Klein-Gordon Equation in 1+1-Dimensions

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We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis of the degree of Master of Science in Physics.

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ABSTRACT

In this study we give a brief introduction to the basic differential equation for zero spin relativistic particle which is called Klein-Gordon equation. Klein-Gordon equation, then, is presented in 1+1-dimensions where we give some exact solutions for the equation with different potentials. The first problem is the particle inside a potential of the form of smooth finite well. We find the exact solutions in terms of the Heun functions. Our second example is a K-G particle inside an infinite well whose wall is moving. This problem is solved for massless particle. Finally, we give the remarks in our conclusion.

Keywords: Klein-Gordon equation, Heun functions, 1+1-dimensions, Particle in infi- nite well.

ÖZ

Bu çalışmada, Klein – Gordon denklemi olarak bilinen, sıfır spine sahip rölativistik parçacık için temel diferansiyel denkleme, kısa ve öz bir giriş yapılmıştır. Klein – Gordon denklemi, farklı potansiyellere sahip bir denklem için, bazı kesin çözümler verdiğimiz 1 + 1 boyutları ile sunulmuştur. İlk soru, pürüzsüz sonlu kuyunun bir formu olan bir potansiyelin ic¸indeki parc¸acıkla ilgilidir. Kesin çözümler, Heun fonksiyonları cinsinden bulunmuştur. İkinci örneğimiz ise duvarı hareket halinde olan sonsuz bir kuyu içindeki KG parçacığıdır. Bu soru kütlesiz parçacık için çözülmüştür. Son olarak, sonuç kısmında yorumlara yer verilmiştir.

Anahtar Kelimeler: Klein - Gordon denklemi, Heun fonksiyonları, 1 + 1 boyutları, sonsuz kuyu içerisindeki parçacık.

DEDICATION

To My Family

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INTRODUCTION

Klein-Gordon equation is the basic equation which is used to describe relativistic particle with zero spin, like Higgs boson H^0. From the field theory the action for a relativistic spinless particle under the electromagnetic field is given by

$$S = \int \sqrt{-g}\, d^4x \mathcal{L} \tag{1.1}$$

in which the Lagrangian density reads as [14, 10]

$$\mathcal{L} = \frac{1}{2m}\left[\left(i\hbar\partial_\mu - \frac{q}{c}A_\mu\right)\bar{\phi}\left(-i\hbar\partial^\mu - \frac{q}{c}A^\mu\right)\phi - m^2c^2\phi\bar{\phi}\right] - \mathcal{F}\,. \tag{1.2}$$

Here m is the rest mass of the particle, Aµ is the four electromagnetic potential with $A^\mu = g^{\mu\alpha}A_\alpha, \phi$ is the Klein-Gordon scalar field with its complex conjugate $\bar{\phi}$ and

$$\mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1.3}$$

the Maxwell invariant with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.4}$$

We note that $g = \det(g_{\mu\nu})$ and in our study the spacetime is flat Minkowski spacetime.

which implies $g_{\mu\nu} = \eta_{\mu\nu}$ and therefore $g = -1$. We start with the variation of the action with respect to $\bar{\phi}$ which yields

$$\partial_\mu \left(\frac{\partial L}{\partial\left(\partial_\mu \bar{\phi}\right)} \right) = \frac{\partial L}{\partial \bar{\phi}}. \tag{1.5}$$

In an explicit calculation one finds

$$\partial_\mu \left[i\hbar \left(-i\hbar\partial^\mu - \frac{q}{c} A^\mu \right) \phi \right] = \left(-\frac{q}{c} A_\mu \right) \left(-i\hbar\partial^\mu - \frac{q}{c} A^\mu \right) \phi - m^2 c^2 \phi \tag{1.6}$$

which after simplification it becomes

$$\left(-i\hbar\partial_\mu - \frac{q}{c} A_\mu \right) \left(-i\hbar\partial^\mu - \frac{q}{c} A^\mu \right) \phi = m^2 c^2 \phi \tag{1.7}$$

or in more convenient form

$$\left(p_\mu - \frac{q}{c} A_\mu \right) \left(p^\mu - \frac{q}{c} A^\mu \right) \phi = m^2 c^2 \phi \tag{1.8}$$

in which

$$p_\mu = -i\hbar\partial_\mu. \tag{1.9}$$

The latter is called Klein-Gordon equation for a massive scalar field which interacts non-minimally with the electromagnetic field. We also add that variation of the action with respect to electromagnetic potential A_μ

$$\partial_\nu \left(\frac{\partial L}{\partial\left(\partial_\nu A_\mu\right)} \right) = \frac{\partial L}{\partial A_\mu} \tag{1.10}$$

which yields the Maxwell's equation i.e., [14, 10]

$$\partial_\nu F^{\nu\mu} = j^\mu \tag{1.11}$$

in which

$$j^\mu = \frac{\partial \mathrm{L}}{\partial A_\mu} = \frac{iq\hbar}{2mc}\left[\bar{\phi}\left(\partial^\mu - \frac{iq}{c\hbar}A^\mu\right)\phi - \phi\left(\partial^\mu + \frac{iq}{c\hbar}A^\mu\right)\bar{\phi}\right]. \tag{1.12}$$

Our concentration in this thesis is on the 1 + 1−dimensional Klein-Gordon equation which implies

$$\left(-i\hbar\partial_\mu - \frac{q}{c}A_\mu\right)\left(-i\hbar\partial^\mu - \frac{q}{c}A^\mu\right)\phi(x,t) = m^2c^2\phi(x,t) \tag{1.13}$$

in which μ = 0, 1. One may open this equation explicitly to find

$$\left(i\hbar\partial_0 + \frac{q}{c}A_0\right)\left(i\hbar\partial^0 + \frac{q}{c}A^0\right)\phi(x,t) + \left(i\hbar\partial_1 + \frac{q}{c}A_1\right)\left(i\hbar\partial^1 + \frac{q}{c}A^1\right)\phi(x,t) = m^2c^2\phi(x,t), \tag{1.14}$$

in which $\partial_0 = \frac{\partial}{c\partial t}$ and $\partial_1 = \frac{\partial}{\partial x}$ while $\partial^0 = \frac{\partial}{c\partial t}$ and $\partial^{1`} = -\frac{\partial}{\partial x}$ (Note that the signature of the spacetime is −2, i.e. + − −−). Now we suppose that $A_\mu = A_\mu(x)$ which implies $\phi(x,t) = \exp\left(-\frac{iEt}{\hbar}\right)u(x)$ and consequently

$$-\left(E + \frac{q}{c}A_0\right)^2 u(x) + \left(i\hbar\partial_1 + \frac{q}{c}A_1\right)\left(i\hbar\partial^1 + \frac{q}{c}A^1\right)u(x) = m^2c^2u(x) \tag{1.15}$$

or after considering $\frac{q}{c}A_0 = -\mathrm{V}(x)$ and $\frac{q}{c}A_1 = W(x)$ the latter becomes

$$\left[\left(i\hbar\partial_1 + W(x)\right)^2 - \left(\mathrm{E} - \mathrm{V}(x)\right)^2 + m^2c^2\right]u(x) = 0. \tag{1.16}$$

In another assumption we set $W(x)=0$ and instead we add a scalar potential coupled minimally with the mass of the particle as

$$\left[\hbar^2c^2\frac{d^2}{dx^2}+\left(E-V(x)\right)^2-\left(mc^2+S(x)\right)^2\right]u(x)=0 \tag{1.17}$$

in which $E=\frac{E}{c}$ and $V=\frac{V(x)}{c}$. In the rest of this thesis we use this equation with certain potentials and we try to find solutions for such system [13, 1, 12, 11, 7, 3, 8,4].

Chapter 2

1+1-DIMENSIONAL KLEIN-GORDON EQUATION

For a spinless relativistic quantum particle under a scalar and a vector potential $S\,(x)$ and (x) , respectively, the time independent Klein-Gordon equation is given by [1]

$$\left\{\hbar^2 c^2 \frac{d^2}{dx^2} + \left[E - V(x)\right]^2 - \left[mc^2 + S(x)\right]^2\right\} u(x) = 0 \tag{2.1}$$

in which m is the mass of the particle, c is the speed of light E is the energy and $u\,(x)$ is the wavefunction of the particle. Our specific choice of the potentials are given as follows [12]:

$$V(x) = 0 \tag{2.2}$$

and

$$S(x) = mc^2 \left(\tanh\left(\frac{x - \frac{L}{2}}{a} \right) - \tanh\left(\frac{x + \frac{L}{2}}{a} \right) \right) \tag{2.3}$$

in which L and a are two non-negative real constants. In Fig. (2.1) we plot $\frac{S(x)}{mc^2}$ for $L = 1$ and different values of a. As it is clear from Fig. (2.1) in the limit of $a \rightarrow 0^+$, the potential $S(x)$ becomes a finite well of width L and depth 2 mc^2.

