Bernoulli and Euler Polynomials

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Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the Degree of

Master of Science
in
Mathematics

Eastern Mediterranean University
July 2013
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

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ABSTRACT

This thesis provides an overview of Bernoulli and Euler numbers. It describes the Bernoulli and Euler polynomials and investigates the relationship between the classes of the two polynomials. It also discusses some important identities using the finite difference calculus and differentiation. The last part of this study is concerned with the Generalized Bernoulli and Euler polynomials. Furthermore, the properties obtained in the second chapter are also examined for the generalized Bernoulli and Euler polynomials in this part of the thesis. The Complementary Argument Theorem, the generating functions, the Multiplication and the Euler-Maclauren Theorems are widely used in obtaining the mentioned results.

Keywords: Bernoulli -Euler Polynomials, Generalized Bernoulli -Euler Polynomials, Finite Difference

ÖZ

Bu çalışmada Bernoulli ve Euler sayıları ile Bernoulli ve Euler polinomları arasındaki ilişkiler incelenmiştir. Bernoulli sayıları için ardışıklık, kapalılık ve üreticilik gibi temel özellikler çalışılmıştır. Bunun yanında Bernoulli ve Euler Polinomları arasındaki ilişkiler incelenip her ikisi için de geçerli olan türev, integral, fark ve simetri özellikleri incelenmiştir. Ayrıca Genelleştirilmiş Bernoulli ve Euler polinomları için de türev, intregral özellikleri çalışılmıştır.

Anahtar Kelimeler: Bernulli-Euler Polinomları, Genelleştirilmiş Bernoulli-Euler Polinomları, Sonlu Fark

DEDICATION

TO MY FATHER

ACKNOWLEDGMENTS

First of all, I would like to express my special thanks to my supervisor Assoc. Prof. Dr Sonuç Zorlu Oğurlu for her patience, supervision, guidance, encouragement and help during the preperation of this thesis. I also would like to thank to my parents and friends specially to Noushin Ghahramanlou for their help, encouragement and love.

Finally, my special thanks go to the members of Department of Mathematics for their advices and help in completing my study at EMU. I am specially thankful to Prof.Dr.Nazım Mahmudov and Asst.Prof.Dr. Arif Akkeleş for their help.

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Chapter 1

BERNOULLI AND EULER NUMBERS

1.1 Bernoulli Numbers

In Mathematics, the Bernoulli numbers B_n are a sequence of rational numbers with important relations to number theory and with many interesting arithmetic properties. We find them in a number of contexts, for example they are closely related to the values of the Riemann zeta function at negative integers and appear in the Euler-Maclaurin formula. The values of the first few Bernoulli numbers are $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, $B_4 = -1/30$ (Some authors use $B_1 = +1/2$ and some write B_n for B_{2n}). After B_1 all Bernoulli numbers with odd index are zero and the non zero values alternate in sign. These numbers appear in the series expansions of trigonometric and are important in number theory and analysis.

The Bernoulli numbers first used in the Taylor series expansions of the tangent and hyperbolic tangent functions, in the formulas for the sum of powers of the first positive integers, in the Euler-Maclaurin formula and in the expression for certain values of the Riemann zeta function.

Bernoulli numbers appeared in 1713 in Jacob Bernoulli's work, who was Johanns Bernoulli's older brother, Euler's teacher and mentor at Bassel's university.

Those numbers were studied at the same time independently by Japanese mathematician Seki Kowa. His discovery was published in 1712 in his work and in his Arts Conjectandi of 1713. Ada Lovelace's note G on the analytical engine from 1842 describes an algorithm for generating Bernoulli numbers with Babbage's machine. Also, the Bernoulli numbers have the distinction of being the main topic of the first computer program.

Bernoulli numbers are still a bit misterious, they appear frequently in studying Gamma function about Euler's constant and people continue to discover new properties and to publish articles about them.

Definition 1.1.1 *Bernoulli numbers has the following closed form expression:*

$$S_m(n) = \sum_{k=1}^{n} k^m = 1^m + 2^m + ... + n^m. \tag{1.1.1}$$

Note that $S_m(0) = 0$ for all $m \geqslant 0$ The equation (1.1.1) can always be written as a polynomial in n of degree $m+1$.

Definition 1.1.2 *The coefficients of the polynomials are related to the Bernoulli numbers by* ***Bernoulli's formula:*** $S_m(n) = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k n^{m+1-k}$,

where $B_1 = +1/2$. Bernoulli's formula can also be stated as

$$S_m(n) = \frac{1}{m+1}\sum_{k=0}^{m}(-1)^k\binom{m+1}{k}B_k n^{m+1-k}.$$

Here are some simple examples of Bernoulli numbers:

Example 1.1.3 *Let* $n \geqslant 0$. *Taking* m *to be* 0 *and* $B_0 = 1$ *gives the natural numbers* $0, 1, 2, 3, ...$

$$0 + 1 + 1 + ... + 1 = \frac{1}{1}(B_0 n) = n.$$

Example 1.1.4 *Let* $n \geqslant 0$. *Taking* m *to be* 1 *and* $B_1 = 1/2$ *gives the triangular numbers* $0, 1, 3, 6, ...$

$$0 + 1 + 2 + ... + n = \frac{1}{2}(B_0 n^2 + 2B_1 n^1) = \frac{1}{2}(n^2 + n).$$

Example 1.1.5 *Let* $n \geqslant 0$. *Taking* m *to be* 2 *and* $B_2 = 1/6$ *gives the square pyramidal numbers* $0, 1, 5, 14, ...$

$$0 + 1^2 + 2^2 + ... + n^2 = \frac{1}{3}(B_0 n^3 + 3B_1 n^2 + 3B_2 n^1) = \frac{1}{3}(n^3 + \frac{3}{2}n^2 + \frac{1}{2}n)$$

where $B_1 = -1/2$.

There are many characterizations of the Bernoulli numbers where each can be used to introduce them. Here are most useful characterizations:

1. **Recursive Definition.** The recursive equation is best introduced in a slightly more general form

$$B_n(x) = x^n - \sum_{k=0}^{n-1} \binom{n}{k} \frac{B_k(x)}{n-k+1}.$$

For $x = 0$, the recursive equation becomes,

$$B_n = [x = 0] = -\sum_{k=0}^{n-1} \binom{n}{k} \frac{B_k}{n-k+1}$$

Also, when $x = 1$,we get the following form

$$B_n = 1 - \sum_{k=0}^{n-1} \binom{n}{k} \frac{B_k}{n-k+1}$$

2. **Explicit Definition.** Starting again with slightly more general formula

$$B_n(x) = \sum_{k=0}^{n} \sum_{\upsilon=0}^{k} (-1)^{\upsilon} \binom{k}{\upsilon} \frac{(x+\upsilon)^n}{k+1}.$$

For $x = 0$, the recursive equation becomes,

$$B_n = \sum_{k=0}^{n} \sum_{\upsilon=0}^{k} (-1)^{\upsilon} \binom{k}{\upsilon} \frac{\upsilon^n}{k+1}.$$

Also, when $x = 1$, we get the following from

$$B_n = \sum_{k=1}^{n+1} \sum_{\upsilon=1}^{k} (-1)^{\upsilon+1} \binom{k-1}{\upsilon-1} \frac{\upsilon^n}{k}.$$

3. **Generating Function.** The general formula for the generating function is given as

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

For $x = 0$, the recursive equation becomes,

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

Also, when $x = 1$, we get the following form

$$\frac{t}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

1.1.1 Worpitzky's Representation for Bernoulli Numbers

The definition to proceed with was developed by Julius Worpitzky in 1883. Besides elementary arithmetic only the factorial function $n!$ and the power function k^m are employed. The signless Worpitzky numbers are defined as

$$W_{n,k} = \sum_{n=0}^{k} (-1)^{n+k} (n+1)^n \frac{k!}{n!(k-n)!}.$$

One can also express $W_{n,k}$ through the Stirling numbers of the second kind as follows:

$$W_{n,k} = k! \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\}.$$

A Bernoulli number is then introduced as an inclusion-exclusion sum of Worpitzky numbers weighted by the sequence $1, 1/2, 1/3, ...$

$$B_n = \sum_{k=0}^{n} (-1)^k \frac{W_{n,k}}{k+1} = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{n=0}^{k} (-1)^n (n+1)^n \binom{k}{n}.$$

1.2 Euler Numbers

In Combinatorics, the Eulerian number $A(n,m)$ is the number of permutations of the numbers 1 to n in which exactly m elements are greater than the previous element. The coefficients of the Eulerian polynomials are given as follows,

$$A_n(x) = \sum_{m=0}^{n} A(n,m)x^{n-m}.$$

This polynomial appears as the numerator in an expression for the generating function of the sequence $1^n, 2^n, 3^n, \cdots$. Other notations for $A(n,m)$ are $E(n,m)$ and $\langle\rangle nm$.

In Number Theory, the Euler numbers are sequence E_n of integers defined by the following Taylor series expansion

$$\frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n, \tag{1.2.1}$$

where $\cosh t$ is the hyperbolic cosine. The Euler numbers appear as a special value of Euler polynomials. The odd indexed Euler numbers are all zero while the even ones have alternating signs. They also appear in the Taylor series expansions of the secant and hyperbolic secant functions. The latter is the function given in equation (1.2.1). We also play important role in Combinatorics, especially when counting the number of alternating permutations of a set with an even number of elements.

John Napier who made common use of the decimal point, was a Scottish mathematician, physicist, astronomer and astrologer. He is best known as the discoverer of logarithms. In 1618, he published a work on logarithms which contained the first reference to the constant e. It was not until Jacob Bernoulli whom to the Bernoulli Principles named after, attempted to find the value of a compound-interest expression. Historically Euler's number was actually expressed as "b" until Euler himself published his work Mechanica who used "e" instead of "b" as the variable. Eventually the letter made its way as the standard notation of Euler's number.

To find Euler numbers $A(n, m)$ one can use the following formula,

$$A(n, m) = (n - m)A(n - 1, m - 1) + (m + 1)A(n - 1, m).$$

Recall that,

$$A(n, m) = A(n, n - m - 1).$$

A closed form expression for $A(n, m)$ is given by,

$$A(n, m) = \sum_{k=0}^{m} (-1)^k \binom{n + 1}{k} (m + 1 - k)^n.$$

1.2.1 Properties of the Euler Numbers

1. It is clear from the combinatorics definition that the sum of the Eulerian numbers for a fixed value of n is the total number of permutations of the

numbers 1 to n, so

$$\sum_{m=0}^{n-1} A(n,m) = n! \quad for\ n \geqslant 1.$$

2. The alternating sum of the Eulerian numbers for a fixed value of n is related to the Bernoulli number B_{n+1} and

$$\sum_{m=0}^{n-1} (-1)^m A(n,m) = \frac{2^{n+1}(2^{n+1}-1)B_{n+1}}{n+1} \quad for\ n \geqslant 1.$$

Other summation properties of the Eulerian numbers are:

$$\sum_{m=0}^{n-1} (-1)^m \frac{A(n,m)}{\binom{n-1}{m}} = 0 \quad \text{for} \quad n \geqslant 2,$$

$$\sum_{m=0}^{n-1} (-1)^m \frac{A(n,m)}{\binom{n}{m}} = (n+1)B_n \quad \text{for} \quad n \geq 2.$$

1.2.2 Identities Involving Euler Numbers

The Euler numbers are involved in the generating function for the sequence of n^{th} powers

$$\sum_{k=1}^{\infty} k^n x^k = \frac{\sum_{m=0}^{n} A(n,m) x^{m+1}}{(1-x)^{n+1}}.$$

Worpitzky's identity expresses x^n as the linear combination of Euler numbers with binomial coefficients:

$$x^n = \sum_{m=0}^{n-1} A(n,m) \binom{x+m}{n}.$$

It follows from Worpitzky's identity that

$$\sum_{k=1}^{N} k^n = \sum_{m=0}^{n-1} A(n,m) \binom{N+1+m}{n+1}.$$

Another interesting identity is given as follows,

$$\frac{ex}{1-xe} = \sum_{n=0}^{\infty} \frac{\sum_{m=0}^{n} A(n,m) x^{m+1}}{(1-x)^{n+1} n!}.$$

Chapter 2

BERNOULLI AND EULER POLYNOMIALS

The classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$ are usually defined by means of the following exponential generating functions:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \qquad |t| < 2\pi,$$

and

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \qquad |t| < \pi \tag{2.0.1}$$

respectively. The classical Bernoulli and Euler polynomials can explicitly be defined as,

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \tag{2.0.2}$$

and

$$E_n(x) = \frac{1}{n+1} \sum_{k=1}^{n+1} (2 - 2^{k+1}) \binom{n+1}{k} B_k x^{n+1-k},$$

respectively, where $B_k := B_k(0)$ is the Bernoulli number for each $k = 0,1,...,n$.

Several interesting properties and relationships involving each of these polynomials and numbers can be found in many books and journals ([1]- [7]) on this subject. Some of these properties are given in the following section.

2.1 Properties of Bernoulli and Euler Polynomials

The purpose of this section is to obtain interesting properties of the Bernoulli and Euler polynomials, and the relationship between these polynomials.

Recently, Cheon ([1]) obtained the results given below:

$$B_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} B_k(x), \quad n \in N_0 \tag{2.1.1}$$

$$E_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} E_k(x), \quad n \in N_0 \tag{2.1.2}$$

$$B_n(x) = \sum_{\substack{k=0 \\ k \neq 1}}^{n} \binom{n}{k} B_k E_{n-k}(x), \qquad (n \in \mathbb{N}_0)\,. \tag{2.1.3}$$

Both (2.1.1) and (2.1.2) are well-known results and are obviously special cases of the following familiar addition theorems:

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k}, \qquad (n \in \mathbb{N}_0) \tag{2.1.4}$$

and

$$E_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) y^{n-k}, \qquad (n \in \mathbb{N}_0)\,. \tag{2.1.5}$$

Furthermore, Cheon's main result (2.1.3) is essentially the same as the following known relationship:

$$B_n(x) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} B_{n-k} E_k(2x), \qquad (n \in \mathbb{N}_0)\,,$$

or equivalently,

$$2^n B_n(\frac{x}{2}) = \sum_{k=0}^{n} \binom{n}{k} B_k E_{n-k}(x), \qquad (n \in \mathbb{N}_0)\,. \tag{2.1.6}$$

These two polynomials have many similar properties (see [1]).

