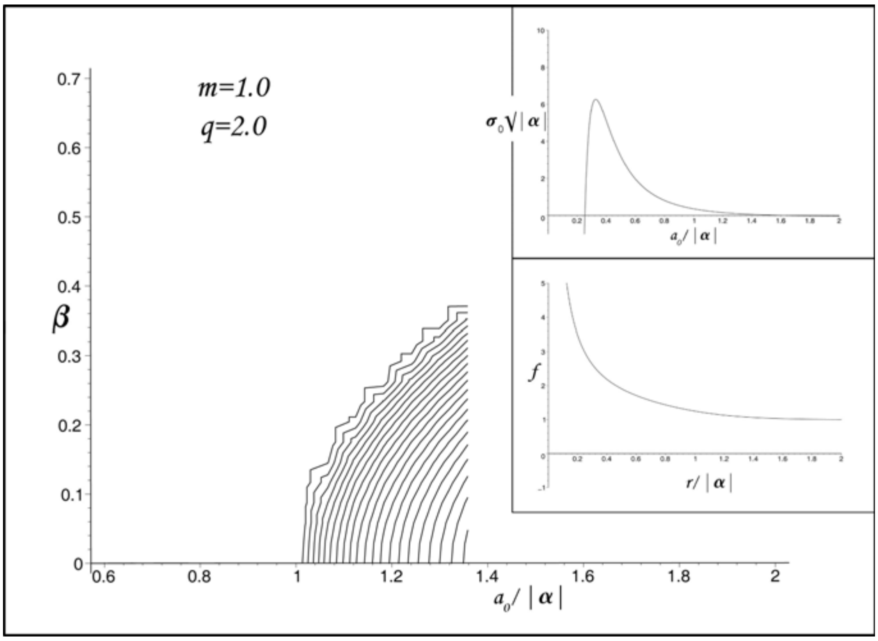

Figure 3.3. The stability region for $m = 1.0$, $q = 2.0$, is seen to shift outward and get smaller.

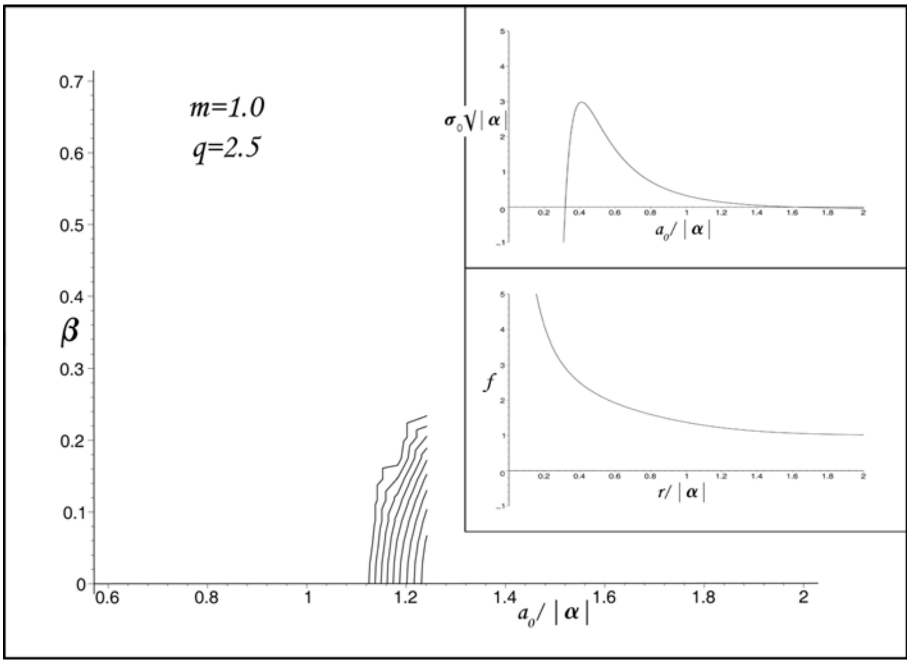

Figure 3.4. For fixed mass $m = 1.0$ but increased charge $q = 2.5$ it is clearly seen that the stability region and the associated energy density both get further reduced.

time domain of stability can be computed from (3.22) as

$$\Delta\tau = \int_{a_{\min}}^{a_{\max}} \frac{d\tilde{a}}{\sqrt{-V\left(\tilde{a}\right)}}. \tag{3.33}$$

From a distant observer's point of view the timespan Δt can be found by using the radial geodesics Lagrangian which admits the energy integral

$$f\left(\frac{dt}{d\tau}\right) = E_{\circ} = const. \tag{3.34}$$

This gives the lifetime of each stability region determined by

$$\Delta t = \frac{1}{E_{\circ}} \int_{a_{\min}}^{a_{\max}} \frac{d\tilde{a}}{f\left(\tilde{a}\right)\sqrt{-V\left(\tilde{a}\right)}}. \tag{3.35}$$

Once $a_{\min}$ ($a_{\max}$) are found numerically, assuming that no zeros of $f\left(\tilde{a}\right)$ and $V\left(\tilde{a}\right)$ occurs for $a_{\min} < a < a_{\max}$, the lifespan of each stability island can be determined. We must admit that the mathematical complexity discouraged us to search for possible metastable region that may be triggered by employing a semi-classical treatment.

CHAPTER 4

NON-ASYMPTOTICALLY FLAT THIN SHWLL WORMHOLES IN EINSTEIN-YANG-MILLS-DILATON GRAVITY

4.1. Overview

The original aim of a spacetime wormhole was to connect two distinct, asymptotically flat (AF) spacetimes, or two distant regions in the same AF spacetime [1]. In making such a short cut travel possible it is crucial that the traveller doesn't encounter event horizons of black holes. A thin-shell may support such a wormhole provided it has the proper source to resist against the gravitational collapse. An exotic matter, which fails to satisfy the energy conditions has been used extensively to provide maintenance of such wormholes. The non-physical source of energy is confined on rather thin spherical shells, simply to invoke justification from quantum theory. More recently, however, it has been shown that without such resort, TSWs can be constructed entirely from normal matter obeying the energy conditions [5, 6, 33]. Further, such wormholes established on realistic matter may be stable against radial, linear perturbations. Clearly this implies that existence of wormholes may be an undeniable reality in our universe.

In $4-$dimensional Einstein-Maxwell (EM) theory it was not possible to employ normal matter in the construction / maintenance of TSWs. In $5-$dimensions, with the Gauss-Bonnet (GB) extension of EM theory [19, 20, 21, 22, 23, 24, 25], it was further proved that stable, TSWs supported by normal matter is possible provided the GB parameter takes negative values, i.e. $\alpha < 0$ [5, 6, 33]. Is this true also for different sources such as Yang-Mills (YM) fields when considered in Einstein-Gauss-Bonnet (EGB) gravity? The answer, to the best of our knowledge, is not in the affirmative.

4.2. Review of Higher Dimensional Einstein-Yang-Mills-Dilaton Gravity

We consider the $d-$dimensional action in the EYMD theory as $(G = 1)$

$$\begin{aligned} S &= -\frac{1}{16\pi}\int_{\mathcal{M}} d^d x\sqrt{-g}\left(R - \frac{4}{d-2}\left(\boldsymbol{\nabla}\Phi\right)^2 + \mathcal{L}\left(\Phi\right)\right), \\ \mathcal{L}\left(\Phi\right) &= -e^{-4\alpha\Phi/(d-2)}\mathbf{Tr}(F^{(a)}_{\lambda\sigma}F^{(a)\lambda\sigma}), \end{aligned} \tag{4.1}$$

where

$$\mathbf{Tr}(.) = \sum_{a=1}^{\frac{(d-1)(d-2)}{2}} (.)\,, \tag{4.2}$$

Φ is the dilaton scalar potential, the parameter α denotes the coupling between dilaton and Yang-Mills (YM) field and as usual R is the Ricci scalar. The YM field $2-$forms $\mathbf{F}^{(a)} = F^{(a)}_{\mu\nu}dx^\mu \wedge dx^\nu$ are given by [9, 11]

$$\mathbf{F}^{(a)} = \mathbf{dA}^{(a)} + \frac{1}{2\sigma}C^{(a)}_{(b)(c)}\mathbf{A}^{(b)} \wedge \mathbf{A}^{(c)} \tag{4.3}$$

in which $C^{(a)}_{(b)(c)}$ are the structure constants, σ is a coupling constant and the YM potential $1-$forms are given by (2.3). The field equations, after varying the action, are given by

$$\mathbf{d}\left(e^{-4\alpha\Phi/(d-2)\star}\mathbf{F}^{(a)}\right) + \frac{1}{\sigma}C^{(a)}_{(b)(c)}e^{-4\alpha\Phi/(d-2)}\mathbf{A}^{(b)} \wedge^{\star} \mathbf{F}^{(c)} = 0, \tag{4.4}$$

$$R_{\mu\nu}=\frac{4}{d-2}\partial_{\mu}\Phi\partial_{\nu}\Phi+2e^{-4\alpha\Phi/(d-2)}\left[\mathbf{Tr}\left(F_{\mu\lambda}^{(a)}F_{\nu}^{(a)\ \lambda}\right)-\frac{1}{2\left(d-2\right)}\mathbf{Tr}(F_{\lambda\sigma}^{(a)}F^{(a)\lambda\sigma})g_{\mu\nu}\right], \tag{4.5}$$

$$\nabla^{2}\Phi=-\frac{1}{2}\alpha e^{-4\alpha\Phi/(d-2)}\mathbf{Tr}(F_{\lambda\sigma}^{(a)}F^{(a)\lambda\sigma}), \tag{4.6}$$

in which $R_{\mu\nu}$ is the Ricci tensor and the hodge star * means duality. As it was shown in Ref. [9, 11], these equations admit black hole solution in the form of

$$ds^{2}=-f\left(r\right)dt^{2}+\frac{dr^{2}}{f\left(r\right)}+h\left(r\right)^{2}d\Omega_{d-2}^{2}, \tag{4.7}$$

where $d\Omega_{d-2}^{2}$ is the line element on S^{d-2} and the solution can be summarized as follows

$$\Phi=-\frac{(d-2)}{2}\frac{\alpha\ln r}{\alpha^{2}+1},\quad h\left(r\right)=Ar^{\frac{\alpha^{2}}{\alpha^{2}+1}},\quad f\left(r\right)=\Xi\left(1-\left(\frac{r_{h}}{r}\right)^{\frac{(d-3)\alpha^{2}+1}{\alpha^{2}+1}}\right)r^{\frac{2}{\alpha^{2}+1}}. \tag{4.8}$$

We abbreviate here

$$\Xi=\frac{(d-3)}{\left(\left(d-3\right)\alpha^{2}+1\right)Q^{2}},\quad A^{2}=Q^{2}\left(\alpha^{2}+1\right),\ r_{h}=\left(\frac{4\left(\alpha^{2}+1\right)M}{\left(d-2\right)\Xi\alpha^{2}A^{d-2}}\right) \tag{4.9}$$

and r_h stands for the radius of event horizon. Here M implies the quasilocal mass (see [11] and the references therein).