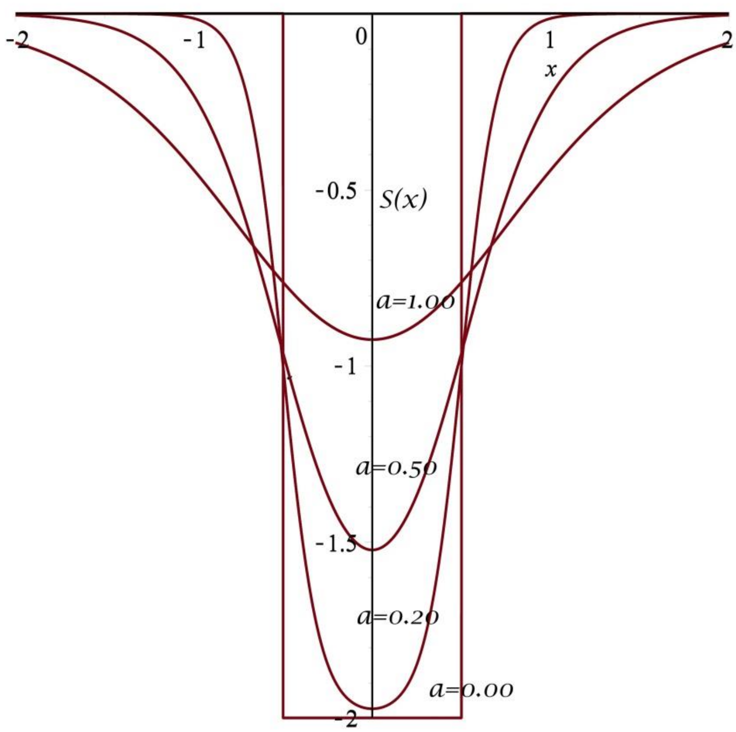

Figure 2.1: Scalar potential $S(x)$ in terms of x for $L = 1$ and $a = 1.00, 0.50$, 0.20 and 0.00. It is clear that when a goes to zero the potential approaches to the square well with width L and depth 2.

Now, we are looking for bound state solutions to Eq. (2.1). To that end, first we define

$$k^2 = \frac{E^2}{\hbar^2 c^2} \tag{2.4}$$

and

$$\alpha^2 = \frac{m^2 c^2}{\hbar^2} \tag{2.5}$$

to reduce the K-G equation as

$$\left[\frac{d^2}{dx^2}+k^2-\alpha^2\left(1+\tanh\left(\frac{x-\frac{L}{2}}{a}\right)-\tanh\left(\frac{x+\frac{L}{2}}{a}\right)\right)^2\right]u(x)=0. \qquad (2.6)$$

Next we introduce

$$z=-\exp\left(-\frac{2x}{a}\right) \qquad (2.7)$$

which by virtue of

$$\frac{d^2}{dx^2}=\frac{d}{dx}\left(\frac{dz}{dx}\frac{d}{dz}\right)=\frac{d^2z}{dx^2}\frac{d}{dz}+\left(\frac{dz}{dx}\right)^2\frac{d^2}{dz^2}, \qquad (2.8)$$

$$\tanh\left(\frac{x-\frac{L}{2}}{a}\right)=\frac{1-\exp\left(-\frac{2x-L}{a}\right)}{1+\exp\left(-\frac{2x-L}{a}\right)}=\frac{1-\frac{z}{z_0}}{1+\frac{z}{z_0}} \qquad (2.9)$$

and

$$\tanh\left(\frac{x+\frac{L}{2}}{a}\right)=\frac{1-\exp\left(-\frac{2x+L}{a}\right)}{1+\exp\left(-\frac{2x+L}{a}\right)}=\frac{1-z_0z}{1+z_0z} \qquad (2.10)$$

in which $z_0=-\exp\left(-\frac{L}{a}\right)$ one finds

$$\left[\frac{4}{a^2}z\frac{d}{dz}+\frac{4}{a^2}z^2\frac{d^2}{dz^2}+k^2-\alpha^2\left(1+\frac{1-\frac{z}{z_0}}{1+\frac{z}{z_0}}-\frac{1-z_0z}{1+z_0z}\right)\right]u(z)=0. \qquad (2.11)$$

Latter equation after some manipulation becomes

$$u''+\frac{1}{z}u'+\left[\frac{a^2\left(k^2-\alpha^2\right)}{4z^2}+\frac{\alpha^2a^2\left(1-z_0^2\right)}{2z\left(z_0+z\right)\left(1+z_0z\right)}\right]u=0$$

in which a prime stands for derivative with respect to z: Let's introduce $\nu^2 = -\frac{a^2\left(k^2-\alpha^2\right)}{4}$ and $\mu^2 = \frac{\alpha^2 a^2\left(1-z_0^2\right)}{2}$ and $u(z) = z^\nu \phi(z)$ such that the main equation (2.1) becomes

$$\phi'' + \frac{2\nu+1}{z}\phi' + \frac{\mu^2}{z\left(z_0+z\right)\left(1+z_0 z\right)}\phi = 0. \tag{2.12}$$

This equation is the so called Heun's differential equation whose general form is given by

$$\phi'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\varepsilon}{z-p}\right)\phi' + \frac{\alpha\beta z - q}{z(z-1)(z-p)}\phi = 0 \tag{2.13}$$

with the condition $\varepsilon = \alpha + \beta - \gamma - \delta + 1.$ The solution is given by

$$\begin{aligned}\phi = {} & C_1 HeunG\left(p, q, \alpha, \beta, \gamma, \delta, z\right) + \\ & C_2 z^{1-\gamma} HeunG\left(p, q-\left(p\delta+\varepsilon\right)\left(\gamma-1\right), \beta-\gamma+1, \alpha-\gamma+1, 2-\gamma, \delta, z\right).\end{aligned} \tag{2.14}$$

Comparing these two equations one finds that with $-\zeta = z / z_0$ Eq. (2.1) becomes

$$\phi''(\zeta) + \frac{2\nu+1}{\zeta}\phi'(\zeta) - \frac{\frac{\mu^2}{z_0^2}}{\zeta(\zeta-1)\left(\zeta - \frac{1}{z_0^2}\right)}\phi(\zeta) = 0 \tag{2.15}$$

which implies $\gamma = 2\nu+1, \delta = \varepsilon = 0, \alpha = 0, \beta = 2\nu, q = \frac{\mu^2}{z_0^2}$ and $p = \frac{1}{z_0^2}$

Therefore the solution of the main equation becomes

$$\phi(z) = C_1 HeunG\left(\frac{1}{z_0^2}, \frac{\mu^2}{z_0^2}, 0, 2\nu, 2\nu+1, 0, -\frac{z}{z_0}\right) + C_2 z^{-2\nu} HeunG\left(\frac{1}{z_0^2}, \frac{\mu^2}{z_0^2}, 0, -2\nu, 1-2\nu, 0, -\frac{z}{z_0}\right). \tag{2.16}$$

Herein C_1 and C_2 are two integration constants.

The first boundary condition to be considered is as follows:

$$\lim_{x\to\infty} u(x) = 0$$

this means indeed

$$\lim_{z\to-\infty} z^{\nu}\phi(z) = 0.$$

Upon considering $HeunG(p,q,\alpha,\beta,\gamma,\delta,0)=1$ we find

$$\lim_{z\to-\infty}\left(z^{\nu}C_1 + C_2 z^{-\nu}\right) = 0$$

which by considering n > 0 it yields $C_2=0$: Therefore the solution becomes

$$\phi(z) = C_1 HeunG\left(\frac{1}{z_0^2}, \frac{\mu^2}{z_0^2}, 0, 2\nu, 2\nu+1, 0, -\frac{z}{z_0}\right).$$

To consider the second boundary condition i.e. $\lim_{x\to-\infty} u(x) = 0$, we need to evaluate

$$\lim_{z\to-\infty} z^{\nu}\phi(z) = 0.$$

This limit is not easy to be calculated unless we transform the HeunG to its other forms.