The following identity will be useful in the sequel.

$$\binom{n}{k}\binom{n-k}{j} = \binom{n}{j+k}\binom{j+k}{k}. \tag{2.1.7}$$

Theorem 2.1.1 *For any integer* $n \geqslant 0$*, we have*

a) $B_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} B_k(x)$

b) $E_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} E_k(x)$

Proof. a) Letting $y = 1$ in equations (2.1.4) and (2.1.5), one can directly obtain the equations (2.1.1) and (2.1.2) respectively. Also, applying (2.0.2) and (2.1.7), we obtain

$$B_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} B_k (x+1)^{n-k}. \tag{2.1.8}$$

Recall the following Binomial expansion

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{2.1.9}$$

Applying (2.1.9) to (2.1.8), one can easily get the following formula

$$B_n(x+1) = \sum_{k=0}^{n} B_k \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} x^j.$$

Using (2.1.7), the above equation becomes,

$$B_n(x+1) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} B_k \binom{n}{j+k} \binom{j+k}{k} x^j.$$

Expanding the last expression gives

$$\begin{aligned} B_n(x+1) \quad &= \quad \binom{n}{0} \left\{ \binom{0}{0} B_0 \right\} + \binom{n}{1} \left\{ \binom{1}{0} B_0 x + \binom{1}{1} B_1 \right\} + \\ &\quad +\ldots + \binom{n}{n} \left\{ \binom{n}{0} B_0 x^n + \binom{n}{1} B_1 x^{n-1} + \ldots + \binom{n}{n} B_n \right\}, \end{aligned}$$

which yields to

$$B_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{n} \binom{k}{j} B_j x^{k-j} = \sum_{k=0}^{n} \binom{n}{k} B_k(x).$$

Hence the theorem (a) is proved.

b) We will now give the proof of part (b) of the theorem. Replacing x with $(x+1)$ in equation (2.0.1), we get,

$$\frac{2e^{(x+1)t}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x+1)\frac{t^n}{n!}. \tag{2.1.10}$$

Also multiplying both sides of equation (2.0.1) by e^t and using the expansion $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$,

$$\begin{aligned}\frac{2e^{xt}e^t}{e^t+1} &= \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}e^t \\ &= \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}\sum_{n=0}^{\infty}\frac{t^n}{n!}.\end{aligned}$$

Now applying Cauchy Product Formula, one can obtain the following relation

$$\begin{aligned}\frac{2e^{xt}e^t}{e^t+1} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} E_k(x)\frac{t^k}{k!}\frac{t^{n-k}}{(n-k)!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} E_k(x)\frac{t^k}{k!}\frac{t^{n-k}}{(n-k)!}\left(\frac{n!}{n!}\right), \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} E_k(x)\binom{n}{k}\frac{t^n}{n!},\end{aligned}$$

which implies

$$E_n(x+1) = \sum_{k=0}^{n} E_k(x)\binom{n}{k},$$

This gives the desired result. ■

2.1.1 Equivalence of Relation (2.1.3) and (2.1.6)

For both Bernoulli and Euler polynomials, the following Multiplication Theorems are well known :

Theorem 2.1.2 *For* $n \in \mathbb{N}_0$ *and* $m \in \mathbb{N}$ *the following relations hold:*

a) $B_n(mx) = m^{n-1} \sum_{j=0}^{m-1} B_n\left(x + \frac{j}{m}\right)$,

b) $$E_n(mx) = \begin{cases} m^n \sum_{j=0}^{m-1} (-1)^j E_n\left(x + \frac{j}{m}\right), & (m = 1, 3, 5, ...) \\ -\frac{2}{n+1} m^n \sum_{j=0}^{m-1} (-1)^j B_{n+1}\left(x + \frac{j}{m}\right), & (m = 2, 4, 6, ...) \end{cases}$$

The above theorem yields the following relationships between these two polynomials when $m = 2$.

$$E_n(2x) = -\frac{2}{n+1} 2^n \left(B_{n+1}(x) - B_n\left(\frac{x}{2}\right)\right).$$

Also letting n to be $n-1$ and x to be $x/2$, we obtain

$$\begin{aligned} E_{n-1}(x) &= \frac{2^n}{n}\left(B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right)\right) & (2.1.11) \\ &= \frac{2}{n}\left[B_n(x) - 2^n B_n\left(\frac{x}{2}\right)\right] \quad n \in N \\ &= \frac{2^n}{n}\left[2^{n-1}\left(B_n\left(\frac{x+1}{2}\right) + B_n\left(\frac{x}{2}\right)\right) - 2^n B_n\left(\frac{x}{2}\right)\right]. & (2.1.12) \end{aligned}$$

Also letting $m = 2$ in part(a), we have

$$B_n(2x) = 2^{n-1}\left(B_n(x) + B_n\left(x + \frac{1}{2}\right)\right).$$

Using (2.1.2) and replacing $2x$ with x, we get

$$B_n(x) = 2^{n-1}\left[B_n\left(\frac{x}{2}\right) + B_n\left(\frac{x+1}{2}\right)\right].$$

From (2.1.12),

$$E_{n-1}(x) = \frac{2}{n}\left(B_n(x) - 2^n B_n\left(\frac{x}{2}\right)\right).$$

Since $B_1 = -\frac{1}{2}$, by separating the second term ($k = 1$) of the sum in (2.1.6), we have

$$\begin{aligned} 2^n B_n\left(\frac{x}{2}\right) &= \binom{n}{1} B_1 E_{n-1}(x) \\ &= -\frac{n}{2} E_{n-1}(x). \end{aligned}$$

Hence,

$$2^n B_n\left(\frac{x}{2}\right) = \sum_{\substack{k=0 \\ k\neq 1}}^{n} \binom{n}{k} B_k E_{n-k}(x) - \frac{n}{2} E_{n-1}(x), \qquad (n \in \mathbb{N}_0)$$

which in the light of the second relation in (2.1.11), immediately yields to (2.1.3).

Chapter 3

THE GENERALIZED BERNOULLI AND EULER POLYNOMIALS

In this chapter we study some properties of two classes of polynomials namely Bernoulli and Euler polynomials which play an important role in the finite calculus. These polynomials have been the object of much research and have been generalized in a very elegant manner by Nörlund.

We shall here approach these polynomials by a symbolic method described by Milne-Thomson (see [4]) by which they arise as generalizations of the simplest polynomials, namely the powers of x. The method is applicable to whole classes of polynomials, including those of Hermite. Considerations of space must limit us to the discussion of only a few of the most interesting relations to which these polynomials give rise.

3.1 The φ Polynomials

ϕ polynomials of degree n and order α are denoted as $\phi_n^{(\alpha)}(x)$ and defined as below :

$$f_\alpha(t)e^{xt+g(t)} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\phi_n^{(\alpha)}(x), \tag{3.1.1}$$

where the uniformly convergent series in t on the right-hand side of (3.1.1) exists for $f_\alpha(t)$ and $g(t)$ in a certain range of x.

Substituting $x = 0$, we get

$$f_\alpha(t)e^{g(t)} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\phi_n^{(\alpha)}(x), \tag{3.1.2}$$

where $\phi_n^{(\alpha)}(x) = \phi_n^{(\alpha)}(0)$ is a ϕ number of order α.

Writing $x + y$ instead of x in (3.1.1) we get obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!}\phi_n^{(\alpha)}(x+y) &= f_\alpha(t)e^{(x+y)t+g(t)}, \\ &= f_\alpha(t)e^{xt}e^{g(t)+yt} \\ &= e^{xt}\sum_{n=0}^{\infty} \frac{t^n}{n!}\phi_n^{(\alpha)}(y). \end{aligned} \tag{3.1.3}$$

Having put the coefficients of t^n, equal on both sides, we have

$$\begin{aligned} \phi_n^{(\alpha)}(x+y) &= \phi_n^{(\alpha)}(y) + x\binom{n}{1}\phi_{n-1}^{(\alpha)}(y) + ... + x^n\binom{n}{n}\phi_0^{(\alpha)}(y) \quad (3.1.4) \\ &= \sum_{k=0}^{v} x^k \phi_{n-k}^{(\alpha)}(y). \end{aligned}$$

Substituting (3.1.4) and applying the Cauchy product formula to (3.1.3), we have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\phi_n^{(\alpha)}(x+y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(xt)^k}{k!}\frac{t^{n-k}}{(n-k)!}\phi_{n-k}^{(\alpha)}(y)$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{n!}{n!}\frac{x^k t^n}{k!(n-k)!}\phi_{n-k}^{(n)}(y)$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} x^k \binom{n}{k}\frac{t^n}{n!}\phi_{n-k}^{(n)}(y).$$

Taking $y = 0$, we obtain

$$\phi_n^{(\alpha)}(x) = \phi_n^{(\alpha)} + x\binom{n}{1}\phi_{n-1}^{(\alpha)} + x^2\binom{n}{2}\phi_{n-2}^{(\alpha)} + ... + x^n\binom{n}{n}\phi_0^{(\alpha)},$$

which shows unless $\phi_0^{(\alpha)} = 0$ that $\phi_n^{(\alpha)}(x)$ is actually of degree n.

Therefore we can write the below equality

$$\phi_n^{(\alpha)}(x) \doteqdot (\phi^{(\alpha)} + x)^n \tag{3.1.5}$$

where after expanding the powers, each index of $\phi^{(\alpha)}$ will be replaced by the corresponding suffix.

In this way ϕ polynomials defined completely by ϕ numbers mentioned in (3.1.2) and also by the equality (3.1.5).

From (3.1.5), we have

$$\frac{d}{dx}\phi_n^{(\alpha)}(x) \doteqdot n(\phi_n^{(\alpha)} + x)^{n-1} = n\phi_{n-1}^{(\alpha)}(x) \tag{3.1.6}$$

and

$$\int_a^x \phi_n^{(\alpha)}(t)dt = \frac{\phi_{n+1}^{(\alpha)}(x) - \phi_{n+1}^{(\alpha)}(a)}{n+1}. \tag{3.1.7}$$

Therefore as we already know differentiation will decrease the degree by one unit and integration will increase it one unit but in both cases we will have no changes in the order. Using $\triangle$ in equation (3.1.1), we will get

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \triangle \beta_n^{(\alpha)}(x) = (e^t - 1) f_\alpha(t) e^{xt+g(t)}. \tag{3.1.8}$$

In a similar way, using $\bigtriangledown$ in (3.1.1), we get

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \bigtriangledown \phi_n^{(\alpha)}(x) = \frac{e^t + 1}{2} f_\alpha(t) e^{xt+g(t)}. \tag{3.1.9}$$

3.2 The β Polynomials

A result from (3.1.8) is that if we take $f_\alpha(t) = t^\alpha (e^t - 1)^{-\alpha}$ in (3.1.1) where α is any integer (either positive, negative or zero), we get a particularly simple class of ϕ polynomials which are called β polynomials and are written as follows

$$\frac{t^\alpha}{(e^t - 1)^\alpha} e^{xt+g(t)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \beta_n^{(\alpha)}(x) \tag{3.2.1}$$

so that from (3.1.8)

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{t^n}{n!} \triangle \beta_n^{(\alpha)}(x) &= t\sum_{n=0}^{\infty} \frac{t^n}{n!} \beta_n^{(\alpha-1)}(x) \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left[\beta_n^{(\alpha)}(x+1) - \beta_n^{(\alpha)}(x)\right] \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} \beta_n^{(\alpha)}(x+1) - \sum_{n=0}^{\infty} \frac{t^n}{n!} \beta_n^{(\alpha)}(x) \\
&= \frac{t^\alpha}{(e^t-1)^\alpha} \left[e^{(x+1)t+g(t)} - e^{xt+g(t)}\right] \\
&= t\sum_{n=0}^{\infty} \frac{t^n}{n!} \beta_n^{(\alpha-1)}(x),
\end{aligned}$$

where

$$\begin{aligned}
\left[\beta_n^{(\alpha)}(x+1) - \beta_n^{(\alpha)}(x)\right] &= \frac{t^\alpha}{(e^t-1)^\alpha} e^{(x+1)t+g(t)} - \frac{t^\alpha}{(e^t-1)^\alpha} e^{xt+g(t)} \\
&= \frac{t^\alpha}{(e^t-1)^\alpha} \left[e^{(x+1)t+g(t)} - e^{xt+g(t)}\right] \\
&= \frac{tt^{\alpha-1}}{(e^t-1)^{\alpha-1}} e^{xt+g(t)} \\
&= t\sum_{n=0}^{\infty} \frac{t^n}{n!} \beta_n^{(\alpha-1)}(x).
\end{aligned}$$

Replacing $n+1$ with n,

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{t^n}{(n-1)!} \beta_{n-1}^{(\alpha-1)}(x) \\
&= \sum_{n=0}^{\infty} \frac{n}{n} \frac{t^n}{(n-1)!} \beta_{n-1}^{(\alpha-1)}(x) \\
&= n\sum_{n=0}^{\infty} \frac{t^n}{n!} \beta_{n-1}^{(\alpha-1)}(x) \\
&= n\beta_{n-1}^{(\alpha-1)}(x).
\end{aligned}$$

Therefore we have

$$\triangle\beta_n^{(\alpha)}(x) = n\beta_{n-1}^{(\alpha-1)}(x). \tag{3.2.3}$$

It is clear that $\triangle$ decreases the order and the degree both by unit one .

Using (3.1.5), we can rewrite (3.2.3) as follows

$$\left(\beta^{(\alpha)} + x + 1\right)^n - \left(\beta^{(\alpha)} + x\right)^n \doteqdot n\left(\beta^{(\alpha-1)} + x\right)^{n-1}.$$

Substituting $x = 0$, we obtain

$$\left(\beta^{(\alpha)} + 1\right)^n - \beta_n^{(\alpha)} \doteqdot n\beta_{n-1}^{(\alpha-1)} \tag{3.2.4}$$

which results a one to one relation between the β numbers of orders α and $\alpha - 1$.

3.3 Definition of Bernoulli Polynomials

The function $e^{xt+g(t)}$ generates the β polynomials of order zero, where if we put $g(t) = 0$ we will obtain the simplest polynomials of this kind e^{xt}. These β polynomials are known as Bernoulli polynomials of order zero which are defined as follows :

$$B_n^{(0)}(x) = x^n. \tag{3.3.1}$$

Therefore we have

$$e^{xt} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n^{(0)}(x) = \sum_{n=0}^{\infty} \frac{(xt)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n^{(0)}(x).$$

Using (3.2.1), we can expand this definition to Bernoulli polynomials of order α given by the identity

$$\frac{t^\alpha e^{xt}}{(e^t-1)^\alpha} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n^{(\alpha)}(x). \tag{3.3.2}$$

Putting $x = 0$, we will obtain

$$\frac{t^\alpha}{(e^t-1)^\alpha} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n^{(\alpha)}.$$

3.3.1 Fundamental Properties of Bernoulli Polynomials

Since β polynomials are ϕ polynomials, so Bernoulli polynomials are also ϕ polynomials. Here are some properties of Generalized Bernoulli polynomials:

$$B_n^{(\alpha)}(x) = \left(B_n^{(\alpha)} + x\right)^n. \tag{3.3.3}$$

$$\frac{d}{dx} B_n^{(\alpha)}(x) = n B_{n-1}^{(\alpha)}(x). \tag{3.3.4}$$

$$\int_a^x B_n^{(\alpha)}(t)dt = \frac{1}{n+1}\left[\beta_{n+1}^{(\alpha)}(x) - \beta_{n+1}^{(\alpha)}(a)\right]. \tag{3.3.5}$$

$$\triangle B_n^{(\alpha)}(x) = nB_{n-1}^{(\alpha-1)}(x). \tag{3.3.6}$$

$$\left(B^{(\alpha)} + 1\right)^n - B_n^{(\alpha)} \doteq nB_{n-1}^{(\alpha-1)}. \tag{3.3.7}$$

Properties (3.3.3), (3.3.4), and (3.3.5) are common in an ϕ polynomials and properties (3.3.6) and (3.3.7) are shared in all β polynomials (see [4]).

For $n \geq \alpha$, repeated applications of property (3.3.6) will give the relation

$$\triangle^{\alpha} B_n^{(\alpha)}(x) = n(n-1)(n-2)...(n-\alpha+1)x^{n-\alpha}.$$

Let us prove the first property. From (3.3.6)

$$\begin{aligned} \triangle^{\alpha-1} \triangle B_n^{(\alpha)}(x) &= \triangle^{\alpha-1} \left(nB_{n-1}^{(\alpha-1)}(x)\right) \\ \triangle^{\alpha-2} \triangle \left(nB_{n-1}^{(\alpha-1)}(x)\right) &= \triangle^{\alpha-2} \left(n(n-1)B_{n-1}^{(\alpha-1)}(x)\right). \end{aligned}$$

Applying $\triangle$, α times yields to

$$\begin{aligned} \triangle^{\alpha} B_n^{(\alpha)}(x) &= n(n-1)(n-2)...(n-\alpha+1)B_{n-\alpha}^{(0)}(x) \\ &= n(n-1)(n-2)...(n-\alpha+1)x^{n-\alpha}. \end{aligned}$$

Here note that $B_n^{(0)}(x) = x^n$. Now if $n < \alpha$, the right-hand will vanish, since we can not have a Bernoulli polynomial of negative degree. Norlund's theory of Bernoulli polynomials arised from relations (3.3.6) and (3.3.7).

There are some useful results gained from (3.3.6) such as the following theorem.