4.3. Dynamic Thin Shell Wormholes in Einstein-Yang-Mills-Dilaton Gravity

Following the method introduced in Sec. 2.3, we consider two copies of of the EYMD spacetime to construct the TSWs in EYMD gravity. Beside those given in

Sec. 2.3 we note that the induced metric on the TSW is given by

$$g_{ij} = \text{diag}\left(-1, h\left(a\right)^2, h\left(a\right)^2 \sin^2\theta_1, h\left(a\right)^2 \sin^2\theta_1 \sin^2\theta_2, ...\right). \tag{4.10}$$

The parametric equation of the hypersurface Σ is given by

$$F\left(r, a\left(\tau\right)\right) = r - a\left(\tau\right) = 0, \tag{4.11}$$

and the normal unit vectors to $M^{\pm}$ defined by

$$n_\gamma = \left(\pm\left|g^{\alpha\beta}\frac{\partial F}{\partial x^\alpha}\frac{\partial F}{\partial x^\beta}\right|^{-1/2}\frac{\partial F}{\partial x^\gamma}\right)_{r=a}, \tag{4.12}$$

are found as follows

$$n_t = \pm\left(\left|g^{tt}\left(\frac{\partial a\left(\tau\right)}{\partial t}\right)^2 + g^{rr}\right|^{-1/2}\frac{\partial F}{\partial t}\right)_{r=a}. \tag{4.13}$$

Upon using

$$\left(\frac{\partial t}{\partial \tau}\right)^2 = \frac{1}{f\left(a\right)}\left(1 + \frac{1}{f\left(a\right)}\dot{a}^2\right), \tag{4.14}$$

it implies

$$n_t = \pm\left(-\dot{a}\right). \tag{4.15}$$

Similarly one finds that

$$n_r = \pm \left(\left| g^{tt} \frac{\partial F}{\partial t} \frac{\partial F}{\partial t} + g^{rr} \frac{\partial F}{\partial r} \frac{\partial F}{\partial r} \right|^{-1/2} \frac{\partial F}{\partial r} \right)_{r=a} = \qquad (4.16)$$

$$\pm \left(\frac{\sqrt{f\left(a\right) + \dot{a}^2}}{f\left(a\right)} \right), \text{ and } n_{\theta_i} = 0, \text{ for all } \theta_i.$$

After the unit $d-$normal, one finds the extrinsic curvature tensor components from the definition

$$K_{ij}^{\pm} = -n_{\gamma}^{\pm} \left(\frac{\partial^2 x^{\gamma}}{\partial \xi^i \partial \xi^j} + \Gamma_{\alpha\beta}^{\gamma} \frac{\partial x^{\alpha}}{\partial \xi^i} \frac{\partial x^{\beta}}{\partial \xi^j} \right)_{r=a}. \qquad (4.17)$$

It follows that

$$K_{\tau\tau}^{\pm} = -n_t^{\pm} \left(\frac{\partial^2 t}{\partial \tau^2} + \Gamma_{\alpha\beta}^{t} \frac{\partial x^{\alpha}}{\partial \tau} \frac{\partial x^{\beta}}{\partial \tau} \right)_{r=a} - n_r^{\pm} \left(\frac{\partial^2 r}{\partial \tau^2} + \Gamma_{\alpha\beta}^{r} \frac{\partial x^{\alpha}}{\partial \tau} \frac{\partial x^{\beta}}{\partial \tau} \right)_{r=a} =$$

$$-n_t^{\pm} \left(\frac{\partial^2 t}{\partial \tau^2} + 2\Gamma_{tr}^{t} \frac{\partial t}{\partial \tau} \frac{\partial r}{\partial \tau} \right)_{r=a} - n_r^{\pm} \left(\frac{\partial^2 r}{\partial \tau^2} + \Gamma_{tt}^{r} \frac{\partial t}{\partial \tau} \frac{\partial t}{\partial \tau} + \Gamma_{rr}^{r} \frac{\partial r}{\partial \tau} \frac{\partial r}{\partial \tau} \right)_{r=a} =$$

$$\pm \left(-\frac{f' + 2\ddot{a}}{2\sqrt{f + \dot{a}^2}} \right). \qquad (4.18)$$

Also

$$K_{\theta_i\theta_i}^{\pm} = -n_{\gamma}^{\pm} \left(\frac{\partial^2 x^{\gamma}}{\partial \theta_i^2} + \Gamma_{\alpha\beta}^{\gamma} \frac{\partial x^{\alpha}}{\partial \theta_i} \frac{\partial x^{\beta}}{\partial \theta_i} \right)_{r=a} = \pm \sqrt{f\left(a\right) + \dot{a}^2} h h'. \qquad (4.19)$$

In sum, we have

$$\left\langle K_{ij} \right\rangle = 2hh' \sqrt{f\left(a\right) + \dot{a}^2} \text{diag} \left(-\frac{f' + 2\ddot{a}}{2hh' \left(f + \dot{a}^2\right)}, 1, \sin^2\theta_1, \sin^2\theta_1 \sin^2\theta_2, ... \right), \qquad (4.20)$$

which implies

$$\left\langle K_i^j \right\rangle = 2\sqrt{f\left(a\right)+\dot{a}^2}\text{diag}\left(\frac{f'+2\ddot{a}}{2\left(f+\dot{a}^2\right)},\frac{h'}{h},\frac{h'}{h},\frac{h'}{h},...\right) \tag{4.21}$$

and therefore

$$K = \text{Trace}\left\langle K_i^j \right\rangle = \left\langle K_i^i \right\rangle = \frac{f'+2\ddot{a}}{\sqrt{f+\dot{a}^2}} + 2\left(d-2\right)\sqrt{f\left(a\right)+\dot{a}^2}\frac{h'}{h}. \tag{4.22}$$

The surface energy-momentum components of the thin-shell are [34, 35, 14, 15, 17, 36, 37]

$$S_i^j = -\frac{1}{8\pi}\left(\left\langle K_i^j \right\rangle - \left\langle K \right\rangle \delta_i^j\right) \tag{4.23}$$

which yield

$$\sigma = -S_\tau^\tau = -\frac{\left(d-2\right)}{4\pi}\left(\sqrt{f\left(a\right)+\dot{a}^2}\frac{h'}{h}\right), \tag{4.24}$$

$$S_{\theta_i}^{\theta_i} = p_{\theta_i} = \frac{1}{8\pi}\left(\frac{f'+2\ddot{a}}{\sqrt{f\left(a\right)+\dot{a}^2}} + 2\left(d-3\right)\sqrt{f\left(a\right)+\dot{a}^2}\frac{h'}{h}\right). \tag{4.25}$$

By substitution one can show that the energy conservation takes the form

$$\nabla_i S^{ij} = \frac{d}{d\tau}\left(\sigma\mathcal{A}\right) + p\frac{d}{d\tau}\left(\mathcal{A}\right) = -\frac{\left(d-2\right)}{4\pi}\frac{h''}{h}\dot{a}\mathcal{A}\sqrt{f\left(a\right)+\dot{a}^2} \neq 0, \tag{4.26}$$

in which $\mathcal{A} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}h\left(a\right)^{d-2}$ is the area of the thin-shell. In other words, due to the exchange with the bulk spacetime, the energy on the shell is not conserved.

Let us note that we adopt the junction conditions of general relativity which can be justified by the fact that the dilaton field Φ and its normal derivative are both continuous across the shell. Similar approach has been followed by different authors [4, 38] which can be justified easily by integrating the dilaton equation (4.6) across the throat radius $a_0 \pm \epsilon$, in the limit as $\epsilon \to 0$. Since the singularity and horizons all reside deliberately at distances $r < a_0$, the contribution from the dilaton field and its derivative to the thin-shell source vanishes. We note that this is different in the case of Brans-Dicke (BD) scalar field, which has structural difference compared with the dilaton field [39, 62]. To say the least among the others, the exponential coupling of dilaton with the gauge field makes it short ranged whereas BD field is long ranged. To calculate the amount of exotic matter needed to construct the traversable wormhole we use the integral (2.35). For a TSW $p_r = 0$ and $\rho = \sigma\delta\left(r-a\right)$, where $\delta\left(r-a\right)$ is the Dirac delta function. In static configuration, a simple calculation gives

$$\Omega = \int_0^{2\pi}\int_0^{\pi}...\int_0^{\pi}\int_0^{\infty}\sqrt{-g}\sigma\delta\left(r-a\right)drd\theta_1 d\theta_2...d\theta_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}h\left(a\right)^{d-2}\sigma\left(a\right). \tag{4.27}$$

We aim now to apply a small radial perturbation around the radius of equilibrium a_0 and to investigate the behavior of the throat under this perturbation. For this perturbation we consider the radial pressure of the thin-shell to be a linear function of the energy density, i.e., (2.38) [31, 18]. Therein p_0 and σ_0 are the radial pressure and energy density of the thin-shell in the static configuration of radius a_0 which are given by

$$\sigma_0 = -\frac{(d-2)}{4\pi}\left(\sqrt{f\left(a_0\right)}\frac{h'\left(a_0\right)}{h\left(a_0\right)}\right), \tag{4.28}$$

$$p_0 = \frac{1}{8\pi}\left(\frac{f'(a_0)}{\sqrt{f(a_0)}} + 2(d-3)\sqrt{f(a_0)}\frac{h'(a_0)}{h(a_0)}\right). \tag{4.29}$$

By substituting (2.38) into (4.26), one finds a first order differential equation for $\sigma(a)$ which is given by

$$\sigma'(a) + (d-2)(\sigma+p)\frac{h'(a)}{h(a)} = \frac{h''(a)}{h'(a)}\sigma(a), \tag{4.30}$$

or equivalently

$$\sigma'(a) + \sigma(a)\left[(d-2)\left(1+\beta^2\right)\frac{h'(a)}{h(a)} - \frac{h''(a)}{h'(a)}\right] = (d-2)\left(\sigma_0\beta^2 - p_0\right)\frac{h'(a)}{h(a)}. \tag{4.31}$$

