Existence of Reissner-Nordström Type Black Hole in f(R) Gravity

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ABSTRACT

We investigate the existence of Reissner-Nordström (RN) type black holes in $f(R)$ gravity. Our emphasis is to derive, in the presence of electrostatic source, the necessary conditions which provide such static, spherically symmetric (SSS) black holes available in $f(R)$ gravity by applying the "*near horizon test*" method. In this method we expand all the unknown functions about the horizon and we obtain zeroth and first terms of these fuctions. We also study the Extremal RN type black hole in this framework. In this thesis we show that it seems impossible to have a closed form of $f(R)$ for these types of black holes. Since, finding the total energy is rather difficult we derive the Misner-Sharp (MS) energy in $f(R)$ gravity by using the properties of black hole thermodynamics.

Keywords: Reissner-Nordström; $f(R)$ Gravity; Black Hole Thermodynamics

ÖZ

$f(R)$ yerçekim modelinde Reissner-Nordström (RN) tipi karadelik çözümlerinin varlığı incelenmektedir. Statik elektrik kaynak durumunda static küresel simetrik çözümlere "*ufuk yanı testi*" uygulayarak gerekli varlık şartları elde edilmiştir. Bilinmeyen fonksiyonlar ufuk civarında açılımlara tabi tutulup sıfır ve birinci mertebeden denklemler türetilmiştir. Özel bir hal olarak Ekstrem RN çözümünün varlığı da incelenmiştir. Bu tip kara deliklerin f(R) fonksiyonları kapalı bir formda elde edilememiştir. Kara delik termodinamiği kullanılarak Misner-Sharp (MS) türü enerji tanımı yöntemimizde esas alınmıştır.

To My Family

ACKNOWLEDGMENTS

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Chapter 1

INTRODUCTION TO f(R) GRAVITY

The Big-Bang theory of cosmology assumes that the universe started from an initial singularity. Very early universe, Early universe, Nucleosynthesis, Matter-Radiation-Equality, Recombination and Structure formation are the main stages that the universe has experienced. At the present epoch we know that the universe is homogenous and isotropic for large scales (larger than 100 Mpc). The cosmic microwave background (CMB) (as observed by the satellites COBE and WMAP), the huge low-redshift galaxy surveys (such as the 2-degree field galaxy redshift survey (2dfGRS)) and the Sloan digital sky survey (SDSS) have convinced most cosmologists that homogeneity and isotropy are, in fact, reasonable assumptions for the universe. On the other hand, the universe is passing through an accelerating phase of expansion which is discovered by high redshift surveys of type Ia Supernovae, the position of acoustic peak from the CMB observations, the size of baryonic acoustic peak, etc[1].

Based on recent Planck's space telescope, modern cosmology claims that total energy density of the universe consist of 4.9% of baryonic mater,26.8% of cold dark mater, and 68.3% of dark energy. This distribution of energy density led the framework for describing these observational data to be proposed. ΛCDM model is one of the invaluable possibilities among other concordance models in which energy budget at the

present epoch is dominated by cold dark matter (CDM) and dark energy in the form of a cosmological constant Λ[1].

Structure formation is one of the most challenging stages in cosmology. Large scale structure (LSS) formation which is due to the tiny perturbations in the very early universe has started when the fractal nature of the universe stoped at a certain scale. In the standard ΛCDM cosmology the very small deviation from uniformity, density fluctuations in the early universe (that grow rapidly due to the inflation) are the cosmic seeds of structure formation. It is determined that baryonic gravitational effect could not create LSS that can be seen in the universe today. These collapsing overdensities, which are primarily composed of dark matter halos, provide the initial potential wells for baryons to condense and begin the process of galaxy formation[2].

There are numerous competing theories and speculations regarding what dark matter might be made of. From astrophysical measurements we can deduce some properties of Dark Matter like non-baryonic, stable against decay, weak interaction, etc. It seems that one of the simplest ways by which the mystery of the dark matter can be solved is to assume that an unknown exotic particle exists[2].

For the dark energy models one can assume that (i) the universe is filled with an exotic fluid with the property of having a negative pressure that dominates in the late time and results in an accelerating expansion, (ii) modification of the matter sector is described by quantum fields instead of perfect fluid[2].

For solving these problems there exists another possible scenario by considering modification of Einstein's equations of gravity (i.e. modified gravity) which is mentionned for the first time by Hermann Weyl in 1919[3].

