Stability of Autonomous and Non Autonomous Differential Equations

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ABSTRACT

In this thesis, we dealt with Autonomous and non Autonomous systems of ordinary differential equations and the stability properties of their solutions were discussed with some basic results. We also discussed and analyzed methods of investigating the stability of nonlinear systems and classified equilibrium points (critical points) of linear systems with respect to their stability. Liapounov's direct method for stability of Autonomous and non Autonomous Equations was analyzed in detail. Some important Ecological applications such as Lotka-Volterra Competition Model and Predator-Prey Model modeled by differential Equations were discussed in details with relevant examples.

Keywords : Autonomous and Non Autonomous differential equations, Stability, Predator-prey Model, Equilibrium points, Liapounov's Direct Method.

ÖZ

Bu tezde, otonom ve otonom olmayan adi diferansiyel denklem sistemleri ve bu sistemlerin çözümlerinin stabilite özellikleri tartışılmıştır. Ayrıca, doğrusal olmayan sistemlerin stabilitesi üzerine bazı metodlar çalışılmış ve analiz edilmiş ve doğrusal sistemlerin stabilite özelliklerine göre denge noktaları sınıflandırılmıstır. Otonom ve otonom olmayan denklemlerin stabilitesi için Lyapounov Direkt metodu detaylı bir şekilde analiz edilmiştir. Son olarak, diferansiyel denklemlerce modellenmiş olan Lotka-Volterra Yarışma modeli ve Predator Prey modeli gibi bazı önemli ekolojik uygulamalar ayrıntılı bir şekilde incelenmiştir.

Anahtar Kelimeler : Otonom ve otonom olmayan diferansiyel denklemler, Stabilite, Predator-Prey Model, Denge Noktaları, Liapounov Direkt Metod

I dedicate this thesis to my late mother, Victoria N. Obi. She taught me perseverance and prepared me to face challenges with humility, strength and faith. She was a constant source of inspiration to my life. Although she is no longer alive to give me strength and support, I always feel her presence that urges me to strive hard to achieve my goals in life.

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INTRODUCTION

There is a striking difference between Autonomous and non Autonomous differential equations. Autonomous equations are systems of ordinary differential equations that do not depend explicitly on the independent variable. Physically, an autonomous system is one in which the parameters of the system do not depend on time. Autonomous systems and dynamical systems are closely related, any system of autonomous equation can be transformed into a dynamical system and by applying some assumptions, we can transform a dynamical system into an autonomous one.

Non autonomous equations depend on the independent variables and the parameters of the systems are time dependent. However, we do not know of a coherent and general theory yet. We can rely on several independent and stability theory. But we cannot combine these facts into a unified approach. Accordingly, perturbations and other subjects can be treated in a case by case analysis which will leave many questions open. The situation worsens if we look into asymptotic properties, because we cannot apply transform theory directly. The available results are usually restricted to problems that are "very close" to an equation with known behavior. (An autonomous one In particular). Any attempt to overcome these shortcomings must face the challenges that can be demonstrated by simple examples refuting many natural conjectures. Such examples concerning the asymptotic behavior and the stability theory are treated in detail. Stability theory deals with the stability of differential equations and their solutions and also the trajectories of dynamical systems under tiny disturbances of original conditions. The heat equation is an example of a stable partial differential equation because little alterations of the original data lead to near same differences in the temperature at a future time.

In general, a system is stable if infinitesimal alterations in the theory bring about near-same changes at the end. The metric used in measuring the perturbations must be specified if we are to claim that a system is stable.

Example 1.1

The equation $y' = (2 - y)y$ is an autonomous equation because the independent variable, (call it x) does not appear explicitly in the equation.

A system of ordinary differential equation is said to be autonomous if it does not depend on time (it doesn't depend on the independent variable) i.e. $\dot{x} = f(x)$. In contrast, non autonomous is when the system of ordinary differential equation depends on time (it depends on the independent variable) i.e. $\dot{x} = f(x, t)$.

Let us consider two-dimensional systems of the form:

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where $x = (x)t$ and $y = y(t)$ are unknown scalar functions, and P and Q together with their first partial derivatives are continuous in some domain Γ of the xy-plane. Such systems are called autonomous because P and Q do not depend on t. If $z = (x, y)$, then (1) is of the form $\dot{z} = f(z) = \big(P(x, y), Q(x, y)\big)$, and the hypothesis guarantees existence and uniqueness of solution.

Below are some reasons for discussing systems of the form (1)

(i) A more complete theory exists than for higher-dimensional systems, and

(ii) The geometry of the plane and that of plane curves is available to throw more light on the discussion.

Furthermore, in many cases the analysis of the important second-order autonomous equation

$\ddot{x} + g(x, \dot{x}) = 0, \quad x = x(t)$ a scalar function,

can be extended considerably by transforming it into the system

$$\dot{x} = y, \qquad \dot{y} = -g(x, y),$$

which is the form (1).

We start by giving some simple properties of solution of (1), and introducing some terminologies.

Theorem 1.1 *If $x = x(t),\ y = y(t),\ r_1 < t < r_2$, is a solution of* (1), *then for any real constant c the functions*

$$x_1(t) = x(t + c), \qquad y_1(t) = y(t + c)$$

are also solutions of (1).

Proof 1.1 By the chain rule for differentiation it follows that;

$\dot{x}_1 = \dot{x}(t + c), \dot{y}_1 = \dot{y}(t + c)$. Since $\dot{x} = P\big(x(t), y(t)\big), \dot{y} = Q\big(x(t), y(t)\big)$, replacing t by $t + c$ gives

$$\dot{x}_1 = P\big(x(t+c), y(t+c)\big) = P(x_1, y_1),$$

$$\dot{y}_1 = Q\big(x(t+c), y(t+c)\big) = Q(x_1, y_1),$$

which implies that x_1 and y_1 are solutions. They are clearly defined on $r_1 - c < t < r_2 - c$.

Remark The above property in most cases does not hold for non autonomous systems; for a example, a solution of

$\dot{x} = x,\ \dot{y} = tx$ is $x(t) = e^t, y(t) = te^t - e^t$, and $\dot{y}(t+c) = (t+c)e^{t+c} \neq tx(t+c)$ unless $c = 0$.

As t varies, a solution $x = x(t), y = y(t)$ of (1) parametrically describes a curve lying in Γ. This curve is called a trajectory of (1).

Theorem 1.2 *At most one trajectory passes through any point*

Proof 1.2 Let $C_1: x = x_1(t), y = y_1(t)$, and $C_2: x = x_2(t), y = y_2(t)$ be distinct trajectories having a common point

$$(x_0, y_0) = \big(x_1(t_1), y_1(t_1)\big) = \big(x_2(t_2), y_2(t_2)\big)$$

Then $t_1 \neq t_2$, since otherwise, the uniqueness of solutions would be contradicted. By the just concluded theorem,

$$x(t) = x_1(t + t_1 - t_2), \qquad y(t) = y_1(t + t_1 - t_2)$$

are solutions, and $\big(x(t_2), y(t_2)\big) = (x_0, y_0)$ implies that $x(t)$ and $y(t)$ must agree respectively with $x_2(t)$ and $y_2(t)$ by uniqueness. This implies that $C_1$and C_2 coincide.

Note with care the difference between solutions and trajectories of (1) : A trajectory is a curve in Γ that is parametrically represented by more than one solution. Thus $x(t),\ y(t)$ and $x(t+c),\ y(t+c),\ c \neq 0$ represent distinct solutions, but they represent the same curve parametrically.

For example, as α varies between 0 and 2π the functions

$$x(t) = \sin(t+\alpha), \quad y(t) = \cos(t+\alpha), \quad -\infty < t < \infty,$$

represent an infinite number of distinct solutions of the system $\dot{x} = y,\ \dot{y} = -x$. They represent the same trajectory, the circle $C: x^2 + y^2 = 1$.

Suppose there exists a solution $x(t) = x_0,\ y(t) = y_0,\ -\infty < t < \infty$ of (1), where x_0 and y_0 are constants. Obviously no trajectory can pass through the point (x_0, y_0), because uniqueness would be violated. Furthermore, we have $\dot{x} = 0 = P(x_0, y_0), \quad \dot{y} = 0 = Q(x_0, y_0),$

Since $x(t)$ and $y(t)$ are solutions. Conversely, if there exists a point (x_0, y_0) in Γ for which $P(x_0, y_0) = Q(x_0, y_0) = 0$, then certainly the functions $x(t) = x_0, y(t) = y_0, -\infty < t < \infty$, are a solution of (1).

Definition 1.1 Any point (x_0, y_0) in Γ at which P and Q both vanish is called *critical point* of (1). Any other point in Γ is called $regular$.

Other names for critical points are singular points, equilibrium state and points of equilibrium, and they may be thought of as points where the motion described by (1) is in a state of rest.

Consider the field of vectors $V(x, y) = (P(x, y),\ Q(x, y))$ with (x, y) in Γ. Then (1) describes the motion of a particle (x, y) whose velocity $(\dot{x}, \dot{y})$ is given by $V(x, y)$ at every point in Γ. Trajectories are fixed paths along which the particle moves independent of its starting point, and critical points are points of equilibrium.

Definition 1.2 : A critical point (x_0, y_0) of (1) is said to be an isolated critical point if a neighborhood of (x_0, y_o) containing no other critical points exists.

We now introduce the notion of stability of an equilibrium point or equivalently, stability of the solution $x(t) = x_0,\quad y(t) = y_0,\quad -\infty < t < \infty,$ of (1).

(Take note that critical point and equilibrium point mean the same thing and will be used interchangeably).

Definition 1.3 Let (x_0, y_0) be an isolated critical point of (1). Then (x_0, y_0) is said to be *stable* if given any $\varepsilon > 0$, $\exists\ \delta > 0$ such that

(i) all trajectories of (1) in the δ-neighborhood of (x_0, y_0) for some $t = t_1$ are defined for $t_1 \leq t < \infty$, and

(ii) if a trajectory satisfies (i) it remains in the ε-neighborhood of (x_0, y_0) for $t > t_1$.

If in addition every trajectory $C\colon x = x(t),\ y = y(t)$ satisfying (i) and (ii) also satisfies

(iii) $\lim_{t\to\infty} x(t) = x_0$ and $\lim_{t\to\infty} y(t) = y_0$,

then (x_0, y_0) is said to be *asymptotically stable*. Finally, an isolated critical point is said to be *unstable* if it is not stable.

The definition of stability states roughly that (x_0, y_0) is stable if once a trajectory enters a small disc containing (x_0, y_0) it remains within a slightly larger disc for all future time. The above definition is sometimes called stability to the right; a similar definition can be given for stability to the left when t approaches $-\infty$

Example 1.2 : The point $(0, 0)$ is the only critical point of the systems

$$(a)\ \dot{x} = y, \quad (b)\ \dot{x} = -x, \quad (c)\ \dot{x} = x,$$

$$\dot{y} = -x, \qquad \dot{y} = -y, \qquad \dot{y} = y.$$

In (a) the trajectories are a family of circles $C: x^2 + y^2 = r^2, 0 < r^2 < \infty$ given by the solutions

$$x(t) = rsin(t + \alpha), \quad y(t) = rcos(t + \alpha)$$

Then (i) and (ii) are satisfied with $r^2 < \delta = \varepsilon$ but (iii) is not; therefore $(0, 0)$ is stable.

In (b) and (c) the trajectories are a family of straight lines $C: y = ({}^{y_0}/_{x_0})x$ as well as the lines $x = 0,\ y = 0$, given by the solutions

$$x(t) = x_0\, e^{\pm(t-t_0)}, \qquad y(t) = y_0\, e^{\pm(t-t_0)},$$

not the both of x_0 and y_0 equal to zero. Here the negative sign is used for (b), the positive sign for (c). For (b) we have (i), (ii), and (iii) satisfied; hence $(0, 0)$ is asymptotically stable. For (c) either $x(t)$ or $y(t)$ or both become infinite as t approaches infinity; hence $(0, 0)$ is unstable.

Chapter 2

AUTONOMOUS SYSTEMS

2.1 Solutions of Autonomous Systems

Consider the two simultaneous differential equations of the form:

$$dx/dt = F(x, y), \quad dy/dt = G(x, y) \tag{1}$$

Let us assume that F and G are continuous and have continuous partial derivatives in some domain D in the xy plane and if $(x_{0\,,}y_0)$ is a point in this domain, then there exists a unique solution $x = \phi(t),\ y = \varphi(t)$ of the system (1) that satisfies the initial conditions:

$$x(t_0) = x_0\,, \qquad y(t_0) = y_0 \tag{2}$$

The solution is defined in some interval $\alpha < t < \beta$ that contains the point t_0.

Notice that the independent variable t is not explicitly visible in equation (1). This type of system is known as an autonomous system. Autonomous systems occur frequently in practice; for example the motion of an un damped pendulum of length L is governed by the differential equation

$$d^2\theta/dt^2 + g/l\, sin\theta = 0 \tag{3}$$

Letting $x = \theta \; and \; y = {}^{d\theta}/_{dt}$, we can rewrite equation (3) as a nonlinear non autonomous system of two equations

$$ {}^{dx}/_{dt} = y , \qquad {}^{dy}/_{dt} = -\left({}^{g}/_{l}\right) sinx \qquad (4) $$

In order to understand this better, we will consider the ecological problem of two competing species.

2.2 Two Competing Species

Suppose there are two similar species competing for a limited food supply, it is known as a competitive interaction, for example, two species of fish in a pond that do not prey on each other but compete for the available food. Let the populations of the two species at time t be $x \; and \; y$. In the absence of species y, the growth of species x is given by the equation below:

$$ {}^{dx}/_{dt} = x(\epsilon_1 - \sigma_1 x) \qquad (5), $$

and in the absence of species x, the growth of species y is given by an equation of the form:

$$ {}^{dy}/_{dt} = y(\epsilon_2 - \sigma_2 y) \qquad (6), $$

However, when both species are present, each will have an effect on the available food supply for the other. This means that they reduce the growth rate and saturation populations of each other. The easiest expression for decreasing the growth rate of species x as a result of the presence of species y is to replace the growth rate factor $\epsilon_1 - \sigma_1 x$ of equation (5) by $\epsilon_1 - \sigma_1 x - \alpha_1 y$, where α_1 is a measure of the degree to which species y interferes with species x. Similarly, in equation (6), we replace $\epsilon_2 - \sigma_2 y$ by $\epsilon_2 - \sigma_2 y - \alpha_2 x$. Thus, we have ; the systems of equations :

$$dx/_{dt} = x(\epsilon_1 - \sigma_1 x - \alpha_1 y) \qquad (7a)$$

$$dy/_{dt} = y(\epsilon_2 - \sigma_2 y - \alpha_2 x) \qquad (7b)$$

The actual values of the positive constants $\epsilon_1,\ \sigma_1,\ \alpha_1,\ \epsilon_2,\ \sigma_2,\ \alpha_2$. depend on the physical problem under consideration.