Theorem 3.3.1 *For any integer $n, \alpha \geqslant 0$, we have*

$$B_n^{(\alpha)}(x+1) = B_n^{(\alpha)}(x) + nB_{n-1}^{(\alpha-1)}(x). \tag{3.3.8}$$

Proof. Putting $x = 0$ in equation (3.3.8) yields to

$$B_n^{(\alpha)}(1) = B_n^{(\alpha)} + nB_{n-1}^{(\alpha-1)}.$$

Using (3.3.5) and (3.3.6), we get

$$\int_x^{x+1} B_n^{(\alpha)}(t)dt = \frac{1}{n+1} \triangle B_{n+1}^{(\alpha)}(x) = B_n^{(\alpha-1)}(x). \tag{3.3.9}$$

Replacing $x+1$ by x and x by a in equation (3.3.5)

$$\begin{aligned}
\int_x^{x+1} B_n^{(\alpha)}(t)dt &= \frac{1}{n+1}\left[\beta_{n+1}^{(\alpha)}(x+1) - \beta_{n+1}^{(\alpha)}(x)\right] \\
&= \frac{1}{n+1} \triangle B_{n+1}^{(\alpha)}(x) \\
&= B_n^{(\alpha-1)}(x)
\end{aligned}$$

and in particular

$$\int_0^1 B_n^{(\alpha)}(t)dt = B_n^{(\alpha-1)}. \tag{3.3.10}$$

■

3.3.2 The Complementary Argument Theorem

Theorem 3.3.2 *If the arguments x and $\alpha - x$ are complementary, then*

$$B_n^{(\alpha)}(\alpha - x) = (-1)^n B_n^{(\alpha)}(x). \tag{3.3.11}$$

Proof. Considering (3.3.2),

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{t^n}{n!} B_n^{(\alpha)}(\alpha - x) &= \frac{t^\alpha e^{(\alpha - x)t}}{(e^t - 1)^\alpha} \\
&= \frac{t^\alpha e^{(\alpha - x)t}}{(e^t - 1)^\alpha} \frac{e^{-\alpha t}}{e^{-\alpha t}} \\
&= \frac{t^\alpha e^{-xt}}{(e^t - 1)^\alpha (e^{-t})^\alpha} \\
&= \frac{t^\alpha e^{-xt}}{(1 - e^{-t})^\alpha} \\
&= \frac{(-t)^\alpha e^{-xt}}{(1 - e^{-t})^\alpha} \\
&= \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} B_n^{(\alpha)}(x),
\end{aligned}$$

equating the coefficients of t^n on both sides of (3.2.1), we prove the theorem. ■

The equation (3.2.1) is called the complementary argument theorem. The theorem holds for any β polynomial with an even function as its generating function. Taking $x = 0$, $n = 2\mu$ in (3.3.11), $B_{2\mu}^{(\alpha)}(\alpha) = B_{2\mu}^{(\alpha)}$, which results $x = \alpha$, $x = 0$ as zeros of $B_{2\mu}^{(\alpha)}(x) - B_{2\mu}^{(\alpha)}$.

In a similar way putting $x = \frac{1}{2}\alpha$, $n = 2\mu + 1$in (3.3.11), we have

$$
\begin{aligned}
B^{(\alpha)}_{2\mu+1}(\alpha - \frac{1}{2}\alpha) &= (-1)^{2\mu+1} B^{(\alpha)}_{2\mu+1}(\frac{1}{2}\alpha) \\
B^{(\alpha)}_{2\mu+1}(\frac{1}{2}\alpha) &= -B^{(\alpha)}_{2\mu+1}(\frac{1}{2}\alpha).
\end{aligned}
$$

Resulting that

$$
B^{(n)}_{2\mu+1}(\frac{1}{2}n) = 0. \tag{3.3.12}
$$

3.3.3 The Relation between Polynomials of Successive Orders

The following theorem gives the relation between the polynomials of successive orders.

Theorem 3.3.3 *For any integer $n, \alpha \geqslant 0$, we have*

$$
B^{(\alpha+1)}_n(x) = \left(1 - \frac{n}{\alpha}\right) B^{(\alpha)}_n(x) + n\left(\frac{x}{\alpha} - 1\right) B^{(\alpha)}_{n-1}(x).
$$

Proof. Differentiating both sides of the equality below

$$
\sum_{n=0}^{\infty} \frac{t^n}{n!} B^{(\alpha)}_n(x) = \frac{t^\alpha e^{xt}}{(e^t - 1)^\alpha} \tag{3.3.13a}
$$

and then multiplying it by t, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{t^n}{(n-1)!}B_n^{(\alpha)}(x)\\
&= \frac{t\left[\left(\alpha t^{\alpha-1}e^{xt}+t^{\alpha}xe^{xt}\right)\left(e^t-1\right)^{\alpha}\right]-\left[tt^{\alpha}e^{xt}\alpha\left(e^t-1\right)^{\alpha-1}e^t\right]}{\left(e^t-1\right)^{2\alpha}}\\
&= \frac{\left[\alpha t^{\alpha}e^{xt}\left(e^t-1\right)^{\alpha}\right]+\left[t^{\alpha+1}xe^{xt}\left(e^t-1\right)^{\alpha}\right]-\left[t^{\alpha+1}e^{t(x+1)}\alpha\left(e^t-1\right)^{\alpha-1}\right]}{\left(e^t-1\right)^{2\alpha}}\\
&= \frac{\alpha t^{\alpha}e^{xt}\left(e^t-1\right)^{\alpha}}{\left(e^t-1\right)^{2\alpha}}+\frac{\left[t^{\alpha+1}xe^{xt}\left(e^t-1\right)^{\alpha}\right]}{\left(e^t-1\right)^{2\alpha}}-\frac{t^{\alpha+1}e^{(x+1)t}\alpha\left(e^t-1\right)^{\alpha-1}}{\left(e^t-1\right)^{2\alpha}}\\
&= \frac{\alpha t^{\alpha}e^{xt}}{\left(e^t-1\right)^{\alpha}}+\frac{xt^{\alpha+1}e^{xt}}{\left(e^t-1\right)^{\alpha}}-\frac{\alpha t^{\alpha+1}e^{(x+1)t}}{\left(e^t-1\right)^{\alpha+1}}\\
&= \alpha\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n^{(\alpha)}(x)+xt\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n^{(\alpha)}(x)-\alpha\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n^{(\alpha+1)}(x+1).
\end{aligned}$$

By equating the coefficients of t^n we have

$$nB_n^{(\alpha)}(x)=\alpha B_n^{(\alpha)}(x)+xnB_{n-1}^{(\alpha)}(x)-\alpha B_n^{(\alpha+1)}(x+1). \tag{3.3.14}$$

Using (3.3.8) we obtain

$$B_n^{(\alpha+1)}(x+1)=B_n^{(\alpha+1)}(x)+nB_{n-1}^{(\alpha)}(x). \tag{3.3.15}$$

Therefore we get

$$B_n^{(\alpha+1)}(x)=\left(1-\frac{n}{\alpha}\right)B_n^{(\alpha)}(x)+n\left(\frac{x}{\alpha}-1\right)B_{n-1}^{(\alpha)}(x) \tag{3.3.16}$$

which demonstrates a relation between Bernoulli polynomials of order α and $\alpha+1$, as required. ■

Now letting $x = 0$, we also have

$$B_n^{(\alpha+1)} = \left(1 - \frac{n}{\alpha}\right) B_n^{(\alpha)} - nB_{n-1}^{(\alpha)}(x).$$

Again putting $x = 0$ in equation (3.3.14), we get

$$B_n^{(\alpha+1)}(1) = \left(1 - \frac{n}{\alpha}\right) B_n^{(\alpha)}, \tag{3.3.18}$$

$$\begin{aligned}
nB_n^{(\alpha)}(0) &= \alpha B_n^{(\alpha)}(0) - \alpha B_n^{(\alpha+1)}(1) \\
nB_n^{(\alpha)} &= \alpha B_n^{(\alpha)} - \alpha B_n^{(\alpha+1)}(1) \\
\alpha B_n^{(\alpha+1)}(1) &= \alpha B_n^{(\alpha)} - nB_n^{(\alpha)} \\
B_n^{(\alpha+1)}(1) &= \frac{(\alpha - n)B_n^{(\alpha)}}{\alpha} \\
&= \left(1 - \frac{n}{\alpha}\right) B_n^{(\alpha)},
\end{aligned}$$

replacing α by $n + \alpha$ in (3.3.18), we have

$$B_n^{(n+\alpha+1)}(1) = \frac{\alpha}{n+\alpha} B_n^{(n+\alpha)}. \tag{3.3.19}$$

3.3.4 Relation of Bernoulli Polynomials to Factorials

Letting $n = \alpha$ in (3.3.16), we obtain

$$\begin{aligned}
B_n^{(\alpha+1)}(x) &= (x-\alpha)B_{\alpha-1}^{(\alpha)}(x) = (x-\alpha)(x-\alpha+1)B_{\alpha-2}^{(\alpha-1)}(x) = \dots \\
&= (x-\alpha)(x-\alpha+1)\dots(x-2)(x-1)B_0^{(1)}(x).
\end{aligned}$$

Putting α for n in equation (3.3.16)

$$\begin{aligned} B_{\alpha}^{(\alpha+1)}(x) &= \left(1-\frac{\alpha}{\alpha}\right) B_{\alpha}^{(\alpha)}(x) + \alpha\left(\frac{x}{\alpha}-1\right) B_{\alpha-1}^{(\alpha)}(x) \\ &= (x-\alpha)B_{\alpha-1}^{(\alpha)}(x). \end{aligned}$$

Now putting $\alpha - 1$ for α at right side,

$$\begin{aligned} B_{\alpha}^{(\alpha+1)}(x) &= (x-\alpha)(x-\alpha+1)B_{\alpha-2}^{(\alpha-1)}(x) = \dots \\ &= (x-\alpha)(x-\alpha+1)...(x-2)(x-1)B_{0}^{(1)}(x). \end{aligned}$$

Therefore

$$B_{\alpha}^{(\alpha+1)}(x) = (x-1)(x-2)\dots(x-\alpha) = (x-1)^{(\alpha)}. \tag{3.3.20}$$

Putting $x+1$ for x in equation (3.3.20),

$$B_{\alpha}^{(\alpha+1)}(x+1) = x(x-1)(x-2)\dots(x-\alpha+1) = x^{\alpha}. \tag{3.3.21}$$

Taking integral from 0 to 1 in both of the above equations and considering (3.3.10), we get

$$\int_0^1 (x-1)(x-2)...(x-\alpha)dx = B_{\alpha}^{(\alpha)}.$$

Using relation between (3.3.10) and (3.3.21) and from (3.3.15)

$$\int_0^1 (x-1)(x-2)\dots(x-\alpha+1)dx = B_{\alpha}^{(\alpha)}(1) = -\frac{1}{\alpha-1}B_{\alpha}^{(\alpha-1)}.$$

Note that the above equation will be the same we put $\alpha = -1$ in (3.3.19).

Using (3.3.4) and differentiating (3.3.20) $\alpha - n$ times ($\alpha \geqslant n$), we obtain,

$$\alpha(\alpha-1)\ldots(\alpha-\alpha+n+1)B_n^{(\alpha+1)}(x) = \frac{d^{\alpha-n}}{dx^{\alpha-n}}(x-1)^{\alpha}$$

which represents an expression for $B_n^{(\alpha+1)}(x)$ as below

$$B_n^{(\alpha+1)}(x) = \frac{n!}{\alpha!}\frac{d^{\alpha-n}}{dx^{\alpha-n}}\left[(x-1)(x-2)\ldots(x-\alpha)\right].$$

In Stirling's and Bessel's interpolation formula, we have the following coefficients:

1. $a_{2s+1}(p) = \binom{p+s}{2s+1}$

2. $a_{2s}(p) = \frac{p}{2s}\binom{p+s-1}{2s-1}$

3. $b_{2s+1}(p) = \frac{p-\frac{1}{2}}{(2s+1)}\binom{p+s-1}{2s}$

4. $b_{2s}(p) = \binom{p+s-1}{2s}$

5. $a_{2s+1}(p) = \frac{1}{(2s+1)!}B_{2s+1}^{(2s+2)}(p+s+1)$

We will now prove the above properties. To prove the property on $a_{2s}(p)$,we rewrite equation (3.3.20) as

$$
\begin{aligned}
B_{\alpha}^{(\alpha+1)}(x) &= (x-1)(x-2)...(x-\alpha) \\
&= \frac{(x-1)(x-2)\ldots(x-\alpha)(x-\alpha-1)\ldots 3.2.1}{(x-\alpha-1)!} \\
&= \frac{(x-1)!}{(x-\alpha-1)!} \\
&= \alpha!\binom{x-1}{\alpha}.
\end{aligned} \tag{3.3.22}
$$

Substitute $x = p+s+1$, $\alpha = 2s+1$, $\alpha+1 = 2s+2$, in (3.3.22) to get

$$
\begin{aligned}
B_{2s+1}^{(2s+2)}(p+s+1) &= (2s+1)!\binom{p+s+1-1}{2s+1} \\
&= (2s+1)!\binom{p+s}{2s+1}.
\end{aligned}
$$

Now, substitute $\alpha = 2s-1$, $\alpha+1 = 2s$, $x = p+s$ in (3.3.22) to get

$$
B_{2s-1}^{(2s)}(p+s) = (2s-1)!\binom{p+s-1}{2s-1}.
$$

Use the above relation below to obtain the desired property

$$
\begin{aligned}
a_{2s}(p) &= \frac{p}{(2s)!}(2s-1)!\binom{p+s-1}{2s-1} \\
&= \frac{p}{2s}\binom{p+s-1}{2s-1}.
\end{aligned}
$$

Let us prove the property (3) on $b_{2s+1}(p)$.Substitute $x = p+s$, $n = 2s$, $n+1 =$ $2s+1$in equation (3.3.22),

$$
B_{2s}^{(2s+1)}(p+s) = (2s)!\binom{p+s-1}{2s}.
$$

Then, use the above relation below,

$$\begin{aligned}
b_{2s+1}(p) &= \frac{p-\frac{1}{2}}{(2s+1)!}B_{2s}^{(2s+1)}(p+s) \\
& \frac{p-\frac{1}{2}}{(2s+1)!}(2s)!\binom{p+s-1}{2s} \\
&= \frac{p-\frac{1}{2}}{(2s+1)}\binom{p+s-1}{2s}
\end{aligned}$$

which proves the desired property.

Next, we prove the property on $b_{2s}(p)$. Using the relation

$$b_{2s}(p) = \frac{1}{(2s)!}B_{2s}^{(2s+1)}(p+s) \tag{3.3.23}$$

and substituting $x = p+s$, $\alpha = 2s$, $\alpha+1 = 2s+1$ in equation (3.3.22), we easily obtain

$$B_{2s}^{(2s+1)}(p+s) = (2s)!\binom{p+s-1}{2s}.$$

Using the above result in (3.3.23),

$$\begin{aligned}
b_{2s}(p) &= \frac{1}{(2s)!}(2s)!\binom{p+s-1}{2s} \\
&= \binom{p+s-1}{2s}.
\end{aligned}$$

Hence the property is obtained.