Chapter 3

K-G MASSLESS PARTICLE IN AN INFINITE WELL WITH MOVING WALL

3.1 Klein-Gordon for Relativistic Spin-0 Particle in a Box

Let’s consider a relativistic spin-zero particles in an infinite box as defined below [6, 2, 5]:

$$V(x) = \begin{cases} 0, & 0 \le x \le L \\ \infty, & elsewhere \end{cases} \tag{3.1}$$

The K-G equation inside the box reads

$$\left(\Box + \frac{m_0^2 c^2}{\hbar^2} \right) \psi(x,t) = 0 \tag{3.2}$$

In which

$$\Box = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \tag{3.3}$$

and m_0 is the rest mass of the particle. Our solution to Eq. (3.2) is given as

$$\psi(x,t)=U(x)T(t). \tag{3.4}$$

Substituting Eq. (3.4) into Eq. (3.2) we get

$$\frac{1}{c^2}U\frac{\partial^2 T}{\partial t^2}-T\frac{\partial^2 U}{\partial x^2}+\frac{m_0^2c^2}{\hbar^2}UT=0. \tag{3.5}$$

Dividing Eq. (3.5) by UT yields

$$\frac{1}{c^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2}-\frac{1}{U}\frac{\partial^2 U}{\partial x^2}+\frac{m_0^2c^2}{\hbar^2}=0. \tag{3.6}$$

Now, one can separate the variables as

$$-\frac{1}{U}\frac{\partial^2 U}{\partial x^2}+\frac{m_0^2c^2}{\hbar^2}=-\frac{1}{c^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2}=\alpha^2 \tag{3.7}$$

in which α^2 is a constant. The time part reads

$$-\frac{1}{c^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2}=\alpha^2 \tag{3.8}$$

or equivalently

$$\frac{\partial^2 T}{\partial t^2}+\omega^2 T=0, \tag{3.9}$$

where $\omega^2=\alpha^2c^2$ and $\omega=\frac{E}{\hbar}$.

The general solution to Eq. (3.9) is given by

$$T = Ae^{i\frac{E}{\hbar}t} + Be^{-i\frac{E}{\hbar}t} \tag{3.10}$$

in which A and B are integration constants.

The space part of the KG equation (3.7) becomes

$$-\frac{1}{U}\frac{\partial^2 U}{\partial x^2} + \frac{m_0^2 c^2}{\hbar^2} = \alpha^2 \tag{3.11}$$

or consequently

$$\frac{\partial^2 U}{\partial x^2} + k^2 U = 0 \tag{3.12}$$

in which

$$k^2 = \alpha^2 - \frac{m_0^2 c^2}{\hbar^2}. \tag{3.13}$$

The general solution to Eq. (3.12) is also given by

$$U = C\ \sin(kx) + D\ \cos(kx) \tag{3.14}$$

in which C and D are integration constants.

The boundary conditions concern only the space part which are given as

$$U\ (x=0) = U\ (x=L) = 0\,. \tag{3.15}$$

The condition at $x = 0$ implies $$D = 0 \tag{3.16}$$

while the condition at $x = L$ gives

$$C \ \sin\left(kL\right) = 0 \tag{3.17}$$

which, eventually yields $$kL = n\pi \tag{3.18}$$

and therefore $$k_n = \frac{n\pi}{L} \ , \ n = 1, 2, 3, ... \tag{3.19}$$

It directs to the energy spectrum of the particle as

$$E_n = \pm\ c \ \ = \pm\ c \ \sqrt{k_n^2 \hbar^2 + m_0^2 c^2} \tag{3.20}$$

or

$$E_n = \pm\ c \ \sqrt{\frac{n^2 \pi^2 \hbar^2}{L^2} + m_0^2 c^2}. \tag{3.21}$$

In addition to the energy spectrum we also find

$$U_n \ \left(x\right) = C_n \ \sin\left(\frac{n\pi}{L} x\right) \tag{3.22}$$

in which C_n is the normalization constant.

Substituting Eq. (3.22) and Eq. (3.10) in to Eq. (3.4) we get [7]

$$\psi_n = Ae^{-i\frac{E_n}{\hbar}t}\sin(k_n x) + Be^{i\frac{E_n}{\hbar}t}\sin(k_n x). \tag{3.23}$$

From Eq. (3.23), we can write the eigenfunctions for the particle and the anti-particle respectively as

$$\psi_n^+ = C_n^+ e^{-i\frac{E_n}{\hbar}t}\sin(k_n x) \tag{3.24}$$

$$\psi_n^- = C_n^- e^{i\frac{E_n}{\hbar}t}\sin(k_n x) \tag{3.25}$$

Note that the particle density r for particle and anti-particle are

$$\rho^{\pm} = \frac{-\hbar e}{2im}\left(\psi^{*\pm}\frac{\partial\psi^{\pm}}{\partial t} - \psi^{\pm}\frac{\partial\psi^{*\pm}}{\partial t}\right). \tag{3.26}$$

From Eq. (3.24) one finds

$$\psi^{*+} = C_n^+ e^{i\frac{E_n}{\hbar}t}\sin(k_n x), \tag{3.27}$$

$$\frac{\partial\psi^+}{\partial t} = C_n^+ e^{-i\frac{E_n}{\hbar}t}\sin(k_n x)\left(\frac{-iE_n}{\hbar}\right), \tag{3.28}$$

and

$$\frac{\partial\psi^{*+}}{\partial t} = C_n^+ e^{i\frac{E_n}{\hbar}t}\sin(k_n x)\left(\frac{iE_n}{\hbar}\right). \tag{3.29}$$

Herein a star $*$ stands for the complex conjugate. Substituting Eq. (3.24), Eq. (3.27), Eq. (3.28), and Eq. (3.29), into Eq. (3.26) yields

$$\rho^{+}=\frac{-\hbar e}{2im}\left[\left|C_n^{+}\right|^2\sin^2\left(k_n x\right)\left(\frac{-iE_n}{\hbar}\right)e^{i\frac{E_n}{\hbar}t}e^{-i\frac{E_n}{\hbar}t}-\left|C_n^{+}\right|^2\sin^2\left(k_n x\right)\left(\frac{iE_n}{\hbar}\right)e^{-i\frac{E_n}{\hbar}t}e^{i\frac{E_n}{\hbar}t}\right], \quad (3.30)$$

or in short

$$\rho^{+}=\frac{-\hbar e}{2im}\left[2\left|C_n^{+}\right|^2\sin^2\left(k_n x\right)\left(\frac{-iE_n}{\hbar}\right)\right], \quad (3.31)$$

and finally

$$\rho^{+}=\frac{\left|C_n^{+}\right|^2 E_n e}{m}\sin^2\left(k_n x\right). \quad (3.32)$$

Following the same steps of the particle density for anti-particle one finds

$$\rho^{-}=-\frac{\left|C_n^{-}\right|^2 E_n e}{m}\sin^2\left(k_n x\right). \quad (3.33)$$

The relativistic normalization condition is given by

$$\int_{-\infty}^{\infty}\rho^{\pm}d^3x=\pm e \quad (3.34)$$

where for the particle

$$\int\rho^{+}d^3x=+e \quad (3.35)$$

and for the anti -particle

$$\int \rho^{-} d^3x = -e. \qquad (3.36)$$

Herein e is the charge of the particle. Substituting the value of particle density for particle Eq. (3.32), in to Eq. (3.35), we get

$$\int_0^L \frac{\left|C_n^+\right|^2 E_n}{m} \sin^2\left(k_n x\right) dx = 1 \qquad (3.37)$$

or

$$\int_0^L \frac{\left|C_n^+\right|^2 E_n}{m} \frac{1}{2}\left[1-\cos\left(2k_n x\right)\right] dx = 1. \qquad (3.38)$$

Then

$$\frac{\left|C_n^+\right|^2 E_n}{m}\left[\int_0^L dx - \int_0^L \cos\left(2k_n x\right) dx\right] = 1, \qquad (3.39)$$

and finally

$$\frac{\left|C_n^+\right|^2 E_n}{m}\left(L\right) = 1 \Rightarrow \left|C_n^{\pm}\right|^2 = \frac{2m}{LE_n} \Rightarrow C_n^{\pm} = \sqrt{\frac{2m}{LE_n}}. \quad (3.40)$$