To determine the constants of equations $(7a)$ and $(7b)$ we set the right hand sides equal to zero.

$$x(\epsilon_1 - \sigma_1 x - \alpha_1 y) = 0 \qquad (8a)$$

$$y(\epsilon_2 - \sigma_2 y - \alpha_2 x) = 0 \qquad (8b)$$

The solutions corresponding to either $x = 0$ or $y = 0$ are ;

$x = 0,\ y = 0\ ;\ x = 0,\ y = {}^{\epsilon_2}/_{\sigma_2}\ y = 0,\ x = {}^{\epsilon_1}/_{\sigma_1}$.

Also, there is a constant solution corresponding to the intersection of the lines

$\epsilon_1 - \sigma_1 x - \alpha_1 y = 0$ and $\epsilon_2 - \sigma_2 y - \alpha_2 x = 0$ if these lines intersect. There are no other constant solutions of equation (7). Geometrically, we can represent these solutions as points in the xy plane ; they are called critical points or equilibrium points. Moreover, in the same xy plane, it is very helpful to visualize a solution of the system (7) as a point (x, y) moving as a function of time. At time $t = 0$ the initial populations of the two species provide an initial point $(x_{0,}y_0)$ in the plane ; then we follow the motion of the point (x, y) representing the populations of the two species at time t as it traces a curve in the plane. We can obtain considerable information about the behavior of solutions of equations $(7a)$ and $(7b)$ without actually solving the problem.

First from equation $(7a)$, we observe that x increases if $\epsilon_1 - \sigma_1 x - \alpha_1 y > 0$, and it decreases if $\epsilon_1 - \sigma_1 x - \alpha_1 y < 0$. Similarly, from equation $(7b)$, y increases if $\epsilon_2 - \sigma_2 y - \alpha_2 x > 0$ and decreases if $\epsilon_2 - \sigma_2 y - \alpha_2 x < 0$. This situation is depicted geometrically in the figure below:

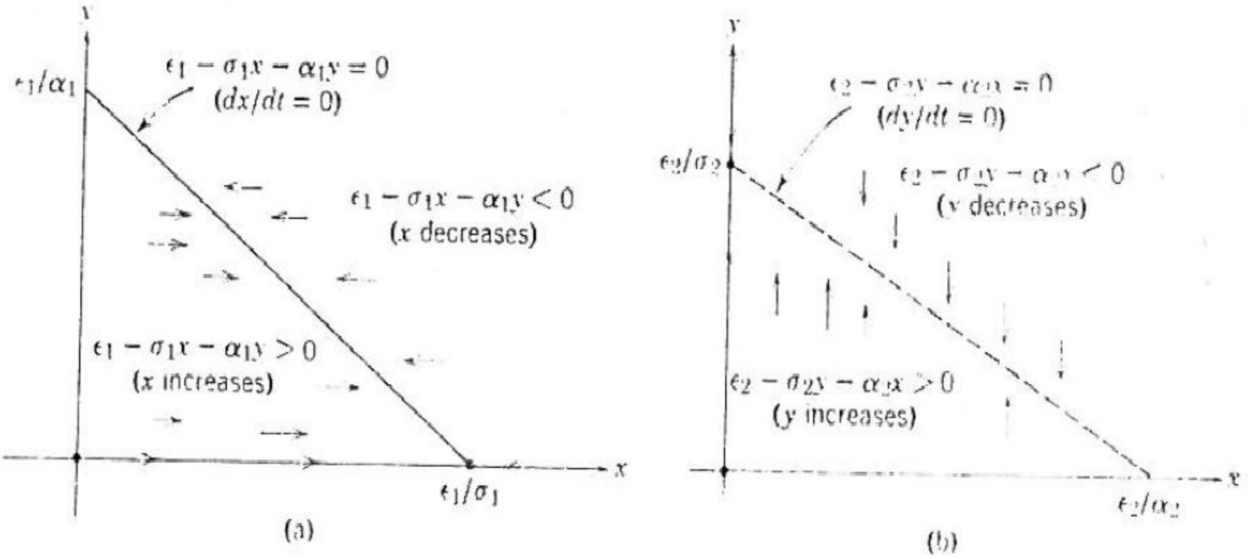

Figure 2.1 : Diagram illustrating two competing species

The critical points are indicated by the darkened dots. In order to see what is happening to the populations simultaneously, we must superimpose the above diagram to have a better understanding.

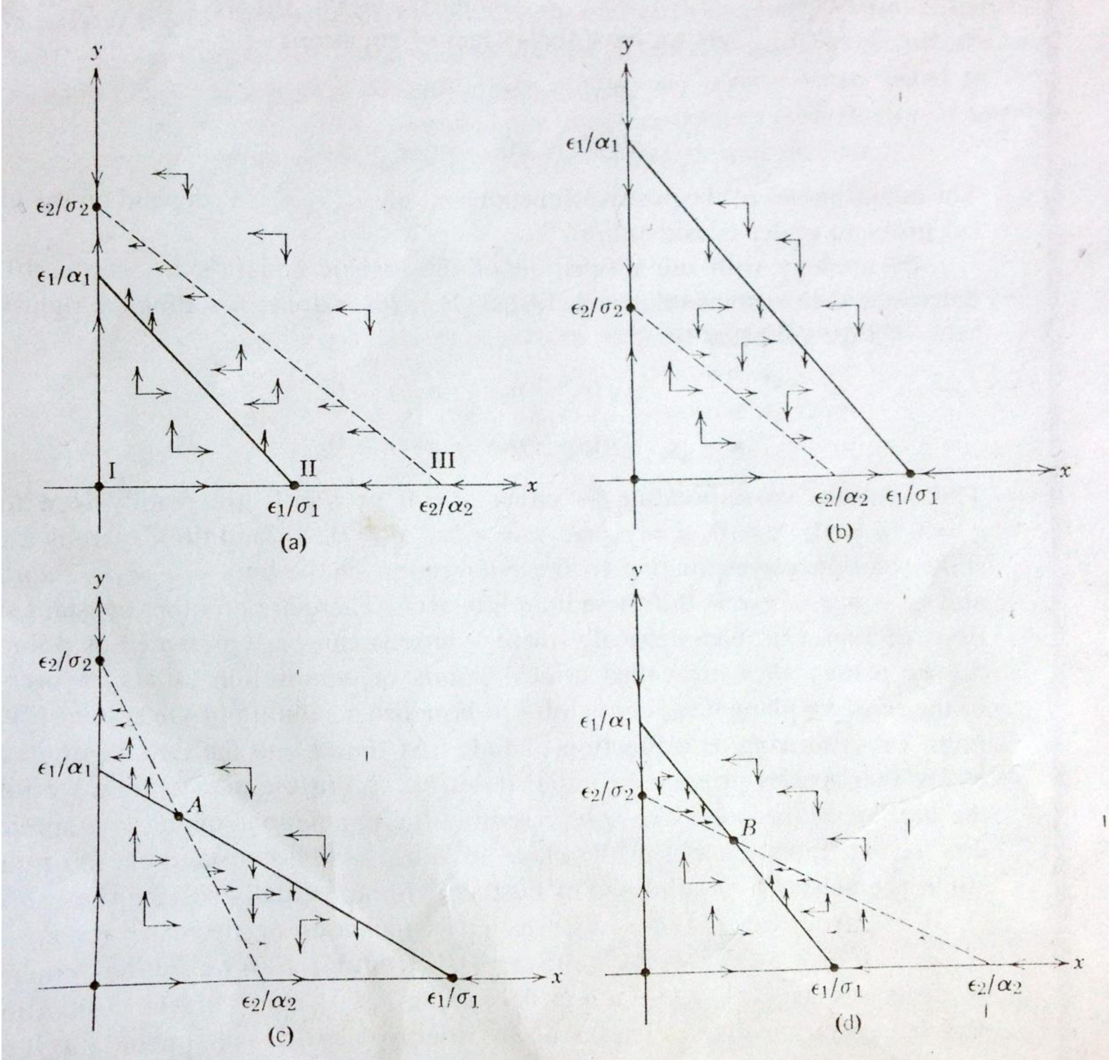

Figure 2.2 : Two Competing Species

For convenience, we will assume that the initial populations x_0 and y_0 are each non zero.

There are four possibilities as shown in Figure 2.2. The critical points are indicated by the darkened dots. We will examine only case (a) and (d) in details. Cases (b) and (c) are similar.

Consider case (a). If the initial populations are in the region I, then both x and y will increase; if the point moves into region II, then species y will continue to increase, but species x will start to decrease. Also, if the initial point is in region III, then both x and y will decrease; if the points moves into region II then x will continue to decrease while y now starts to increase. This suggests, for populations initially reasonably close to $(0, {}^{\epsilon_2}/_{\sigma_2})$ that the point (x, y) representing the populations at time t approaches the critical point $(0, {}^{\epsilon_2}/_{\sigma_2})$ as $t \to \infty$. This is shown in Figure 2.3a for several different initial states. This situation corresponds to the extinction of population x with population y reaching an equilibrium state of size ${}^{\epsilon_2}/_{\sigma_2}$.

One might ask if the point $\left(0, {}^{\epsilon_1}/_{\alpha_1}\right)$ is also a possible limiting state, since populations that start close to this point may seem it, as $t \to \infty$. The answer is no. In region I the point (x, y) moves away from the y axis while moving upward and in region II, while moving towards the y axis the point (x, y) still moves upwards. Moreover, note that $\left(0, {}^{\epsilon_1}/_{\alpha_1}\right)$ is not a critical point ; that is $x = 0,\ y = {}^{\varepsilon_1}/_{\alpha_1}$ is not a solution of equation (7). The other critical points in figure 2.2a are (0,0) and $({}^{\epsilon_1}/_{\alpha_1}, 0)$. However, an inspection of figure 2.3 shows that shows that a solution (x, y) starting from nonzero values (x_0, y_0) cannot approach either of these points as $t \to \infty$.

Consider case (d). An examination of figure 2.2d suggests that the population point (x, y) will move towards the intersection of the two straight dividing lines as t increases. These is shown schematically in figure 2.3b for several different initial states. In this case, both species can coexist with equilibrium populations given by the coordinates of the critical point B.

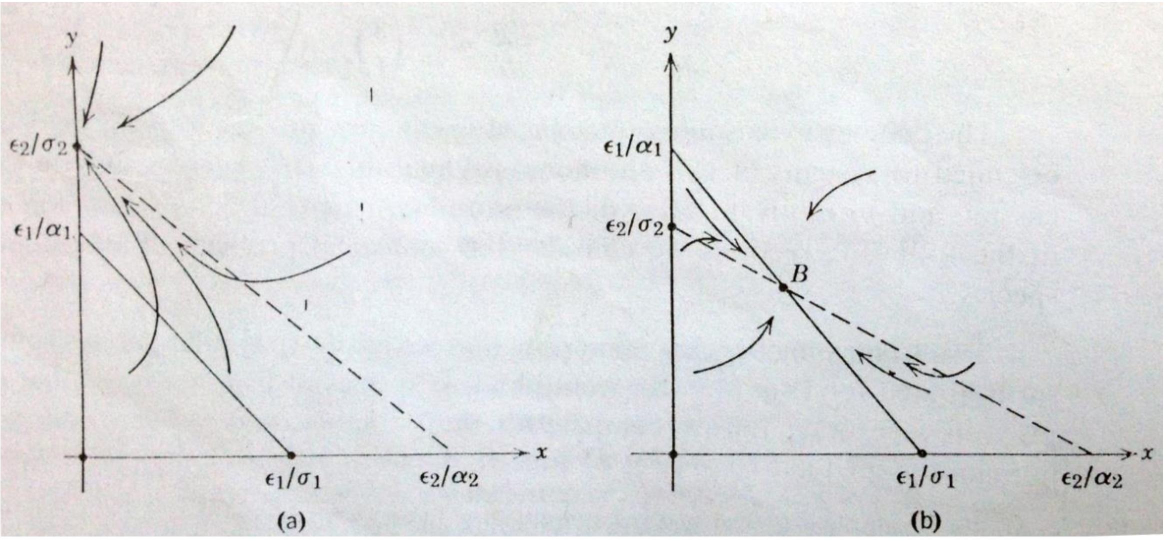

Figure 2.3 : Two Competing Species

Problem:

Determine the critical points of the system below;

$$dx/dt = x - x^2 - xy\,, \qquad dy/dt = \frac{1}{2}y - \frac{1}{4}y^2 - \frac{3}{4}xy.$$

Solution :

We factorize the right hand side and set it to equal zero.

$$x(1 - x - y) = 0.$$

$$y\left(\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x\right) = 0.$$

The solution corresponding to $x = 0$ and $y = 0$ are;

$$x = 0, \qquad y = 0\,;\; x = 0, \qquad y = 2\,;\; y = 0, \qquad x = 1.$$

2.3 Linear Systems−Constant Coefficient

In this section we will consider the linear system

$$\dot{x} = ax + by, \qquad \dot{y} = cx + dy \tag{2}$$

where a, b, c and d are real constants. Therefore we may let Γ be the entire xy-plane, and so all solutions are uniquely defined on $-\infty < t < \infty$. Hence we can discuss the behavior of trajectories in the *phase plane* of (2).

Why discuss systems of the form (2) ? First of all, a complete description of the phase plane can be given, since solutions of (2) can be determined explicitly. Secondly, many systems can be expressed in the form

$$\dot{x} = ax + by + \varepsilon_1(x, y),$$

$$\dot{y} = cx + dy + \varepsilon_2(x, y).$$

If ε_1 and ε_2 are sufficiently small in the neighborhood of a critical point, we would hope that the behavior of trajectories is local like that of (2). Thus we need to know about the linear systems.

The point $(x_0, y_0) = (0, 0)$ is a critical point of (2), and we will assume there are no other critical points. This is equivalent to assuming that $ad - bc \neq 0$.

The characteristic polynomial associated with (2) is

$$\det(A - pI) = \det\begin{pmatrix} a-p & b \\ c & d-p \end{pmatrix}$$

$$= p^2 - (a+d)p + (ad - bc),$$

whose roots are given by

$$\lambda_1, \lambda_2 = \frac{1}{2}\left[(a-d) \pm \sqrt{(a-d)^2 + 4bc}\right].$$

Since we are only interested in the behavior of trajectories, we will only need to know the nature of the roots λ_i.