Differentiating each of these coefficients m times with respect to p and then putting $p=0$ in $a_{2s+1}(p)$ and $a_{2s}(p)$ and also putting $p=\frac{1}{2}$ in $b_{2s+1}(p)$ and

$b_{2s}(p)$, we get

$$D^m a_{2s+1}(0) = \frac{1}{(2s-m+1)!} B_{2s-m+1}^{(2s+2)}(s+1). \qquad (3.3.24)$$

From (3.3.4),

$$\begin{aligned}
\frac{d}{dp} a_{2s+1}(p) &= \frac{d}{dp}\left[\frac{1}{(2s+1)!} B_{2s+1}^{(2s+2)}(p+s+1)\right], \\
D\left(a_{2s+1}(p)\right) &= \frac{2s+1}{(2s+1)!} B_{2s}^{(2s+2)}(p+s+1) \\
&= \frac{1}{(2s)!} B_{2s}^{(2s+2)}(p+s+1).
\end{aligned}$$

Differentiating once more,

$$\begin{aligned}
D^2\left(a_{2s+1}(p)\right) &= D\left[\frac{1}{(2s)!} B_{2s}^{(2s+2)}(p+s+1)\right] \\
&= \frac{2s}{(2s)!} B_{2s-1}^{(2s+2)}(p+s+1) \\
&= \frac{1}{(2s-1)!} B_{2s-1}^{(2s+2)}(p+s+1).
\end{aligned}$$

Repeatedly, we obtain

$$D^m\left(a_{2s+1}(p)\right) = \frac{1}{(2s-m+1)!} B_{2s-m+1}^{(2s+2)}(p+s+1). \qquad (3.3.25)$$

Now putting $p = 0$ in (3.3.25),

$$D^m\left(a_{2s+1}(0)\right) = \frac{1}{(2s-m+1)!} B_{2s-m+1}^{(2s+2)}(s+1). \qquad (3.3.26)$$

Now, let us obtain the relation,

$$D^m a_{2s}(0) = \frac{m}{2s(2s-m)!} B_{2s-m}^{(2s)}(s). \tag{3.3.27}$$

From (3.3.4),

$$\frac{d}{dp}\left(a_{2s}(p)\right) = \frac{d}{dp}\left[\frac{1}{(2s)!} p B_{2s-1}^{(2s)}(p+s)\right],$$

thus

$$\begin{aligned}
D(a_{2s}(p)) &= \frac{1}{(2s)!}\left[B_{2s-1}^{(2s)}(p+s) + p(2s-1)B_{2s-2}^{(2s)}(p+s)\right]. \\
D^2(a_{2s}(p)) &= \frac{1}{(2s)!} \\
&\times \left[(2s-1)B_{2s-2}^{(2s)}(p+s) + (2s-1)B_{2s-2}^{(2s)}(p+s)\right. \\
&+ \left. p(2s-1)(2s-2)B_{2s-3}^{(2s)}(p+s)\right] \\
&= \frac{1}{(2s)!}\left[2(2s-1)B_{2s-2}^{(2s)}(p+s) + p(2s-1)(2s-2)B_{2s-3}^{(2s)}(p+s)\right] \\
&= \frac{1}{(2s)!}(2s-1)\left[2(B_{2s-2}^{(2s)}(p+s)) + p(2s-2)B_{2s-3}^{(2s)}(p+s)\right] \\
&= \frac{1}{(2s)(2s-2)!}\left[2(B_{2s-2}^{(2s)}(p+s)) + p(2s-2)B_{2s-3}^{(2s)}(p+s)\right].
\end{aligned}$$

Repeatedly,

$$D^m(a_{2s}(p)) = \frac{1}{(2s)(2s-m)!}\left[m(B_{2s-2}^{(2s)}(p+s)) + p(2s-2)B_{2s-3}^{(2s)}(p+s)\right].$$

Now place $p = 0$ in the last equation

$$D^m(a_{2s}(0)) = \frac{m}{(2s)(2s-m)!} B_{2s-2}^{(2s)}(s).$$

Whence the result. The next result is about the mth derivative of the coefficient $b_{2s+1}(p)$ at $p = \frac{1}{2}$ which is,

$$D^m b_{2s+1}(\frac{1}{2}) = \frac{m}{(2s+1)(2s-m+1)!} B_{2s-m+1}^{(2s+1)}(s+\frac{1}{2}). \tag{3.3.28}$$

From property (3)

$$\begin{aligned}
\frac{d}{dp} b_{2s+1}(p) &= \frac{d}{dp}\left(\frac{p-\frac{1}{2}}{(2s+1)!} B_{2s}^{(2s+1)}(p+s)\right) \\
D\left(b_{2s+1}(p)\right) &= \frac{1}{(2s+1)!}\left[B_{2s}^{(2s+1)}(p+s) + \left(p-\frac{1}{2}\right)(2s)B_{2s-1}^{(2s+1)}(p+s)\right] \\
D^2\left(b_{2s+1}(p)\right) &= \frac{1}{(2s+1)!}\left[(2s)B_{2s-1}^{(2s+1)}(p+s) + (2s)B_{2s-1}^{(2s+1)}(p+s)\right] \\
&+ \frac{1}{(2s+1)!}\left[\left(p-\frac{1}{2}\right)(2s)(2s-1)B_{2s-2}^{(2s+1)}(p+s)\right] \\
&= \frac{1}{(2s+1)!}(2s)\left[2B_{2s-1}^{(2s+1)}(p+s) + \left(p-\frac{1}{2}\right)(2s-1)B_{2s-2}^{(2s+1)}(p+s)\right] \\
&= \frac{1}{(2s+1)(2s-1)!}\left[2B_{2s-1}^{(2s+1)}(p+s) + \left(p-\frac{1}{2}\right)(2s-1)B_{2s-2}^{(2s+1)}(p+s)\right]
\end{aligned}$$

Repeatedly,

$$\begin{aligned}
D^m\left(b_{2s+1}(p)\right) &= \frac{1}{(2s+1)(2s-m+1)!} \\
&\times \left[mB_{2s-m+1}^{(2s+1)}(p+s) + \left(p-\frac{1}{2}\right)(2s-m+1)B_{2s-m}^{(2s+1)}(p+s)\right].
\end{aligned}$$

We will put $p = \frac{1}{2}$ in last equation to get,

$$\begin{aligned} D^m\left(b_{2s+1}(\frac{1}{2})\right) &= \frac{1}{(2s+1)(2s-m+1)!} \\ &\quad \times \left[mB^{(2s+1)}_{2s-m+1}(\frac{1}{2}+s) + \left(\frac{1}{2}-\frac{1}{2}\right)(2s-m+1)B^{(2s+1)}_{2s-m}(\frac{1}{2}+s)\right] \\ &= \frac{m}{(2s+1)(2s-m+1)!}B^{(2s+1)}_{2s-m+1}(\frac{1}{2}+s). \end{aligned}$$

To prove the relation below

$$D^m b_{2s}(\frac{1}{2}) = \frac{1}{(2s-m)!}B^{(2s+1)}_{2s-m}(s+\frac{1}{2}), \tag{3.3.29}$$

we use equation (3.3.23) to have,

$$\begin{aligned} \frac{d}{dp}\left(b_{2s}(p)\right) &= \frac{d}{dp}\left(\frac{1}{(2s)!}B^{(2s+1)}_{2s}(p+s)\right) \\ D\left(b_{2s}(p)\right) &= \frac{1}{(2s)!}(2s)B^{(2s+1)}_{2s-1}(p+s) \\ D^2\left(b_{2s}(p)\right) &= \frac{1}{(2s)!}\left[(2s)(2s-1)B^{(2s+1)}_{2s-2}(p+s)\right] \\ &= \frac{(2s)(2s-1)}{(2s)(2s-1)(2s-2)!}B^{(2s+1)}_{2s-2}(p+s) \\ &= \frac{1}{(2s-2)!}B^{(2s+1)}_{2s-2}(p+s). \end{aligned}$$

Repeatedly,

$$D^m\left(b_{2s}(p)\right) = \frac{1}{(2s-m)!}B^{(2s+1)}_{2s-m}(p+s).$$

We will now put $p = \frac{1}{2}$ in last equation to obtain

$$D^m\left(b_{2s}(\frac{1}{2})\right) = \frac{1}{(2s-m)!}B^{(2s+1)}_{2s-m}(\frac{1}{2}+s),$$

which gives the desired relation.

From equation (3.3.26),

$$D^m a_{2s+1}(0) = \frac{1}{(2s-m+1)!} B_{2s-m+1}^{(2s+2)}(s+1)$$

and

$$D^{2m} a_{2s+1}(0) = \frac{1}{(2s-2m+1)!} B_{2s-2m+1}^{(2s+2)}(s+1).$$

Using (3.3.12)

$$D^{2m} a_{2s+1}(0) = 0.$$

From equation (3.3.27),

$$D^m a_{2s}(0) = \frac{m}{2s(2s-m)!} B_{s-m}^{(2s)}(s)$$

and

$$D^{2m+1} a_{2s}(0) = \frac{2m+1}{2s(2s-2m-1)!} B_{2s-2m-1}^{(2s)}(s).$$

Using (3.3.12), we obtain

$$D^{2m+1} a_{2s}(0) = 0.$$

By equation (3.3.28), $D^m b_{2s+1}(\frac{1}{2})$ and $D^{2m} b_{2s+1}(\frac{1}{2})$ become,

$$D^m b_{2s+1}(\frac{1}{2}) = \frac{m}{(2s+1)(2s-m+1)!} B^{(2s+1)}_{2s-m+1}(s+\frac{1}{2})$$

and

$$D^{2m} b_{2s+1}(\frac{1}{2}) = \frac{2m}{(2s+1)(2s-2m+1)!} B^{(2s+1)}_{2s-2m+1}(s+\frac{1}{2}),$$

respectively.

Now from equation (3.3.12),

$$D^{2m} b_{2s+1}(\frac{1}{2}) = 0.$$

From (3.3.29),

$$D^m b_{2s}(\frac{1}{2}) = \frac{1}{(2s-m)!} B^{(2s+1)}_{2s-m}(s+\frac{1}{2})$$

and

$$D^{2m+1} b_{2s}(\frac{1}{2}) = \frac{1}{(2s-2m+1)!} B^{(2s+1)}_{2s-2m+1}(s+\frac{1}{2}).$$

Using (3.3.12), we get

$$D^{2m+1} b_{2s}(\frac{1}{2}) = 0.$$

3.3.4.1 The Integral of the Factorial

An important function in the theory of numerical integration is the $\chi(x)$ function :

$$\begin{aligned} \chi(x) &= \int_{1-k}^{x} (y-1)(y-2)...(y-2n+1)dy \quad (3.3.30) \\ &= \int_{1-k}^{x} B_{2n-1}^{(2n)}(y)dy. \quad (3.3.31) \end{aligned}$$

The above integral results with the following expression,

$$\int_{1-k}^{x} B_{2n-1}^{(2n)}(y)dy = \frac{1}{2n}\left[B_{2n}^{(2n)}(x) - B_{2n}^{(2n)}(1-k)\right], \quad (3.3.32)$$

where k is either zero or unit. Considering the Complemantary Argument Theorem, substituting $n = 2n$, $\alpha = 2n$ in (3.3.11) and using (3.3.30), we have

$$\begin{aligned} B_{2n}^{(2n)}(2n-(-k+1)) &= (-1)^{2n}B_{2n}^{(2n)}(1-k) \\ &= B_{2n}^{(2n)}(1-k). \end{aligned}$$

Also, substituting the above equalities in (3.3.30), we have

$$\begin{aligned} \chi(2n+k-1) &= \frac{B_{2n}^{(2n)}(2n+k-1) - B_{2n}^{(2n)}(1-k)}{2n} \\ &= \frac{B_{2n}^{(2n)}(1-k) - B_{2n}^{(2n)}(1-k)}{2n} \\ &= 0. \end{aligned}$$

Resulting that $\chi(2n+k-1) = \chi(1-k) = 0$.

Using Complementary Argument Theorem mentioned in (3.3.11) and letting $x = 1-k$, we also have the following relation

$$\begin{aligned} B_{2n+1}^{(2n+1)}(2n+1-x) &= (-1)^{2n+1} B_{2n+1}^{(2n+1)}(1-k) \\ &= -B_{2n+1}^{(2n+1)}(1-k). \end{aligned}$$

That is,

$$\begin{aligned} \int_{1-k}^{2n+k-1} 1 \cdot \chi(x)dx &= -\int_{1-k}^{2n+k-1} x(x-1)(x-2)\ldots(x-2n+1)dx \\ &= \frac{B_{2n+1}^{(2n+1)}(2-k) + B_{2n+1}^{(2n+1)}(1-k)}{2n+1}. \end{aligned} \tag{3.3.33}$$

Rewriting the left hand side of the above equation leads to the following integral,

$$\int_{1-k}^{2n+k-1} 1 \cdot \chi(x)dx = \int_{1-k}^{2n+k-1} \int_{1-k}^{x} (y-1)(y-2)...(y-2n+1)dydx.$$

Letting $u = \chi(x)$ and $dv = 1dx$ and using integration by parts method we have,

$$\begin{aligned} \textstyle\int_{1-k}^{2n+k-1} 1 \cdot \chi(x)dx = &\ \Big[x \cdot \textstyle\int_{1-k}^{x}(y-1)(y-2)\ldots(y-2n+1)dy \,\Big|_{1-k}^{2n+k-1} \\ - &\ \textstyle\int_{1-k}^{2n+k-1} x(x-1)(x-2)\ldots(x-2n+1)dx \\ = &\ [x \cdot \chi(x)] \,|_{1-k}^{2n+k-1} - \textstyle\int_{1-k}^{2n+k-1} x(x-1)(x-2)\ldots(x-2n+1)dx \\ = &\ [(2n+k-1) \cdot \chi\,(2n+k-1) \cdot \chi\,(1-k)] \\ - &\ \textstyle\int_{1-k}^{2n+k-1} x(x-1)(x-2)\ldots(x-2n+1)dx. \end{aligned}$$

Since $\chi(2n+k-1) = \chi(1-k) = 0$,

$$\int_{1-k}^{2n+k-1} 1 \cdot \chi(x)dx = -\int_{1-k}^{2n+k-1} x(x-1)(x-2)\ldots(x-2n+1)dx.$$

Substituting $2n$ instead of n in (3.3.21), the above integral will be equal to the integral given below

$$\int_{1-k}^{2n+k-1} 1.\chi(x)dx = -\int_{1-k}^{2n+k-1} B_{2n}^{(2n+1)}(x+1)dx.$$

From (3.3.5),

$$-\int_{1-k}^{2n+k-1} B_{2n}^{(2n+1)}(x+1)dx = -\frac{1}{2n+1}\left[B_{2n+1}^{(2n+1)}(2n+k) - B_{2n+1}^{(2n+1)}(2-k)\right],$$

which proves the relation (3.3.33).

It is easy to verify that $B_{2n-1}^{(2n-1)} = \int_0^1 (y-1)(y-2)...(y-2n+1)dy$ is negative, while

$$B_{2n-1}^{(2n-1)}(1) = \int_1^2 (y-1)(y-2)...(y-2n+1)dy$$

is positive.

Continuing in this way, when υ is an integer, $0 \leq \upsilon \leq 2n$,

$$(-1)^{\upsilon+1} B_{2n-1}^{(2n-1)}(\upsilon) \geqslant 0.$$

At this point we will prove the following inequality,

$$(-1)^{\upsilon} B_{2n-1}^{(2n-1)}(\upsilon-1) \geqslant (-1)^{\upsilon+1} B_{2n-1}^{(2n-1)}(\upsilon), \qquad 1 \leq \upsilon \leq n-1. \qquad (3.3.35)$$

From (3.3.9), we have

$$B_{2n-1}^{(2n-1)}(\upsilon-1)=\int_{\upsilon-1}^{\upsilon}(y-1)(y-2)\dots(y-2n+1)dy, \tag{3.3.36}$$

and

$$\begin{aligned}
B_{2n-1}^{(2n-1)}(\upsilon) &= \int_{\upsilon}^{\upsilon+1}(y-1)(y-2)\dots(y-2n+1)dy \\
&= \int_{\upsilon-1}^{\upsilon} y(y-1)(y-2)\dots(y-2n+2)dy \\
B_{2n-1}^{(2n-1)}(\upsilon) &= -\int_{\upsilon-1}^{\upsilon}\frac{y}{2n-y-1}(y-1)(y-2)\dots(y-2n+1)dy. \quad (3.3.37)
\end{aligned}$$

Since $\upsilon-1\le y\le \upsilon$ and $\upsilon\le n-1$, we have $y\le n-\frac{1}{2}$. Thus $y/(2n-y-1)$ is positive and less than unity. Considering (3.3.36) and (3.3.37) it is clear that the absolute value of the integrand of (3.3.37) is less than the absolute value of the integrand of (3.3.36), which results (3.3.35).According to the above explanations we will show that the function $\chi(x)$ has a fixed sign whenever

$$1-k\le x\le 2n+k-1.$$

Let $\upsilon-1\le x\le \upsilon$. If $\upsilon-1\le y\le \upsilon$, then the sign of the integrand of (3.3.30) will not change. Hence,

$$\int_{1-k}^{\upsilon-1}(y-1)(y-2)\dots(y-2n+1)dy\le\chi(x)\le\int_{1-k}^{\upsilon}(y-1)(y-2)\dots(y-2n+1)dy.$$

If we rewrite each integral using the intervals $(1-k,2-k)$, $(2-k,3-k)$,. . ., we will realize that $\chi(x)$ is between the following two sums.

$$B_{2n-1}^{(2n-1)}(1-k)+B_{2n-1}^{(2n-1)}(2-k)+\ldots+B_{2n-1}^{(2n-1)}(\upsilon-2),$$

$$B_{2n-1}^{(2n-1)}(1-k)+B_{2n-1}^{(2n-1)}(2-k)+\ldots+B_{2n-1}^{(2n-1)}(\upsilon-1).$$

According to the Complementary Argument theorem, we will consider $\upsilon \leq n$, when the terms satisfy this condition.