Thus, the eigenfunctions and the Eigen energies can be written as

$$\psi_n^{\pm}\left(x,t\right) = \sqrt{\frac{2m}{LE_n}} \sin\left(\frac{n\pi}{L} x\right) e^{\pm i \frac{E_n}{\hbar} t} \qquad (3.41)$$

$$E_n = c\sqrt{\frac{n^2\pi^2\hbar^2}{L^2} + m_0^2c^2} \tag{3.42}$$

$$\psi_n^{\pm}(x) = \sqrt{\frac{2m}{LE_n}}\, sin\left(\frac{n\pi}{L}x\right). \tag{3.43}$$

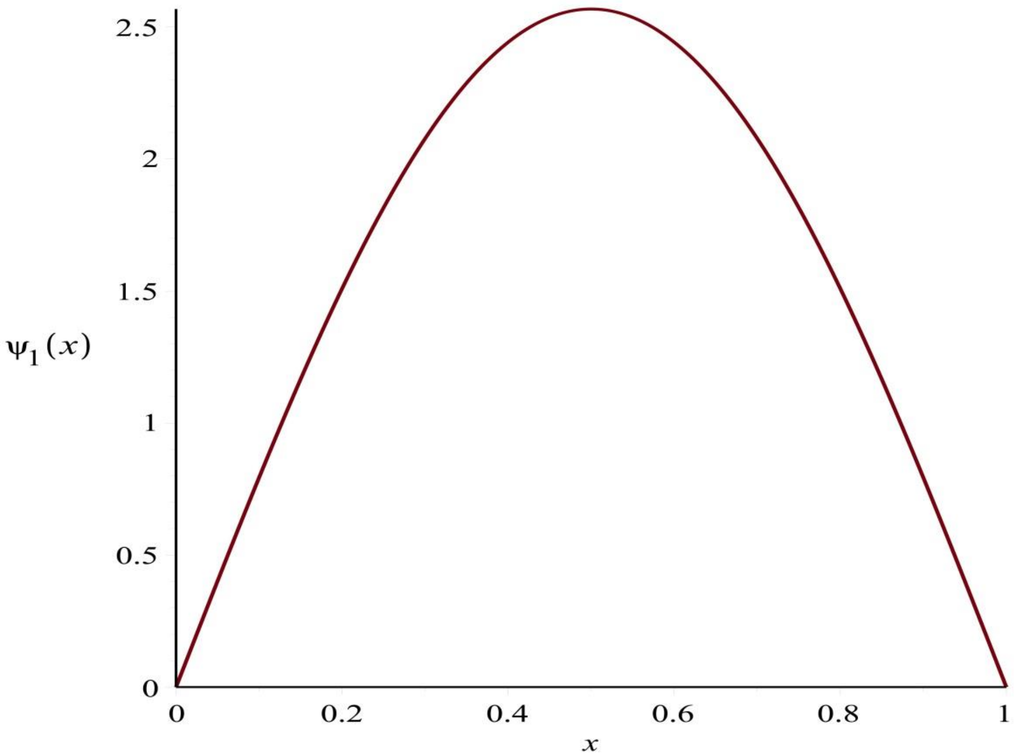

Figure 3.1: The Klein-Gordon field ψ_1 of a massive particle inside an infinite well. The ground-state of the particle is taken from the equation (3.43) with $n = 1$. We note that unlike the case of non-relativistic wave function in which $<\psi_1|\psi_1>= 1$, in this case i.e., K-G field the equation of normalization becomes (3.34).

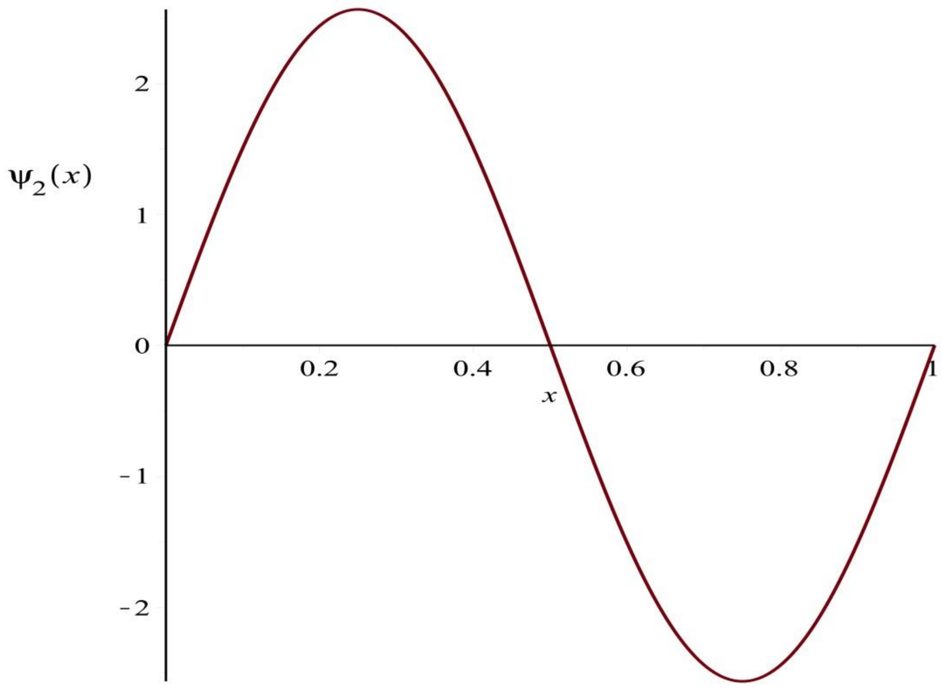

Figure 3.2: The first excited state of the K-G massive particle inside an infinite square well.

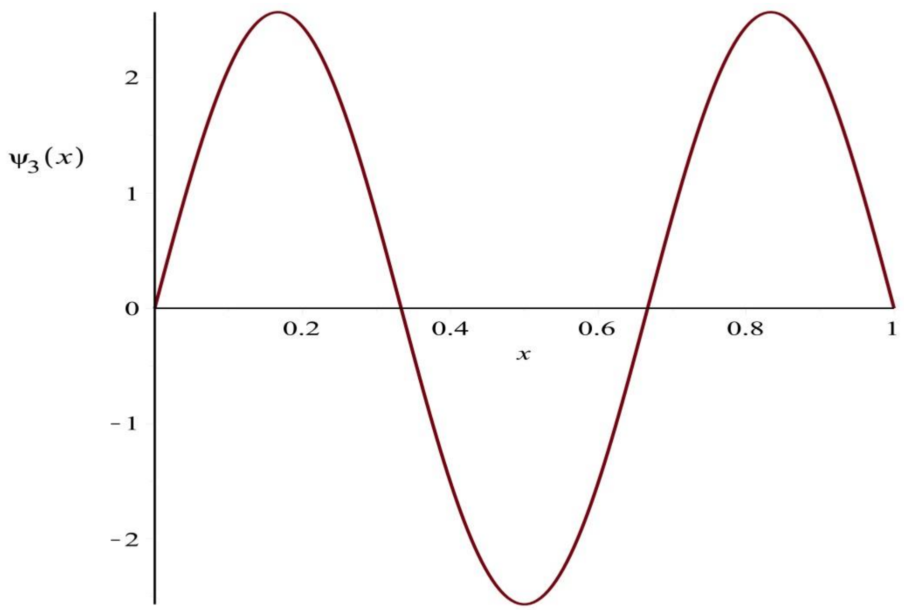

Figure 3.3: The second excited state of the K-G massive particle inside an infinite square well.

3.2 K-G particle in an Infinite square-well with a moving wall

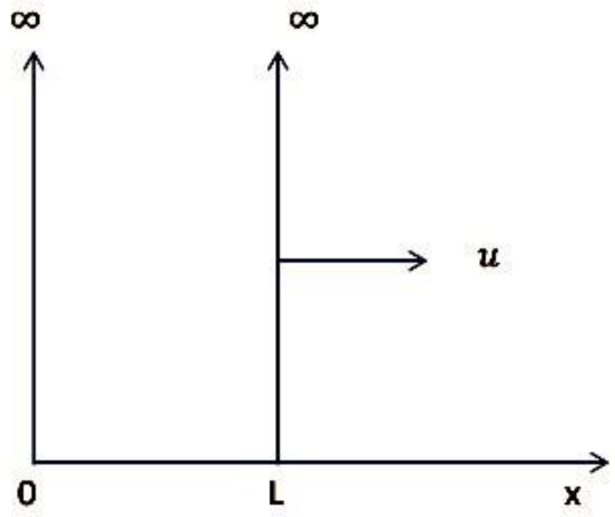

Figure 3.4: Infinite well potential with a moving wall. Here the left wall is fixed at $x = 0$ and the other wall is moving with a constant velocity u in $+x$-direction.