To simplify the description of the behavior of trajectories near the critical point $(0,0)$, it will often be useful to perform a linear transformation of the form

$$\xi = \alpha x + \beta y, \qquad \eta = \gamma x + \delta y, \qquad \alpha\delta - \beta\gamma \neq 0.$$

The point $(x,y) = (0,0)$ is mapped into $(\xi,\eta) = (0,0)$, and conversely. Furthermore, such a transformation will only result in a rotation and a magnification or shrinking of trajectories, but will not distort their essential behavior near $(0,0)$.

Case I : λ_1, λ_2 are real, distinct, and neither is zero:

$$(a-d)^2 + 4bc > 0.$$

The transformation

$$\xi = cx + (\lambda_1 - a)y, \quad \eta = cx + (\lambda_2 - a)y$$

transforms (2) into the system

$$\dot{\xi} = \lambda_1 \xi, \qquad \dot{\eta} = \lambda_2 \eta$$

For instance, since $ad - bc = \lambda_1\lambda_2$ and $\lambda_1 + \lambda_2 = a + d$, we have $\dot{\xi} = c\dot{x} + (\lambda_1 - a)\dot{y} = cax + cby + (\lambda_1 - a)(cx + dy)$

$= \lambda_1 cx + (-\lambda_1\lambda_2)y + \lambda_1 dy = \lambda_1 cx + \lambda_1(\lambda_1 - a)y = \lambda_1\xi,$

and similarly for η.

Therefore, to simplify the discussion we may as well consider the system

$$\dot{x} = \lambda_1 x, \qquad \dot{y} = \lambda_2 y, \quad \lambda_1 \neq \lambda_2, \quad \lambda_1\lambda_2 \neq 0, \tag{3}$$

where $\lambda_1$and λ_2 are real. The solutions are of the form

$$x(t) = c_1 e^{\lambda_1 t},\ \ y(t) = c_2 e^{\lambda_2 t}$$

where $c_1$and c_2 are arbitrary real constants.

(a) λ_1, λ_2 have the same sign : $ad\ - bc > 0$;

(i) both roots are negative : $a + d < 0$.

If $\lambda_1 < \lambda_2 < 0$, then, as t approaches infinity, (x, y) approaches $(0,0)$, and y/x, the slope of the trajectories near the origin, becomes infinite. If $c_1 = 0$ we have the rectilinear trajectory

$$C: x = 0, y = c_2 e^{\lambda_2 t},$$

and similarly, if $c_2 = 0$.

In this case, we say $(0, 0)$ is a $stable\ node$ and the phase plane of (3) looks like the following diagram, in which the arrows denote the direction of increasing time. The diagram will be rotated ninety degrees if $\lambda_1 < \lambda_2 < 0$.

For the corresponding phase plane of (2), the only essential changes in the diagram could consist a rotation, and possibly the rectilinear trajectories will no longer be perpendicular. Evidently $(0, 0)$ is $asymptotically\ stable$.

(ii) Both roots are positive : $a + d > 0$.

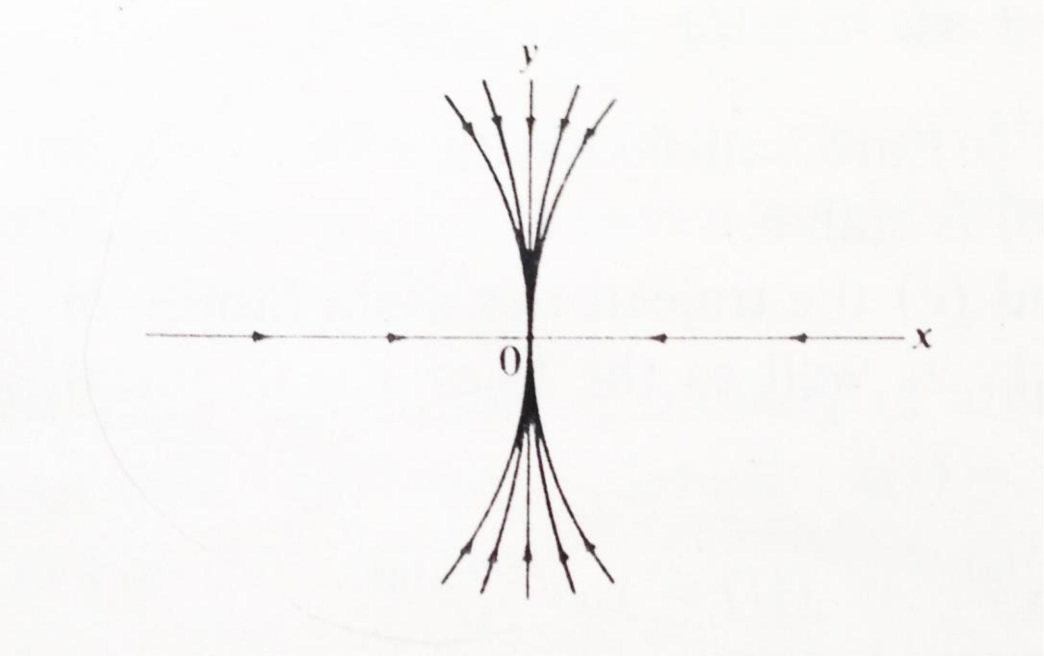

Figure 2.4 : Phase Plane of an Asymptotically Stable Critical Point

Then, if $0 < \lambda_1 < \lambda_2$, the diagram is the same with the arrows reversed. In this case $(0,0)$ is an $unstable\ node$.

(b) λ_1, λ_2 have different sign : $ad - bc < 0$.

If $\lambda_2 < 0 < \lambda_1$, then the rectilinear trajectories are

$$C: x = 0, y = c_2 e^{\lambda_2 t},$$

which approaches (0, 0) as t approaches ∞, and

$$C: x = c_1 e^{\lambda_1 t}, \qquad y = 0,$$

which becomes infinite. If $c_1 > c_2$, $c_2 < 0$ then (x, y) approaches $(\infty, 0)$ as t approaches ∞ or (x, y) approaches $(0, -\infty)$ as t approaches $-\infty$. A similar analysis can be made for the other possible values of c_1 and c_2.

In this case, we say that $(0, 0)$ is a *saddle point* and it is obviously unstable. The phase plane of the system (2) will resemble that below except for the possibility of a rotation and change of direction of the trajectories.

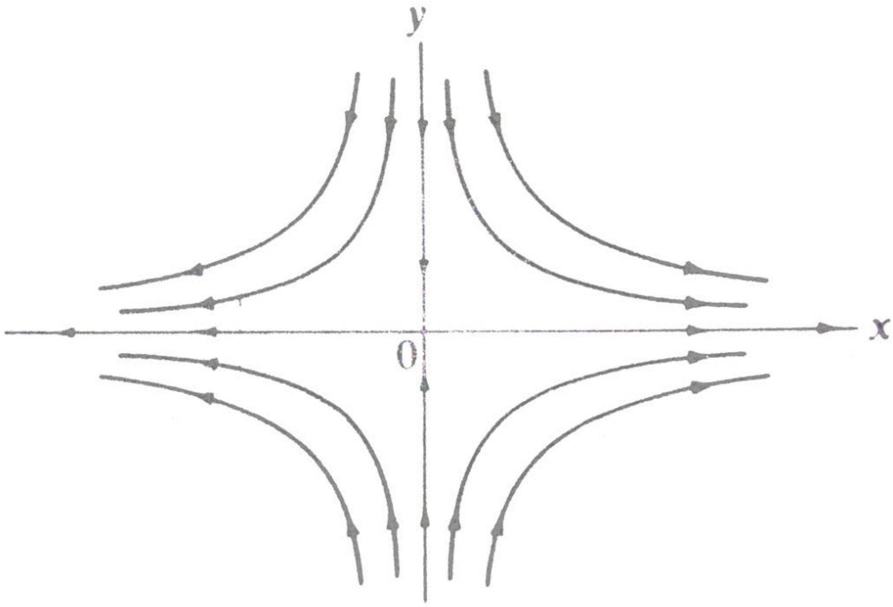

Figure 2.5 : Phase Plane of a Saddle Point

Example 2.3.1

Investigate the type and stability of the systems

$(a)\ \dot{x} = 3x + y$.

$\dot{y} = 4x - 2y,$

$(b)\ \ \dot{x} = 2x + y,$

$\dot{y} = 4x - 2y,$

$(c)\ \ \dot{x} = 2x + 3y,$

$\dot{y} = x + y,$

Solution 2.3.1

$(a-d)^2+4bc>0$ in all cases. The equilibrium point $(0,0)$ is a stable node for (a), since $ad-bc>0$ and $a+d<0$; for (b), it is an unstable node, since $ad-bc>0$ and $a+d>0$. In (c), we have $ad-bc<0$, so $(0,0)$ is a saddle point.

Case II : λ_1, λ_2 are complex conjugate:

$$(a-d)^2+4bc<0.$$

We may therefore assume that $\lambda_1 = u+iv$ and $\lambda_2 = u-iv$,

where u, v are real numbers. The transformation

$$\xi = cx+(u-a)y, \qquad \eta = vy,$$

transforms (2) into the system

$$\dot{\xi} = u\xi - v\eta, \dot{\eta} = v\xi + u\eta.$$

Therefore we will consider the system

$$\dot{x} = ux - vy, \qquad \dot{y} = vx + uy,$$

where u and v are real.

(a) λ_1, λ_2 are imaginary: $a+d=0$.

Then $\lambda_1 = iv, \lambda_2 = -iv, u = 0$ and (4) becomes

$$\dot{x} = -vy, \;\; \dot{y} = vx,$$

whose solutions are

$$x(t) = c_1 \cos(vt + \alpha), \qquad y(t) = c_1 \sin(vt + \alpha),$$

and the trajectories are a family of circles

$$C = x^2 + y^2 = {c_1}^2.$$

In this case (0,0) is called a *center* and is stable but not asymptotically stable. The corresponding trajectories for the system (2) will be a family of ellipses. Note in (2) that if $y = 0$, then $\dot{y} = cx$, which indicates that the direction of increasing time is clockwise if $c < 0$ and counterclockwise if $c > 0$.

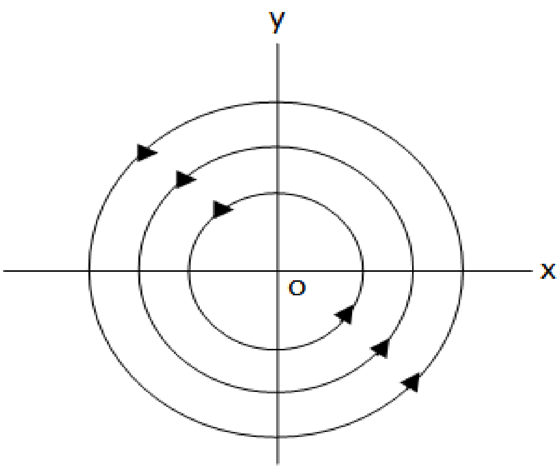

Figure 2.6 : Phase Plane of a Stable Center

(b) λ_1, λ_2 are complex : $a + d \neq 0$.

Then solutions of (4) are

$$x(t) = c_1 e^{ut} \cos(vt + \alpha), \qquad y(t) = c_1 e^{ut} \sin(vt + \alpha)$$

and the trajectories are a family of spirals

$$C: x^2 + y^2 = {c_1}^2 e^{2ut}.$$

The critical point (0,0) is called a $spiral\,point$ or $focus$ and is asymptotically stable if $a + d = u < 0$, and unstable if $u > 0$. As before, the direction of increasing time is determined by the sign of c.

Trajectories have no limiting direction, since $y/z = tan(vt + \alpha)$ has no limit as t becomes infinite.

Examples 2.3.2

For the systems

(a) $\dot{x} = -x + 3y,$

$\dot{y} = -2x + y,$

(b) $\dot{x} = x - 2y,$

$\dot{y} = 3x - 3y,$

(c) $\dot{x} = 2x + 2y,$

$\dot{y} = -x + 3y,$

we have $(a + d)^2 + 4bc < 0$. The critical point $(0, 0)$ is a center $(a + d = 0)$, a stable spiral point $(a + d < 0)$, and an unstable spiral point $(a + d > 0)$, respectively. For (a) and (c) the direction of increasing time is clockwise $(c < 0)$, whereas for (b) it is counterclockwise $(c < 0)$.

Case III :

$$\lambda_1 = \lambda_2 = \frac{(a + d)}{2} \neq 0; \;\; (a - d)^2 + 4bc = 0.$$

This is the case of a double root $\lambda_1 = \lambda_2 = u$, where $u = {}^{(a+d)}/_{2} \neq 0$ since $ad - bc \neq 0$.

(a) A special subcase arises when $b = c = 0$ in (2), which then becomes the system $\dot{x} = ux,\ \dot{y} = uy$, whose solutions are of the form

$$x(t) = c_{1e^{ut}}, \ \ y(t) = c_{2e^{ut}}$$

The trajectories are then a family of straight lines $C: y = (c_2/c_1)x$ as well as the lines $x = 0$ and $y = 0$.

Then (0,0) is called a $proper\ node$ and is asymptotically stable if $a + d < 0$, whereas it is unstable if $a + d > 0$.

(b) In the general case, we may assume that $b \neq 0$. Then the transformation

$$\xi = \frac{(a-d)}{2b}x + y, \qquad \eta = \frac{1}{b}x,$$

transforms (2) into the system

$$\dot{\xi} = \frac{a+d}{2}\xi, \quad \dot{n} = \frac{a+d}{2}\eta,$$

(If $b = 0, c \neq 0$, then (2) is essentially in this form.) Therefore we may as well consider the system

$$\dot{x} = ux, \quad \dot{y} = (c_1 t + c_2)e^{ut}.$$

If $c_1 = 0$, the rectilinear trajectory is the y-axis. Otherwise all trajectories are asymptotic to the y-axis since y/x becomes infinite as t approaches infinity.

In this case the equilibrium point (0,0) is called a *node* or an *improper node*; it is asymptotically stable if $a + d > 0$.

Examples 2.3.3

The systems

(a) $\dot{x} = -2x,$ $\quad$ (b) $\dot{x} = 8x - y,$

$\dot{y} = -2y,$ $\quad$ $\dot{y} = 4x + 4y,$

represent a stable proper node and an unstable node (improper node), respectively.

Remark *Given the system*

$$\dot{x} = ax + by, \qquad \dot{y} = cx + dy, \;\; ad - bc \neq 0,$$

where a, b, c and d are real, then (0,0) *is an isolated equilibrium point and it is said to be*

(i) stable if the roots of the characteristic polynomial are purely imaginary,

(ii) asymptotically stable if the roots are negative and real, or

(iii) unstable if the roots positive and real.