Considering (3.3.35) the absolute magnitude of the terms in the above sums are in descending order and their signs alternate. Thus the sign of each sum is the same as the sign of the first term, namely, $B_{2n-1}^{(2n-1)}(1-k)$.

Following the above applications we proved that $B_{2n}^{(2n)}(x)-B_{2n}^{(2n)}$ has no zeros in $0 \leq x \leq 2n$, and

$$\left|B_{2n-1}^{(2n-1)}(1-k)\right| \geq \left|B_{2n-1}^{(2n-1)}(2-k)\right|, \qquad \text{for } k=0 \text{ or } 1.$$

3.3.4.2 Expansion of x^{α} in Powers of x

Having differentiated (3.3.21) p times, we have

$$\frac{d^p}{dx^p}x^{(\alpha)}=n^{(p)}B_{\alpha-p}^{(\alpha+1)}(x+1).$$

To prove the above relation, we differentiate (3.3.21) with respect to x, that is

$$
\begin{aligned}
B_{\alpha}^{(\alpha+1)}(x+1) &= x(x-1)(x-2)\ldots(x-\alpha+1) = x^{(\alpha)} \\
D\left(B_{\alpha}^{(\alpha+1)}(x+1)\right) &= \alpha B_{\alpha-1}^{(\alpha+1)}(x+1) \\
D^2 &= D\left(\alpha B_{\alpha-1}^{(\alpha+1)}(x+1)\right) = \alpha(\alpha-1)B_{\alpha-2}^{(\alpha+1)}(x+1) \\
D^3 &= D^2\left(\alpha(\alpha-1)B_{\alpha-2}^{(\alpha+1)}(x+1)\right) = \alpha(\alpha-1)(\alpha-2)B_{\alpha-3}^{(\alpha+1)}(x+1).
\end{aligned}
$$

Repeatedly,

$$
D^p = D^{p-1}\left(\alpha(\alpha-1)B_{\alpha-2}^{(\alpha+1)}(x+1)\right) = \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-p+1)B_{\alpha-p}^{(\alpha+1)}(x+1).
$$

Hence,

$$
\frac{d^p}{dx^p}x^{(\alpha)} = \alpha^{(p)}B_{\alpha-p}^{(\alpha+1)}(x+1).
$$

Now, letting $x=0$, using (3.3.18) and (3.3.21), we obtain

$$
\begin{aligned}
\left[\frac{d^p}{dx^p}x^{(\alpha)}\right]_{x=0} &= \alpha^{(p)}B_{\alpha-p}^{(\alpha+1)}(1) \\
&= \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-p+1).(\alpha-p)!}{(\alpha-p)!}B_{\alpha-p}^{(\alpha+1)}(1) \\
&= \frac{\alpha!}{(\alpha-p)!}B_{\alpha-p}^{(\alpha+1)}(1) \\
&= \frac{\alpha!}{(\alpha-p)!}\frac{p}{\alpha}B_{\alpha-p}^{(\alpha)}.
\end{aligned}
$$

By applying Maclaurin's Theorem on $x^{(\alpha)}$, we have

$$
x^{(\alpha)} = \sum_{p=0}^{\alpha}\frac{x^p}{p!}\frac{\alpha!}{(\alpha-p)!}\frac{p}{\alpha}B_{\alpha-p}^{(\alpha)} = \sum_{p=0}^{\alpha}\binom{\alpha-1}{p-1}x^pB_{\alpha-p}^{(\alpha)}.
$$

3.3.4.3 Expansion of x^{υ} in Factorials

By Newton's Interpolation formula which is

$f(x) = f(a)+p\triangle f(a)+\binom{p}{2}\triangle^2 f(a)+\binom{p}{3}\triangle^3 f(a)+...+\binom{p}{n-1}\triangle^{n-1} f(a)+\binom{p}{n}\omega^n f^{(n)}(\xi)$

and knowing that $B_n^{(\alpha)}(x+h)$ is a polynomial of degree n from (3.3.6), we get

$$\begin{aligned} B_n^{(\alpha)}(x+h) &= B_n^{(\alpha)}(h) + \sum_{s=1}^{\upsilon} \frac{x^{(s)}}{s!} \triangle^s B_n^{(\alpha)}(h) \qquad (3.3.38) \\ &= \sum_{s=0}^{n} \binom{n}{s} x^{(s)} B_{n-s}^{(\alpha-s)}(h). \end{aligned}$$

By putting $h = 0$ in (3.3.38), we have a factorial series for $B_n^{(\alpha)}(x)$,

$$B_n^{(\alpha)}(x) = \sum_{s=0}^{n} \binom{n}{s} x^{(s)} B_{n-s}^{(\alpha-s)}.$$

One can easily see that, having $n = 0$ gives $B_n^{(0)}(x) = x$ which yields to the following required expansion

$$x^n = \sum_{s=0}^{n} \binom{n}{s} x^{(s)} B_{n-s}^{(-s)}. \qquad (3.3.40)$$

Operating $\triangle^{\alpha}$ on (3.3.40) ($\alpha \leq n$), we will obtain the differences of zeros having taken $x = 0$,

$$\triangle^{\alpha} 0^n = \frac{n!}{(n-\alpha)!} B_{n-\alpha}^{(-\alpha)}.$$

Taking $\alpha = n+1$, replacing h by $h+1$ in (3.3.38) and using (3.3.21) leads to,

$$(x+h)^n = \sum_{s=0}^{n} \binom{n}{s} x^{(s)} h^{(n-s)}$$

which will be used in follows

$$\begin{aligned} B_n^{(n+1)}(x+h+1) &= \sum_{s=0}^{n} \binom{n}{s} x^{(s)} B_{n-s}^{(n+1-s)}(h+1) \\ &= \sum_{s=0}^{n} \binom{n}{s} x^{(s)} B_{n-s}^{(n-s+1)}(h+1) \\ &= \sum_{s=0}^{n} \binom{n}{s} x^{(s)} h^{(n-s)}. \end{aligned}$$

The above relation is the well known Vandermonde's theorem in factorials and demonstrates an analogou to the Binomial Theorem as below

$$(x+h)^n = \sum_{s=0}^{n} \binom{n}{s} x^s h^{n-s}.$$

Also interchanging x and h in (3.3.38), we get

$$\frac{B_n^{(\alpha)}(x+h) - B_n^{(\alpha)}(x)}{h} = \sum_{s=1}^{n} \binom{n}{s} (h-1)^{(s-1)} B_{n-s}^{(\alpha-s)}(x). \qquad (3.3.41)$$

Taking limit when $h \to 0$ we will get the derivative of $B_n^{(\alpha)}(x)$ on the left-hand side

$$nB_{n-1}^{(\alpha)}(x) = \sum_{s=1}^{n} \binom{n}{s} (-1)^{(s-1)} (s-1)! B_{n-s}^{(\alpha-s)}(x)$$

we will use (3.3.4) and (3.3.41) and the definition of derivative,

$$\begin{aligned}
\frac{B_n^{(\alpha)}(x+h)-B_n^{(\alpha)}(x)}{h} &= \sum_{s=1}^{n}\tbinom{n}{s}(h-1)^{(s-1)}B_{n-s}^{(\alpha-s)}(x) \\
\frac{d}{dx}B_n^{(\alpha)}(x) &= \sum_{s=1}^{n}\tbinom{n}{s}(h-1)^{(s-1)}B_{n-s}^{(\alpha-s)}(x) \\
nB_{n-1}^{(\alpha)}(x) &= \sum_{s=1}^{n}\tbinom{n}{s}(-1)^{(s-1)}(s-1)!B_{n-s}^{(\alpha-s)}(x).
\end{aligned}$$

Specially, when $x=0$

$$nB_{n-1}^{(\alpha)} = \sum_{s=1}^{n}\tbinom{n}{s}(-1)^{(s-1)}(s-1)!B_{n-s}^{(\alpha-s)}.$$

3.3.4.4 Generating Functions of Bernoulli Numbers

Using Binomial Theorem, we have

$$\begin{aligned}
(1+t)^{x-1} &= \sum_{n=0}^{x-1}\frac{(x-1)(x-2)\dots(x-n)}{n!}t^n = \sum_{n=0}^{\infty}\frac{t^n}{n!}B_n^{(n+1)}(x) \\
&= \sum_{n=0}^{x-1}\tbinom{x-1}{n}t^n(1)^{\alpha-n} \\
&= \sum_{n=0}^{x-1}\tbinom{x-1}{n}t^n = \sum_{n=0}^{x-1}\frac{(x-1)!}{n!(x-1-n)!}t^n \\
&= \sum_{n=0}^{x-1}\frac{(x-1)(x-2)\dots(x-n)(x-1-n)!}{n!(x-1-n)!}t^n \\
&= \sum_{n=0}^{x-1}\frac{(x-1)(x-2)\dots(x-n)}{n!}t^n \\
&= \sum_{n=0}^{x-1}\frac{t^n}{n!}B_n^{(n+1)}(x).
\end{aligned}$$

Having differentiated the above equation α times with respect to x, we get

$$(1+t)^{x-1}\left[\log(1+t)\right]^{\alpha} = \sum_{n=\alpha}^{\infty} \frac{t^n}{(\alpha-n)!} B_{n-\alpha}^{(n+1)}(x).$$

Letting $x=1$ and then dividing by t^{α}, we obtain

$$\begin{aligned} \left[\frac{\log(1+t)}{t}\right]^{\alpha} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n^{(\alpha+n+1)}(1) && (3.3.42) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\alpha}{\alpha+n} B_n^{(\alpha+n)}. && (3.3.43) \end{aligned}$$

$$\begin{aligned} (1+t)^{x-1}\left[\log(1+t)\right]^{\alpha} &= \sum_{n=\alpha}^{\infty} \frac{t^n}{(n-\alpha)!} B_{n-\alpha}^{(n+1)}(x) \\ \frac{(1+t)^{1-1}\left[\log(1+t)\right]^{\alpha}}{t^{\alpha}} &= \sum_{n=\alpha}^{\infty} \frac{\frac{t^n}{(n-\alpha)!} B_{n-\alpha}^{(n+1)}(1)}{t^{\alpha}} \\ \left[\frac{\log(1+t)}{t}\right]^{\alpha} &= \sum_{v=0}^{\infty} \frac{t^n}{t^n(n-\alpha)!} B_{n-\alpha}^{(n+1)}(1) \\ &= \sum_{n=0}^{\infty} \frac{t^{n-\alpha}}{(n-\alpha)!} B_{n-\alpha}^{(n+1)}(1) \\ &= \sum_{n=0}^{\infty} \frac{t^{n+n-n}}{(n+\alpha-\alpha)!} B_{n+\alpha-\alpha}^{(n+\alpha+1)}(1) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n^{(n+\alpha+1)}(1) \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\alpha}{\alpha+n} B_n^{(\alpha+n)}. \end{aligned}$$

In particular, for $\alpha = 1$,

$$\frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!} B_n^{(n+1)}.$$

Integrating $(1+t)^{x-1}$ α times with respect to x from x to $x+1$ and using (3.3.9), we have

$$\frac{(1+t)^{x-1}t^{\alpha}}{[\log(1+t)]^{\alpha}}=\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n^{(n-\alpha+1)}(x).$$

Taking $x=0$, we get

$$\frac{t^{\alpha}}{(1+t)\,[\log(1+t)]^{\alpha}}=\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n^{(n-\alpha+1)}.$$

In a particular case where $\alpha=1$, we get the generating function of $B_n^{(\alpha)}$ numbers as below

$$\frac{t}{(1+t)\log(1+t)}=\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n^{(n)}.$$

Now putting $x=1$, we obtain

$$\left[\frac{t}{\log(1+t)}\right]^{\alpha}=\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n^{(n-\alpha+1)}(1)$$

resulting that the equation (3.3.42) holds for negative α's as well.

In particular for $\alpha=1$, we will obtain the generating function of the numbers $B_n^{(n)}(1)$,

$$\frac{t}{\log(1+t)}=\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n^{(n)}(1). \tag{3.3.44}$$

Using (3.3.19) on (3.3.44), we get

$$\frac{t}{\log(1+t)} = 1 + \frac{1}{2}t - \sum_{n=2}^{\infty} \frac{t^n}{n!} \frac{B_n^{(n-1)}}{n-1}.$$

Below the first ten of the $B_n^{(n)}$ numbers are listed :

$$\begin{aligned}
B_1^{(1)} &= -\frac{1}{2}, & B_6^{(6)} &= \frac{19087}{84}, \\
B_2^{(2)} &= \frac{5}{6}, & B_7^{(7)} &= -\frac{36799}{24}, \\
B_3^{(3)} &= -\frac{9}{4}, & B_8^{(8)} &= \frac{1070017}{90}, \\
B_4^{(4)} &= \frac{251}{30}, & B_9^{(9)} &= -\frac{2082753}{20}, \\
B_5^{(5)} &= -\frac{475}{12}, & B_{10}^{(10)} &= \frac{134211265}{132}.
\end{aligned}$$

3.3.5 Bernoulli Polynomials of the First Order

In the rest of the thesis we will write $B_n(x)$ instead of $B_n^{(1)}(x)$, having in mind that the order is one. Therefore from (3.3.2), we get the below function as the generating function of the Bernoulli polynomials :

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(x). \tag{3.3.45}$$

The generating function of Bernoulli numbers, B_n of the first order are shown as below

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n. \tag{3.3.46}$$

Using the properties given in the subsection (3.2.1), the following properties are satisfied

$$B_n(x) \doteqdot (B+x)^n. \tag{3.3.47}$$

Putting $\alpha = 0$ in equation (3.3.3)

$$B_n^{(0)}(x) = \left(B^{(0)} + x\right)^n,$$

hence

$$B_n(x) \doteqdot (B+x)^n.$$

Now, we will prove the following property.

$$(B+1)^n - B_n \doteqdot 0, \qquad n = 2, 3, 4, \dots \ . \tag{3.3.48}$$

We put $x = 0$, $\alpha = 0$ in equation (3.3.15),

$$\begin{aligned}
B_n^{(0+1)}(0+1) &= B_n^{(0+1)}(0) + nB_{n-1}^{(0)}(0) \\
B_n(1) &= B_n + nB_{n-1}^{(0)} \\
B_n^{(1)}(1) - B_n^{(1)} &= 0 \\
(B+1)^n - B_n &\doteqdot 0, \qquad n = 2, 3, 4, \dots .
\end{aligned}$$

Another property is related with differentiation.

$$\frac{d}{dx}B_n(x) = nB_{n-1}(x). \tag{3.3.49}$$

To prove this, we will differentiate both sides of equation (3.3.45) with respect to x.