This equation, for the case of EYMD wormhole introduced before can be expressed as

$$r\sigma'(a) + \xi_1\sigma(a) = \xi_2 \tag{4.32}$$

in which

$$\xi_1 = \frac{1+(d-2)\alpha^2(1+\beta^2)}{\alpha^2+1}, \tag{4.33}$$

$$\xi_2 = \frac{(d-2)\alpha^2(\sigma_0\beta^2 - p_0)}{\alpha^2+1}. \tag{4.34}$$

This equation admits a solution in the form of

$$\sigma(a) = \frac{\xi_2}{\xi_1} + \left(\sigma_0 - \frac{\xi_2}{\xi_1}\right)\left(\frac{a_0}{a}\right)^{\xi_1}, \tag{4.35}$$

which obviously at $a = a_0$ is nothing but σ_0. In terms of the metric function $f(a)$,

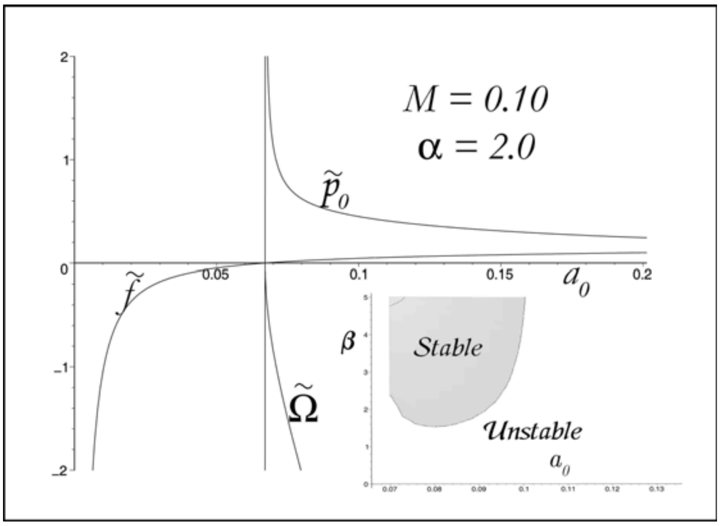

Figure 4.1. The plot of $\tilde{f}(a_0) = Q^2 f(a_0)$, $\tilde{\Omega}(a_0) = \Omega(a_0)/Q^2$ and pressure $\tilde{p}_0 = p_0|Q|$, for $d = 5$. The shaded inscribed part shows the stable regions (i.e. $V''(a_0) > 0$) in the (β, a_0) diagram for $M = 0.1$,$\alpha = 2.0$.

in a dynamic case $\sigma(a)$ was given by (4.24) which after equating with the latter expression in (4.35) we obtain (2.41). Here the potential function $V(a)$ is given by

$$V(a) = f(a) - \frac{16\pi^2 a^2}{(d-2)^2}\left(\frac{\alpha^2+1}{\alpha^2}\right)^2\left[\frac{\xi_2}{\xi_1} + \left(\sigma_0 - \frac{\xi_2}{\xi_1}\right)\left(\frac{a_0}{a}\right)^{\xi_1}\right]^2. \tag{4.36}$$

We notice that $V(a)$, and more tediously $V'(a)$, both vanish at $a = a_0$. The stability requirement for equilibrium reduces therefore to the determination of the regions in which $V''(a_0) > 0$. For this reason we shall proceed through numerical analysis to see whether stability regions/ islands develop and under what conditions. Let us note that our figures refer to $d = 5$, for $d > 5$ we observed that no significant changes take place. In Fig.s 4.1-4.5 we show the stability regions in terms of β and wormhole throat a_0.

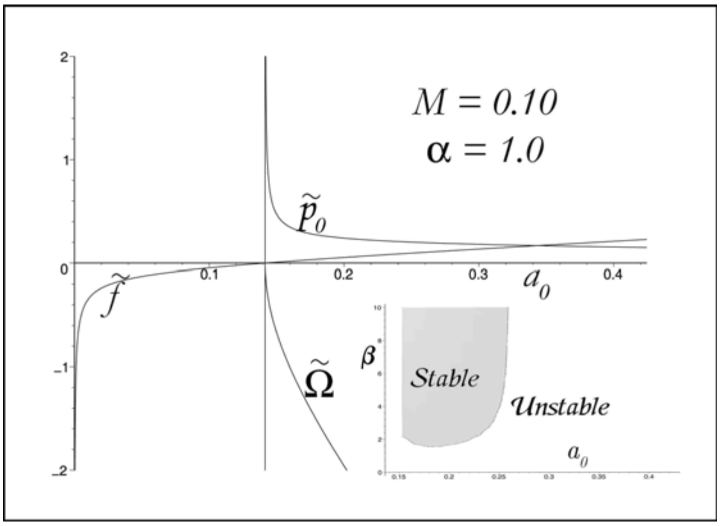

Figure 4.2. The plot of $\tilde{f}(a_0) = Q^2 f(a_0)$, $\tilde{\Omega}(a_0) = \Omega(a_0)/Q^2$ and pressure $\tilde{p}_0 = p_0|Q|$, for $d = 5$. The shaded inscribed part shows the stable regions (i.e. $V''(a_0) > 0$) in the (β, a_0) diagram for $M = 0.10$,$\alpha = 1.0$.

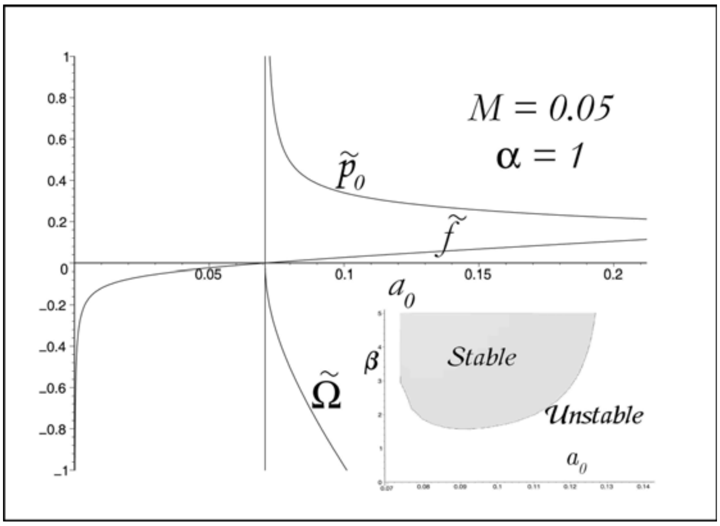

Figure 4.3. The plot of $\tilde{f}(a_0) = Q^2 f(a_0)$, $\tilde{\Omega}(a_0) = \Omega(a_0)/Q^2$ and pressure $\tilde{p}_0 = p_0|Q|$, for $d = 5$. The shaded inscribed part shows the stable regions (i.e. $V''(a_0) > 0$) in the (β, a_0) diagram for $M = 0.05$,$\alpha = 1.0$. It is seen that the Ω behavior doesn't differ much in this range of parameters.

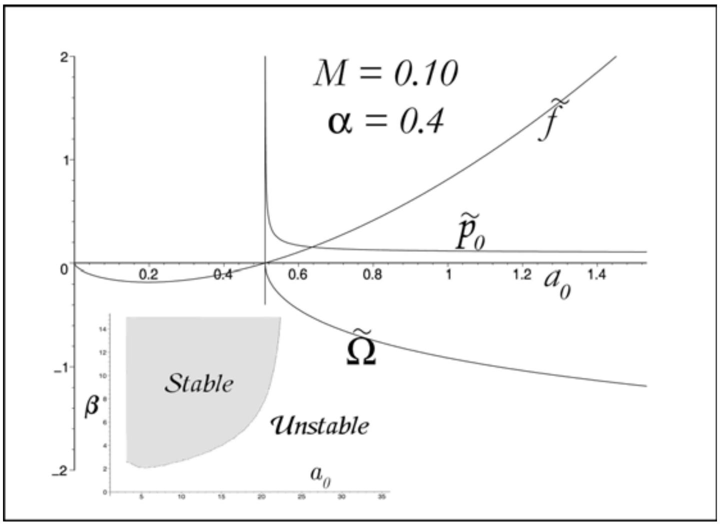

Figure 4.4. Similar plots for $\tilde{f}\left(a_0\right)$, $\tilde{\Omega}\left(a_0\right)$ and pressure $\tilde{p}_0$, (again for $d=5$), for $M=0.10$, $\alpha=0.4$.

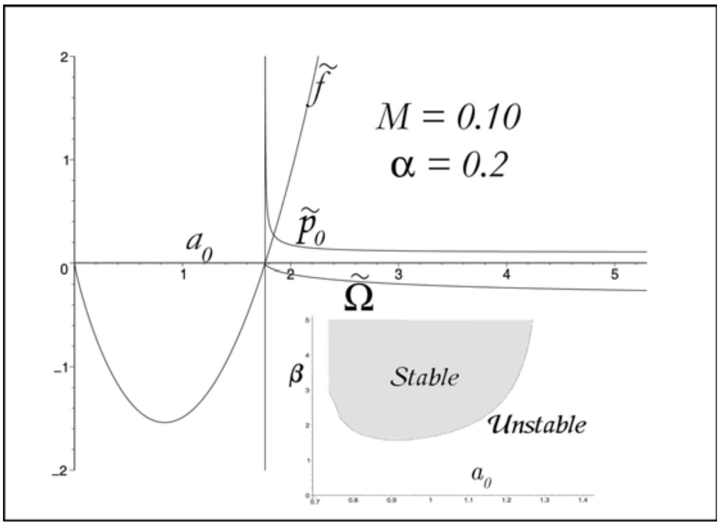

Figure 4.5. Similar plots for $\tilde{f}\left(a_0\right)$, $\tilde{\Omega}\left(a_0\right)$ and pressure $\tilde{p}_0$, (again for $d=5$), for $M=0.10$, $\alpha=0.2$. Decreasing α values improves $\tilde{\Omega}\left(a_0\right)$ slightly which still lies in the $(-)$ domain. Smaller α implies also less pressure ($\tilde{p}_0$) on the shell. Stable regions ($V''\left(a_0\right)>0$) are shown, versus (β, a_0) in dark.