Modified gravity, in which the origin of inflation is considered purely geometrical, may explain several fundamental cosmological problems. For instance, expansion of the universe may be described by modified gravity especially by f(R) gravity. Indeed, it also explains naturally the unification of earlier and later cosmological epochs as the manifestation of a different role of gravitational terms relevant at the small and large curvature as it happens in the model with negative and positive powers of curvature. Moreover, expansion of the universe can solve the coincidence problem. By considerating string/M-theory, same type of modified gravity can be anticipated[4].

On the other hand, modified gravity may describe dark matter completely. It may be helpful in high energy physics. As an example, it can be useful in solving the hierarchy or gravity-GUTs unification problems. Finally, modified gravity may pass the local tests and cosmological bounds[4].

Thus, these reasons show how this field is rich, invaluable and fruitful in application to many aspects of gravity and cosmology.

In this thesis, we wish to look at $f(R)$ gravity from a different angle which was introduced by Bergliaffa and Nunes[5].Our approach is to extract information for our unknown quantities from the geometrical behavior near the horizon in order to declare horizon as a physical reality. This approach which gives us the existence conditions

for the relevant quantities maybe called as a *"near horizon test"*[6].

In this test, for specific static and spherically symmetric (SSS) black holes we consider that there is an arbitrary $f(R)$ gravity model. Then, we use the Taylor expansion

$$F(r) = F(r_0) + F'(r_0)(r - r_0) + O\left((r - r_0)^2\right), \qquad (1.1)$$

in series of the distance to the horizon for all unknown functions we have, to take an account matematically the strong gravity existing near the event horizon. Consequently, when we substitute back all series into the equations of motion we shall obtain a necessary condition that the $f(R)$ must satisfy for the existence of the SSS black hole solution.

Indeed, *"near horizon test"* makes the strong restriction that we cant propose arbitrarily any polynomial forms of $f(R)$ as the representative black holes.

Beside these, different aspects of additional external sources have been previously discussed (some examples of $f(R)$ black hole with charge are given in[7, 8, 9, 10, 11, 12, 13]), which makes the principal aim of the present thesis. We consider an external static electric field as source and adopt the Reissner-Nordström (RN)-type black hole within $f(R)$ gravity. then we derive an infinite series representation for the near-horizon behavior of our metric functions. The exact determination of the constant coefficients in the series is theoretically possible, at least in the leading orders. The addition of further external sources beside electromagnetism will naturally make the problem more complicated. An equally simple case is the extremal RN black hole

which is also considered in our study.

Based on our work[14] this thesis is organized as follows. In Chapter 2 we review the concept of action in General Relativity and $f(R)$ gravity. In Chapter 3 we investigate the necessary conditions for the existence of a RN-type/Extremal RN-type black holes in $f(R)$ gravity. Thermodynamics properties and in particular the Misner-Sharp (MS) Energy for such black holes are presented in Chapter 4. The thesis ends with Conclusion which appears in Chapter 5.

Chapter 2

THEORETICAL FRAMEWORK

In this Chapter we shall review the concept of action Lagrangian in the general relativity and $f(R)$ gravity. However, since we are familiar with this concept in classical physics we will discuss it for future use.

2.1 Introduction

In the classical mechanics the action is defined as

$$S = \int L(q,\dot{q})dt, \tag{2.1}$$

and Hamilton's principle claims that the trajectory of a body, described by the Lagrangian $L(q,\dot{q})$ should satisfy

$$\delta S = 0, \tag{2.2}$$

or

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0. \tag{2.3}$$

The Lagrangian's definition is not very different in GR than in classical mechanics. The main difference between classical and relativistic Lagrangians lies in the fact that in GR, we have a curved space-time, and so, we must associate a Lagrangian to the vacuum space.