2.4 Nonlinear Systems

We will now apply the previous analysis given for linear systems to systems of the form

$$\dot{x} = P(x, y) = ax + by + \varepsilon_1(x, y),$$

$$\dot{y} = Q(x, y) = cx + dy + \varepsilon_2(x, y),$$

where we assume that

(i) $P,\ Q$, and their first partial derivatives are continuous in some neighborhood of $(0,0)$,

(ii) $ad - bc \neq 0$, and

(iii) $\lim_{r\to 0} \frac{\varepsilon_1(x,y)}{r} = 0,\ i = 1,2$, where $r = \sqrt{x^2 + y^2}$.

This implies that (0,0) is a critical point of $(7a)$ and$(7b)$, and given the systems $(7a)$ and $(7b)$ satisfying (i),(ii), and (iii), we will say that (0,0) is a simple critical point of $(7a)$ and $(7b)$.

Definition 2.4.1 Suppose that $C: x = x(t), y = y(t)$ is a trajectory of $(7a)$ and $(7b)$; then we may represent it as

$$C: r = r(t), \quad \omega = \omega(t), \quad r(t) > 0,$$

where

$$x(t) = r(t) cos\omega(t), \qquad y(t) = r(t) sin\omega(t).$$

Assume there is a neighborhood U of the simple critical point $(0,0)$ of $(7a\&b)$ in which

(i) all trajectories are defined on $t_0 < t < \infty$ or $-\infty < t < t_0$ for some t_0 ;

(ii) $\lim_{t\to\infty} r(t) = 0$ or $\lim_{t\to-\infty} r(t) = 0$.

Then $(0,0)$ is said to be

(a) a spiral point if $\lim_{t\to\infty} |\omega(t)| = \infty$ or $\lim_{t\to-\infty} |\omega(t)| = \infty$ for all trajectories in U,

(b) a node if $\lim_{t\to\infty} \omega(t) = C$, a constant, for all trajectories in U, or

(c) a proper node if it is a node, and for every constant C there is a trajectory satisfying $\lim_{t\to\infty} \omega(t) = C$.

Definition 2.4.2 The simple equilibrium point (0,0) of $(7a$ & $b)$ is called

(a) a center if there exists a neighborhood of (0,0) containing countably many closed trajectories, each containing (0,0) and whose diameters tend to zero,

(b) a saddle point if there are two trajectories approaching (0,0) along opposite directions, and all other trajectories close to either of them and to (0,0) tend away from them as t becomes infinite.

Example 2.4.1

(a) The motion of a simple pendulum is governed by the equation

$$\ddot{\theta} + 2k\dot{\theta} + qsin\theta = 0, \;\; k > 0, \qquad q > 0,$$

and by the substitution $x = 0$, $\dot{x} = y$ this becomes the system

$$\dot{x} = y, \quad \dot{y} = -2ky - qsinx,$$

which can be written as

$$\dot{x} = y, \qquad \dot{y} = -qx - 2ky + q(x - sinx).$$

It's critical points are $(\pm n\pi, 0)$, $n = 0, 1, 2, \ldots$, and the term

$\varepsilon(x, y) = q(x - sinx)$ satisfies the required assumptions near $x = 0$, so (0,0) is a simple critical point.

We therefore consider the system

$$\dot{x} = y, \quad \dot{y} = -qx - 2ky,$$

which has an isolated singularity at $(0, 0)$. If for example, we assume that $q > k^2$, then $(0, 0)$ is a stable spiral point of the linear system, and hence it is a stable spiral point of the given system.

If we make the change of variable $\theta = \varphi + \pi$, we arrive at the equation

$$\ddot{\varphi} + 2k\dot{\varphi} - q\varphi = 0, \quad k > 0, q > 0,$$

and a similar analysis shows that $(0, 0)$ is a saddle point of the corresponding system. Therefore $(\pi, 0)$ is a saddle point of the original system, and the phase plane of the pendulum equation might look like the diagram below.

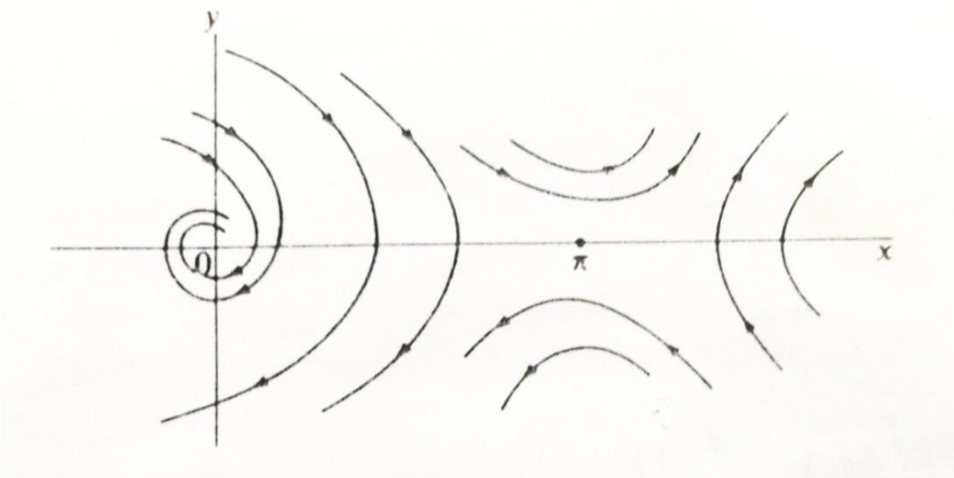

Figure 2.7 : Phase Plane of a Pendulum Equation

(b) The system

$$\dot{x} = y, \quad \dot{y} = 2x - x^2,$$

has critical points at $(0,0)$ and $(2,0)$. The first is a simple critical point ($\varepsilon(x,y) = x^2$) and is a saddle point. By making a change of variable $x = z + 2$, we obtain the system

$$\dot{z} = y, \quad \dot{y} = -2z - z^2.$$

For the corresponding linear system, The point $(0,0)$ is a center, a trajectory C passing through the positive x-axis near $(0,0)$ must intersect the negative x −axis. But the last system is unchanged if we replace y by $-y$ and t by $-t$, which implies that C is closed. Therefore (0,0) is a center, so (2,0) is a center for the original system.

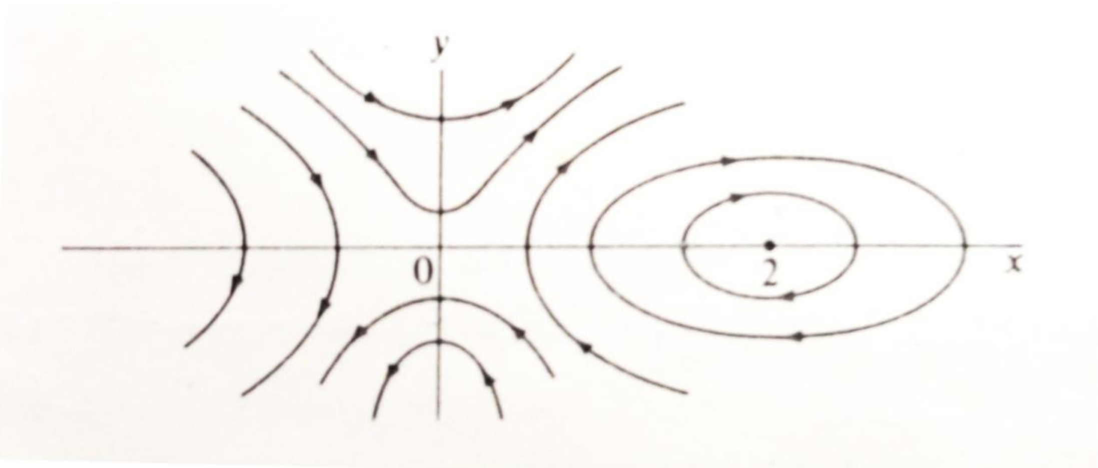

Figure 2.8 : Phase Plane containing critical points, nonintersecting trajectories and cycles.

Note that the phase plane of the last system contains all three ingredients: critical points, nonintersecting trajectories, and cycles.

Chapter 3

STABILITY OF NON AUTONOMOUS EQUATIONS

3.1 Stability Theory

The theory of stability is concerned with the stability of solutions of differential equations and trajectories of dynamical systems subjected to infinitesimal disturbances of the original state. The thermal equation is an example of a stable partial differential equation, as slight alterations from the original data, causes small changes in the temperature at a later time. In general, a solution is stable when small changes in the hypothesis result in corresponding changes in the conclusion. Most of the features however of the qualitative hypothesis of dynamical systems and differential equations are based on the asymptotic properties of solutions, what will happen to the system after a very long time has elapsed. The easiest type of the characteristics is shown by periodic orbits and critical points. Should a said orbit be well comprehended, one is free to ask if a slight variation in the original state will bring about the same characteristic. The theory of stability takes care of these questions: will a "close by" orbit remain indefinitely close to a known orbit? Or will it converge to the known orbit? If it will remain in position, then it is *stable*, but in the second case, it is *asymptotically stable*, or *attracting*.

Stability implies that the solutions and trajectories change only very little as a result of small perturbations. caused by disturbances. In general, perturbations of the original form in certain paths where orbits are in the same direction, cause the trajectory to attract (asymptotically approach) one another and in different directions, they tend to repulse each other. Trajectories can also be moving in directions that they neither converge or repulse each other. In this case, the theory of stability does not proffer enough knowledge concerning the dynamics.

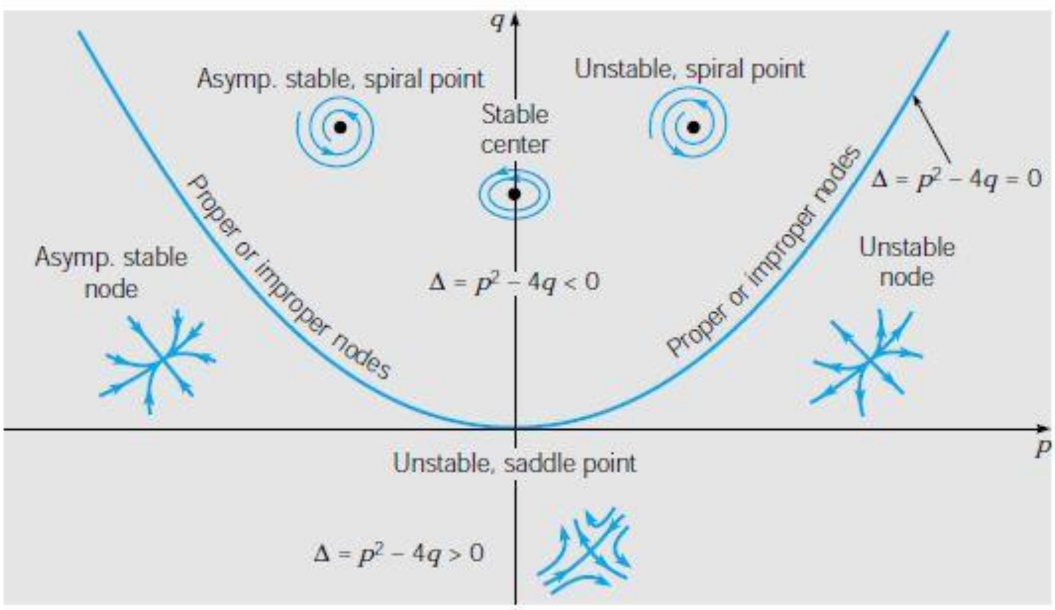

Figure 3.1 : Stability Diagram

3.2 Stability of Solutions

Consider the general 1st order differential equation:

$$\dot{x} = f(t,x) \qquad (1)$$

Where $x = x(t) = (x_1(t), \dots x_n(t))$ is an unknown n−dimensional vector function and we assume that

$f(t,x) = (f_1(t,x), \dots f_n(t,x))$, is defined and continuous in

$$\Gamma = \{(t,x) |\ r_1 < t < \infty, \|x\| < a\}$$

Recall that if $x = (x_1, \dots, x_n)$, then by the norm of $\|x\|$, we mean

$$\|x\| = \sum_{i=1}^{n} |x|.$$

A solution of (1) satisfying $x(t_0) = x_0$ will be denoted by $x(t) = x(t; t_0, x_0)$.

Definition 3.2.1

Let $x(t) = x(t; t_0, x_0)$ be a solution of (1) satisfying :

(I) $x(t)$ is defined on $t_0 \leq t < \infty$, and

(II) the point $t, x(t)$ belongs to Γ for $t \geq t_0$

Then $x(t)$ is said to be stable if :

(a) $\exists\, \gamma > 0$ s.t every solution $x(t; t_0, x_1)$ satisfies (I) and (II) whenever $\|x - x_0\| < \gamma$, and

(b) given $\varepsilon > 0, \exists\, a\, \delta > 0, s.t\ \|x - x_0\| < \delta$ implies

$$\|x(t; t_0, x_0) - x(t; t_0, x_1)\| < \varepsilon, t_0 \leq t < \infty.$$

A solution that is not stable is said to be unstable.

Definition 3.2.2

The solution $x(t) = x(t; t_0, x_0)$ of (1) is *asymptotically stable* if in addition to it being stable, $\exists\ p > 0,\ 0 < p \leq \gamma\ s.t\ \|x_0 - x_1\| < p$ implies that

$$\lim_{t\to\infty} \|x(t; t_0, x_0) - x(t; t_0, x_1)\| < 0$$

Examples 3.2.1

(a) All solutions of the equation $\dot{x} = 0$ are stable, since

$\|x(t;t_0,x_0) - x(t;t_0,x_1)\| = \|x_0 - x_1\|$, $\quad -\infty < t < \infty$, but no solution is asymptotically stable.

(b) Every solution of the equation $\dot{x} = -tx$ is asymptotically stable, since;

$$\|x(t;t_0,x_0) - x(t;t_0,x_1)\| = \|x_0 - x_1\| exp\frac{1}{2}({t_0}^2 - t^2), \qquad -\infty < t < \infty.$$

(c) The solution $x = 0$ of the equation $\dot{x} = x^2$ is unstable, since for $t_0, x_0 > 0$, the solution $x(t; t_{0,} x_0) = ({x_0}^{-1} + t_0 - t)^{-1}$ fails to exist at $t = {x_0}^{-1} + t_0$.

3.3 Stability for linear systems

$$\dot{x} = A(t)x \qquad (2)$$

Consider the logistic equation ${}^{dA}/_{dt} = \epsilon A - \sigma A^2, \quad \epsilon > 0,\ \sigma > 0$. We have seen that the constant solutions (critical points) $A = 0$ and $A = {}^{\epsilon}/_{\sigma}$ play a crucial part in analyzing this differential equation. In this problem, we will discuss an analytical method, rather than the geometric arguments of the text, for analyzing the stability of these solutions.

Let A_E be any constant solution (critical point) of the logistic equation. Suppose that this equation is very slightly perturbed, what happens? We write $A(t) = A_E + u(t)$ where $u(0)$ is very small and ask what happens to u as $t \to \infty$.