(Note that we use scalings $\Omega = Q^2\tilde{\Omega}$, $f = \frac{\tilde{f}}{Q^2}$ and $p_0 = \frac{1}{|Q|}\tilde{p}_0$ in terms of the YM charge Q and we plot $(\tilde{\Omega}, \tilde{f}, \tilde{p}_0)$). We also show $\tilde{\Omega}$ in terms of a_0 for different values of α. As stated before, our wormhole is supported entirely by negative energy, $\tilde{\Omega} < 0$ on the shell. For changing dilatonic parameter α the change in $\tilde{\Omega}$ is visible in the plots shown.

For fixed mass, decreasing α improves $\tilde{\Omega}$ toward zero line but yet in the $(-)$ domain. The pressure on the shell increases with increasing α. It is noticed from the dark stability regions that no stable region forms for $|\beta| < 1$, i.e. below the speed of light. Consideration of dimensions $d > 5$, doesn't change the behaviors of $d = 5$ much, this can be seen by further plots which we shall ignore in these studies.

CHAPTER 5

THIN SHELL WORMHOLE IN HOFFMANN-BORN-INFELD THEORY

5.1. Overview

It is a well-known fact by now that non-linear electrodynamics (NED) with variations formulations has therapeutic effects on the divergent results that arise naturally in linear Maxwell electrodynamics. The theory introduced by Born and Infeld (BI) in 1930s [41] constitutes the most prominent member among the healing power of singularities, however, drawbacks were not completely eliminated from the theory. One such serious handicap was pointed out by Born's co-workers shortly after the introduction of the original BI theory. This concerns the double-valued degenerace of the displacement vector $\vec{D}\left(\vec{E}\right)$ as a function of the electric field $\vec{E}$ [42]. That is, for the common value of $\vec{E}$ the displacement D undergoes a branching which from physical grounds was totally unacceptable. To overcome this particular problem, Hoffmann and Infeld and Rosen, both published successive papers on this issue [41, 42, 43]. Specifically, the model Lagrangian proposed by Hoffmann and Infeld (HI) contained a logarithmic term with remarkable consequences. It removed, for instance, the singularity that used to arise in the Cartesian components of the $\vec{E}$. Being unaware of this contribution by HI, and after almost 70 years, we have rediscovered very recently the ubiquitous logarithmic term of Lagrangian while in attempt to construct a model of elementary particle in Einstein-NED theory [44]. In our model the spacetime is divided into two regions: the inner region consists of the Bertotti-Robinson (BR) [45] spacetime while the outer region is a Reissner-Nordström (RN) type spacetime. The radius of our particle coincides with the horizon of the RN-type black hole solution whereas inner BR part represents a singularity-free uniform electric field region. The two regions and the NED are glued together at the horizon on which the appropriate boundary gave not only a feasible geometrical model of a particle but remarkably resolved also the double-valued property of the displacement vector. In other words, with our technique $\vec{D}\left(\vec{E}\right)$ turns automatically into a single-valued function. In this chapter we wish to make further use of the Hoffmann-Born-Infeld (HBI) Lagrangian in general relativity, more specifically, in constructing regular black holes and TSWs. We extend our model also to $5-$dimensional Gauss-Bonnet (GB) theory and search for the possibility of wormholes dominated by ordinary matter rather than exotic matter. It turns out, as our analysis supports, that the GB modification is not much promising in this regard. However, a non-black hole solution with asymptotically flat regions supported by the TSW against linear perturbations is tested and stable region is obtained.

5.2. Review of the Hoffmann-Born-Infeld Approach in General Relativity

Singularity for classical charged elementary particles leads to infinite self electromagnetic energy. This should be removed from the Maxwell theory of charged particles and in this regard Born and Infeld (BI) introduced a non-linear electrodynamics such that successfully they solved the problem in some senses [41]. Briefly, we can summarize their proposal in curved spacetime by considering a spherically symmetric pure electric particle (i.e., an electron) described by the line element

$$ds^2 = -f\left(r\right)dt^2 + \frac{1}{f\left(r\right)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{5.1}$$

They aimed to have a non-singular electric field (with unit charge) as (we use $c = \hbar = k_B = 8\pi G = \frac{1}{4\pi\epsilon_\circ} = 1$)

$$E_r = \frac{q}{\sqrt{q^2b^2 + r^4}}, \text{ (} b = \text{constant, the BI parameter and } q = \text{constant charge)} \tag{5.2}$$

which means that the Maxwell $2-$form is

$$\mathbf{F} = E_r dt \wedge dr. \tag{5.3}$$

The corresponding action is

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\mathcal{L}\left(F,{}^\star F\right), \tag{5.4}$$

in which $F = F_{\mu\nu}F^{\mu\nu}, {}^\star F = F_{\mu\nu}{}^\star F^{\mu\nu}$ and * stands for duality (here we only consider the electric field such that ${}^\star F = 0$). Easily one finds the Maxwell equation modified into

$$d\left(\mathcal{L}_F\,{}^\star\mathbf{F}\right) = 0, \;\left(\mathcal{L}_F = \frac{\partial\mathcal{L}}{\partial F}\right) \tag{5.5}$$

which reveals

$$d\left(\mathcal{L}_F\;E_r r^2\sin\theta d\theta\wedge d\varphi\right) = 0, \tag{5.6}$$

or

$$\mathcal{L}_F \; E_r = \frac{c}{r^2}. \tag{5.7}$$

Since $F = F_{\mu\nu}F^{\mu\nu} = -2\;{E_r}^2$ and $r^2 = \sqrt{q^2\left(\frac{1-b^2{E_r}^2}{{E_r}^2}\right)} = \sqrt{-q^2\left(\frac{2+b^2F}{F}\right)}$it yields

$$\mathcal{L}_F \;\; = c\sqrt{\frac{2}{2+b^2F}} \tag{5.8}$$

where $c =$constant of integration which is identified as the charge q. Solution for the Lagrangian, after adjusting the constants takes the form of

$$\mathcal{L} = \frac{4}{b^2}\left(1-\sqrt{1+\frac{b^2F}{2}}\right), \tag{5.9}$$

i.e. the BI Lagrangian.

This example gives an idea of how simple it is to find a Lagrangian which yields a non-singular electric field, but the question was whether this much was enough. B. Hoffmann and L. Infeld [42] shortly after the BI non-linear Lagrangian, pointed this problem out and tried to get rid of any possible difficulties.

In [42] the authors remarked that although the electric field becomes finite at $r = 0$ it yields a discontinuity, for instance in E_x. To quote from [42] ”It is evident that any finite value for E_r at $r = 0$ will lead to a discontinuity of this type”. Accordingly, their proposal alternative to the BI Lagrangian can be summarized as follows. The simplest non-singular electric field which takes zero value at $r = 0$ can be written as

$$E_r = \frac{qr^2}{(q^2b^2+r^4)}, \tag{5.10}$$

so that r^2 in terms of F is

$$r^2 = q\frac{1 \pm \sqrt{1 - 4b^2E_r^2}}{2E_r} = q\frac{1 \pm \sqrt{1 + 2b^2F}}{\sqrt{-2F}} \tag{5.11}$$

where $+$ and $-$ stand for $r^4 > q^2b^2$ and $r^4 < q^2b^2$, respectively. From (5.8) we find

$$\mathcal{L}_F = \frac{2c}{1 \pm \sqrt{1 + 2b^2F}} \tag{5.12}$$

where the positive branch leads to the Lagrangian

$$\mathcal{L}_+ = -\frac{2}{b^2}\left(k + \alpha\epsilon_+ - \ln \epsilon_+\right) \tag{5.13}$$

with $\alpha = 1$, $k = \ln 2 - 2$ and $\epsilon_+ = 1 + \sqrt{1 + 2b^2F}$. Let us note that we wrote the Lagrangian in this form to show consistency with [42]. Again we remind that the constant c has been chosen in such a way that $\lim_{b\to 0} \mathcal{L} = -F$, which is the Maxwell limit. In analogy, the negative branch gives

$$\mathcal{L}_- = -\frac{2}{b^2}\left(k + \alpha\epsilon_- - \ln |\epsilon_-|\right) \tag{5.14}$$

where $\epsilon_- = 1 - \sqrt{1 + 2b^2F}$. It should be noted that, here one does not expect the Maxwell limit as b goes to zero. In fact, since $\mathcal{L}_-$ is defined for $r^4 < q^2b^2$, automatically b can not be zero unless r also goes to zero in which, the case $\mathcal{L}_-$ becomes meaningless.

Having $\mathcal{L}_+$ for $r^4 > q^2b^2$ and $\mathcal{L}_-$ for $r^4 < q^2b^2$ imposes $(\mathcal{L}_+ = \mathcal{L}_-)_{r^4=q^2b^2}$ which is satisfied, as it should. Also at $r^4 = q^2b^2$, one gets $E_r = \frac{q}{2b}$ which is the maximum value that E_r may take.

Based on the criticisms made in [42], as mentioned above, we see that this Lagrangian removes the discontinuity in, say, E_x. So shall we adopt this Lagrangian for further results? The answer was given few years later by N. Rosen [43], which was negative. The crux of the problem lies in the relation between E_r and D_r. Let us go back to the previous case (5.10) once more. It is known from non-linear electrodynamics [41, 42, 43] that

$$D_r = \mathcal{L}_F E_r = \frac{q}{r^2} \tag{5.15}$$

which is singular at $r = 0$. Of course, being singular for D_r does not matter; the problem arises once we consider D_r as a function of E_r. In this way at $r = 0$, $E_r = 0$ and $D_r = \infty$, and once $r = \infty$ again $E_r = 0$, but $D_r = 0$. This means that D_r in terms of E_r is double-valued(i.e., $D_r \left(E_r \left(r = 0\right) = 0\right) = \infty$ and $D_r \left(E_r \left(r = \infty\right) = 0\right) = 0$). Concerning this objection Rosen suggested to reject this Lagrangian and instead he recommended that the Lagrangian should be a function of the potentials. For the detail of his work we suggest Ref. [43], but here we wish to draw attention to a recent paper we published [44] which gives a different solution to this problem. Before we give the detail of the solution we admit that during the time of working on [44] we were not aware about this problem and we did not know the Hoffmann-Infeld (HI) form of Lagrangian. In certain sense, we have rediscovered anew a Lagrangian of 70 years old, from the hard way.! Our aim in that work was based on some different papers such as [45, 46]. Recently, indirectly from [47, 48] we became aware about the history of this problem. with these remarks, therefore, firstly we wish to pay tribute to all its historic, eminent originators, and secondly, to draw attention to the importance of such a Lagrangian.