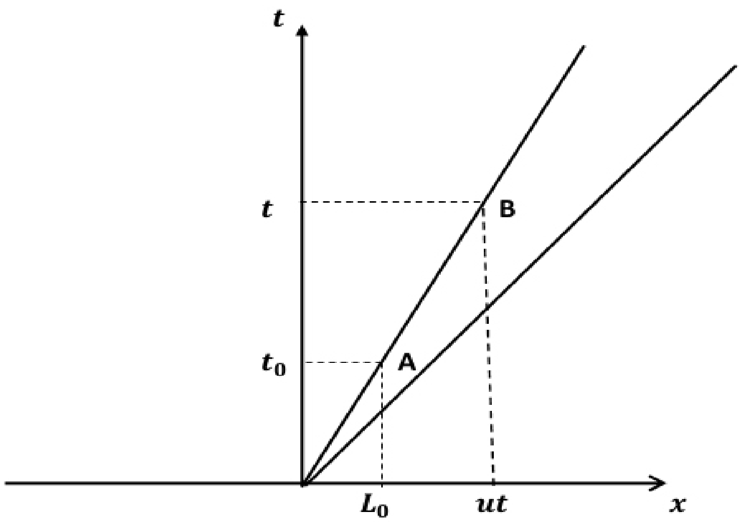

Figure 3.5: In this figure we show the light cone (the lower line) and the world-line of the moving wall. The points A and B are the corresponding location of the moving wall at $t = 0$ and $t > 0$.

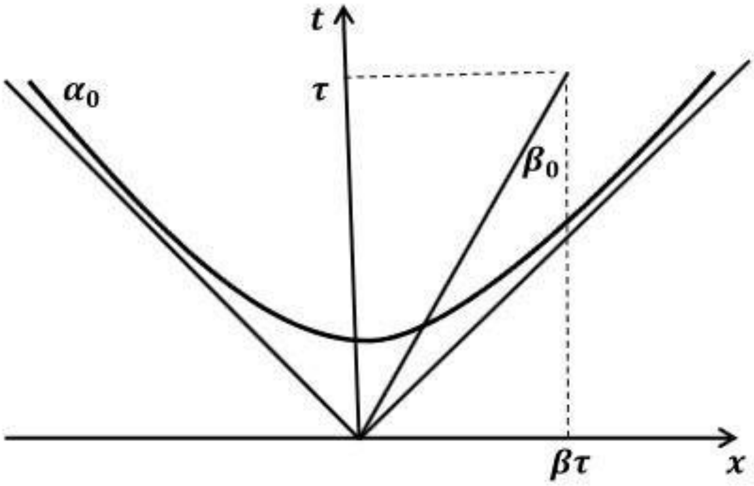

Figure 3.6: In this figure, the new coordinate is shown. The curved line is $\alpha\ = constant$ and the other line is $\beta\ =\ constant$. Also we plot the light cone clearly.

In this section we consider the relativistic massless particle in an infinite square well potential with a moving wall by a transformation from $(x\ -\ t)$ coordinate to $(\alpha\ -\ \beta)$ coordinate (Hyperbolic coordinates) [6].

As it is shown in the figure (3.4), the right wall of the well is moving with a constant speed. A transformation of the form

$$\alpha\beta = t + x \tag{3.44}$$

and

$$\frac{\alpha}{\beta} = t - x \tag{3.45}$$

Maps our problem from x-t coordinate to $\alpha-\beta$ coordinate. Both space time are 1+1-dimensional and flat which after we set $c=1$, the line element in x-t coordinate is given by

$$ds^2 = dt^2 - dx^2 \tag{3.46}$$

The K-G equation for massless particle reads as

$$\square\psi = 0 \tag{3.47}$$

in which

$$\square = -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}. \tag{3.48}$$

Derivative of Eq. (3.44), and Eq. (3.45), yield

$$\alpha\beta = t + x \Rightarrow d\alpha\beta + \alpha d\beta = dt + dx \tag{3.49}$$

and

$$\frac{\alpha}{\beta} = t - x \Rightarrow \frac{d\alpha}{\beta} - \frac{d\beta}{\beta^2}\alpha = dt - dx. \tag{3.50}$$

Next, using above one finds

$$\left(d\alpha\beta + \alpha d\beta\right)\left(\frac{d\alpha}{\beta} - \frac{d\beta}{\beta^2}\alpha\right) = \left(dt + dx\right)\left(dt - dx\right) \Rightarrow \ d\alpha^2 - \frac{\alpha^2}{\beta^2}d\beta^2 = (\ dt^2 - dx^2)$$

(3.51)

Therefore the line element transforms as

$$ds^2 = d\alpha^2 - \frac{\alpha^2}{\beta^2} d\beta^2 \qquad (3.52)$$

whose metric tensor is given by

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{\alpha^2}{\beta^2} \end{bmatrix} \qquad (3.53)$$

and its inverse becomes

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{\beta^2}{\alpha^2} \end{bmatrix}. \qquad (3.54)$$

Next, we transform the K-G equation from x-t space time to $\alpha - \beta$ space time. The standard form of the K-G equation for a massless particle reads

$$\Box \psi(\alpha, \beta) = 0 \qquad (3.55)$$

in which

$$\Box = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\ \partial^\mu \right) \qquad (3.56)$$

and

$$-g = \left| g_{\mu\nu} \right| = \frac{\alpha^2}{\beta^2}.$$

Finally one finds

$$\Box = \frac{\beta}{\alpha}\,\partial_\alpha\left(\frac{\alpha}{\beta}\,\partial^\alpha\right) + \frac{\beta}{\alpha}\partial_\beta\left(\frac{\alpha}{\beta}\,\partial^\beta\right), \qquad (3.57)$$

which after lowering the indices

$$\Box = \frac{\beta}{\alpha}\partial_\alpha\left(\frac{\alpha}{\beta}\partial_\alpha\right) + \frac{\beta}{\alpha}\partial_\beta\left[\frac{\alpha}{\beta}\left(-\frac{\beta^2}{\alpha^2}\right)\partial_\beta\right] \qquad (3.58)$$

we get

$$\Box = \frac{1}{\alpha}\partial_\alpha\left(\alpha\,\partial_\alpha\right) - \frac{\beta}{\alpha^2}\partial_\beta\left(\beta\,\partial_\beta\right). \qquad (3.59)$$

Equation (3.59) is the d'Alembert operator in $\alpha - \beta$ coordinates. The K-G equation becomes

$$\frac{1}{\alpha}\partial_\alpha\left(\alpha\,\partial_\alpha\right)\psi - \frac{\beta}{\alpha^2}\partial_\beta\left(\beta\,\partial_\beta\right)\psi = 0 \qquad (3.60)$$

or more clear

$$\frac{1}{\alpha}\frac{\partial}{\partial\alpha}\left(\alpha\frac{\partial}{\partial\alpha}\right)\psi - \frac{\beta}{\alpha^2}\frac{\partial}{\partial\beta}\left(\beta\frac{\partial}{\partial\beta}\right)\psi = 0. \qquad (3.61)$$

The solution of the Eq. (3.61) can be written as

$$\psi = R(\alpha)\phi(\beta) \tag{3.62}$$

which after substituting Eq. (3.62) into Eq. (3.61), we get

$$\frac{\phi}{\alpha}\frac{\partial}{\partial\alpha}\left(\alpha\frac{\partial R}{\partial\alpha}\right) - \frac{R\beta}{\alpha^2}\frac{\partial}{\partial\beta}\left(\beta\frac{\partial\phi}{\partial\beta}\right) = 0. \tag{3.63}$$

Dividing Eq. (3.63) by $R\phi$, and multiply by α^2 yields

$$\frac{\alpha}{R}\frac{\partial}{\partial\alpha}\left(\alpha\frac{\partial R}{\partial\alpha}\right) - \frac{\beta}{\phi}\frac{\partial}{\partial\beta}\left(\beta\frac{\partial\phi}{\partial\beta}\right) = 0. \tag{3.64}$$