(a) Derive the differential equation satisfied by $u(t)$.

(b) If $u(0)$ is very small, then at least initially, $u^2 \ll u$. This is known as linearization.

(c) If $u(t) \rightarrow \infty$ as $t \rightarrow \infty$ the constant solution is said to be linearly unstable; On the other hand, if $u(t) \rightarrow 0$ as $t \rightarrow \infty$ the constant solution is said to be linearly stable. (actually asymptotically stable). Show that the constant solution $A_E = 0$ is linearly unstable and that the constant solution $A_E = {}^{\epsilon}/_{\sigma}$ is linearly stable. Note that the stability characteristics may be modified by the inclusion of the nonlinear term involving u^2 which has been neglected. That is why the stability is referred to as linear stability or linear instability.

Theorem 3.3.1

All solutions of (2) *are stable if and only if they are bounded*.

Proof 3.3.1 If all solutions of (2) are bounded, then ∃ a constant m s.t $\|\varphi(t)\| < m$.

Given any $\varepsilon > 0$, then $\|x_0 - x_1\| < {}^{\varepsilon}/_{m}$ implies that:

$$\|x(t; t_0, x_0) - x(t; t_0, x_1)\| = \|\varphi(t)(x_0 - x_1)\| \leq m\|x_0 - x_1\| < \varepsilon,$$

and hence all solutions are stable

Conversely;

If all solutions are stable, the solution $x(t; t_{0,}0) \equiv 0$ is stable ; ∴ given $\varepsilon > 0$, $\exists\ \delta > 0\ s.t\ \|x_1\| < \delta$ implies $\left\|0 - x(t;\ t_{0,}\ x_1)\right\| = \|\varphi(t)x_1\| < \varepsilon$.

In particular, we can let x_1 be the vector with ${}^{\delta}/_{2}$ in the i^{th} place and zero elsewhere. Then ;

$\|\phi(t)x_1\| = \|\phi_i(t)\|\ {}^{\delta}/_{2} < \varepsilon$, where $\varphi_i(t)$ is the i^{th} column of $\varphi(t)$, and hence $\|\varphi(t)\| = k$.

Therefore, for any solution we have;

$\|x(t;\ t_0, x_0)\| = \|\phi(t)x_0\| < k\ \|x_0\|$, and hence all solutions are bounded.

Definition 3.3.1

The solution $x(t; t_0, x_0) = x(t)$ is said to be uniformly stable if, given $\varepsilon > 0, \exists\ \delta > 0$ s.t any solution $x_1(t)$ satisfying $\|x(t_1) - x_1(t_1)\| < \delta$ for some $t_1 > t_0$ exists and satisfies $\|x(t) - x_1(t)\| < \varepsilon$ for $t \geq t_1$. Note the distinction between stability and uniform stability. In the former, a solution remains in ε-neighborhood of $x(t; t_0, x_0)$ if it is close to the point x_0 at time t_0; other solutions may enter and leave the ε-neighborhood at later times. In the case of uniform stability, once a solution enters the ε−neighborhood of $x(t; t_0, x_0)$ it remains there. In the determination of stability, the number δ no longer depends on t_0.

Example 3.3.1 Consider the equation $\dot{x} = a(t)x,\ a(t)$ continuous on $0 \leq t < \infty$. Then; $x(t; t_1, x_1) = x_1 \exp\left[\int_{t_1}^{t} a(s)ds\right]$ The solution $x(t) \equiv 0$ is uniformly stable if and only if the quantity $|x_1| \exp\left[\int_{t_1}^{t} a(s)ds\right]$ can be made uniformly small for sufficiently small value of $|x|$. Therefore, $x(t) \equiv 0$ is uniformly stable if and only if $\exp\left[\int_{t_1}^{t} a(s)ds\right]$ is bounded above for $t \geq t_1 \geq 0$.

3.4 Stability for Almost Linear Systems.

We have referred to the notions of instability, stability and asymptotic stability of a solution of the autonomous system

$$dx/dt = F(x, y), \qquad dy/dt = G(x, y) \tag{1}$$

We will now give mathematical meaning to these concepts and explore it's consequences by considering an illustrative example.

A critical point $x = x_0,\ y = y_0$ of the autonomous equation (1) is stable if, given any $\varepsilon > 0,\ \exists$ a δ such that every solution $x = \phi(t),\ y = \varphi(t)$ of (1), where $t = 0$ satisfies

$$\{[\phi(0) - x_0]^2 + [\varphi(0) - y_0]^2\}^{1/2} < \delta, \tag{2}$$

exists and satisfies

$$\{[\phi(t) - x_0]^2 + [\varphi(t) - y_0]^2\}^{1/2} < \varepsilon \tag{3}$$

for $t \geq 0$. This is systematically depicted in figure $9.21a$ and $9.21b$.

A critical point (x_0, y_0) is asymptotically stable if in addition to being stable, there exists a $\delta_0,\ 0 < \delta_0 < \delta,$ s.t if a solution $x = \phi(t),\ y = \varphi(t)$ satisfies

$$\{[\phi(0) - x_0]^2 + [\varphi(0) - y_0]^2\}^{1/2} < \delta_0 \tag{4}$$

Then

$$\lim_{t\to\infty} \phi(t) = x_0, \quad \lim_{t\to\infty} \varphi(t) = y_0 \tag{5}$$

Trajectories that start "extremely near" to (x_0, y_0) must not just stay "near" but will at the end approach (x_0, y_0) as t goes to infinity. This is what happens in the trajectory in fig 9.21a, though not for that in fig 9.21b. Also notable is the fact that stability is a lesser requirement compared to asymptotic stability because a critical point needs to be stable for us to then decide whether it is asymptotically stable or not. However, the limit condition (5), known to be a crucial characteristic of asymptotic stability, doesn't on it's own mean ordinary stability. Of course, illustrations can be made, showing that all the trajectories approach (x_0, y_0) as $t \to \infty$, but for which (x_0, y_0) is not a stable equilibrium point. The only requirement is a group of trajectories with individual trajectories that begin arbitrarily close to (x_0, y_0), then fall back to an arbitrarily far distance prior to approaching (x_0, y_0) as t goes to infinity. Critical points that are not stable are termed unstable.

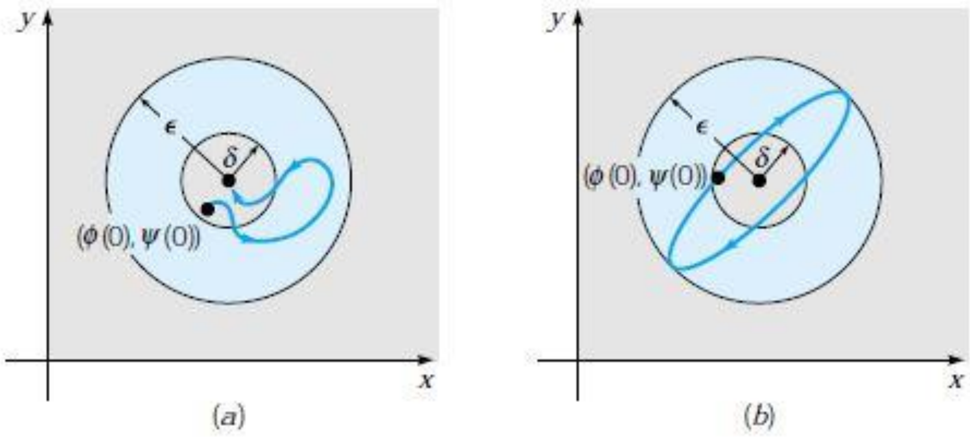

Figure 3.2 : (a) Asymptotic Stability. (b) Stability

Given the linear system below

$$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy. \tag{6}$$

with $ad - bc \neq 0$, the type and stability of the equilibrium point (0,0) as a function of the roots $r_1, r_2 \neq 0$ of the characteristic equation

$$r^2 - (a + d)r + ad - bc = 0 \tag{7}$$

are listed in the table below :

Table 3.1 : Type and Stability the critical point (0,0) as a function of the roots $r_1, r_2 \neq 0$ of the characteristic equation

Roots of the Characteristic Equation	Types of Critical Point	Stability
$r_1 > r_2 > 0$	*IN*	*Unstable*
$r_1 < r_2 < 0$	*IN*	*AS*
$r_2 < 0 < r_1$	*SP*	*Unstable*
$r_1 = r_2 > 0$	*Proper or IN*	*Unstable*
$r_1 = r_2 < 0$	*Proper or IN*	*Unstable*
$r_1, r_2 = \lambda \pm i\mu$	*SP*	
$\lambda > 0$		*Unstable*
$\lambda < 0$		*AS*
$r_1 = i\mu, r_2 = -i\mu$	*Center*	*Stable*

IN = Improper node; PN = Proper node; SP = Saddle point; AS = Asymptotically stable

The theorem below is a summary of the stability characteristic.

Theorem 3.4.1 *The equilibrium point (0,0) of the linear system (6) is said to be*

(i) asymptotically stable if the roots $r_1,\ r_2$ *of the characteristic equation (7) are real and negative or have negative real parts;*

(ii) stable, but not asymptotically stable, if r_1 *and* r_2 *are pure imaginary;*

(iii) unstable if r_1 *and* r_2 *are real and any of them is positive, or if they have positive real parts.*

To give an accurate and thorough proof, it is of great importance to show how to compute δ for any given ϵ for equation (3) to be satisfied, and for asymptotic stability, to compute δ_0 for equation (5) to be satisfied. This kind of detailed analysis will not be taken. Observe that if an equilibrium point of the linear system (6) is asymptotically stable, not only will trajectories that begin near the equilibrium point move close to the equilibrium point, but all trajectories will approach to the equilibrium point because all solutions are linear combinations of $e^{r_1 t}$ and $e^{r_2 t}$. In such a scenario, the equilibrium point is referred to as being globally asymptotically stable. This property of linear systems does not hold for nonlinear systems in general. Most times, a relevant, real problem in taking into consideration an asymptotically stable equilibrium point of a nonlinear system is to approximate the original state (initial conditions) that made the equilibrium state to be asymptotically stable. This set of initial conditions is known as the region of *asymptotic stability* for the equilibrium point. As an alternative, we may want to find out if the equilibrium point is asymptotically stable for an already stated set of initial conditions.

Let us now relate the solution for the linear system (6) to the nonlinear system.

$$\frac{dx}{dt} = ax + by + F_1(x, y), \qquad (8a)$$

$$\frac{dy}{dt} = cx + dy + G_1(x, y), \qquad (8b)$$

Remember that this type of system is almost linear around the origin. While discussing, we will not include the phrase "near the origin", because it is obvious that we are referring to the neighborhood of the critical (equilibrium) point (0,0).

Let us assume that (0,0) is a critical point of systems $(8a)$ and $(8b)$ and that $ad - bc \neq 0$. Let us also assume that F_1 and G_1 have continuous partial derivatives and are small near the origin in the sense that $F_1(x,y)/r \to 0$ and $G_1(x,y)/r \to 0$, where $r = (x^2 + y^2)^{1/2}$. Recall that such a system is said to be almost linear in the neighborhood of the origin. In our discussion, we will not mention the phrase "near the origin," since it is clear that we are referring to the neighborhood of the critical point $(0,0)$.

As an example, the system

$$\frac{dx}{dt} = x - x^2 - xy, \qquad (9a)$$

$$\frac{dy}{dt} = \frac{1}{2}y - \frac{1}{4}y^2 - \frac{3}{4}xy, \qquad (9b)$$

satisfies the given conditions. Here $a = 1,\ b = 0,\ c = 0,\ d = \frac{1}{2},\ F_1(x,y) = -x^2 - xy$, and $G_1(x,y) = -\frac{1}{4}y^2 - \frac{3}{4}xy$. To show that $F_1(x,y)/r \to 0$ as $r \to 0$, let $x = rcos\theta, y = rsin\theta$. Then

$$\frac{F_1(x,y)}{r} = \frac{-r^2cos^2\theta - r^2\sin\theta\cos\theta}{r} = -r(cos^2\theta\ + \cos\theta\sin\theta)$$

$$\to 0 - - - - (10)$$

as $r \to 0$. The argument that $G_1(x,y)/r \to 0$ as $r \to 0$ is similar.

The stability and type of the critical point of the almost linear systems (8a) and (8b) are related closely to the stability and type of the equilibrium point of the corresponding linear system (6).

Theorem 3.4.2 Let $r_1 and\ r_2$ be roots of the characteristic equation (7) of the linear system (6) corresponding to the almost linear system (8). Then the type and stability of the equilibrium point (0,0) of the almost linear system (8) and the linear system (6) are given in the table below.

Table 3.2 : Type and Stability of the Critical point (0,0) of the almost linear System (8) and the Linear System (6)

r_1, r_2	**Linear System**		**Almost Linear System**	
	Type	*Stability*	*Type*	*Stability*
$r_1 > r_2 > 0$	*IN*	*Unstable*	*IN*	*Unstable*
$r_1 < r_2 < 0$	*IN*	*AS*	*IN*	*AS*
$r_2 < 0 < r_1$	*SP*	*Unstable*	*SP*	*Unstable*
$r_1 = r_2 > 0$	*PN or IN*	*Unstable*	*PN, IN or SpP*	*Unstable*
$r_1 = r_2 < 0$	*PN or IN*	*AS*	*PN, IN or SpP*	*AS*
$r_1 = r_2 = \lambda \pm i\mu$				
$\lambda > 0$	*SpP*	*Unstable*	*SpP*	*Unstable*
$\lambda < 0$	*SpP*	*AS*	*SpP*	*AS*
$r_1 = i\mu, r_2 = -i\mu$	*C*	*Stable*	*C or SpP*	*Indeterminate*

IN = Improper node; PN = Proper node; SP = Saddle point; SpP = Spiral point; C = Center.

At this stage, the proof to this theorem is extremely tough to provide, and we shall take the result of the theorem without proof. Importantly, the theorem states that for x and y close to zero, the nonlinear terms $F_1(x, y)$ and $G_1(x, y)$ are insignificant and have no effect on the nature and stability of the critical point as is determined by the linear terms, save in two crucial cases: r_1 and r_2 pure imaginary, and r_1 and r_2 equal and real. We should note that small variations in the coefficients of the linear system (6), and also in the roots r_1 and r_2 are capable of changing the stability and nature of the critical point in these two crucial scenarios alone. If $r_1$and r_2 be pure imaginary, a little disturbance is capable of altering the stable center into an unstable spiral point or asymptotically stable or have it remain as a center. If r_1 =r_2, small perturbations have no effect on the stability of the equilibrium point, but however may alter the node into a spiral point. It is sensible to anticipate that the little nonlinear terms in equation (8a) and (8b) could bring about same results at least in the two aforementioned scenarios. This is true, but the actual relevance of theorem 3.4.2 is that in every other case, the small nonlinear terms do not change the stability or type of the equilibrium point. Save for the two crucial scenarios, the type and stability of the equilibrium point of the nonlinear system (8a) and (8b) can be found from a review of an easier linear system (6).