Returning to the problem, we see that in the case of the HI Lagrangian they used two different forms for inside and outside of the typical particle in order to keep the spacetime spherically symmetric, static Reissner-Nordström (RN) type. This is understandable since in 1930s RN solution was one of the best known solution whereas the Bertotti-Robinson (BR) [13, 14, 15] solution was yet unknown. The latter, i.e., (BR), constitutes a prominent inner substitute to (RN) as far as Einstein-Maxwell solutions are concerned and resolves the singularity at $r = 0$, which caused HI to worry about [42]. As we gave the detail of such a choice in Ref. [44], one can choose $\mathcal{L}_+ = -\frac{2}{b^2}\left(k + \alpha\epsilon_+ - \ln\epsilon_+\right)$ for all regions (i.e., $r \geq \sqrt{qb} =$ the radius of our particle and $r \leq \sqrt{qb}$). For outside we adopted a RN type spacetime while for inside we had to choose a BR type spacetime. Accordingly one finds

$$E_r = \begin{cases} \frac{1}{2b}, & r \leq \sqrt{qb} \\ \frac{qr^2}{(q^2b^2+r^4)}, & r \geq \sqrt{qb} \end{cases} \tag{5.16}$$

and consequently

$$D_r = \begin{cases} \frac{1}{b}, & r \leq \sqrt{qb} \\ \frac{q}{r^2}, & r \geq \sqrt{qb} \end{cases} \tag{5.17}$$

which clearly reveals that D_r is not a double valued function of E_r any more. We note that in matching the two spacetime the Lanczos energy-momentum tensor [18] was employed. Let us add that this is not the unique choice, so that the opposite choice also is possible. That is, a RN type space time for $r \leq \sqrt{qb}$ and a BR type spacetime for $r \geq \sqrt{qb}$. In this latter choice the Lagrangian is $\mathcal{L}_- = -\frac{2}{b^2}\left(k + \alpha\epsilon_- - \ln|\epsilon_-|\right)$

everywhere, which yields

$$E_r = \begin{cases} \frac{qr^2}{(q^2b^2+r^4)}, & r \leq \sqrt{qb} \\ \frac{1}{2b}, & r \geq \sqrt{qb} \end{cases} \tag{5.18}$$

and

$$D_r = \begin{cases} \frac{q}{r^2}, & r \leq \sqrt{qb} \\ \frac{1}{b}, & r \geq \sqrt{qb} \end{cases} \tag{5.19}$$

is again not double-valued. In Ref. [44] we studied in detail the first case alone. Obviously, the second case also can be developed into a model of elementary particle.

5.3. A Different Aspect of the Hoffmann-Born-Infeld Spacetime

Once more we start from the HBI Lagrangian

$$\mathcal{L} = \begin{cases} \mathcal{L}_-, & r \leq \sqrt{qb} \\ \mathcal{L}_+, & r \geq \sqrt{qb} \end{cases} . \tag{5.20}$$

where b is a free parameter such that

$$\lim_{b\to 0} \mathcal{L} = \lim_{b\to 0} \mathcal{L}_+ = -F \tag{5.21}$$

and

$$\lim_{b\to\infty} \mathcal{L} = \lim_{b\to\infty} \mathcal{L}_- = 0 \tag{5.22}$$

which are the RN and S limits, respectively. A solution to the Einstein equations which gives the correct limits may be written as

$$f(r) = 1 - \frac{2m}{r} + \frac{q^2}{3r_\circ^4} r^2 \ln\left(\frac{r^4}{r^4 + r_\circ^4}\right) - \frac{q^2\sqrt{2}}{3rr_\circ}\left[\tan^{-1}\left(\frac{\sqrt{2}r}{r_\circ} + 1\right) + \tan^{-1}\left(\frac{\sqrt{2}r}{r_\circ} - 1\right)\right] - \frac{q^2\sqrt{2}}{6rr_\circ}\ln\left[\frac{r^2 + r_\circ^2 - \sqrt{2}rr_\circ}{r^2 + r_\circ^2 + \sqrt{2}rr_\circ}\right] + \frac{\sqrt{2}q^2\pi}{rr_\circ}, \tag{5.23}$$

here $r_\circ = \sqrt{qb}$and m is the correspondence mass of S and RN spacetime. One can easily show that

$$\lim_{b\to 0} f(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2} \tag{5.24}$$

and

$$\lim_{b\to\infty} f(r) = 1 - \frac{2m}{r}. \tag{5.25}$$

It is interesting to observe that although the ADM mass of HBI solution is still m but the effective mass depends on charge and HBI parameter, i.e.,

$$m_{eff} = m - \frac{q^2\pi}{\sqrt{2}r_\circ}. \tag{5.26}$$

Here one may set the effective mass to zero (note that the ADM mass of the HBI is not zero and survives with metric indirectly) i.e.,

$$m_{ADM} = m_{regular} = \frac{q^2\pi}{\sqrt{2qb}} \tag{5.27}$$

to get a regular metric function whose Kretschmann and Ricci scalars are regular at any point. It must be noticed that this is not the case of regular solution mentioned in [42], i.e. in contrast to [42] our HBI black hole is not massless.

5.4. Thermodynamics of Hoffmann-Born-Infeld Black Hole

In this section we investigate some thermodynamical properties of the HBI black hole. To do so firstly we find the horizon of the BH by equating metric function to zero, and finding the black hole effective mass in terms of the horizon radius

$$m_{eff} = \frac{r_h}{2}\left(1 + \frac{q^2}{3r_\circ^4} r_h^2 \ln\left(\frac{r_h^4}{r_h^4 + r_\circ^4}\right) - \mathfrak{O}\right), \quad r_h > r_\circ. \tag{5.28}$$

in which

$$\mathfrak{O} = \frac{q^2\sqrt{2}}{3r_h r_\circ}\left[\tan^{-1}\left(\frac{\sqrt{2}r_h}{r_\circ} + 1\right) + \tan^{-1}\left(\frac{\sqrt{2}r_h}{r_\circ} - 1\right)\right] - \frac{q^2\sqrt{2}}{6r_h r_\circ}\ln\left[\frac{r_h^2 + r_\circ^2 - \sqrt{2}r_h r_\circ}{r_h^2 + r_\circ^2 + \sqrt{2}r_h r_\circ}\right] \tag{5.29}$$

Hawking temperature in terms of the event horizon radius is given by

$$T_H = \frac{1}{4\pi r_h}\left(1 - \frac{q^2 r_h^2}{r_\circ^4}\ln\left(1 + \frac{r_\circ^4}{r_h^4}\right)\right). \tag{5.30}$$

Also the heat capacity which is defined as

$$C_q = T_H\left(\frac{\partial S(r)}{\partial T_H}\right)_q, \tag{5.31}$$

can be written as

$$C_q = \frac{\pi r_h^2\left(r_h^4 + r_\circ^4\right)\left(q^2 r_h^2 \ln\left(\frac{r_h^4}{r_h^4 + r_\circ^4}\right) + r_\circ^4\right)}{q^2 r_h^2\left(r_h^4 + r_\circ^4\right)\ln\left(\frac{r_h^4}{r_h^4 + r_\circ^4}\right) - r_\circ^8 + \left(4q^2 - r_h^2\right) r_h^2 r_\circ^4} \tag{5.32}$$

which clearly nominator is positive and roots of denominator give the possible phase transitions.

5.5. Thin Shell Wormhole in $4-$Dimensions

Here we follow the standard method of constructing a thin shell wormhole introduced in Sec. 2.3 [49]. To do so we take two copies of HBI spacetime, and from each manifold we remove the following $4-$dimensional submanifold

$$\Omega_{1,2} \equiv \left\{ r_{1,2} \leq a \middle| a > \sqrt{qb} \right\} \tag{5.33}$$

in which a is a constant and b is the HBI parameter introduced before. In addition, we restrict our free parameters (any) to keep our metric function non-zero and positive for $r > \sqrt{b}$. In order to have a complete manifold we define a manifold $\mathcal{M} = \Omega_1 \cup \Omega_2$ which its boundary is given by the two timelike hypersurfaces

$$\partial\Omega_{1,2} \equiv \left\{ r_{1,2} = a \middle| a > \sqrt{qb} \right\}. \tag{5.34}$$

After identifying the two hypersurfaces, $\partial\Omega_1 \equiv \partial\Omega_2$, the resulting manifold will be geodesically complete [5, 6, 31] which possesses two asymptotically flat regions connected by a traversable Lorantzian wormhole. The throat of the wormhole is at $\partial\Omega$. The induced metric on $\partial\Omega$ reads

$$ds^2_{ind} = -d\tau^2 + a(\tau)^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \tag{5.35}$$

where τ states the proper time on the hypersurface $\partial\Omega$. Lanczos equations reads

$$S^i_j = -\frac{1}{8\pi} \left(\langle K^i_j \rangle - \langle K \rangle \delta^i_j \right), \tag{5.36}$$

which leads the surface stress-energy tensor

$$S_j^i = diag\left(-\sigma, p_\theta, p_\phi\right). \tag{5.37}$$

Here σ, and $p_\theta = p_\phi$ are the surface-energy density and the surface pressure respectively. A detail study shows [17] that

$$\sigma = -\frac{1}{2\pi a}\sqrt{f\left(a\right) + \dot{a}^2} \tag{5.38}$$

and

$$p_\theta = p_\phi = -\frac{1}{2}\sigma + \frac{1}{8\pi}\frac{2\ddot{a} + f'\left(a\right)}{\sqrt{f\left(a\right) + \dot{a}^2}}. \tag{5.39}$$

Also the conservation equation gives

$$\frac{d}{d\tau}\left(\sigma a^2\right) + p\frac{d}{d\tau}\left(a^2\right) = 0 \tag{5.40}$$

or

$$\dot{\sigma} + 2\frac{\dot{a}}{a}\left(p + \sigma\right) = 0. \tag{5.41}$$

The total amount of the exotic matter for constricting the TSW is given by

$$\Omega = \int\left(\rho + p\right)\sqrt{-g}d^3x. \tag{5.42}$$

Here $\rho = \delta\left(r - a\right)\sigma\left(a\right)$ and $d^3x = drd\theta d\phi$ and therefore

$$\Omega = 4\pi a^2\sigma\left(a\right) = -2a\sqrt{f\left(a\right)}. \tag{5.43}$$