That is,

$$\begin{aligned}
\frac{d}{dx}\left(\frac{te^{xt}}{e^t-1}\right) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n(x)\right) \\
\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{d}{dx}\left(B_n(x)\right) &= \frac{tte^{xt}}{e^t-1} \\
&= t\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n(x) \\
&= \sum_{n=0}^{\infty}\frac{t^{n+1}}{n!}B_n(x) \\
&= \sum_{n=0}^{\infty}(n+1)\frac{t^{n+1}}{(n+1)!}B_n(x) \\
&= \sum_{n=0}^{\infty}n\frac{t^n}{(n)!}B_{n-1}(x).
\end{aligned}$$

Thus

$$\frac{d}{dx}\left(B_n(x)\right) = nB_{n-1}(x).$$

The following is the integral representation of $B_n(x)$

$$\int_a^x B_n(t)dt = \frac{1}{n+1}\left[B_{n+1}(x) - B_{n+1}(a)\right]. \tag{3.3.50}$$

By integrating both sides of equation (3.3.45), we have

$$\begin{aligned}
\int_a^x \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(t)\right) dt &= \int_a^x \left(\frac{te^{ty}}{e^t - 1}\right) dy \\
\sum_{n=0}^{\infty} \frac{t^n}{n!} \int_a^x B_n(t) dt &= \frac{e^{xt}}{e^t - 1} - \frac{e^{at}}{e^t - 1} \\
&= \sum_{n=0}^{\infty} \frac{t^{n-1}}{n!} B_n(x) - \sum_{n=0}^{\infty} \frac{t^{n-1}}{n!} B_n(a) \\
&= \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!} B_{n+1}(x) - \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!} B_{n+1}(a) \\
&= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left[\frac{1}{n+1} \left(B_{n+1}(x) - B_{n+1}(a)\right)\right].
\end{aligned}$$

Thus,

$$\int_a^x B_n(t) dt = \frac{1}{n+1} \left[B_{n+1}(x) - B_{n+1}(a)\right].$$

Next, consider the difference operator.

$$\triangle B_n(x) = nx^{n-1}. \tag{3.3.51}$$

From (3.3.15),

$$\begin{aligned}
B_n^{(\alpha+1)}(x+1) &= B_n^{(\alpha+1)}(x) + nB_n^{(\alpha+1)}(x) \\
B_n^{(\alpha+1)}(x+1) - B_n^{(\alpha+1)}(x) &= nB_n^{(\alpha+1)}(x) \\
\triangle B_n^{(\alpha+1)}(x) &= nB_n^{(\alpha+1)}(x).
\end{aligned}$$

Putting $\alpha = 0$,

$$\begin{aligned}\triangle B_n^{(1)}(x) &= nB_n^{(1)}(x) \\ \triangle B_n(x) &= nx^{n-1}.\end{aligned}$$

Below the first seven polynomials are listed

$$\begin{aligned}
B_0(x) &= 1, \\
B_1(x) &= x - \frac{1}{2}, \\
B_2(x) &= x^2 - x + \frac{1}{6}, \\
B_3(x) &= x(x-1)(x-2) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \\
B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}, \\
B_5(x) &= x(x-1)(x-\frac{1}{2})(x^2 - x - \frac{1}{3}) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x, \\
B_6(x) &= x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}.
\end{aligned}$$

Also the values for the first seven numbers are :

B_0	B_1	B_2	B_3	B_4	B_5	B_6
1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$

.

3.3.5.1 A Summation Problem

By (3.3.50) and (3.3.51), we have

$$\begin{aligned}\int_s^{s+1} B_n(x)dx &= \frac{1}{n+1}\left[B_{n+1}(s+1) - B_{n+1}(s)\right] \\ &= s^n.\end{aligned}$$

One can easily show the obove relation as follows:

$$\begin{aligned}
\int_s^{s+1} B_n(x)dx &= \frac{1}{n+1}\left[B_{n+1}(s+1) - B_{n+1}(s)\right] \\
&= \frac{1}{n+1} \triangle B_{n+1}(s) \\
&= \frac{1}{n+1}(n+1)s^n \\
&= s^n.
\end{aligned}$$

Also,

$$\sum_{s=1}^{\alpha} s^n = \int_0^{\alpha+1} B_n(x)dx = \frac{1}{n+1}\left[B_{n+1}(\alpha+1) - B_{n+1}\right].$$

As an example for $n = 3$, we have

$$\begin{aligned}
\sum_{s=1}^{\alpha} s^3 &= \frac{1}{4}\left[B_4(\alpha+1) - B_4\right] \\
&= \frac{1}{4}\left[(\alpha+1)^4 - 2(\alpha+1)^3 + (\alpha+1) - \frac{1}{30} - \left(-\frac{1}{30}\right)\right] \\
&= \frac{1}{4}\left[(\alpha+1)^4 - 2(\alpha+1)^3 + (\alpha+1)\right] \\
&= \left[\frac{1}{2}\alpha(\alpha+1)\right]^2.
\end{aligned}$$

3.3.5.2 Bernoulli Numbers of the First Order

(3.3.46) results

$$\frac{t}{2} + \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n = \frac{t}{2} \cdot \frac{e^t+1}{e^t-1} \tag{3.3.52}$$

we will use (3.3.46)

$$\begin{aligned}
\frac{t}{2}+\sum_{n=0}^{\infty}\frac{t^n}{n!}B_n &= \frac{t}{2}+\frac{t}{e^t-1}\\
&= \frac{t(e^t-1)+2t}{2(e^t-1)}\\
&= \frac{t(e^t-1+2)}{2(e^t-1)}\\
&= \frac{t}{2}\cdot\frac{e^t+1}{e^t-1}.
\end{aligned}$$

Since changing t to $-t$ does not make change the function on the right-hand side, the function on the right is even. Therefore the expansion above does not contain any odd powers of t and hence

$$B_{2\mu+1}=0, \qquad \mu>0, \tag{3.3.53}$$

$$B_1=-\frac{1}{2}.$$

Replacing t by $2t$ in (3.3.52) we get

$$\begin{aligned}
\frac{2t}{2}+\sum_{n=0}^{\infty}\frac{(2t)^n}{n!}B_n &= \frac{2t}{2}\cdot\frac{e^{2t}+1}{e^{2t}-1}\\
&= t\cdot\frac{e^{2t}+1}{e^{2t}-1}\\
&= 1+t+2tB_1+\frac{2^2t^2}{2!}B_2+\frac{2^3t^3}{3!}B_3+\dots\ .
\end{aligned}$$

From (3.3.53)

$$t+\sum_{n=0}^{\infty}\frac{2^nt^n}{n!}B_n=1+t+\frac{2^2t^2}{2!}B_2+\frac{2^4t^4}{4!}B_4\dots\ .$$

Writing it instead of t we get

$$t \coth = 1 - \frac{2^2 t^2}{2!} B_2 + \frac{2^4 t^4}{4!} B_4 - \dots \ . \tag{3.3.54}$$

Similarly, one can easily obtain expansions for $\csc t$ and $\tan t$ as follows.

$$\begin{aligned}
\csc t &= \cot \frac{1}{2} t - \cot t, \\
\tan t &= \cot t - 2 \cot 2t \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_{2n} t^{2n-1}.
\end{aligned}$$

We have another expansion in partial fractions

$$\pi t \coth \pi t = 1 + 2t^2 \sum_{n=1}^{\infty} \frac{1}{t^2 - n^2}. \tag{3.3.55}$$

Rearranging these series and comparing with the coefficients of t^{2p} in (3.3.55), and also considering the series for $\pi t \coth \pi t$ in (3.3.54),

$$(-1)^{p-1} \frac{(2\pi)^p B_{2p}}{2(2p)!} = \sum_{\alpha=1}^{\infty} \frac{1}{\alpha^{2p}}. \tag{3.3.56}$$

It is clear to see that the summation on the right hand side of (3.3.56) lies between 1 and 2. Thus, as p increases, B_{2p} increases rapidly and also the Bernoulli numbers alternate in sign. Furthermore, we have

$$(-1)^{p-1} B_{2p} > 0.$$

In order to express the Bernoulli numbers using determinants, we use (3.3.48)

$$
\begin{aligned}
\frac{1}{2!}+\frac{B_1}{1!} &= 0,\\
\frac{1}{3!}+\frac{1}{2!}\frac{B_1}{1!}+\frac{1}{1!}\frac{B_2}{2!} &= 0,\\
\vdots \quad &\vdots \quad \vdots\\
\frac{1}{(\alpha+1)!}+\frac{1}{\alpha!}\frac{B_1}{1!}+\frac{1}{(\alpha-1)!}\frac{B_2}{2!}+\ldots+\frac{1}{2!}\frac{B_{n-1}}{(\alpha-1)!}+\frac{B_\alpha}{\alpha!} &= 0,
\end{aligned}
$$

Solving the above equations, for $\frac{(-1)^\alpha B_\alpha}{\alpha!}$, we have the following determinant

:

$$
\begin{vmatrix}
\frac{1}{2!} & 1 & 0 & 0 & \cdots & 0\\
\frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \cdots & 0\\
\frac{1}{4!} & \frac{1}{3!} & 1 & 0 & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots\\
\frac{1}{(\alpha+1)!} & \frac{1}{\alpha!} & \frac{1}{(\alpha-1)!} & \frac{1}{(\alpha-2)!} & \cdots & \frac{1}{2!}
\end{vmatrix}.
$$

We have

$$
\begin{aligned}
B_n(x)+\frac{1}{2}nx^{n-1} &\doteqdot (x+B)^n+\frac{1}{2}nx^{n-1}\\
&= x^n+\tbinom{n}{2}x^{n-2}B_2+\tbinom{n}{4}x^{n-4}B_4+\ldots,
\end{aligned}
$$

so that $B_n(x)+\frac{1}{2}nx^{n-1}$ is an even function when n is even and odd function whenever n is odd, since $B_{2\mu+1}=0,\ \ \mu>0$.

3.3.5.3 The Euler-Maclaurin Theorem for Polynomials

Suppose that $P(x)$ is an polynomial of degree α.

Using (3.3.48) and (3.3.51) we have,

$$\begin{aligned} nx^{n-1} &= \triangle B_n(x) \\ &= B_n(x+1) - B_n(x) \\ &= (B+x+1)^n - (B+x)^n, \end{aligned}$$

resulting that

$$P'(x) \doteqdot P(x+B+1) - P(x+B), \tag{3.3.57}$$

and therefore

$$\begin{aligned} P'(x+y) &\doteqdot P(x+y+B+1) - P(x+y+B) \\ &\doteqdot P(x+1+B(y)) - P(x+B(y)). \end{aligned} \tag{3.3.58}$$

Applying Taylor's Theorem, we have

$$P(x+B(y)) \doteqdot P(x)+B_1(y)P'(x)+\frac{1}{2!}B_2(y)P''(x)+...+\frac{1}{\alpha!}B_\alpha(y)P^{(\alpha)}(x). \tag{3.3.59}$$

Substituting equation in (3.3.58) and (3.3.59), we will get the Euler-Maclaurin

Theorem for polynomials as follows:

$$
\begin{aligned}
P'(x+y) &\doteqdot P(x+1+B(y)) \\
&\quad - \left[P(x) + B_1(y)P'(x) + \frac{1}{2!}B_2(y)P''(x) + \ldots + \frac{1}{\alpha!}B_\alpha(y)P^{(\alpha)}(x)\right] \\
&= \Bigg[P(x+1) + B_1(y)P'(x+1) + \frac{1}{2!}B_2(y)P''(x+1) \\
&\quad + \ldots + \frac{1}{\alpha!}B_\alpha(y)P^{(\alpha)}(x+1)\Bigg] \\
&\quad - \left[P(x) + B_1(y)P'(x) + \frac{1}{2!}B_2(y)P''(x) + \ldots + \frac{1}{\alpha!}B_\alpha(y)P^{(\alpha)}(x)\right] \\
&= [P(x+1) - P(x)] + B_1(y)\left[P'(x+1) - P'(x)\right] \\
&\quad + \frac{1}{2!}B_2(y)\left[P''(x+1) - P''(x)\right] + \ldots + \\
&\quad + \frac{1}{\alpha!}B_\alpha(y)\left[P^{(\alpha)}(x+1) - P^{(\alpha)}(x)\right] \\
&= \triangle P(x) + B_1(y) \triangle P'(x) + \frac{1}{2!}B_2(y) \triangle P''(x) + \ldots + \frac{1}{\alpha!}B_\alpha(y) \triangle P^{(\alpha)}(x)
\end{aligned}
$$

which gives us the Euler-Maclaurin theorem for polynomials.

In the special case when $y = 0$,

$$
P'(x) = \triangle P(x) + B_1 \triangle P'(x) + \frac{1}{2!}B_2 \triangle P''(x) + \frac{1}{4!}B_4 \triangle P^{(iv)}(x) + \ldots + \frac{1}{\alpha!}B_\alpha \triangle P^{(\alpha)}(x), \tag{3.3.60}
$$

where $B_3, B_5, B_7, \ldots$ all vanish. Now considering $P(x)$ as :

$$
P(x) = \int_x^a \phi(t)dt.
$$

Integrating both sides of equation (3.3.60) we get,

$$
\phi(x) = \int_x^{x+1} \phi(t)dt + B_1 \triangle \phi(x) + \frac{1}{2!}B_2 \triangle \phi'(x) + \frac{1}{4!}B_4 \triangle \phi'''(x) + \ldots \tag{3.3.61}
$$

Since $B_1 = -\frac{1}{2}$, we have the following equation for any polynomial $\phi(x)$

$$\int_x^{x+1} \phi(t)dt = \frac{1}{2}\left[\phi(x+1) + \phi(x)\right] - \frac{1}{2!}B_2 \triangle \phi'(x) - \frac{1}{4!}B_4 \triangle \phi'''(x) - ... \ .$$

One can easily prove the above equation by substituting

$$\begin{aligned}
\int_x^{x+1} \phi(t)dt &= -\left[B_1 \triangle \phi(x) + \frac{1}{2!}B_2 \triangle \phi'(x) + \frac{1}{4!}B_4 \triangle \phi'''(x) + ...\right] \\
&= -B_1 \triangle \phi(x) - \frac{1}{2!}B_2 \triangle \phi'(x) - \frac{1}{4!}B_4 \triangle \phi'''(x) - \dots \\
&= \frac{1}{2}\left[\phi(x+1) - \phi(x)\right] - \frac{1}{2!}B_2 \triangle \phi'(x) - \frac{1}{4!}B_4 \triangle \phi'''(x) - \dots \ .
\end{aligned}$$

After a finite number of terms, the series on the right-hand side will terminate.

The equation (3.3.57) shows that

$$u(x) \doteqdot P(x+B) \doteqdot P(B(x)) \tag{3.3.62}$$

is the polynomial solution of the difference equation below :

$$\triangle u(x) = P'(x). \tag{3.3.63}$$

Thus, as an example consider

$$u(x) = \frac{1}{4}B_4(x) - B_3(x) + B_1(x) + c,$$

$$\triangle u(x) = x^3 - 3x^2 + 1,$$

having c as an arbitrary constant.

We use (3.3.63), (3.3.62) and (3.3.47)

$$\triangle u(x) == P'(x) = x^3 - 3x^2 + 1.$$

We then integrate both sides to get

$$\begin{aligned}
P(x) &= \frac{x^4}{4} - x^3 + x + c \\
P(x+B) &= u(x) = \frac{(x+B)^4}{4} - (x+B)^3 + (x+B) + c \\
&= \frac{1}{4}B_4(x) - B_3(x) + B_1(x) + c.
\end{aligned}$$

Replacing c by an arbitrary periodic function $\varpi(x)$, we will get the general solution as :

$$\varpi(x+1) = \varpi(x).$$

3.3.5.4 The Multiplication Theorem

Considering m as a positive integer in (3.3.45), we have,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{s=0}^{m-1} B_n\left(x + \frac{s}{m}\right) &= \sum_{m=0}^{m-1} \frac{te^{\left(x+\frac{s}{m}\right)t}}{e^t - 1} \\
&= \frac{te^{xt}(e^t - 1)}{(e^t - 1)(e^{\frac{t}{m}} - 1)} = \frac{m\frac{t}{m}e^{mx\frac{t}{m}}}{e^{\frac{t}{m}} - 1} \\
&= \sum_{n=0}^{\infty} m.\frac{t^n}{m^n} B_n(mx).
\end{aligned}$$

Thus,

$$B_n(mx) = m^{n-1} \sum_{s=0}^{m-1} B_n\left(x + \frac{s}{m}\right).$$

The above result is the well-known multiplication theorem for Bernoulli polynomials of order one.