Though the critical point and linear system have the same type, the trajectories of the of the almost linear and the corresponding linear system may show remarkable discrepancies in form. Nevertheless, it can be explained that the gradients where trajectories "go into" or "go out of" the equilibrium point is accurately represented by the linear equations.

We will show some illustrations of these ideas by considering the motion of a damped pendulum and problems in ecology.

3.5 Damped Pendulum

Let us examine the motion of a damped pendulum whose the damping and it's speed are proportional to each other (see figure below).

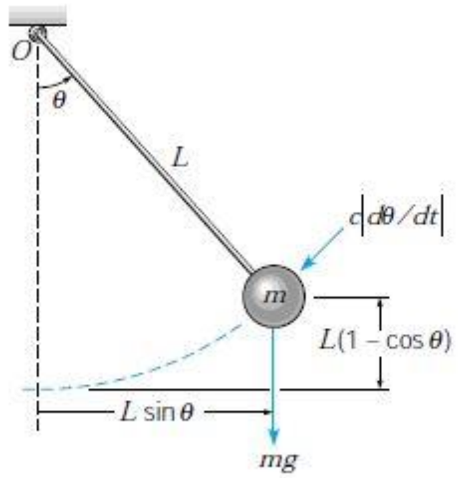

Figure 3.3 : An Oscillating Pendulum

The governing equation is given by

$$ml^2\frac{d^2\theta}{dt^2} + cl\frac{d\theta}{dt} + mgl\sin\theta = 0, \tag{11}$$

where the damping constant $c > 0$. Putting $x = 0$ and $y = d\theta/dt$ gives the system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\frac{g}{l}\sin x - \frac{c}{ml}y. \tag{12}$$

The point $x = 0, y = 0$ is an equilibrium point of the system (12). As a result of the mechanism of damping, it is expected any small motion about $\theta = 0$ to decay in amplitude. Hence, the critical point (0,0) should be asymptotically stable. In order to prove this, system (12) should be rewritten as

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\frac{g}{l}x - \frac{c}{ml}y - \frac{g}{l}(sinx - x). \tag{13}$$

$(sinx - x)/r \to 0$ as $r \to 0$, this means that the system (13) is an almost linear system; therefore theorem 3.4.2 can be applied. The roots of the characteristic equation of the corresponding linear system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -\frac{g}{l}\sin x - \frac{c}{ml}y. \tag{14}$$

are

$$r_1, r_1 = \frac{-c/ml \pm \sqrt{(c/ml)^2 - 4g/l}}{2}. \tag{15}$$

1. If $(c/ml)^2 - 4g/l > 0$ the roots are real, unequal and negative. The equilibrium point (0,0) is asymptotically stable and an improper node of the linear system (14) and of the almost linear system (13).

2. If $(c/ml)^2 - 4g/l = 0$ the roots are equal, real and negative. The equilibrium point (0,0) is an asymptotically stable node of the linear system (14). It may either be an asymptotically stable spiral point or asymptotically stable node of the almost linear system (13).

3. If $(c/ml)^2 - \frac{4g}{l} < 0$ the roots are complex with real and negative parts. The equilibrium point (0,0) is asymptotically stable and a spiral point of the almost linear system (13) and the linear system (14).

Apart from the equilibrium points (0,0), the almost linear system (12) has the equilibrium points $= n\pi, y = 0, n = \pm 1, \pm 2, \pm 3, \ldots$ corresponding to $\theta = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots, d\theta/dt = 0$. We once again expect (from figure 3.2) that the points that corresponds to $\theta = \pm 2\pi, \pm 4\pi, \ldots$ are asymptotically stable spiral points, and the points corresponding to $\theta = \pm\pi, \pm 3\pi, \ldots$ are unstable saddle points. Consider the equilibrium point $x = \pi,\ y = 0$. To check the stability of the point, we let

$$x = \pi + u, \qquad y = 0 + v. \tag{16}$$

Substituting for x and y in Equation (12), and using the fact that $\sin(\pi + u) = -\sin u$, we get

$$\frac{du}{dt} = v, \qquad \frac{dv}{dt} = -\frac{c}{ml}v + \frac{g}{l}\sin u. \tag{17}$$

Our area of interest is in analyzing the equilibrium point $u = v = 0$ of the system (17). The second of Equation (17) can be rewritten as

$$\frac{dv}{dt} = \frac{g}{l}u - \frac{c}{ml}v + \frac{g}{l}(\sin u - u). \tag{18}$$

Obviously the system (13) is the same with the first of equation (17) and equation (18), the only exception is that $-g/l$ is replaced by g/l. This means that it is an almost linear system and the roots of the characteristic equation of the corresponding linear system are given by

$$r_1, r_1 = \frac{-c/ml \pm \sqrt{(c/ml)^2 + 4g/l}}{2}. \tag{19}$$

One of r_1 and r_2 is positive and the other is negative. Therefore, the equilibrium point $x = \pi, y = 0$ is an unstable saddle point of both the almost linear system and the linear system as expected.

3.6 Ecological Applications

We will consider two problems in ecology: Competing species and Predator-Prey.

3.6.1 Lotka-Volterra Competition Model

Previously, we showed that a model for the competition between two species with population densities x and y leads to the differential equations:

$$dx/dt = x(\varepsilon_1 - \sigma_1 x - \alpha_1 y), \qquad (1a)$$

$$dy/dt = y(\varepsilon_2 - \sigma_2 y - \alpha_2 x), \qquad (1b)$$

where the parameters $\varepsilon_1, \sigma_1, \dots, \alpha_2$ are positive. As we saw then, we can analyze these equations by dividing the phase plane into regions according to the sign of dx/dt and dy/dt and then drawing typical trajectories.

Let us now obtain a more precise understanding of what happens by using the theory of almost linear systems

We start by considering the following specific example:

$$dx/dt = x(1 - x - y), \qquad (2a)$$

$$dy/dt = y\left(\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x\right). \qquad (2b)$$

Let us take x and y as the population densities of two bacteria competing with each other for the same supply of food. We ask whether there are equilibrium states that might be reached, or whether a periodic growth and decay will be observed, and how such possibilities depend on the initial state of the two cultures.

The equilibrium points of the system (2) are the solutions of the nonlinear algebraic equations

$$x(1 - x - y) = 0, \qquad (3a)$$

$$y\left(\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x\right) = 0 \qquad (3b)$$

Clearly, one of the solutions is $x = y = 0$; a second solution is $x = 1,\ y = 0$; and a third solution is $x = 0,\ y = 2$. Finally, if $x \neq 0,\ y \neq 0$, we obtain from equation (3) the system

$$x + y = 1,$$

$$3x + y = 2,$$

which has the solution $x = \frac{1}{2},\ y = \frac{1}{2}$. These four points in the xy plane are the only critical points of the system (2). We will consider each separately.

$\boldsymbol{x = 0,\ y = 0}$. This corresponds to a state in which both bacteria die as a result of their competition. From equation (2), the corresponding linear system is given as

$$dx/dt = x, \qquad dy/dt = \frac{1}{2}y, \qquad (4)$$

and the roots of the characteristic equation are 1 and $\frac{1}{2}$. Thus the origin is an unstable improper node (A fixed point for which the stability solution has positive Eigen values).

The solution $x = 0,\ y = 0$ of the interaction problem will not occur in reality.

$\boldsymbol{x = 1}$, $\boldsymbol{y = 0}$**.** Clearly, this corresponds to a state in which bacteria x survives the competition but bacteria y does not. To examine this critical point, let $x = 1 + u$ and $y = 0 + v$. Substituting for x and y in equation (2) and simplifying, we obtain

$$du/_{dt} = -u - v - u^2 - uv, \qquad (5a)$$

$$dv/_{dt} = -\frac{1}{4}v - \frac{1}{4}v^2 - \frac{3}{4}uv, \qquad (5b)$$

Systems $(5a \;\&\; 5b)$ are almost linear systems. The corresponding linear system is $du/_{dt} = -u - v,\ dv/_{dt} = {-v}/_{4}$ and the roots of the characteristics equation are $-\frac{1}{4}$ and -1. The general solution is

$$u = -\tfrac{4}{3}Ae^{-t/4} + Be^{-t}, \qquad v = Ae^{-t/4}.$$

where A and B are arbitrary. Thus $x = 1,\ y = 0$ is an improper node that is asymptotically stable. If the initial values of x and y are sufficiently close to $x\ = 1$, $y = 0$ the interaction will lead finally to that state.

For this critical point, we will indicate how the trajectories of the linear system behave in the neighborhood of $x = 1,\ y = 0$. It is clear that $u \to 0,\ v \to 0$ as $t \to \infty$ so that all of the trajectories enter the critical point $(1, 0)$ as $t \to \infty$. For $A = 0$ we have $x = 1 + u = 1 + Be^{-t}$ and $y = v = 0$ so one pair $(B < 0$ and $B > 0)$ of trajectories enters along the $x\ axis$. For A ≠0 we can compute the slope at any point on a trajectory by taking noting that

$$\frac{dy}{dx} = \frac{dv/_{dt}}{d(1+u)/_{dt}} = \frac{-\frac{1}{4}Ae^{-t/4}}{\frac{1}{3}Ae^{-t/4} - Be^{-t}} = \frac{-\frac{1}{4}A}{\frac{1}{3}A - Be^{-3t/4}}.$$

Particularly, as we approach the critical point ($t \to \infty$) along any trajectory with $A \neq 0$, we see that ${dy}/_{dx} \to -\frac{3}{4}$. Thus all the trajectories except one pair enter the critical point along a line with slope $-\frac{3}{4}$

$\boldsymbol{x = 0,\ y = 2.}$ The analysis is exactly similar to that for the critical point $x = 1,\ y = 0$. The critical point $x = 0,\ y = 2$ is also an improper node that is asymptotically stable. In this case, bacteria y survives, but bacteria x does not.

$\boldsymbol{x = {1}/_{2},\ y = {1}/_{2}}$. This critical point corresponds to a mixed equilibrium state or coexistence in the competition between the two bacteria cultures. To check the type of this equilibrium point, we let $x = \frac{1}{2} + u,\ y = \frac{1}{2} + v$. Substituting for x and y in equation (2) we obtain:

$$ {du}/_{dt} = \frac{1}{2}u - \frac{1}{2}v - u^2 - uv, \qquad (6a) $$

$$ {dv}/_{dt} = -\frac{3}{8}u - \frac{1}{8}v - \frac{1}{4}v^2 - \frac{3}{4}uv \qquad (6b) $$

The system (6) is an almost linear system, and the roots of the characteristic equation of the corresponding linear system are $(-5 \pm \sqrt{57})/16$. Since these roots are of opposite sign and are real, the critical point $(\frac{1}{2}, \frac{1}{2})$ is an unstable saddle point. (Two distinct real Eigen values with opposite sign). One pair of trajectories enters the critical point; the other recedes from it.

It can be shown by considering the general solution of the corresponding linear system that the slope of the pair entering the trajectories as $(x,y) \to (\frac{1}{2}, \frac{1}{2})$ is $(\sqrt{57}-3)/8 \cong 0.57$.

An illustrative sketch of what the trajectories might look like in the neighborhood of each critical point is shown in figure 9.27a. We are only interested in x and y positive. Since trajectories cannot cross other trajectories and since the x and y axis are trajectories, it follows that a trajectory that starts in the first quadrant must stay in the first quadrant, and a trajectory that starts in other quadrant cannot enter the first quadrant. Also, we accept without proof two facts that follow from advanced theory: (i) The system (2) does not have any periodic solutions, that is, it does not have trajectories that are closed curves; and (ii) a trajectory that is not a closed curve must either enter a critical point or go off to infinity as t→∞. But consider what is happening for x and y large. The nonlinear terms $-(x^2+xy)$ and $-\frac{1}{4}(y^2+3xy)$ in the first and second of equations $(2a$ & $b)$, respectively, outweigh the linear terms. Since they are negative, dx/dt and dy/dt are negative for x and y large. Thus for large x and y, the direction of motion on any trajectory is inward. The trajectories cannot escape to infinity. Eventually they must head toward one of the two stable nodes. In figure 9.27b, if the initial values of x and y are in region **I**, then x wins the competition; if the initial values are in region **II**, then y wins. "Peaceful coexistence" is not possible unless the initial point lies exactly on the dividing trajectory. Of particular interest would be the determination of the dividing trajectories that enter the saddle point $\left(\frac{1}{2}, \frac{1}{2}\right)$ which separate regions **I** and **II**.

3.6.2 Lotka -Volterra Predator-Prey

As a second example, let us consider the predator-prey problem. Here, we study an ecological situation involving two species, one of which preys on the other (does not compete with it for food but preys on it), while the other lives on a different source of food. An example is foxes and rabbits in a closed forest; the foxes prey on the rabbits, the rabbits live on vegetation in the forest. Other examples are bass in a lake as predators and sunfish as prey, and lady bugs as predators and aphids (insects that suck the juice of plants) as prey. Let $\boldsymbol{x}(\boldsymbol{t})$ and $\boldsymbol{y}(\boldsymbol{t})$ be the populations of prey and predator respectively, at time $\boldsymbol{t}$.

Let us build a simple model of interaction and make the following assumptions:

1. The prey grows without bound in the absence of the predator. Thus $dx/_{dt} = ax,$ $a > 0$ for $y = 0$.

2. The predator dies out in the absence of the prey. Thus $dy/_{dt} = -cy,\ c > 0,$ for $x = 0$.

3. The increase in the number of predators dependents wholly on the food supply (the prey) and the prey are consumed at a rate proportional to the number of encounters between predators and prey. For example, if the number of prey is doubled, the number of encounters is doubled. Encounters increase the number of predators and decrease the number of prey. A fixed proportion of prey is killed in each encounter, and the rate at which the population of the predator grows is enhanced by a factor proportional to the amount of prey consumed.

As a consequence, we have the following equations

$$dx/_{dt} = ax - bxy = x(a - by), \qquad (7a)$$

$$dy/_{dt} = -cy + dxy = y(-c + dx). \qquad (7b)$$

The constants $a, b, c,$ and d are positive; a and c are the growth rate of the prey and the death rate of the predator, respectively, and b and d are the measures of the effect of the interaction between the two species. Equations $(7a \,\&\, b)$ are known as the Lokta-Volterra equations. They were developed in papers by Lokta in 1925 and Volterra in 1926. Although these equations are simple, they characterize a wide class of problems.