Concerning to the HBI metric function [44]

$$f(r) = 1 - \frac{2m}{r} + \frac{q^2}{3r_\circ^4} r^2 \ln\left(\frac{r^4}{r^4 + r_\circ^4}\right) - \mathfrak{W}, \qquad r > r_\circ, \tag{5.44}$$

in which

$$\mathfrak{W} = \frac{q^2\sqrt{2}}{3rr_\circ}\left[\tan^{-1}\left(\frac{\sqrt{2}r}{r_\circ} + 1\right) + \tan^{-1}\left(\frac{\sqrt{2}r}{r_\circ} - 1\right)\right] - \mathfrak{F}, \tag{5.45}$$

$$\mathfrak{F} = \frac{q^2\sqrt{2}}{6rr_\circ} \ln\left[\frac{r^2 + r_\circ^2 - \sqrt{2}rr_\circ}{r^2 + r_\circ^2 + \sqrt{2}rr_\circ}\right] + \frac{\sqrt{2}q^2\pi}{rr_\circ}, \tag{5.46}$$

and $r_\circ = \sqrt{qb}$. It is not difficult to show that we obtain the S and RN case in the limits

$$\lim_{b\to\infty} f(r) = 1 - \frac{2m}{r} \equiv f_S(r), \tag{5.47}$$

and

$$\lim_{b\to 0} f(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2} \equiv f_{RN}(r). \tag{5.48}$$

In order to investigate the stability of the TSW we start with the thin shell's equation of motion Eq. (2.41) in which the thin shell's potential is given by

$$V(a) = f(a) - (2\pi a\sigma(a))^2 \tag{5.49}$$

which after expansion around $a_\circ$ (which requires $V(a_\circ) = V'(a_\circ) = 0$) up to the second order one gets

$$V(a) \cong \frac{1}{2} V''(a_\circ)(a - a_\circ)^2 . \tag{5.50}$$

By considering (5.49) we get

$$V''(a_\circ) = f''_\circ - \frac{f'^2_\circ}{2f_\circ} - \frac{1+2\beta_\circ}{a_\circ^2}(2f_\circ - a_\circ f'_\circ) \tag{5.51}$$

in which $\beta_\circ = \beta(a_\circ)$ and $\beta(a) = \partial p/\partial\sigma$. The stability condition easily read

$$\text{for } 2f_\circ \gtrless a_\circ f'_\circ, \qquad 1+2\beta_\circ \lessgtr \frac{a_\circ^2}{2f_\circ}\left(\frac{2f''_\circ f_\circ - f'^2_\circ}{2f_\circ - a_\circ f'_\circ}\right). \tag{5.52}$$

5.6. $5-$Dimensional Hoffmann-Born-Infeld Black Hole

In order to extend the $4-$dimensional HBI black solution to $5-$dimensions we choose our action as

$$S = \frac{1}{2}\int dx^5 \sqrt{-g}\left\{-4\Lambda + R + \mathcal{L}(\mathcal{F})\right\}, \tag{5.53}$$

where

$$\mathcal{L} = \begin{cases} \mathcal{L}_-, & r \leq \sqrt{qb} \\ \mathcal{L}_+, & r \geq \sqrt{qb} \end{cases} \tag{5.54}$$

and the nonlinear Maxwell equation leads to

$$E_r = \frac{qr^3}{(q^2b^2 + r^6)}. \tag{5.55}$$

Variation of the action yields the field equations as

$$G_{\mu}^{\ \nu} + 2\Lambda\delta_{\mu}^{\ \nu} = T_{\mu}^{\ \nu}, \ \ T_{\mu}^{\ \nu} = \frac{1}{2}\left(\mathcal{L}\delta_{\mu}^{\ \nu} - 4\mathcal{L}_{\mathcal{F}}F_{\mu\lambda}F^{\nu\lambda}\right), \tag{5.56}$$

which clearly gives $T_{t}^{\ t} = T_{r}^{\ r} = \left(\frac{1}{2}\mathcal{L} - \mathcal{L}_{\mathcal{F}}\mathcal{F}\right)$, stating that $G_{t}^{\ t} = G_{r}^{\ r}$ and $T_{\theta_i}^{\ \theta_i} = \frac{1}{2}\mathcal{L}$. Now we introduce our line element

$$ds^2 = -(\chi - r^2 H(r))dt^2 + \frac{1}{(\chi - r^2 H(r))}dr^2 + r^2 d\Omega_3^2 \tag{5.57}$$

to cover both topological and non-topological black hole solutions [49]. Our choice of $g_{tt} = -\left(g_{rr}\right)^{-1}$ is a direct result of $G_{t}^{\ t} = G_{r}^{\ r}$ up to a constant coefficient which we set it to be one. Einstein tensor components are given by

$$G_{t}^{\ t} = G_{r}^{\ r} = -\frac{3}{2r^3}\left(r^4 H(r)\right)', \quad G_{\theta_i}^{\ \theta_i} = -\frac{1}{2r^2}\left(r^4 H(r)\right)'' \tag{5.58}$$

which, after using the theorem given in [49] one gets

$$H(r) = \frac{\Lambda}{3} + \frac{4m}{(d-2)\, r^{d-1}} - \frac{1}{(d-2)\, r^{d-1}}\int r^{d-2} T_t^t dr. \tag{5.59}$$

With the energy-momentum tensor component

$$T_t^t = \frac{1}{b^2}\ln\left(\frac{r^6}{b^2 q^2 + r^6}\right) = -\frac{1}{b^2}\ln\left(1 + \frac{b^2 q^2}{r^6}\right) \tag{5.60}$$

the metric function reads

$$f(r) = \chi - \frac{\Lambda}{3}r^2 - \frac{4m}{3r^2} - \frac{q^2\sqrt{3}}{6r^2 r_\circ^2}\tan^{-1}\left(\frac{1}{\sqrt{3}}\left[\frac{2r^2}{r_\circ^2} - 1\right]\right) - \mathfrak{X}, \tag{5.61}$$

where

$$\mathfrak{X} = \frac{q^2\sqrt{2}}{12r^2r_\circ^2}\ln\left|\frac{r^4 + r_\circ^4 - r^2r_\circ^2}{r^4 + r_\circ^4 + 2r^2r_\circ^2}\right| + \frac{\sqrt{3}q^2\pi}{12r^2r_\circ^2} \tag{5.62}$$

$r_\circ^6 = b^2q^2$ and m is the ADM mass of the black hole. One observes that this solution in two extremal limits for b yields

$$\lim_{b\to 0} f\left(r\right) = \chi - \frac{\Lambda}{3}r^2 - \frac{4m}{3r^2} + \frac{q^2}{3r^4}, \quad \lim_{b\to\infty} f\left(r\right) = \chi - \frac{\Lambda}{3}r^2 - \frac{4m}{3r^2}. \tag{5.63}$$

Also in the sense of usual ADM mass, represented by m, if one adjusts

$$m_{ADM} = m_{regular} = \frac{\sqrt{3}q^2\pi}{16r_\circ^2} \tag{5.64}$$

the divergent term$\sim \frac{1}{r^2}$ in $f\left(r\right)$ vanishes and one gets a regular 5-dimensional black hole solution.

Our action in 5-dimensional HBIGB theory of gravity is given by

$$S = \frac{1}{2}\int dx^5\sqrt{-g}\left\{-4\Lambda + R + \alpha\mathcal{L}_{GB} + \mathcal{L}\left(\mathcal{F}\right)\right\} \tag{5.65}$$

where $\mathcal{L}_{GB} = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ and α is the GB parameter. We use the theorem given in reference [49] to get black hole solution in HBI-Gauss-Bonnet theory of gravity in $5-$dimensions. If we use spherical symmetry spacetime with line element (5.57) then

$$H\left(r\right) + 4\alpha H\left(r\right)^2 = \frac{\Lambda}{3} + \frac{4m}{3r^4} - \frac{2}{3r^4}\int r^3T_t^t dr, \tag{5.66}$$

and one finally obtains

$$f_{\pm}\left(r\right)=\chi+\frac{r^{2}}{8\alpha}\mathfrak{B} \tag{5.67}$$

where

$$\mathfrak{B}=\left\{1\pm\sqrt{1+16\alpha\left(\frac{\Lambda}{3}+\frac{4m_{eff}}{3r^{4}}+\mathfrak{Z}\right)}\right\} \tag{5.68}$$

in which

$$\mathfrak{Z}=\frac{q^{2}\sqrt{3}}{6r^{4}r_{\circ}^{2}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\left[\frac{2r^{2}}{r_{\circ}^{2}}-1\right]\right)+\frac{q^{2}\sqrt{2}}{12r^{4}r_{\circ}^{2}}\ln\left|\frac{r^{4}+r_{\circ}^{4}-r^{2}r_{\circ}^{2}}{r^{4}+r_{\circ}^{4}+2r^{2}r_{\circ}^{2}}\right| \tag{5.69}$$

$r_{\circ}^{6}=b^{2}q^{2}$ and $m_{eff}=m-\frac{\sqrt{3}q^{2}\pi}{16r_{\circ}^{2}}$. One can check that the following limits are obtained

$$\lim_{b\rightarrow0}f_{\pm}\left(r\right)=\chi+\frac{r^{2}}{8\alpha}\left\{1\pm\sqrt{1+16\alpha\left(\frac{\Lambda}{3}+\frac{4m}{3r^{4}}-\frac{q^{2}}{3r^{6}}\right)}\right\}, \tag{5.70}$$

$$\lim_{b\rightarrow\infty}f_{\pm}\left(r\right)=\chi+\frac{r^{2}}{8\alpha}\left\{1\pm\sqrt{1+16\alpha\left(\frac{\Lambda}{3}+\frac{4m}{3r^{4}}\right)}\right\}, \tag{5.71}$$

$$\lim_{\alpha\rightarrow0}f_{-}\left(r\right)=\chi-\frac{\Lambda}{3}r^{2}-\frac{4m_{eff}}{3r^{2}}-\frac{q^{2}\sqrt{3}}{6r^{2}r_{\circ}^{2}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\left[\frac{2r^{2}}{r_{\circ}^{2}}-1\right]\right)-\mathfrak{H} \tag{5.72}$$

$$\mathfrak{H}=\frac{q^{2}\sqrt{2}}{12r^{2}r_{\circ}^{2}}\ln\left|\frac{r^{4}+r_{\circ}^{4}-r^{2}r_{\circ}^{2}}{r^{4}+r_{\circ}^{4}+2r^{2}r_{\circ}^{2}}\right|, \tag{5.73}$$

as expected.