Next, one can separate the variable as

$$\frac{\alpha}{R}\frac{\partial}{\partial\alpha}\left(\alpha\frac{\partial R}{\partial\alpha}\right) = \frac{\beta}{\phi}\frac{\partial}{\partial\beta}\left(\beta\frac{\partial\phi}{\partial\beta}\right) = -k^2 \tag{3.65}$$

in which k^2 is a separation constant. The α part reads

$$\alpha\frac{\partial}{\partial\alpha}\left(\alpha\frac{\partial R}{\partial\alpha}\right) + k^2 R = 0 \tag{3.66}$$

while the β part of the equation (3.65) becomes

$$\beta\frac{\partial}{\partial\beta}\left(\beta\frac{\partial\phi}{\partial\beta}\right) + k^2\phi = 0. \tag{3.67}$$

Now from equation (3.66)

$$\alpha^2 \frac{d^2R}{d\alpha^2} + \alpha \frac{dR}{d\alpha} + k^2 R = 0 \tag{3.68}$$

has a solution of the form $R = \alpha^m$, which implies .

$$\frac{dR}{d\alpha} = m\alpha^{m-1} \tag{3.69}$$

and

$$\frac{d^2R}{d\alpha^2} = m(m-1)\alpha^{m-2}. \tag{3.70}$$

Substituting in Eq. (3.68), one gets

$$\alpha^2 m(m-1)\alpha^{m-2} + \alpha m\alpha^{m-1} + k^2\alpha^m = 0 \Rightarrow m(m-1)\alpha^m + m\alpha^m + k^2\alpha^m = 0 \tag{3.71}$$

dividing Eq. (3.71) by α^m to obtain

$$m(m-1) + m + k^2 = 0 \Rightarrow m^2 + k^2 = 0 \Rightarrow m = \pm ik. \tag{3.72}$$

Then the solution of $R(\alpha)$ becomes

$$R = A\,\alpha^{ik} + B\,\alpha^{-ik} \tag{3.73}$$

or simply

$$R = A\,e^{(ik\ln\alpha)} + B\,e^{(-ik\ln\alpha)}. \qquad (3.74)$$

The inverse transformation

$$\beta^2 = \frac{t+x}{t-x}. \qquad (3.75)$$

For the left wall, at point $\mathrm{t = t_0}$ and $x = 0$; Eq. (3.75) implies

$$\beta_L^2 = \frac{t+x}{t-x} \Rightarrow \beta_L^2 = \frac{t_0}{t_0} = 1 \Rightarrow \beta_L = 1. \qquad (3.76)$$

For right wall, $x = ut$ and $x = L = L_0 + u\left(t - t_0\right)$. Then Eq. (3.75) also implies

$$\beta_R^2 = \frac{t + L_0 + u\ \left(t - t_0\right)}{t - L_0 - u\ \left(t - t_0\right)}. \qquad (3.77)$$

Substituting the value of $\mathrm{L_0 = ut_0}$ in Eq. (3.77), we get

$$\beta_R^2 = \frac{t+ut}{t-ut} = \frac{t\ \left(1+u\right)}{t\ \left(1-u\right)} = \frac{1+u}{1-u}. \qquad (3.78)$$

Therefore the boundary conditions, simply reads $\phi(\beta_L) = \phi(\beta_R) = 0.$

$$\beta^2 \frac{d^2\phi}{d\beta^2} + \beta \frac{d\phi}{d\beta} + k^2\phi = 0. \qquad (3.79)$$

Has a solution of the form $\phi(\beta) = \beta^q$ which yields

$$\frac{d\ \phi}{d\beta} = q\beta^{q-1}, \qquad (3.80)$$

and

$$\frac{d^2\phi}{d\beta^2} = q\left(q-1\right)\beta^{q-2}\ . \qquad (3.81)$$

Substituting in Eq. (3.79), one gets

$$\beta^2\{q\ \left(q-1\right)\beta^{q-2}\} + \beta(q\beta^{q-1}) + k^2\beta^q = 0 \Rightarrow q\ \left(q-1\right)\beta^q + q\beta^q + k^2\beta^q = 0 \quad (3.82)$$

dividing Eq. (3.82) by β^q , we obtain

$$q\ \left(q-1\right) + q + k^2 = 0 \Rightarrow q^2 + k^2 = 0 \Rightarrow q = \pm ik. \qquad (3.83)$$

Then the solution of $\phi(\beta)$ becomes

$$\phi(\beta) = C\beta^{ik} + D\beta^{-ik} \qquad (3.84)$$

thus

$$\phi(\beta) = C\, e^{(ik\ln\beta)} + D\, e^{(-ik\ln\beta)}. \tag{3.85}$$

Hence, the general solution yields

$$\psi(\alpha,\beta) = R(\alpha)\,\phi(\beta) \tag{3.86}$$

The first boundary condition for $\beta = \beta_L = 1$, $\psi\left(\alpha, \beta = 1\right) = 0$, $R\left(\alpha\right)\phi\left(\beta = 1\right) = 0$,

then, $\phi\left(\beta = 1\right) = 0$. Then Eq. (3.85) becomes

$$C + D = 0 \Rightarrow C = -D \tag{3.87}$$

putting Eq. (3.87) in to Eq. (3.85) we get

$$\phi(\beta) = C\ \left(\ e^{(ik\ln\beta)} - e^{(-ik\ln\beta)}\right) \tag{3.88}$$

$$\phi(\beta) = 2iC\ \sin\left(k\ln\beta\right) \tag{3.89}$$

where $\tilde{C} = 2iC$, then

$$\phi(\beta) = \tilde{C}\sin\left(k\ln\beta\right). \tag{3.90}$$

Second boundary condition when $\beta = \beta_R$, $\psi\left(\alpha, \beta = \beta_R\right) = 0,$

$R\left(\alpha\right)\phi\left(\beta_R\right) = 0 \Rightarrow \phi\left(\beta_R\right) = 0$. Then Eq. (3.90) reads

$$\tilde{C}\ \sin\left[k\ln\beta_R\right] = 0 \tag{3.91}$$

$$k\ln\beta_R = n\,\pi \Rightarrow k_n = \frac{n\pi}{\ln\beta_R}, \quad n = 1, 2, 3, ... \tag{3.92}$$

Substituting $k_n = \dfrac{n\pi}{\ln\beta_R}$ into Eq. (3.90), we get

$$\phi(\beta) = \tilde{C}\ \sin\left[\frac{n\pi}{\ln\beta_R}\ln\beta\right] \tag{3.93}$$

where $\beta_R = \sqrt{\dfrac{1+u}{1-u}}$.

Thus, $R(\alpha) = A\exp\left[ik\ln\alpha\right] + B\exp\left[-ik\ln\alpha\right]$, $\phi(\beta) = \tilde{C}\ \sin\left[\dfrac{n\pi}{\ln\beta_R}\ln\beta\right]$.

Putting $R\left(\alpha\right)$ *and* $\phi\left(\beta\right)$) into Eq. (3.62) .

Then the general solution becomes

$$\psi_n = \tilde{A}e^{ik_n\ln\alpha}\sin\left[\frac{n\pi}{\ln\beta_R}\ln\beta\right] + \tilde{B}e^{-ik_n\ln\alpha}\sin\left[\frac{n\pi}{\ln\beta_R}\ln\beta\right] \tag{3.94}$$

where $\tilde{A} = A\,\tilde{C}$, $\tilde{B} = B\,\tilde{C}$.