What happens for given initial values of $x > 0$ and $y > 0$? Will the predators eat all of their prey and in turn die out, will the predators die out because of a too low level of prey and then the prey grow without bound ? Will an equilibrium state be reached, or will a cyclic fluctuation of prey and predator occur ?

The equilibrium points of the equations (7a&b) are the solutions of

$$x(a - by) = 0, \qquad (8a)$$

$$y(-c + dx) = 0. \qquad (8b)$$

These solutions are

$$x = 0,\ y = 0 \quad \text{and} \quad x = c/_{d},\ y = a/_{b} \qquad (9)$$

We will examine the predator-prey model (7a&b) in the neighborhood of each critical point. The stability of the critical points tells us how the two species interact. The system is nonlinear, so we will linearize it to determine the stability of each equilibrium point.

The Jacobian matrix is given by

$$J = \begin{bmatrix} {\partial f_1}/{\partial x} & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix} = \begin{bmatrix} a - by & -bx \\ dy & -c + dx \end{bmatrix}$$

where $f_1(x, y) = ax - bxy$ and $f_2(x, y) = -cy + dxy$.

For $(0,0)$,

$$J = \begin{bmatrix} a - by & -bx \\ dy & -c + dx \end{bmatrix}_{(0,0)} = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix},$$

The linearized system is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

with characteristic equation $(a - \lambda)(-c - \lambda) = 0$. The eigenvalues are $\lambda_1 = a > 0$ and $\lambda_2 = -c < 0$. The critical point $(0,0)$ is a unstable saddle point. This case is not important since the critical solution $x(t) \equiv 0,\ y(t) \equiv 0$ corresponds to the extinction of both species.

The important case here is the coexistence of the two species, this depends on the stability of the nonzero critical point $(c/d,\ a/b)$. So let us check the stability of the critical point $(c/d,\ a/b)$. We have

$$J = \begin{bmatrix} a - by & -bx \\ dy & -c + dx \end{bmatrix}_{(c/d, a/b)} = \begin{bmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix}$$

Using the substitution

$$u = x - \frac{c}{d},$$

$$v = y - \frac{a}{b},$$

we obtain a corresponding critical point $(0,0)$ for $(c/d,\ a/b)$. Hence the linearized system is

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

The characteristic equation is $\lambda^2 + ac = 0$, i.e., the eigenvalues are $\lambda_1, \lambda_2 = \pm i\sqrt{ac}$. Since the roots of the characteristic equation are pure imaginary, the critical point is a stable center of the linear system. The trajectories of the linear system are closed curves corresponding to the solutions that are periodic in time. They do not approach or recede from the critical point. The trajectories can be shown in the following way:

Divide equation $(7b)$ by $(7a)$ we get

$$\frac{dy}{dx} = \frac{y(-c + dx)}{x(a - by)}.$$

By separating the variable we obtain

$$\int \frac{a - by}{y} dy = \int \frac{-c + dx}{x} dx,$$

so we have

$$a \ln y - by = -c \ln x + dx + \ln C$$

where C is the constant of integration. We cannot solve the equation explicitly for x in terms of y or for y in terms of x. The equation defines closed curve around the equilibrium point $(c/d,\ a/b)$. This means that the critical point is a stable center. Therefore the critical solution $x(t) \equiv \frac{c}{d}$ and $y(t) \equiv \frac{a}{b}$ shows that both populations (the predator and the prey) coexist in the same environment without extinction.

One criticism of the Volterra-Lotka predator-prey model is that the prey will grow without bound in the absence of the predator. This can be corrected by allowing the natural inhibiting effect that an increasing population has on the growth rate of the population; for example, by modifying equation (7a) so that when $y = 0$, it reduces to a logistic equation for x. The models of predator-prey and two competing species discussed here can be modified to allow for the effect of time delays; statistical and probabilistic effects can also be included. Finally, we mention that there are *discrete analogs* of each of the problems we have discussed corresponding to species that breed only at certain times. The mathematics of the discrete problems are often interesting and some of the results are unexpected.

We conclude with a warning. Using only elementary phase theory for one and two nonlinear ordinary differential equations, we have been able to illustrate several of the fundamental principles of simple ecological systems. But one should avoid being misled, ecology is not this simple.

Example 3.6.1 Discuss the predator - prey system that is modeled by the equations below

$$\frac{dx}{dt} = 5x - x^2 - xy = x(5 - x - y) \qquad (1)$$

$$\frac{dy}{dt} = xy - 2y = y(x - 2) \qquad (2)$$

Solution 3.6.1 By equating the right hand side of the equation to zero,

$$f_1(x, y) = x(5 - x - y) = 0 \qquad (3)$$

$$f_2(x, y) = y(x - 2) = 0 \qquad (4)$$

we obtain fixed points $(0,0), (5,0)$ and $(2,3)$. We linearize the given system to obtain the Jacobian matrix

$$J = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix} = \begin{bmatrix} 5 - 2x - y & -x \\ y & x - 2 \end{bmatrix}$$

For (0,0), the linearized system is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J_{(0,0)} \begin{bmatrix} x \\ y \end{bmatrix},$$

that is,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The characteristic equation is $(5 - \lambda)(-2 - \lambda) = 0$, i.e., $\lambda_1 = 5 > 0$ and $\lambda_2 = -2 < 0$.

Hence, (0,0) is an unstable saddle point.

For (5,0), we use the following substitution

$$u_1 = x - 5,$$

$$u_2 = y,$$

to get a linearized system with the equilibrium point (0,0),

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = J_{(5,0)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

so that

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

The characteristic equation is $(-5 - \lambda)(3 - \lambda) = 0$. The roots are $\lambda_1 = -5 < 0$ and $\lambda_2 = 3 > 0$, i.e., (5,0) is an unstable saddle point.

For (2,3), the suitable solution is

$$v_1 = x - 2,$$

$$v_2 = y - 3.$$

Then (0,0) is the equilibrium point for (2,3), and the corresponding system is

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = J_{(2,3)} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

that is,

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

The characteristic equation is $\lambda^2 + 2\lambda + 6 = 0$ with the complex roots $\lambda_1 = -1 + i\sqrt{5}$ and $\lambda_2 = -1 - i\sqrt{5}$. Since $Re(\lambda) < 0$, the critical point (2,3) is a spiral point and it is asymptotically stable.

We can successfully come to the conclusion that for any given pair of initial values x_0, y_0, the two species coexist with the population densities approaching the constant values $x(t) \equiv 2$ and $y(t) \equiv 3$.

3.7 Liapounov's Direct Method for Non Autonomous Systems

In this section, we discuss another approach known as $\boldsymbol{Liapounov's\ second}$ method or $\boldsymbol{direct\ method}$. We refer to the method as direct method because no prior knowledge of the solution of the system of differential equations is required. Rather, conclusions about the stability or instability of a critical point are obtained by constructing a suitable auxiliary function. For example, an estimate of the extent of the region of asymptotic stability of a critical point. We showed how the stability of a critical point of an almost linear system can usually be determined from a study of the corresponding linear system. However, we cannot draw a conclusion when the equilibrium point is a center of the corresponding linear system. Examples of this situation are the predator-prey problem and the undamped pendulum discussed earlier. Also, it may be important to check the region of asymptotic stability for an asymptotically stable critical point; that is, the domain such that all solutions starting within that domain approach the critical point. In addition, Liapounov's direct method can also be used to study systems of equations that are not almost linear.

Liapounov's direct method is basically a generalization of the physical principles that for a conservative system (i) If the potential energy is a local minimum, then the rest position is stable otherwise it is unstable, and (ii) During any motion, the total energy is a constant. For the illustration of these concepts, we again consider the undamped pendulum (a conservative mechanical system), which is governed by the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} sin\theta = 0. \tag{1}$$

The corresponding system of first order equations is

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\frac{g}{l}\sin x, \tag{2}$$

where $x = \theta$ and $y = {d\theta}/{dt}.$ Omitting an arbitrary constant, the potential energy U is the work done in lifting the pendulum above its lower position, namely $mgl(1 - cos\theta)$. Hence

$$u(x, y) = mgl(1 - \cos x). \tag{3}$$

The critical points of the system (2) are $x = \pm n\pi, y = 0, n = 0, 1, 2, 3, ...,$

corresponding to $\theta = \pm n\pi, {d\theta}/{dt} = 0$. Physically, we expect that the points

$x = 0, y = 0;\ \ x = \pm 2\pi, y = 0; ...$ corresponding to $\theta = 0,\ \pm 2\pi, ...$ for which the pendulum bob is vertical with the weight down will be stable; and that the points

$x = \pm\pi, y = 0;\ x = \pm 3\pi, ...$ for which the pendulum bob is vertical with the weight up will be unstable. This agrees with the statement (i), for at the former points U is a minimum equal to zero, and at the latter points U is a maximum equal to $2mgl$.

Next we consider the total energy V, which is the sum of the potential energy U and the kinetic energy $\frac{1}{2}ml^2({d\theta}/{dt})^2$. In terms of x and y

$$V(x, y) = mgl(1 - cosx) + \tfrac{1}{2}ml^2y^2. \qquad (4)$$

On a trajectory corresponding to a solution $x = \phi(t),\; y = \varphi(t)$ of Eqs. (2), V can be considered as a function of t. The derivative of $V[\phi(t), \varphi(t)]$ is called the rate of change of V following the trajectory. By the chain rule

$$\frac{dV[\phi(t), \varphi(t)]}{dt} = V_x[\phi(t), \varphi(t)]\frac{d\phi(t)}{dt} + V_u[\phi(t), \varphi(t)]\frac{d\varphi(t)}{dt}$$

$$= (mgl \sin x)\frac{dx}{dt} + ml^2y\frac{dy}{dt}, \qquad (5)$$

where it is understood that $x = \phi(t), y = \varphi(t)$. But dx/dt can be obtained in terms of x and y from Eqs. (2). Substituting in Eqs. (5) for dx/dt and dy/dt, we find that $dV/dt = 0$. Hence V is a constant along any trajectory of the system (2), which agrees with earlier remark (ii) that the total energy is constant during any motion of a conservative system.

It is necessary to note that at any point (x, y) the rate of change of V along the trajectory was computed without $actually\, solving$ the system (2). It is this fact precisely that allows us to use Liapounov's direct method for systems whose solutions we do not know, and hence makes it such an important technique.

At the stable critical points, $x = \pm 2n\pi, y = 0, n = 0, 1, 2, ...,$ the energy V is zero. If the initial state, say (x_1, y_1), of the pendulum is sufficiently near a stable critical point, then the energy $V(x_1, x_2)$ is small, and the motion (trajectory) associated with this energy stays sufficiently close to the critical point. It can be shown that if $V(x_1, y_1)$ is sufficiently small, then the trajectory is closed and contains the critical point. For example, suppose that (x_1, y_1) is near (0,0) and that $V(x_1, y_1)$ is sufficiently small, the equation of the trajectory with energy $V(x_1, y_1)$ is

$$V(x,y) = mgl(1 - cosx) + \frac{1}{2}ml^2y^2 = V(x_1, y_1).$$

For x small we have $1 - cosx = 1 - \left(1 - {x^2}/{2!} + \ ...\right) \cong {x^2}/{2.}$ Thus the equation of the trajectory is approximately

$$\frac{1}{2}mglx^2 + \frac{1}{2}ml^2y^2 = V(x_1, y_1)$$

or

$$\frac{x^2}{2V(x_1, y_1)/mgl} + \frac{y^2}{2V(x_1, y_1)/ml^2} = 1.$$

This is an ellipse enclosing the critical point (0,0); the smaller (x_1, y_1) is, the smaller are the major and minor axes of the ellipse. Physically, the closed trajectory corresponds to a solution that is periodic in time.

If damping is present, however, it is natural to expect that the amplitude of the motion decays in time and that the stable critical point (center) becomes an asymptotically stable critical point (spiral point). This can almost be argued from a consideration of dV/dt. For the damped pendulum, the total energy is still given by equation (4), but now from equation (12)

$dx/dt = y$ and $dy/dt = -\left(\frac{g}{l}\right)\sin x - (c/lm)y$. Substituting for dx/dt and dy/dt in Eq. (5) gives ${dV}/{dt} = -cly^2 \leq 0$. Thus the energy is non increasing along any trajectory and, except for the line $y = 0$, the motion is such that the energy decreases, and hence each trajectory must approach a point of minimum energy, a stable equilibrium point. If $dV/dt < 0$ instead of $dV/dt \leq 0$, it is reasonable to expect that this would be true for all trajectories that start sufficiently close to the origin.

To pursue this idea further, consider the autonomous system

$$\frac{dx}{dt} = F(x,y), \qquad \frac{dy}{dt} = G(x,y), \tag{6}$$

and suppose that the point $x = 0,\ y = 0$ is an asymptotically stable critical point. Then there exist some domain D containing $(0,0)$ such that every trajectory that starts in D must approach the origin as $t \to \infty$. Suppose that there exist an energy function V such that $V(x,y) \geq 0$ for (x,y) in D with $V = 0$ only at the origin. Since each trajectory in D approaches the origin as $t \to \infty$, then following any particular trajectory, V decreases to zero as t approaches infinity. The type of result we want to prove is essentially the converse: if, on every trajectory, V decreases to zero as t increases, then the trajectories must approach the origin as $t \to \infty$, and hence the origin is asymptotically stable. First, however, it is necessary to make some definitions.

Let V be defined on some domain D that contains the origin. Then V is said to be $positive\ definite$ on D if $V(0,0) = 0$ and $V(x,y) > 0$ for all other points in D. Similarly, V is said to be $negative\ definite$ on D if $V(0,0) = 0$ and $V(x,y) < 0$ for all other points in D. If the inequalities $>$ and $<$ are replaced by $\geq$ and $\leq$, then V is said to be $positive\ semidefinite$ and $negative\ semidefinite$ respectively.

We emphasize that in speaking of a positive definite, (negative definite, . . .) function a domain D containing the origin, the function must be zero at the origin in addition to satisfying the proper inequality at all other points in D

Example 3.7.1 The function

$$V(x, y) = \sin(x^2 + y^2)$$

is positive definite on $x^2 + y^2 < \pi/2$ since $V(0,0) = 0$ and $V(x, y) > 0$ for

$0 < x^2 + y^2 < \pi/2$. However, the function

$$V(x, y) = (x + y)^2$$

is only positive semidefinite since $V(x, y) = 0$ on the line $y = -x$.