To allow for the analysis of radial perturbations we let the throat radius vary with the proper time: $a = a(\tau)$. As a consequence of the generalized Birkhoff theorem the geometry will still be described by (5.1), for any $r > a(\tau)$. The resulting expressions for the energy density and pressures for a generic metric function $f(r)$ turn out to be (2.29) and (2.30) with $d = 5$, $\tilde{\alpha} = 2\alpha$ and

$$f(a) = 1 + \frac{a^2}{4\alpha}\left(1 - \sqrt{1 + \frac{8\alpha}{a^4}\left(\frac{2M}{\pi} - \frac{Q^2}{3a^2}\right)}\right). \tag{5.74}$$

We note that in our notation a 'dot' denote derivative with respect to the proper time τ and a 'prime' with respect to the argument of the function. For simplicity, we set the cosmological constant to zero. By a simple substitution one can show that, the conservation equation (2.32).

In what follows we shall study small radial perturbations around a radius of equilibrium a_0. To this end we adapt a linear relation between p and σ as (2.38). By virtue of the latter equation we express the energy density in the form (2.40). We notice that $V(a)$, and more tediously $V'(a)$, both vanish at $a = a_0$. The stability requirement for equilibrium reduces therefore to the determination of $V''(a_0) > 0$. Of course, $V(a)$ is complicated enough for an immediate analytical result. For this reason we shall proceed through numerical calculation (Fig.s 5.1 and 5.2) to see whether stability regions/ islands develop or not. Since the hopes for obtaining TSWs with ordinary matter when $\alpha > 0$, have already been dashed, we shall investigate only the case for $\alpha < 0$.

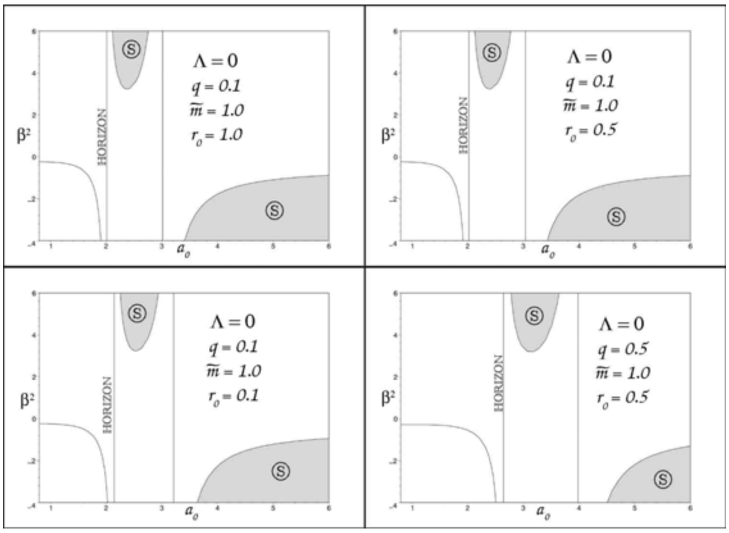

Figure 5.1. The stability regions (shown dark) for various set of parameters due to the inequalities given in Eq. (5.52). These regions correspond to the cases of $V''(\breve{a}_0, \beta) > 0$.

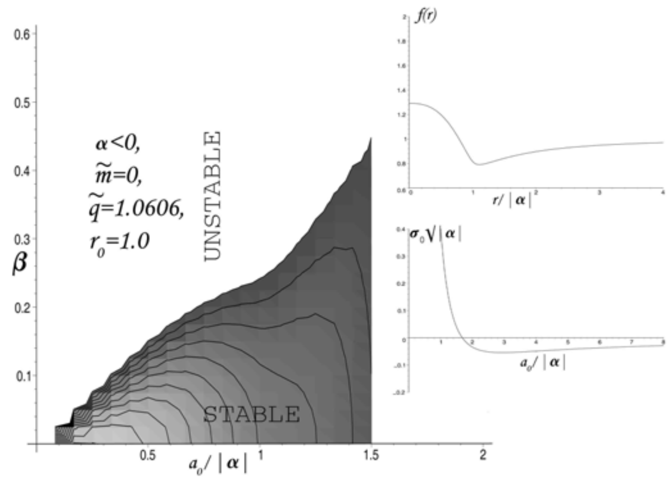

Figure 5.2. The stability region (i.e. $V''(\breve{a}_0, \beta) > 0$) for the chosen parameters, $r_\circ = 1.00$, $q = 0.75$ and $\tilde{m} = 0$). This is given as a projection into the plane with axes β and $\frac{a_0}{|\alpha|}$. The plot of the metric function $f(r)$ and energy density σ are also inscribed in the figure.

CHAPTER 6

THIN SHELL WORMHOLE WITH A GENERALIZED CHAPLYGIN GAS IN EINSTEIN-MAXWELL-GAUSS-BONNET GRAVITY

6.1. Overview

Among other aspects the foremost challenging problems related to Thin Shell Wormholes (TSW) [2, 54, 55, 5, 6, 30] are, i) positivity of energy density , and ii) stability against symmetry preserving perturbations. To overcome these problems recently there have been various attempts in Einstein-Gauss-Bonnet (EGB) gravity with Maxwell and Yang-Mills sources [9, 31, 18, 32, 56, 57, 28, 53, 12]. Specifically, with the negative Gauss-Bonnet (GB) parameter $(\alpha < 0)$ we obtained stable TSW, obeying a linear equation of state, against radial perturbations [2, 54, 55, 5, 6, 30]. By linear equation of state it is meant that the energy density (σ) and surface pressure p satisfy a linear relation. To respond the other challenge, however, i.e. the positivity of the energy density $(\sigma > 0)$, we maintain still a cautious optimism. To be realistic, only in the case of Einstein-Yang-Mills-Gauss-Bonnet (EYMGB) theory and in a finely-tuned narrow band of parameters we were able to beat both of the above stated challenges [9, 31, 18, 32, 56, 57, 28, 53, 12]. Our stability analysis with the unfortunate negative energy density was extended further to cover non-asymptotically flat (NAF) dilatonic solutions [58].

In this Chapter we show that stability analysis of TSW extends to the case of a generalized Chaplygin gas which has already been considered within the context of Einstein-Maxwell TSWs [5, 6, 3, 61]. Due to the accelerated expansion of our universe a repulsive effect of a Chaplygin gas has been considered widely in recent times . From the same token therefore it would be interesting to see how a generalized Chaplygin gas supports a TSW against radial perturbations in Gauss-Bonnet (GB) gravity. For this purpose we perturb the thin-shell radially and reduce the equation into a particle in a potential well problem with zero total energy. The stability amounts to the determination of the positive domain for the second derivative of the potential. We obtain plots that provides us such physical regions indicating stable wormholes. For technical reasons we restrict ourselves only to the $5-$dimensional plots.

6.2. Stability

The $d-$dimensional Einstein-Maxwell-Gauss-Bonnet (EMGB) action without cosmological constant has the form

$$S=\frac{1}{16\pi G}\int\sqrt{|g|}d^{d}x\left(R+\alpha\mathcal{L}_{GB}-\frac{1}{4}\mathcal{F}\right). \tag{6.1}$$

where G is the $d-$dimensional Newton constant, $\mathcal{F}=F_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant and α is the Gauss-Bonnet (GB) parameter with Lagrangian (2.8). Variation of S with respect to $g_{\mu\nu}$ yields the EMGB field equations given in (3.2). Our static spherically symmetric metric ansatz will be as (2.1).

Construction of the TSW in the static spherically symmetric spacetime follows the standard procedure used before in Sec. 2.3. The black hole solution of the EMGB field equations (with $\Lambda = 0$) is given by, with $\tilde{\alpha} = (d-3)(d-4)\alpha$ [20, 25]

$$f_{\pm}(r) = 1 + \frac{r^2}{2\tilde{\alpha}}\left(1 \pm \sqrt{1 + 4\tilde{\alpha}\left(\frac{2M}{8\pi r^{d-1}} - \frac{Q^2}{2(d-2)(d-3)r^{2(d-2)}}\right)}\right) \tag{6.2}$$

in which M is an integration constant related to the ADM mass of the BH and Q is the electric charge of the BH. The corresponding electric field $2-$form is given by

$$\mathbf{F} = \frac{Q}{r^{2(d-2)}}dt \wedge dr. \tag{6.3}$$

The components of energy momentum tensor on the thin-shell are given by (2.29) and (2.30). The static configuration of the expressed equations comes to be as (2.37) and (2.39). Our aim in the sequel is to perturb the throat of the TSW radially around the equilibrium radius a_0. To do this, we assume that the equation of state is in the form of a generalized Chaplying gas [61], i.e.,

$$p = \left(\frac{\sigma_0}{\sigma}\right)^{\nu} p_0 \tag{6.4}$$

in which $\nu \in [0, 1]$ is a free parameter and σ_0/p_0 correspond to σ/p at the equilibrium radius a_0. We plug in the latter expression into the conservation energy equation (2.32) to find a closed form for the dynamic tension on the thin-shell after perturbation as follows

$$\sigma(a) = \sigma_0\left[\left(\frac{a_0}{a}\right)^{(1+\nu)(d-2)} + \frac{p_0}{\sigma_0}\left(\left(\frac{a_0}{a}\right)^{(1+\nu)(d-2)} - 1\right)\right]^{\frac{1}{1+\nu}}. \tag{6.5}$$

Equating this with the one found in Eq. (2.29), i.e.,

$$-\frac{\sqrt{f_{\pm}(a)+\dot{a}^{2}}\,(d-2)}{8\pi}\left[\frac{2}{a}-\frac{4\tilde{\alpha}}{3a^{3}}\left(f_{\pm}(a)-2\dot{a}^{2}-3\right)\right]= \tag{6.6}$$

$$\sigma_0\left[\left(\frac{a_0}{a}\right)^{(1+\nu)(d-2)}+\frac{p_0}{\sigma_0}\left(\left(\frac{a_0}{a}\right)^{(1+\nu)(d-2)}-1\right)\right]^{\frac{1}{1+\nu}}$$

from which one solves for the $\dot{a}^2$ as

$$\dot{a}^{2}+V(a)=0, \tag{6.7}$$

The intricate potential here is given by (2.42)-(2.44). It can be checked that at the equilibrium radius a_0 we have both $V(a_0)=0$ and $V'(a_0)=0$ satisfied.By making an expansion of the potential in the vicinity of a_0 and setting $y=a-a_0$ in the Eq. (6.7) and differentiating once more with respect to the proper time we obtain the perturbation equation as

$$\ddot{y}+\frac{1}{2}V''(a_0)\,y=0 \tag{6.8}$$

The stability condition reduces for this reason to the requirement whether $V''(a_0)>0$ holds. Therefore we search for the regions in our parameters to satisfy this constraint on $V''(a_0)$.