Letting $x = 0$, we get

$$\sum_{s=1}^{m-1} B_n\left(\frac{s}{m}\right) = -\left(1 - \frac{1}{m^{n-1}}\right) B_n.$$

Therefore, if $m = 2$,

$$B_n\left(\frac{1}{2}\right) = -\left(1 - \frac{1}{2^{n-1}}\right) B_n, \qquad n = 1, 2, \dots \ .$$

3.3.5.5 Bernoulli Polynomials in the Interval $(0, 1)$

Recall that

$$B_{2n}(1-x) = B_{2n}(x)$$

and

$$B_{2n+1}(1-x) = -B_{2n+1}(x). \tag{3.3.64}$$

We use the relation $B_n(1-x) = (-1)^n B_n(x)$ and write $2n+1$ for n to get,

$$\begin{aligned} B_{2n+1}(1-x) &= (-1)^{2n+1} B_{2n+1}(x) \\ &= -B_{2n+1}(x). \end{aligned}$$

So, the zeros of $B_{2n}(x) - B_{2n}$ are 0 and 1. We must prove that they are the only zeros in $[0,1]$.

Letting $x = \frac{1}{2}$ in (3.3.64), we get $B_{2n+1}\left(\frac{1}{2}\right) = 0$ and also using (3.3.64) we realize that $B_{2n+1}(x)$ is symmetric around $x = \frac{1}{2}$, thus for $n < 0$, $B_{2n+1}(x)$ has the zeros $0, \frac{1}{2}, 1$. Our aim is to prove that these are the only zeros in $[0,1]$.

Therefore, including $n = \mu > 0$, we suppose that both statements are true.

$B_{2\mu+2}(x) - B_{2\mu+2}$, vanishes at $x = 0$, $x = 1$. Also for $0 < x < 1$, its' minimum or maximum occurs only at $x = \frac{1}{2}$.

Since

$$D\left[B_{2\mu+2}(x) - B_{2\mu+2}\right] = (2\mu+2)\, B_{2\mu+1}(x). \tag{3.3.65}$$

Therefore, it cannot vanish in $(0,1)$.

In a similar way,

$$DB_{2\mu+3}(x) = (2\mu+3)\left[B_{2\mu+2}(x) - B_{2\mu+2}\right] + (2\mu+3)\, B_{2\mu+2}$$

where in the interval $0 < x < \frac{1}{2}$ the above equation can vanish at most once.

So $B_{2\mu+3}(x)$ cannot vanish in $0 < x < \frac{1}{2}$ and consequently can not vanish in $\frac{1}{2} < x < 1$ by (3.3.64).

The properties will now follow by induction.

Recall that $(-1)^{n+1} B_{2n} > 0$. $(-1)^{n+1} B_{2n+1}(x)$ will have the same sign as its derivative whenever x is sufficiently small and positive which means that it will have the same sign as $(-1)^{n+1} B_{2n}$, which also has the same sign as $(-1)^{n+1} B_{2n}$ and is positive. Hence

$$(-1)^{n+1} B_{2n+1}(x) > 0, \qquad 0 < x < \frac{1}{2}.$$

As x increases from 0 to $\frac{1}{2}$, from (3.3.65) $(-1)^{\mu+1} \left(B_{2\mu+2}(x) - B_{2\mu+2}\right)$ will also exceed 0, and therefore is positive. Since the expression above vanishes only at 0 and 1, we get

$$(-1)^{\mu+1} \left(B_{2\mu+2}(x) - B_{2\mu+2}\right) > 0, \qquad 0 < x < 1.$$

3.3.6 The η Polynomials

(3.1.9) is another method for generalizing polynomials. Writing $f_\alpha(t) = 2^\alpha (e^t + 1)^{-\alpha}$ a new class of polynomials will be formed which are named η polynomials and are denoted by

$$\frac{2^\alpha e^{xt+g(t)}}{(e^t+1)^\alpha} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \eta_n^{(\alpha)}(x), \tag{3.3.66}$$

so that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \nabla \eta_n^{(\alpha)}(x) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \eta_n^{(\alpha-1)}(x). \tag{3.3.67}$$

Replacing η for ϕ and writing $f_\alpha(t) = 2^\alpha (e^t+1)^{-\alpha}$ in equation (3.1.9)

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!} \nabla \eta_n^{(\alpha)}(x) &= \frac{e^t+1}{2} \frac{2^\alpha}{(e^t+1)^n} e^{xt+g(t)} \\ &= \frac{2^{\alpha-1}}{(e^t+1)^{\alpha-1}} e^{xt+g(t)}. \end{aligned}$$

From (3.3.66),

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!} \nabla \eta_n^{(\alpha)}(x) &= \frac{2^{\alpha-1}}{(e^t+1)^{\alpha-1}} e^{xt+g(t)} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \eta_n^{(\alpha-1)}(x). \end{aligned}$$

Therefore we have

$$\nabla \eta_n^{(\alpha)}(x) = \eta_n^{(\alpha-1)}(x). \tag{3.3.68}$$

As we can see, ∇ decreases the order by one unit and makes no changes in the degree.

Using (3.1.5), we get

$$\left(\eta^{(\alpha)} + x + 1\right)^n + \left(\eta^{(\alpha)} + x\right)^n \doteqdot 2\left(\eta^{(\alpha-1)} + x\right)^n \tag{3.3.69}$$

where η satisfies the recurrence relation below

$$\left(\eta^{(\alpha)}+1\right)^{n}+\eta_n^{(\alpha)} \doteqdot 2\eta_n^{(\alpha-1)}. \tag{3.3.70}$$

3.3.7 Definition of Euler Polynomial

Letting $g(t)=0$ and $\alpha=0$ in the generating function, we get the simplest η polynomials with e^{xt} as their generating function. These η polynomials are the powers of x. These polynomials are also called Euler polynomials of order zero. Therefore

$$E_n^{(0)}(x)=x^n \tag{3.3.71}$$

and

$$e^{xt}=\sum_{n=0}^{\infty}\frac{t^n}{n!}E_n^{(0)}(x),$$

which is simply obtained as follows

$$\begin{aligned} e^{xt} &= \sum_{n=0}^{\infty}\frac{(xt)^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{x^n t^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{t^n}{n!}E_n^{(0)}(x). \end{aligned}$$

Here $E_n^{(0)}(x)$ is the Euler polynomial of degree n and order zero.

The Euler polynomials of order α are defined as below using (3.3.66),

$$\frac{2^{\alpha}e^{xt}}{(e^t+1)^{\alpha}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} E_n^{(\alpha)}(x). \tag{3.3.72}$$

Generally Euler numbers are the values of $E_n^{(\alpha)}(0)$. In order to avoid confusion with Nörlund's notation for polynomials, we will use the notations below as Nörlund did well

$$E_n^{(\alpha)}(0) = 2^{-n} C_n^{(\alpha)}. \tag{3.3.73}$$

Thus we give the generating function for the C numbers

$$\begin{aligned} \frac{2^{\alpha}}{(e^t+1)^{\alpha}} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{1}{2^n} C_n^{(\alpha)} \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} E_n^{(\alpha)}(x). \end{aligned}$$

We put $x = 0$ in equation (3.3.72) and (3.3.73),

$$\begin{aligned} \frac{2^{\alpha}e^{0.t}}{(e^t+1)^{\alpha}} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} E_n^{(\alpha)}(0) \\ \frac{2^{\alpha}}{(e^t+1)^{\alpha}} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{1}{2^n} C_n^{(\alpha)}. \end{aligned}$$

Substituting $x = \frac{1}{2}\alpha$ in $2^n E_n^{(\alpha)}(x)$, we get the Euler numbers of order α, $E_n^{(\alpha)}$ as,

$$E_n^{(\alpha)} = 2^n E_n^{(\alpha)}(\frac{1}{2}\alpha).$$

Euler numbers with an odd suffix vanish as we shown in (3.3.73).

3.3.7.1 Fundamental Properties of Euler Polynomials

As we know Euler polynomials are η polynomials and hence ϕ polynomials. Therefore

$$E_n^{(\alpha)} \doteqdot \left(\frac{1}{2}C^{(\alpha)} + x\right)^n \tag{3.3.74}$$

and

$$\frac{d}{dx}E_n^{(\alpha)}(x) = nE_{n-1}^{(\alpha)}(x). \tag{3.3.75}$$

We differentiate both sides of equation (3.3.72),

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{d}{dx}E_n^{(\alpha)}(x) &= \frac{d}{dx}\left(\frac{2^\alpha e^{xt}}{(e^t+1)^\alpha}\right) \\
&= \frac{t2^\alpha e^{xt}}{(e^t+1)^\alpha} \\
&= t\sum_{n=0}^{\infty}\frac{t^n}{n!}E_n^{(\alpha)}(x) \\
&= \sum_{n=0}^{\infty}\frac{t^{n+1}}{n!}E_n^{(\alpha)}(x).
\end{aligned}$$

We put $(n+1)$ instead of n to obtain the relation (3.3.75),

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{d}{dx}E_n^{(\alpha)}(x) &= \sum_{n=0}^{\infty}(n+1)\frac{t^{n+1}}{(n+1)!}E_n^{(\alpha)}(x) \\
&= \sum_{n=0}^{\infty}n\frac{t^n}{n!}E_{n-1}^{(\alpha)}(x).
\end{aligned}$$

Let us prove the following equation representing the integral representation for generalized Euler polynomials

$$\int_a^x E_n^{(\alpha)}(t)dt = \frac{1}{n+1}\left[E_{n+1}^{(\alpha)}(x) - E_{n+1}^{(\alpha)}(a)\right]. \tag{3.3.76}$$

We will integrate both sides of equation (3.3.72)

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_a^x E_n^{(\alpha)}(t)dt &= \int_a^x \frac{2^\alpha e^{yt}}{(e^t+1)^\alpha}dy \\
&= \frac{2^\alpha}{(e^t+1)^\alpha}\frac{1}{t}\left[e^{xt} - e^{at}\right] \\
&= \frac{1}{t}\frac{2^\alpha e^{xt}}{(e^t+1)^\alpha} - \frac{1}{t}\frac{2^\alpha e^{at}}{(e^t+1)^\alpha} \\
&= \frac{1}{t}\sum_{n=0}^{\infty}\frac{t^n}{n!}E_n^{(\alpha)}(x) - \frac{1}{t}\sum_{n=0}^{\infty}\frac{t^n}{n!}E_n^{(\alpha)}(a) \\
&= \sum_{n=0}^{\infty}\frac{t^{n-1}}{n!}E_n^{(\alpha)}(x) - \sum_{n=0}^{\infty}\frac{t^{n-1}}{n!}E_n^{(\alpha)}(a).
\end{aligned}$$

We replace $n-1$ with n,

$$\begin{aligned}
& \sum_{n=0}^{\infty}\frac{t^n}{(n+1)!}E_{n+1}^{(\alpha)}(x) - \sum_{n=0}^{\infty}\frac{t^n}{(n+1)!}E_{n+1}^{(\alpha)}(a) \\
&= \frac{1}{n+1}\left[\sum_{n=0}^{\infty}\frac{t^n}{n!}E_{n+1}^{(\alpha)}(x) - \sum_{n=0}^{\infty}\frac{t^n}{n!}E_{n+1}^{(\alpha)}(a)\right], \\
\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_a^x E_n^{(\alpha)}(t)dt &= \frac{1}{n+1}\left[\sum_{n=0}^{\infty}\frac{t^n}{n!}E_{n+1}^{(\alpha)}(x) - \sum_{n=0}^{\infty}\frac{t^n}{n!}E_{n+1}^{(\alpha)}(a)\right].
\end{aligned}$$

Thus, from (3.3.69) and (3.3.73),

$$\nabla E_n^{(\alpha)}(x) = E_n^{(\alpha-1)}(x) \tag{3.3.77}$$

and

$$\left(\frac{1}{2}C^{(\alpha)}+1\right)^n+\frac{1}{2^n}C_n^{(\alpha)} \doteqdot \frac{2}{2^n}C_n^{(\alpha-1)}.$$

Replacing E for η in equation (3.3.70) proves (3.3.74).

Theorem 3.3.4 *a)* $\left(E^{(\alpha)}+1\right)^n+E_n^{(\alpha)} \doteqdot 2E_n^{(\alpha-1)}$

b) $E_n^{(\alpha)}(1)+E_n^{(\alpha)}=2E_n^{(\alpha-1)}$

c) $\left(\frac{1}{2}C^{(\alpha)}+1\right)^n+\frac{1}{2^n}C_n^{(\alpha)} \doteqdot \frac{2}{2^n}C_n^{(\alpha-1)}$.

Using (3.3.73) and (3.1.5),

$$\left(C^{(\alpha)}+2\right)^n+C_n^{(\alpha)} \doteqdot 2C_n^{(\alpha-1)}. \tag{3.3.78}$$

Using (3.3.77) repeatedly,

$$\nabla^\alpha = E_n^{(\alpha)}(x)=x^n \tag{3.3.79}$$

since $E_n^{(0)}(x)=x^n$.

Applying ∇ to both sides of equation (3.3.77) once more,

$$\begin{aligned}\nabla^2 E_n^{(\alpha)}(x) &= \nabla E_n^{(\alpha-1)}(x)\\ &= E_n^{(\alpha-2)}(x).\end{aligned}$$

Applying difference operator consecutively gives,

$$\begin{aligned}\nabla^{\alpha} E_n^{(\alpha)}(x) &= E_n^{(\alpha-\alpha)}(x)\\ &= E_n^{(0)}(x)\\ &= x^n.\end{aligned}$$

Again using (3.3.77)

$$E_n^{(\alpha)}(x+1) = 2E_n^{(\alpha-1)}(x) - E_n^{(\alpha)}(x), \tag{3.3.80}$$

$$\begin{aligned}E_n^{(\alpha-1)}(x) &= \nabla E_n^{(\alpha)}(x)\\ &= \frac{E_n^{(\alpha)}(x+1) + E_n^{(\alpha)}(x)}{2},\\ 2E_n^{(\alpha-1)}(x) &= E_n^{(\alpha)}(x+1) + E_n^{(\alpha)}(x).\end{aligned}$$

Since

$$E_n^{(\alpha)} = 2^n E_n^{(\alpha)}\left(\frac{\alpha}{2}\right) \doteq \left(\frac{1}{2}C^{(\alpha)} + \frac{\alpha}{2}\right)^n 2^n, \tag{3.3.81}$$

using (3.3.74) we will get

$$E^{(n)} \doteq n + C^{(n)}.$$

Equation (3.3.81) yields to the following,

$$\begin{aligned} E_n^{(\alpha)} &= 2^n \left(\frac{1}{2}C^{(\alpha)} + \frac{\alpha}{2}\right)^n = 2^n \left(\frac{1}{2}\left(C^{(\alpha)} + \alpha\right)\right)^n \\ &= 2^n \frac{1}{2^n}\left(C^{(\alpha)} + \alpha\right)^n \\ &= \left(C^{(\alpha)} + n\right)^n. \end{aligned}$$

Therefore

$$\begin{aligned} E_n^{(\alpha)}(x) &\doteqdot \left(x - \frac{1}{2}\alpha + \frac{1}{2}E^{(\alpha)}\right)^n, \\ E_n^{(n)}\left(\frac{\alpha + x}{2}\right) &\doteqdot \left(\frac{x + E^{(\alpha)}}{2}\right)^n. \end{aligned} \tag{3.3.82}$$

Letting $x = 1$ and $x = -1$ in terns and then adding them up, we have

$$\begin{aligned} \left(E^{(\alpha)} + 1\right)^n + \left(E^{(\alpha)} - 1\right)^n &\doteqdot 2^n E_n^{(\alpha)}\left(\frac{\alpha + 1}{2}\right) + 2^n E_n^{(\alpha)}\left(\frac{\alpha - 1}{2}\right) \\ &\doteqdot 2^{n+1} \nabla E_n^{(\alpha)}\left(\frac{\alpha - 1}{2}\right) \\ &\doteqdot 2^{n+1} E_n^{(\alpha-1)}\left(\frac{\alpha - 1}{2}\right) \\ &\doteqdot 2E_n^{(\alpha-1)}. \end{aligned}$$

3.3.7.2 The Complementary Argument Theorem

As we know x and $n - x$ are called complementary. We will now show that the following equation holds

$$E_n^{(\alpha)}(\alpha - x) = (-1)^n E_n^{(\alpha)}(x). \tag{3.3.83}$$

Using (3.3.72) and putting $\alpha - x$ for x we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{t^n}{n!} E_n^{(\alpha)}(\alpha - x) &= \frac{2^n e^{(\alpha - x)t}}{(e^t + 1)^n} \\
&= \frac{2^\alpha e^{\alpha t} e^{-xt}}{(e^t + 1)^\alpha} \cdot \left(\frac{e^{-\alpha t}}{e^{-\alpha t}} \right) \\
&= \frac{2^\alpha e^{-xt}}{(e^t + 1)^\alpha (e^{-t})^\alpha} \\
&= \frac{2^\alpha e^{-xt}}{(1 + e^{-t})^\alpha} \\
&= \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} E_n^{(\alpha)}(x).
\end{aligned}$$

Equating the coefficients of t^n, (3.3.83) will be proved. Equation (3.3.83) is the Complementary Argument Theorem. This theorem holds for any η polynomial with an even generating function $g(t)$.