Let's look at Eq. (3.44) and Eq. (3.45). Dividing Eq. (3.44), by Eq. (3.45), yields

$$\beta^2 = \frac{t+x}{t-x} \tag{3.95}$$

substituting Eq. (3.95), in to Eq. (3.44),we get

$$\alpha^2 = t^2 - x^2 \tag{3.96}$$

putting Eq. (3.95), and Eq. (3.96), in to Eq. (3.94), we get

$$\psi_n = \tilde{A}e^{ik_n \ln\left(t^2-x^2\right)^{\frac{1}{2}}} \sin\left[k_n \ \ln\left(\frac{t+x}{t-x}\right)^{\frac{1}{2}}\right] + \tilde{B}e^{-ik_n \ln\left(t^2-x^2\right)^{\frac{1}{2}}} \sin\left[k_n \ \ln\left(\frac{t+x}{t-x}\right)^{\frac{1}{2}}\right], \tag{3.97}$$

or

$$\psi(t,x) = \tilde{A}\, e^{\frac{ik_n}{2}\ln\left(t^2-x^2\right)} \ \sin\left[\frac{k_n}{2}\ln\left(\frac{t+x}{t-x}\right)\right] + \tilde{B}\, e^{\frac{-ik_n}{2}\ln\left(t^2-x^2\right)} \ \sin\left[\frac{k_n}{2}\ln\left(\frac{t+x}{t-x}\right)\right]. \tag{3.98}$$

The continuity equation is given by (note that we set from the beginning $c = 1$)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.\vec{J} = 0 \tag{3.99}$$

in which

$$\rho = \frac{\hbar i\, e}{2m}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right), \tag{3.100}$$

and

$$J = \frac{\hbar i}{2m}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right). \tag{3.101}$$

By integrating both sides we get

$$\int \frac{\partial \rho}{\partial t} d^3x = -\int \vec{\nabla}.\vec{J}\, d^3x \tag{3.102}$$

or

$$\frac{\partial}{\partial t}\int \rho d^3x = -\oint \vec{J}.\hat{n} dA = 0. \tag{3.103}$$

For $J = 0$ the normalization condition becomes

$$\int \rho d^3x = \pm\, e. \tag{3.104}$$

The eigenfunction with respect to α and β is given as

$$\psi_n^{\pm} = A_n\, e^{(\pm i k_n \ln \alpha)} \sin\ (k_n \ln \beta). \tag{3.105}$$

Then, the eigenfunction with respect to t and x is given as

$$\psi_n^{\pm} = A_n\, e^{\left[\pm i \frac{k_n}{2}\ \ln\left(t^2 - x^2\right)\right]} \sin\left[\frac{k_n}{2}\ \ln\left(\frac{t+x}{t-x}\right)\right], \tag{3.106}$$

We know that for non-relativistic quantum particle the orthogonality implies

$$\langle \psi_n | \psi_m \rangle = \delta_{nm},$$

or

$$\int \psi_n^* \psi_m d^3x = \delta_{nm} \tag{3.107}$$

but for a relativistic K-G quantum particle it becomes ($\hbar = c = m = 1$)

$$\int i \left(\psi_n^* \frac{\partial \psi_m}{\partial t} - \psi_m \frac{\partial \psi_n^*}{\partial t} \right) d\beta = \delta_{nm}, \tag{3.108}$$

in which $\mathrm{e} = 1$. Therefore

$$i \int_{\beta_L}^{\beta_R} \frac{\alpha}{\beta} \, d\beta \, \left(\psi_n^* \, \partial_\alpha \psi_m \; - \psi_m \, \partial_\alpha \psi_n^* \right) = \delta_{nm} \tag{3.109}$$

With

$$\psi_m^+ = A_m \exp\left[\pm i k_m \ln \alpha\right] \sin\left[k_m \ln \beta\right] \tag{3.110}$$

$$\psi_m^- = A_m^- \exp\left[-i k_m \ln \alpha\right] \sin\left[k_m \ln \beta\right]. \tag{3.111}$$

Taking conjugate of Eq. (3.110), we find

$$\psi_n^{*(-)} = A_n^- \exp\left[i k_n \ln \alpha\right] \sin\left[k_n \ln \beta\right] \tag{3.112}$$

and a derivative of Eq. (3.111) implies

$$\frac{\partial \psi_m^-}{\partial \alpha} = A_m^- \exp\left[-i k_m \ln \alpha\right] \sin\left[k_m \ln \beta\right] \left(\frac{-i k_m}{\alpha} \right). \tag{3.113}$$

Also, derivative of Eq. (3.112) yields

$$\frac{\partial \psi_n^{*(-)}}{\partial \alpha} = A_n^- \exp\left[ik_n \ln \alpha\right]\sin\left[k_n \ln \beta\right]\left(\frac{ik_n}{\alpha}\right). \tag{3.114}$$

Therefore by using Eq. (3.109) one can find the normalization constant A.

Now, by substituting of Eqs. (3.111), (3.112), (3.113), to Eq. (3.109) one finds

$$i\alpha\int_{\beta_L}^{\beta_R}\frac{d\beta}{\beta}\left\{A_n^{*(-)}e^{ik_n\ln\alpha}\sin\left[k_n\ln\beta\right]\times A_m^-\left(\frac{-ik_m}{\alpha}\right)e^{-ik_m\ln\alpha}\sin\left[k_m\ln\beta\right]\right\}$$

$$-\left\{A_m^{*(-)}e^{ik_m\ln\alpha}\sin\left[k_m\ln\beta\right]\times A_n^{(-)}\left(\frac{ik_n}{\alpha}\right)e^{-ik_n\ln\alpha}\sin\left[k_n\ln\beta\right]\right\}=\delta_{nm} \tag{3.115}$$

Here there exist two cases:

1. **The first case: For $m = n$.**

In this case Eq. (3.115) becomes

$$i\alpha\int_{\beta_L}^{\beta_R}\frac{d\beta}{\beta}\left|A_n^-\right|^2\left(\frac{-2ik_n}{\alpha}\right)\sin^2\left[k_n\ln\beta\right]=1 \tag{3.116}$$

or

$$2k_n\left|A_n^-\right|^2\int_{\beta_L}^{\beta_R}\sin^2\left[k_n\ln\beta\right]\frac{d\beta}{\beta}=1, \tag{3.117}$$

we know that

$$k_n = \frac{n\pi}{\ln \beta_R} \Rightarrow \ln \beta_R = \frac{n\pi}{k_n}, \tag{3.118}$$

and let's change the variable according to

$$\frac{n\pi}{k_n} = \xi, \text{ then } \ln \beta = \xi \text{ and } \frac{d\beta}{\beta} = d\xi. \tag{3.119}$$

Therefore the limit of integral becomes $\beta = 1 \Rightarrow \xi = 0$, for $\beta = 0$, then $\xi = \frac{n\pi}{k_n}$

which given

$$2k_n \left|A_n^-\right|^2 \int_0^{\frac{n\pi}{k_n}} \sin^2\left[k_n \xi\right] d\xi = 1 \tag{3.120}$$

or consequently

$$2k_n \left|A_n^-\right|^2 \int_0^{\frac{n\pi}{k_n}} \frac{1}{2}\left[1 - \cos\left(2k_n \xi\right)\right] d\xi = 1. \tag{3.121}$$

The latter equation yields

$$k_n \left|A_n^-\right|^2 \left[\xi - \frac{1}{2k_n}\sin\left(2k_n \xi\right)\right]_0^{\frac{n\pi}{k_n}} = 1 \tag{3.122}$$

and finally

$$k_n \left|A_n^-\right|^2 \left(\frac{n\pi}{k_n}\right) = 1 \qquad \Rightarrow \left|A_n^-\right|^2 = \frac{1}{n\pi}. \tag{3.123}$$

Thus, the normalization constant for particle

$$\left|A_n^-\right| = \frac{1}{\sqrt{n\pi}} \tag{3.124}$$

while the normalization constant for anti-particle is

$$\left|A_n^+\right| = \frac{-1}{\sqrt{n\pi}}. \tag{3.125}$$

2. The second case: For $m \neq n$.

Now, we show that the wave functions of Eq. (3.115) are orthonormal i.e.

$$\int_{\beta_L}^{\beta_R} \left(\sin\left[k_n \ln\beta\right]\sin\left[k_m \ln\beta\right]\right)\frac{d\beta}{\beta} = 0. \qquad (3.126)$$

To show that let's use same change of variables.

$$k_n = \frac{n\pi}{\ln\beta_R}, \textit{then } \ln\beta_R = \frac{n\pi}{k_n}, \textit{let}; \frac{n\pi}{k_n} = \xi \rightarrow \ln\beta = \xi, \textit{then } \frac{d\beta}{\beta} = d\xi$$

$$\int_{\ln\beta_L=0}^{\ln\beta_R}\left[\sin\left(\frac{n\pi}{\ln\beta_R}\xi\right)\sin\left(\frac{m\pi}{\ln\beta_R}\xi\right)\right]d\xi = 0. \qquad (3.127)$$