Let us also consider the function

$$\dot{V}(x, y) = V_x(x, y)F(x, y) + V_y(x, y)G(x, y). \qquad (7)$$

We choose this notion because $\dot{V}(x, y)$ can be identified as the rate of change of V along the trajectory of the system (6) that passes through the point (x, y). That is, if $x = \phi(t), y = \varphi(t)$ is a solution of the system (6), then

$$\frac{dV[\phi(t), \varphi(t)]}{dt} = V_x[\phi(t), \varphi(t)]\frac{d\phi(t)}{dt} + V_y[\phi(t), \varphi(t)]\frac{d\varphi(t)}{dt}$$

$$= V_x(x, y)F(x, y) + V_y(x, y)G(x, y)$$

$$\dot{V}(x, y).$$

The function $\dot{V}$ is sometimes referred to as the derivative of V with respect to the system (6).

We now state two Liapounov theorems, the first dealing with stability, and the second with instability.

Theorem 3.7.1 *Suppose that the autonomous system (6) has an isolated critical point at the origin. If there exists a function* V *that is continuous and has first partial derivatives, is positive definite, and for which the function* $\dot{V}$*, given by* $Eq.(7)$ *is negative definite on some domain* D *in the* xy *plane containing* $(0,0)$*, then the origin is asymptotically stable critical point. If* $\dot{V}$*, is negative semidefinite, then the origin is a stable critical point.*

Theorem 3.7.2 *Let the origin be an isolated critical point of the autonomous system* (6). *Let* V *be a function that is continuous and has continuous first partial derivatives. Suppose that* $V(0,0) = 0$ *and that in every neighborhood of the origin there is at least one point at which* V *is positive (negative). Then if there exists a domain* D *containing the origin such that the function* $\dot{V}$ *as given by* $Eq.(7)$ *is positive definite (negative definite) on* D*, then the origin is an unstable critical point.*

The function V is called a *Liapounov function*. It is important to note that the difficulty in using these theorems is that they tell us nothing about how to construct a Liapounov function, assuming that one exists. In cases where the autonomous system (6) represents a physical problem, it is natural first to consider the actual total energy function of the system as a possible Liapounov function. However, we emphasize that the two theorems above are applicable in cases where the concept of physical energy is not pertinent. In such cases, a judicious trial-and-error approach may be necessary.

Now consider the second part of theorem 5.1, that is, the case $\dot{V}(x, y) \leq 0$. Let $c \geq 0$ be a constant and consider the curve in the xy plane given by the equation $V(x, y) = c$. For $c = 0$ the curve reduces to the single point $x = 0, y = 0$. However, for $c > 0$ and sufficiently small, it can be shown by using the continuity of V that we will obtain a closed curve containing the origin as illustrated in the diagram below.

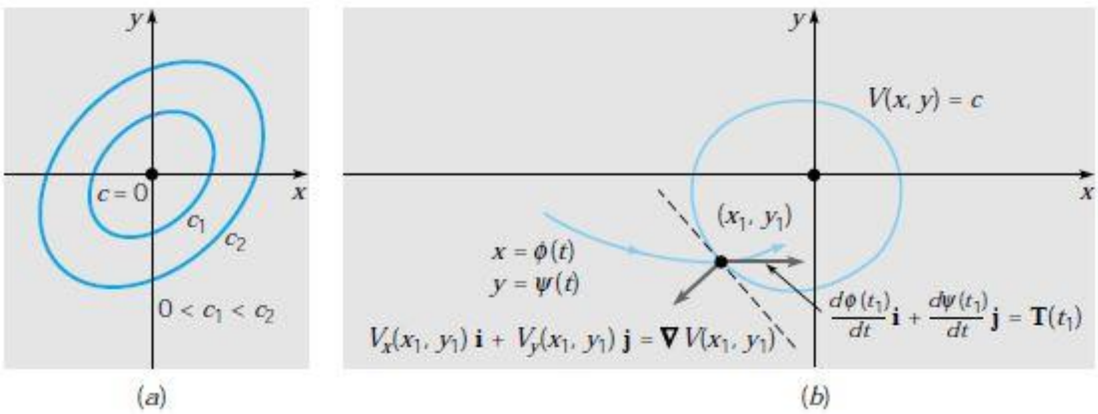

Figure 3.4 : Geometrical Interpretation of Liapounov's Method

There may, of course, be other curves in the xy plane corresponding to the same value of c, but they are not of interest. Further, again by continuity, as c gets smaller and smaller, the closed curves $V(x, y) = c$ enclosing the origin shrink to the origin. We will show that a trajectory starting inside of a closed curve $V(x, y) = c$ cannot cross to the outside. Thus, given a circle of radius ϵ about the origin, by taking c sufficiently small, we can ensure that every trajectory starting inside of the closed curve $V(x, y) = c$ stays within the circle of radius ϵ; indeed it will stay within the closed curve $V(x, y) = c$ itself. Therefore the origin will be a stable critical point.

A geometric proof of Theorem 3.7.2 follows by somewhat similar arguments. Briefly, suppose that $\dot{V}$ is positive definite, and suppose that given any circle about the origin, there is an interior point (x_1, y_1) at which $V(x_1, y_1) > 0$. Consider a trajectory that starts at (x_1, y_1). Along this trajectory it follows from equation (8) that V must increase, since $\dot{V}(x,y) > 0$; furthermore, since $V(x_1, y_1) > 0$ the trajectory cannot approach the origin because $V(0,0) = 0$. This shows that the origin cannot be asymptotically stable. By exploiting further the fact $\dot{V}(x,y) > 0$, it is possible to show that the origin is an unstable point; however, we will not pursue this argument.

To illustrate the use of Theorem 3.7.1, we consider the question of the stability of the critical point (0,0) of the undamped pendulum Equation (2). While the system (2) is almost linear, the point(0,0) is a center of the corresponding linear system, so no conclusion can be drawn from Theorem 5.0. Since the mechanical system is conservative, it is natural to suspect that the total energy function V given by Eq. (4) will be a Liapounov function. For example, if we take D to be the domain $-\pi/2 < x < \pi/2, -\infty < y < \infty$, then V is positive definite. As we have seen $\dot{V}(x,y) = 0$, so it follows from the second part of theorem 5.1 that the critical point (0,0) of Equation (2) is a stable critical point.

From a practical point of view one is more interested in asymptotic stability. The theorem below gives the simplest result in dealing with this.

Theorem 3.7.3 Let the origin be an isolated critical point of the autonomous system (6). Let the function V be continuous and have continuous first partial derivatives. If there is a bounded domain D_k containing the origin where $V(x,y) < K$, V is positive definite, and $\dot{V}$ is negative definite, then every solution of equation (6) that starts at a point in D_k approaches the origin as t approaches infinity.

In other words, the theorem says that if $x = \phi(t), y = \varphi(t)$ is the solution of equation (6) for initial data lying in D_k then (x, y) approaches the critical point (0,0) as $t \to \infty$. Thus D_k gives a region of asymptotic stability : of course, it may not be the entire region of asymptotic stability. This theorem is proved by showing that (i) there are no periodic solutions of the system (6) in D_k , and (ii) there are no other critical points in D_k . It then follows that trajectories starting in D_k cannot escape and, hence, must tend to the origin as t tends to infinity.

Theorems 3.7.1 and 3.7.2 gives sufficient conditions for stability and instability respectively. However, these conditions are not necessary, nor does our failure to determine a suitable Liapounov function mean that there is not one. Unfortunately, there are no general methods for the construction of Liapounov function for special classes of equations. One simple result from elementary algebra, which is often useful in constructing positive definite or negative definite functions, is stated without proof in the theorem below.

Theorem 3.7.4 The function

$$V(x, y) = ax^2 + bxy + cy^2 \qquad (9)$$

is positive definite if, and only if ,

$$a > 0 \quad and \quad 4ac - b^2 > 0, \qquad (10)$$

and is negative definite if, and only if,

$$a < 0 \quad and \quad 4ac - b^2 > 0 \qquad (11)$$

The theorem is illustrated in the example below.

Example 3.7.2 Show that the critical point (0,0) of the autonomous system

$$\frac{dx}{dt} = -x - xy^2 \qquad (12)$$

is asymptotically stable.

Solution : We try to construct a Liapounov function of the form (10).

Then $V_x(x,y) = 2ax + by, \quad V_y(x,y) = bx + 2cy$ so

$$\dot{V}(x,y) = (2ax + by)(-x - xy^2) + (bx + 2cy)(-y - yx^2)$$

$$= -[2a(x^2 + x^2y^2) + b(2xy + xy^3 + yx^3) + 2c(y^2 + x^2y^2)].$$

If we choose $b = 0$, and a and c to be any positive numbers, then $\dot{V}$ is negative definite and V is positive definite by the above theorem. Thus by Theorem 5.1, the origin is an asymptotically stable critical point.

3.8 Stability Analysis by Liapounov Method

For autonomous systems, we can use Liapounov's method to determine the stability of the zero solution. We will investigate a general autonomous system

$$\dot{x} = X(x,y),$$

$$\dot{y} = Y(x,y).$$

with the equilibrium points (0,0).

Definition3.8.1 (Topographical system). Define a family of curves

$$V(x,y) = \alpha, \;\; \alpha > 0$$

with the following properties :

i. $V(x,y)$ is continuous on a connected neighborhood of D of the origin and $\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}$ are continuous on D except possibly at the origin.

ii. $V(0,0) = 0$ and $V(x,y) > 0$ for all $(x,y) \in D$.

iii. There exists $\mu > 0$ such that for all $\alpha, 0 < \alpha < \mu$,

$$V(x,y) = \alpha, \quad (x,y) \in D$$

uniquely determines a simple closed curve τ_α around the origin.

These curves are known as a topographic system.

point in R.

Theorem 3.8.2 *Consider the topographic curve τ defined by*

$$V(x,y) = \alpha, \ \ \alpha > 0$$

in D. Suppose that

$$\dot{V}(x,y) \leq 0$$

in this domain. If H is a half-path starting at a point P inside τ, then H can never escape from this closed region determined by τ.

Here,

$$\dot{V}(x,y) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = X\frac{\partial V}{\partial x} + Y\frac{\partial V}{\partial y}.$$

Chapter 4

CONCLUSION

In this thesis, we dealt with autonomous and non autonomous systems of ordinary differential equations. We gave examples of these systems of equations and explained in detail, their differences. We also discussed methods of investigating the stability of non linear systems and gave the various types of stabilities, like asymptotic stability, uniform stability, etc, and also gave examples of stable solutions and unstable solutions.

We saw how ecological applications such as Lotka-Volterra Competition Model and Predator-Prey Model can be modeled by differential equations, we gave examples of competition model and predator-prey models and we saw what happens to the species as a result of the presence or absence of the other species.

We also studied and analyzed Liapounov's direct method for stability of autonomous and non autonomous differential equations. We showed how the stability of a critical point of an almost linear system can be determined from a study of the corresponding linear system. We are unable to draw a conclusion when the equilibrium point is a center of the corresponding linear system. Examples of this situation are the predator-prey problem and the undamped pendulum that was discussed exhaustively in Chapter 3.

Let us give some results on the behavior of a Predator-Prey System.

Given the system

$$\dot{x} = x\alpha(x) - yV(x),$$

$$\dot{y} = yK(x),$$

where x is the density of the prey population and y is the density of the predator population. We assume that

i. α is smooth with $\alpha'(x) < 0, x \geq 0$ and $\alpha(0) > 0 > \lim_{x\to+\infty} \alpha(x)$;

ii. K and V are nonnegative and increasing, $K(0) = 0 = V(0)$.

Consider the model

$$\dot{x} = f_1(x, y) \qquad (1a)$$

$$\dot{y} = f_2(x, y) \qquad (1b)$$

where

$$f_1(x, y) = \alpha x\left(1 - \frac{x}{k}\right) - \frac{\beta xy}{\varepsilon y + x}, \quad x^2 + y^2 > 0,$$

$$f_2 = -\frac{y(\gamma + \delta y)}{1 + y} + \frac{\beta xy}{\varepsilon y + x}, \quad x^2 + y^2 > 0,$$

$\alpha > 0$, $\varepsilon > 0$ are the rate of growth the prey population in the absence of the predators and environmental limitations (i.e. the prey grows without bound in the absence of the predators). On the other hand, in the absence of the prey, the population of predator reduces. The death rate of the predator depends on y and can be expressed by the formula

$$E(y) = \frac{\gamma + \delta y}{1 + y},$$

where $\gamma < \delta$.

This model is different from other predator-prey systems by the predator mortality rate since mortality is not an unbounded or constant function. It increases with the predator population.

The system $1a$ & $1b$ can be written in polar coordinates using the transformations

$$x = r(\theta)\cos\theta \quad \text{and} \quad y = r(\theta)\sin\theta.$$

Then we notice that $f_i \in C^1\left(\mathbb{R}_0{}^+, \mathbb{R}\right)$ for $i = 1,2$. Hence the solution 1 exists and is unique.

If we consider f_1 and f_2 in the form

$$f_1(x,y) = xM_1(x,y) = x\left[\alpha\left(1 - \frac{x}{k}\right) - \frac{\beta y}{\varepsilon y + x}\right],$$

$$f_2(x,y) = yM_2(x,y) = y\left[-\frac{(\gamma + \delta y)}{1 + y} + \frac{\beta x}{\varepsilon y + x}\right],$$

The following can be derived easily.

i. M_1 and M_2 are smooth functions so the positive quadrant is an invariant region.

ii. $\frac{\partial m_1}{\partial y} = -\frac{\beta x}{(\varepsilon y + x)^2} < 0$ and $\frac{\partial M_2}{\partial x} = \frac{\beta \varepsilon y}{(\varepsilon y + x)^2} > 0$, where $x, y > 0$. That is, it is a predator-prey system with predator y and prey .

REFERENCES

[1] David A. Sanchez, *Ordinary Differential Equations and Stability Theory; An Introduction*, Dover Publications, Inc., New York, 1968.

[2] William E. Boyce and Richard C. Diprima, *Elementary Differential Equations and Boundary Value Problems*, 3rd ed., John Wiley and Sons, Inc., New York, 1984.

[3] R. Bellman, *Stability Theory of Ordinary Differential Equations,* McGraw-Hill, New York, 1953.

[4] L. Cesari, *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*, 2nd ed., Academic Press, New York, 1963

[5] W. Coppel, *Stability and Asymptotic Behavior of Differential Equations*, Heath, Boston, 1965.

[6] J. LaSalle and S. Lefschetz, *Stability by Liapounov's Direct Method with Applications*, Academic Press, New York, 1961.

[7] S. Kovacs, K. Kiss, M. Farkas, *Qualitative Behavior of a Ratio-Dependent Predator-Prey System, Nonlinear Analysis: Real World Applications*, 2009.

[8] G. Birkhoff and G. Rota, *Ordinary Differential Equations,* Ginn, Boston, 1962.

[9] J. L. Brenner, *Problems in Differential Equations*, 2nd ed., W. H. Freeman and Company, San Fransico and London, 1966.