Taking $x = 0$ in (3.3.83) for $n = 2\mu$, we get

$$E_{2\mu}^{(\alpha)}(\alpha) = E_{2\mu}^{(\alpha)}(0) = 2^{-2\mu} C_{2\mu}^{(\alpha)}.$$

Putting $x = 0$ and $n = 2\mu$ in equation (3.3.83) and using equation (3.3.73) leads to the following

$$\begin{aligned}
E_{2\mu}^{(\alpha)}(\alpha) &= (-1)^{2\mu} E_{2\mu}^{(\alpha)}(0) \\
&= E_{2\mu}^{(\alpha)}(0) \\
&= 2^{-2\mu} C_{2\mu}^{(\alpha)}.
\end{aligned}$$

Therefore at $x = 0$ and $x = \alpha$, $E_{2\mu}^{(\alpha)}(x) - 2^{-2\mu} C_{2\mu}^{(\alpha)}$ has zeros.

Again letting $x = \frac{1}{2}\alpha$ and $n = 2\mu + 1$ in (3.3.83), we get

$$E_{2\mu+1}^{(\alpha)} = -E_{2\mu+1}^{(\alpha)},$$

which will result $E_{2\mu+1}^{(\alpha)} = 0$. Thus Euler numbers with odd suffixes vanish.

3.3.7.3 Euler Polynomials of Successive Orders

Define,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} E_n^{(\alpha)}(x) = \frac{2^\alpha e^{xt}}{(e^t + 1)^\alpha}. \tag{3.3.84}$$

Differentiating both sides with respect to t and then multiplying by t, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{t^n}{(n-1)!} E_n^{(\alpha)}(x) &= \frac{2^\alpha \left[xte^{xt}\left(\left(e^t+1\right)^\alpha\right) - \alpha\left(e^t+1\right)^{\alpha-1} e^t te^{xt}\right]}{\left(e^t+1\right)^{2\alpha}} \\
&= \frac{2^\alpha xte^{xt}}{\left(e^t+1\right)^\alpha} - \frac{2^\alpha \alpha te^{t(x+1)}}{\left(e^t+1\right)^{\alpha+1}} \\
&= x\sum_{n=0}^{\infty} \frac{t^n}{(n-1)!} E_{n-1}^{(\alpha)}(x) - \frac{1}{2}\alpha \sum_{n=0}^{\infty} \frac{t^n}{(n-1)!} E_{n-1}^{(\alpha+1)}(x+1).
\end{aligned}$$

By equating the coefficients of t^{n+1}, we have

$$E_{n+1}^{(\alpha)}(x) = xE_n^{(\alpha)}(x) - \frac{1}{2}\alpha E_{n-1}^{(\alpha+1)}(x+1).$$

Using (3.3.80)

$$E_n^{(\alpha+1)}(x+1) = 2E_n^{(\alpha)}(x) - E_n^{(\alpha+1)}(x).$$

Thus,

$$E_n^{(\alpha+1)}(x) = \frac{2}{\alpha} E_{n+1}^{(\alpha)}(x) + \frac{2}{\alpha}(\alpha - x) E_n^{(\alpha)}(x). \tag{3.3.85}$$

Taking $x = 0$ in equation (3.3.85), we get the relation,

$$C_n^{(\alpha+1)} = \frac{1}{\alpha} C_{n+1}^{(\alpha)} + 2C_n^{(\alpha)}.$$

3.3.8 Euler Polynomials of the First Order

For order one in Euler numbers we will use $E_n(x)$ instead of $E_n^{(1)}(x)$. Let us recall the following definitions.

Consider,

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} \frac{t^n}{n!} E_n(x), \qquad \frac{2}{e^t + 1} = \sum_{v=0}^{\infty} \frac{t^n}{n!} \frac{1}{2^n} C_n. \tag{3.3.86}$$

Putting $x = \frac{1}{2}$ and $\alpha = 1$ in equation (3.3.86),

$$\begin{aligned} \frac{2e^{\frac{1}{2}t}}{e^t + 1} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{E_n}{2^n} \\ \frac{2e^{\frac{1}{2}t}}{e^t + 1} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} E_n\left(\frac{1}{2}\right). \end{aligned} \tag{3.3.87}$$

Substituting $\alpha = 1$, $x = 0$ in (3.3.82), we have

$$\begin{aligned} E_n\left(\frac{1}{2}\right) &\doteqdot \left(\frac{E}{2}\right)^n \\ &\doteqdot \frac{E_n}{2^n}. \end{aligned}$$

Replacing $E_n\left(\frac{1}{2}\right)$ with $\frac{E_n}{2^n}$ in the above equation

$$E_n(x) = \left(\frac{1}{2} + x\right)^n, \qquad (C+2)^n + C_n \doteqdot 0, \qquad n > 0.$$

Putting $\alpha = 1$ in equation (3.3.78),

$$\begin{aligned}(C+2)^n + C_n &\doteqdot 2C_n^{(0)} \\ &\doteqdot 2.2^n.0^n \\ &\doteqdot 0.\end{aligned}$$

Also,

$$\nabla E_n(x) = x^n \tag{3.3.88}$$

$$DE_n(x) = nE_{n-1}(x) \tag{3.3.89}$$

$$E_n(1-x) = (-1)^n E_n(x). \tag{3.3.90}$$

Below, the first seven Euler polynomial are fixed.

$$\begin{aligned}
E_0(x) &= 1, \\
E_1(x) &= x - \frac{1}{2}, \\
E_2(x) &= x(x-1), \\
E_3(x) &= \left(x - \frac{1}{2}\right)\left(x^2 - x - \frac{1}{2}\right), \\
E_4(x) &= x(x-1)\left(x^2 - x - 1\right), \\
E_5(x) &= \left(x - \frac{1}{2}\right)\left(x^4 - 2x^3 - x^2 + 2x + 1\right), \\
E_5(x) &= x(x-1)\left(x^4 - 2x^3 - 2x^2 + 3x + 3\right).
\end{aligned}$$

Also,

$$\begin{array}{ccccccc}
E_0 & E_2 & E_4 & E_6 & E_8 & E_{10} & E_{12} \\
1 & -1 & 5 & -61 & 1385 & -50521 & 2702765
\end{array}.$$

Example 3.3.5 *Use equality (3.3.90) to evaluate* $\sum_{s=1}^{\alpha}(-1)^s s^n$.

$$\begin{aligned}
\sum_{s=1}^{\alpha}(-1)^s s^n &= \sum_{s=1}^{\alpha}(-1)^s \nabla E_n(s) \\
&= \frac{1}{2}\sum_{s=1}^{\alpha}(-1)^s\left[E_n(s+1) + E_n(s)\right] \\
&= \frac{1}{2}\left[-E_n(2) - E_n(1) + E_n(3) + E_n(2) + \ldots + (-1)^{\alpha}E_n(\alpha+1) + (-1)^{\alpha}E_n(\alpha)\right] \\
&= \frac{1}{2}(-1)^{\alpha}E_n(\alpha+1) - \frac{1}{2}E_n(1).
\end{aligned}$$

3.3.8.1 Euler Numbers of the First Order

Substituting t by $2t$ in (3.3.87), we have

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} E_n.$$

Putting $2t$ for t

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!} E_n &= \frac{2e^t}{e^{2t} + 1} \\ &= \frac{e^t\,(2)}{e^t\,(e^t + e{-}^t)} \\ &= \frac{2}{e^t + e^{-t}}. \end{aligned}$$

Therefore

$$\sec ht = \sum_{n=0}^{\infty} \frac{t^n}{n!} E_n = 1 + \frac{t^2}{2!} E_2 + \frac{t^4}{4!} E + ...,$$

taking it instead of t, we have

$$\sec t = 1 - \frac{t^2}{2!} E_2 + \frac{t^4}{4!} E - ... \,. \tag{3.3.91}$$

Rearranging the expansion

$$\frac{\pi}{4 \cos \frac{\pi x}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{(2n+1)^2 - x^2}$$

and equating the coefficient of x^{2p} in the above equation with the coefficient of x^{2p} in series for $\frac{1}{4}\pi \sec\frac{\pi x}{2}$obtained from (3.3.91), we get

$$(-1)^p = \frac{E_{2p}}{2^{2p+2}(2p)!}\pi^{2p+1} = 1 - \frac{1}{3^{2p+1}} + \frac{1}{5^{2p+1}} - \frac{1}{7^{2p+1}} + \dots\ .$$

It is easy to see that Euler numbers increase and alternate in sign.

Using the method given in the subsection "Bernoulli Numbers of the First Order", we obtain the determinant below for $(-1)^\alpha E_{2\alpha}\ /\ (2\alpha)!$

$$\begin{bmatrix} \frac{1}{2!} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{4!} & \frac{1}{2!} & 1 & 0 & \cdots & 0 \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{1}{(2\alpha)!} & \frac{1}{(2\alpha-2)!} & \frac{1}{(2\alpha-4)!} & \frac{1}{(2\alpha-6)!} & \cdots & \frac{1}{\alpha!} \end{bmatrix}.$$

Regarding the numbers C_n and using (3.3.86), we have the odd function below

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\frac{C_n}{2^n} - 1 = \frac{e^t - 1}{e^t + 1} = -\tanh\frac{1}{2}t.$$

Therefore all the numbers $C_{2\mu} = 0$, $\mu > 0$. Writing $2t$ for t, we get

$$\tanh t = t - \frac{t^3}{3!}C_3 - \frac{t^5}{5!}C_5 - \frac{t^7}{7!}C_7 - \dots,$$

again as we did before, substituting t by it

$$\tan t = t + \frac{t^3}{3!}C_3 - \frac{t^5}{5!}C_5 + \frac{t^7}{7!}C_7 - \dots\ .$$

Equating the corresponding coefficients in the above series and the series in the subsection "Bernoulli Numbers of the first order", we get

$$C_{2n-1} = -\frac{2^{2n}(2^{2n}-1)}{2n}B_{2n}.$$

Since for $\mu > 0$, $C_{2\mu} = 0$ we have

$$\begin{aligned} E_n(x) - x^n &\doteqdot \left(x + \frac{1}{2}C\right)^n - x^n \\ &= \binom{n}{1}\frac{x^{n-1}}{2}C_1 + \binom{n}{3}\frac{x^{n-3}}{2^3}C_3 + \dots, \end{aligned}$$

resulting that $E_n(x) - x^n$ is an even function when n is odd and is odd when n is even.

3.3.8.2 Boole's Theorem for Polynomials

Putting $\alpha = 1$ in equation (3.3.82), we have

$$E_n(x) \doteqdot \left(x + \frac{1}{2}E - \frac{1}{2}\right)^n.$$

Thus

$$2x^n = 2\nabla E_n(x) \doteqdot \left(x + 1 + \frac{1}{2}E - \frac{1}{2}\right)^n + \left(x + \frac{1}{2}E - \frac{1}{2}\right)^n,$$

and if $P(x)$ is a polynomial

$$2P(x) \doteqdot P\left(x+1+\frac{1}{2}E-\frac{1}{2}\right)+P\left(x+\frac{1}{2}E-\frac{1}{2}\right). \tag{3.3.92}$$

Replacing x by $x+y$,

$$\begin{aligned}2P(x+y) &\doteqdot P\left(x+y+1+\frac{1}{2}E-\frac{1}{2}\right)+P\left(x+y+\frac{1}{2}E-\frac{1}{2}\right)\\ &\doteqdot P\left(x+y+1+E(y)\right)+P\left(x+E(y)\right).\end{aligned}$$

Now applying Taylor's Theorem

$$P\left(x+E(y)\right) \doteqdot P(x)+E_1(y)P'(x)+\frac{1}{2!}E_2(y)P''(x)+\dots\ .$$

Hence

$$P(x+y)=\nabla P(x)+E_1(y)\nabla P'(x)+\frac{1}{2!}E_2(y)\nabla P''(x)+\dots, \tag{3.3.93}$$

which leads to Boole's Theorem. Letting $x=0$, we get an expansion for $P(y)$ in terms of Euler's polynomials.

From the subsection "The Complementary Argument Theorem" ,

$$E_{2s}(1)=E_{2s}(0)=2^{-2s}C_{2s}=0$$

and

$$E_{2s+1}(1) = -E_{2s+1}(0) = -2^{-2s}C_{2s}.$$

Taking $y = 1$ in (3.3.93), we obtain

$$P(x+1)-P(x) = -C_1\nabla P'(x) - \frac{1}{3!2^2}C_3\nabla P^{''}(x) - \frac{1}{5!2^4}C_5\nabla P^{v}(x) - \dots . \tag{3.3.94}$$

Using (3.3.92), we get a solution for

$$\nabla u(x) = P(x) \tag{3.3.95}$$

as below

$$u(x) \doteqdot P\left(x + \frac{1}{2}E - \frac{1}{2}\right) \doteqdot P\left(E(x)\right).$$

For example the equation

$$\nabla u(x) = x^3 + 2x^2 + 1$$

has the solution below

$$u(x) = E_3(x) + 2E_2(x) + 1.$$

In order to get a general solution we can add the above solution with an arbitrary periodic function $\pi(x)$ such that $\pi(x+1) = -\pi(x)$.

Chapter 4

CONCLUSION

In this thesis an overview of Bernoulli and Euler numbers is provided. In Chapter 2, the Bernoulli and Euler polynomials are described and the relationship between these classes of polynomials are investigated. Some important identities involving differentiation, integration, summation are discussed using the basics of Finite Difference Calculus and Differentiation. The last part of this study is concerned with the Generalized Bernoulli and Euler polynomials. Furthermore, the properties obtained in the second chapter are also examined for the Generalized Bernoulli and Euler polynomials in this part of the thesis. The Complementary Argument Theorem, the generating functions, the Multiplication and the Euler-Maclauren Theorems are widely used in obtaining the results given in Chapter 2 and Chapter 3.

REFERENCES

[1] Cheon, G. S., A Note on the Bernoulli and Euler Polynomials, Applied Mathematics Letters, Vol. 16, 365-368, 2003.

[2] Pan, H., and Sun, Z.W., New identities involving Bernoulli and Euler polynomials, Journal of Combinatorial Theory, Series A 113 (2006) 156-175.

[3] Chen, K. W., Algorithms for Bernouli numbers and Euler Numbers, Journal of Integer Sequences, Vol. 4(2001), Article 01.1.6.

[4] Milne-Thomson, L. M., The Calculus of Finite Differences, Macmillan and Co., Limited St. Martin's Street, London, 1933.

[5] Srivastava, H. M., and Pinter, A., Remarks on Some Relationships Between the Bernoulli and Euler Polynomials, Applied Mathematics Letters, Vol 17. 375-380, 2004.

[6] Sun, Z. W., Introduction to Bernoulli and Euler Polynomials, A lecture given in Taiwan on June 6, 2002.

[7] http://en.wikipedia.org/wiki/Bernoulli_number.

[8] http:/en.wikipedia.org/wiki/Eulerian_number.

[9] http://en.wikipedia.org/wiki/Bernoulli_polynomials.

[10] http://oeis.org/wiki/Eulerian_polynomials.

[11] http://dlmf.nist.gow/24.