Using the relation $\sin(\theta)\sin(\gamma) = \frac{1}{2}\left[\cos(\theta-\gamma) - \cos(\theta+\gamma)\right]$

and $\theta = \frac{n\pi}{\ln\beta_R}\xi$, $\gamma = \frac{m\pi}{\ln\beta_R}\xi$ one finds

$$\int_{\ln\beta_L=0}^{\ln\beta_R}\frac{1}{2}\left[\cos\left(\frac{(n-m)\pi}{\ln\beta_R}\xi\right) - \cos\left(\frac{(n+m)\pi}{\ln\beta_R}\xi\right)\right]d\xi \qquad (3.128)$$

which after some manipulation

$$\frac{1}{2}\left[\frac{\ln\beta_R}{(n-m)\pi}\sin\left(\frac{(n-m)\pi}{\ln\beta_R}\xi\right) - \frac{\ln\beta_R}{(n+m)\pi}\sin\left(\frac{(n+m)\pi}{\ln\beta_R}\xi\right)\right]_0^{\ln\beta_R} = 0. \qquad (3.129)$$

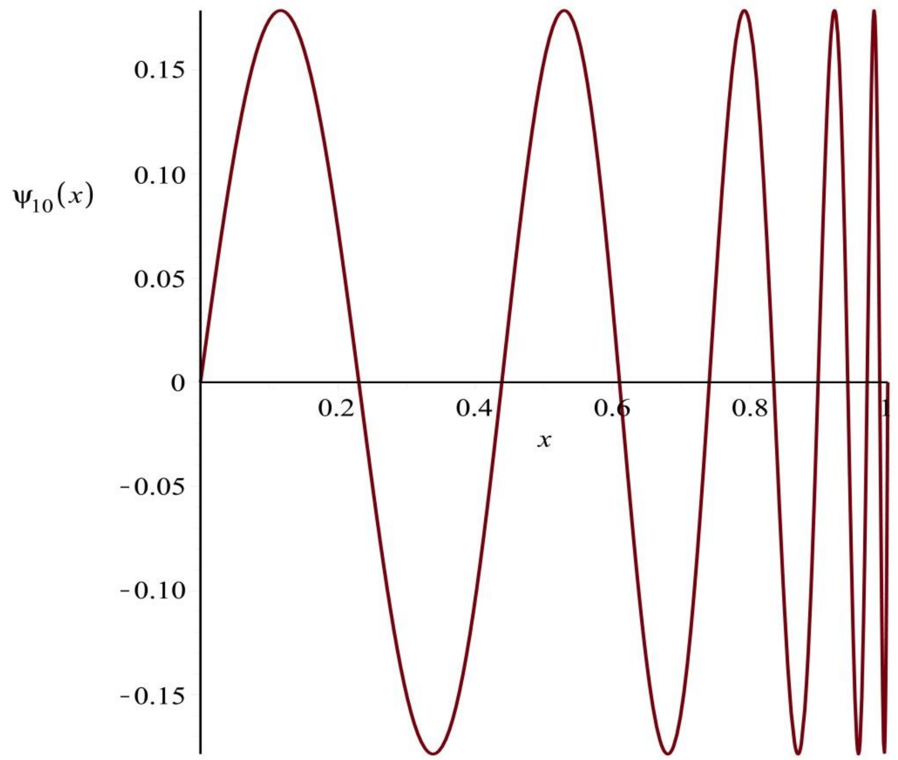

Figure 3.7: The K-G field inside an infinite well with moving wall Eq. (3.106). In this figure we have set n=10.

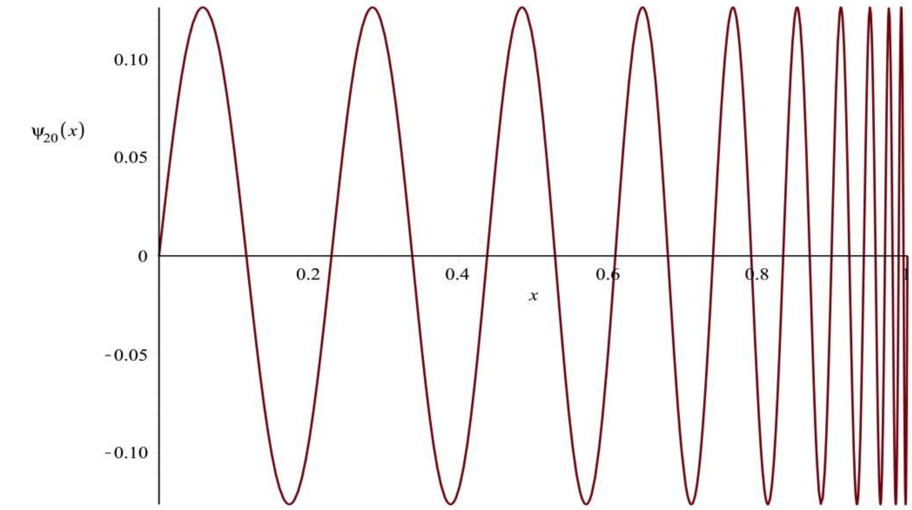

Figure 3.8: The K-G field inside an infinite well with moving wall Eq. (3.106). In this figure we have set $n = 20$.

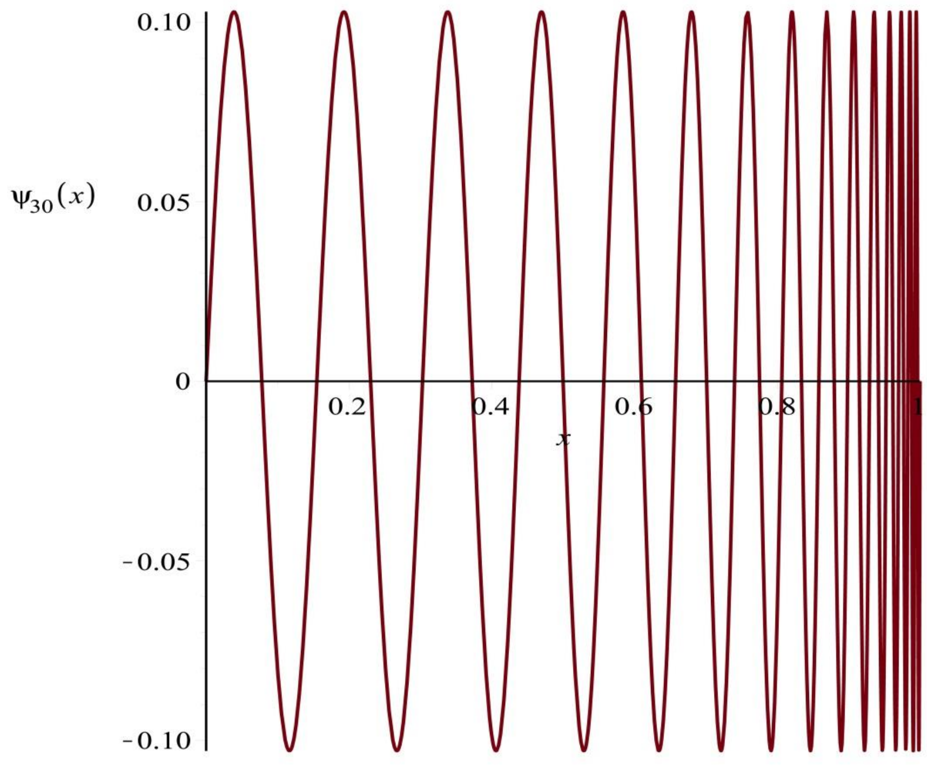

Figure 3.9: The K-G field inside an infinite well with moving wall Eq. (3.106). In this figure we have set n=30.

Chapter 4

CONCLUSION

$1 + 1 -$dimensional K-G equation attracted intensive attentions in the literature [4]. This is due to its application in understanding the nature of the relativistic particle via the possible exact solutions which may not be available in higher dimensions. Although it is always possible to find 1+1-dimensional K-G equation from its general form which can be extracted using the Lagrange density, an explicit derivation has not been given by the authors of those papers. Our first aim in this thesis was to fill this gap and introduce a straight way through this equation. After we introduced the mentioned equation we tried to find some exact solutions for that which helps to understand it better i.e., the relativistic zero-spin quantum particle under certain potentials has been studied. We found analytical solution for the K-G equation. Our first example was about a particle inside one-dimensional smooth finite well whose bounded solutions are given in terms of Heun functions [9]. Our second example is about a relativistic massless particle inside an infinite well with its wall moving. This problem was studied before and we only revisited it due to its interesting features. We finally concluded this thesis.

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