Fig. 6.1 represents a $5-$dimensional plot for the negative branch black hole solution (6.2) with two horizons. For the parameters shown a projection of the three-dimension plot into the $\nu-a_0$ plane is given for a stable region, i.e. $V''(a_0)>0.$

Fig. 6.2 depicts a similar plot for the positive branch solution (6.2) which does not correspond to a black hole case. Both pictures reveal the fact that in the stability

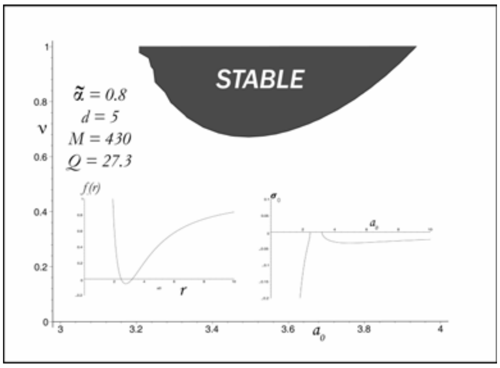

Figure 6.1. The $5-$dimensional plot of stability region $V''(a_0) > 0$, for the chosen parameters and negative branch black hole solution (6.2) versus the parameters ν and a_0. The plots of the black hole metric $f_-(r)$ versus r and the energy density σ_0 versus a_0 for the same parameters are given too. This figure shows that the TSW, under the conditions indicated, is supported by exotic matter but for certain values of ν and a_0 it is stable under a radial perturbation.

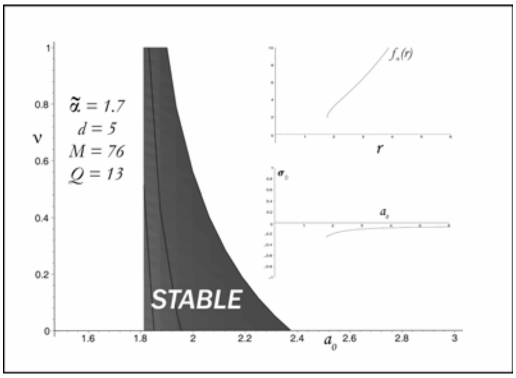

Figure 6.2. Same plot as Fig. 6.1 with the bulk metric $f_+(r)$. The plots of the bulk metric $f_+(r)$ versus r and energy density σ_0 versus a_0 for the same parameters are given.

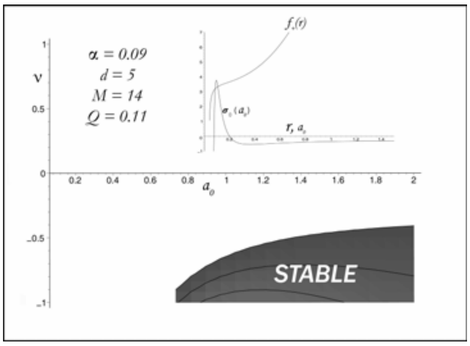

Figure 6.3. The $5-$dimensional plot of stability region $V''(a_0) > 0$, for a different set of parameters than those in Fig. 6.2, with positive branch bulk solution (6.2) versus the parameters ν and a_0. The plots of the bulk metric $f_+(r)$ versus r and energy density σ_0 versus a_0 for the same parameters are also given. This figure displays that the TSW under the conditions indicated on the figure is supported by normal matter but its stability region moves out of the Chaplygin gas region i.e. $\nu < 0$.

domain we have $\sigma_0 < 0$. Finally in Fig. 6.3 we plot the positive branch solution (6.2) with different parameters than those of Fig. 6.2. This figure displays that if we arrange the parameters to have positive energy or normal matter for the TSW, the stability region moves out of the Chaplygin gas region.

CHAPTER 7

CONCLUSION

In this thesis our main concern is stability analysis of thin-shell wormholes (TSW) in various extensions of Einstein's general relativity. Sources considered are linear Maxwell field, Yang-Milles (YM) field. Hoffmann extension of non-linear Born-Infeld electrodynamics dilaton and generalized Chaplygin gas. In each case we found first an exact black hole solution so that the throat radius of the TSW doesn't cross the horizon. This is a requirement in general for any black hole space time if a wormhole is to be constructed with a safe passage. The energy- momentum of the thin-shell is localized on a narrow layer specified by a Dirac delta function. This is an advantage for TSW in comparison with the other wormholes. The equation of state on the thin-shell is chosen either as a linear relation between pressure and density or more generally in the case of a Chaplygin gas. The basic rule is to provide a positive energy density and repulsive force that will encounter gravitational attraction and collapse. The equation of motion for the radius of thin-shell reduces in each case to the simple equation $\dot{a}^2 + V(a) = 0$, where $a(\tau)$ is the radius as a function of proper time and $V(a)$ is a rather complicated potential. Perturbation around the fixed radius a_0 for $y = a - a_0$ yields

$$\ddot{y} + \frac{1}{2}V''(a_0)y = 0$$

This describes an oscillatory motion around a_0 , provided $V''(a_0) > 0$, which implies stability.

We search therefore through numerical analysis region for which $V''(a_0) > 0$, holds in the projective parameter space. For the case of the generalized Chaplygin gas stability condition is satisfied but only with exotic matter, i.e. negative energy density. In the other cases we managed to identity physical matter source together with $V"(a_0) > 0$, so that physical TSW becomes feasible. We observed, in particular that the contribution of Gauss-Bonnet term with quadratic invariants acts positively in obtaining physical TSWs. To be more precise, we give a short conclusion for each chapter.

Chapter two: We have investigated the possibility of thin-shell wormholes in EYMGB theory in higher ($d \geq 5$) dimensions with particular emphasis on stability against spherical, linear perturbations and normal (i.e. non-exotic) matter. For this purpose we made use of the previously obtained solutions that are valid in all dimensions. The case $d = 5$ is considered separately from the cases $d > 5$ because the solution involves a logarithmic term apart from the power-law dependence. For $d = 5$ we observe (Fig. 2.2) the formation of a narrow band of positive energy region that attains a stable wormhole only for $\alpha < 0$. On the contrary, for $\alpha > 0$ although a large region of stability (i.e. $V''(a_0) > 0$) forms, the energy turns out to be exotic. For $d > 5$ also, we have more or less a similar picture. That is, whenever the GB parameter $\alpha > 0,$ negative energy shows itself versus the stability requirements. We have analyzed the cases $d = 6, 7$ and 8 as examples. Our technique is powerful enough to apply in any higher dimensions, however, for technical reasons we had to be satisfied with these selected dimensions. We must admit also that for non-spherical perturbations a similar analysis remains to be seen. In our study we were able also to observe a stability region which employs $0 < \beta < 1$, which can be interpreted as a case corresponding to less than the speed of light. In summary, we state that formation of stable, positive energy thin-shell wormholes in EYMGB are possible only with a GB parameter $\alpha < 0$. Without the GB term whatever source is available the situation is always worse. The indispensable character of the GB parameter toward useful wormhole constructions invites naturally the Lovelock hierarchy for which GB term constitutes the first member.

Chapter 3: Our numerical analysis shows that for $\alpha < 0$, and specific ranges of mass and charge the $5-$dimensional EMGB thin-shell wormholes with normal matter can be made stable against linear, radial perturbations. The fact that for $\alpha > 0$ there is no such wormholes is well-known. The magnitude of α is irrelevant to the stability analysis. This reflects the universality of wormholes in parallel with black holes, i.e., the fact that they arise at each scale. Stable regions develop for each set of finely-tuned parameters which determine the lifespan of each such region. Beyond those regions instability takes the start. Our study concerns entirely the exact EMGB gravity solution given in. It is our belief that beside EMGB theory in different theories also such stable, normal-matter wormholes are abound, which will be our next venture in this line of research.

Chapter 4: In this Chapter we employed dilaton field beside YM field to investigate the reality of such TSWs. In doing this it should be remembered that the strong coupling of dilaton turns the spacetime into a non-asymptotically flat (NAF) one. This is a digression from the original idea of a wormhole since we have to revise the advantages of an AF spacetime. We remind, however, that we have already enough familiarity with the cases of NAF spacetimes, the best known one being the de (anti)-Sitter. For this reason we extended the concept of a wormhole in an AF spacetime to a NAF one through the prominent source of a dilaton. Such a wormhole provides traversability from one NAF to another NAF spacetime. For this purpose we used $d-$dimensional solutions of Einstein-Yang-Mills-dilaton (EYMD) theory found recently [9, 11], and investigate thin-shell construction in such geometries. It turns out that in the dilatonic thin-shells the required source must be exotic. Our findings show, against our expectation that neither the YM charge, nor the dilatonic parameter have significant effect on the negative energy density of the thin-shell.

Chapter 5: The original non-linear BI electrodynamics aimed at removing point-like singularities and resulting divergences. This, however, didn't resolve the dauble-valuedness in the displacement vector $\vec{D}(\vec{E})$ as a function of the electric field. This was the main motivation for emergence of Hoffmann's version of the BI type Lagrangian, which contained an ubiquitous logarithmic term. We have shown that such a supplementary term in the Lagrangian has benefits also when employed in general relativity. Firstly, it removes the double-valuedness in $\vec{D}(\vec{E}),$ as observed / proposed seven decades before. Secondly, by cutting and gluing (pasting) method we obtain new black hole spacetimes. This may be developed into a finite, geometrical model of elementary particles as addressed in. Lastly, as we have emphasized in the present paper, the HBI type Lagrangian can be used in wormhole construction. These wormholes have the attractive features of being supported only by normal matter. The technical complexity in the metric function seems to be the price paid in introducing new parameters / extensions to general relativity. In spite of the emerging complexity we made use of the novelties without obscuring the real physics. Further, the thin-shell wormhole obtained in the EHBIGB gravity can be made stable. Although, for simplicity we have plotted the potential for $d = 5$, we can speculate that this behavior is generic upon suitable choice of parameters involved. We have evidences from other theories that the behaviors of $d > 5$ do not differ much from the $d = 5$ case. This is upon finely-tuned parameters and an intricate potential function which is required to have positive second derivative. From these feats it is hoped that the logarithmic Lagrangian due originally to Hoffmann and Infeld will draw attention from various circles of field theorists for further applications.

Chapter 6: For a generalized Chaplygin gas obeying the equation of state $p = \left(\frac{\sigma_0}{\sigma}\right)^{\nu} p_0$, we have found stable regions for TSWs, within physically acceptable range of parameters in EMGB gravity. The energy-density, however, turns out to be negative to suppress such a TSW as a prominent candidate.

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