We know that the curved spacetime is defined by metric tensor $g_{\mu\nu}$, therefore the Lagrangian should be related to $g_{\mu\nu}$ and its derivatives[15]. Also the Lagrangian must depend on the Riemman and Ricci tensors which provide the information about the curved spacetime. So these constraints lead us to use Ricci scalar in the Lagrangian. Using Ricci scalar in the Lagrangian raises two problems. First, Ricci scalar associates with the second order drivatives of the metric tensor. So we cannot write the Lagrangian in the form of 1.3 . Second, is that integrated function must be invariant. Because of this we add another term to make it invariant. So, one can write the form of the Lagrangian as

$$L = \sqrt{-g}R. \tag{2.4}$$

Finally, we derive the simplest form of Lagrangian which contains all needed properties. Now we are able to write the form of action in four dimensional vacuum spacetime as

$$S = \frac{1}{2\kappa} \int L d^4x, \tag{2.5}$$

where $\kappa = 8\pi G$. So, by using 1.2 we can write Einstein equations with variation respect to $g_{\mu\nu}$ in vacuum. Also, by adding matter term S_m to the action we can write the

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Einstein equation in presence of matter which is known as the Einstein-Hilbert action.

2.2 $f(R)$ Gravity Actions

In the $f(R)$ gravity where the $f(R)$ is the function of Ricci scalar the Lagrangian is written as

$$L = \sqrt{-g} f(R). \tag{2.6}$$

The reason that we use the $f(R)$ gravity as a function of Ricci scalar is only because of the simplicity. Also, we know that $f(R)$ action includes some main properties of higher order gravities. In the rest of this section we will review the different types of $f(R)$ gravities[16].

2.2.1 Metric $f(R)$ Gravity

The action in the vacuum for this type of $f(R)$ gravity is

$$S = \frac{1}{2\kappa} \int \sqrt{-g} f(R) d^4 x + S_m, \tag{2.7}$$

where S_m stands for the physical source. By taking variation with respect to $g_{\mu\nu}$ we can derive the equations of motion

$$F R_\mu^\nu - \frac{1}{2} f \delta_\mu^\nu - \nabla_\mu \nabla^\nu F + \delta_\mu^\nu \Box F = \kappa T_\mu^\nu, \tag{2.8}$$

where T_μ^ν is the physical energy-momentum tensor[16].

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2.2.2 Palatini $f(R)$ Gravity

Palatini method was proposed as the candidate for inflation shortly after metric $f(R)$ theories were proposed. In the Palatini method not only $g_{\mu\nu}$ but also Christoffel symbols $\Gamma^{\rho}_{\mu\nu}$ are independent variables. As a result, we have two independent Ricci scalars. Form of action in this method is

$$S = \frac{1}{2\kappa} \int \sqrt{-g} f(\tilde{R}) d^4 x + S_m, \tag{2.9}$$

after some manipulation one can find the field equations as

$$f'(\tilde{R})\tilde{R}_{\alpha\beta} - \frac{f(\tilde{R})}{2} g_{\alpha\beta} = \kappa T_{\alpha\beta}, \tag{2.10}$$

$$\tilde{\nabla}_{\gamma}(\sqrt{-g} f'(\tilde{R}) g^{\alpha\beta}) - \tilde{\nabla}_{\delta}(\sqrt{-g} f'(\tilde{R}) g^{\delta(\beta)}) \delta^{\beta)}_{\gamma}, \tag{2.11}$$

where the matter part of action does not depend on the Christoffel symbols $\Gamma^{\rho}_{\mu\nu}$[16].

2.2.3 Metric-Affine $f(R)$ Gravity

The only diffrence between this approach and Palatini approach is that the matter part of action depends on the Christoffel symbol. This leads to a torsion associated with matter, and to a modern revival of torsion theories. These were originally introduced within a non-cosmological context, with the spin of elementary particles coupling to the torsion. Metric-affine $f(R)$ gravity still needs to construct many concepts and definitions specially for cosmological application[17].

Chapter 3

NECESSARY CONDITIONS FOR BLACK HOLES IN f(R)

GRAVITY

In this Chapter we shall derive coditions for Reissner-Nordström (RN)-type and Extremal RN-type black hole in f(R) gravity by using "*near horizon test*" where the action is given by

$$S = \int \sqrt{-g}(\frac{f(R)}{2\kappa} - \frac{\mathcal{F}}{4\pi})d^4x, \tag{3.1}$$

in which $\mathcal{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $\kappa = 8\pi G$.

3.1 RN-type Black Hole

We choose RN-type black hole metric as

$$ds^2 = -e^{-2\Phi}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{3.2}$$

in which Q and M are integration constants that represent the charge and the mass of the black hole, respectively. The real unknown fuction $\Phi = \Phi(r)$ is well behaved everywhere and dies off at large *r* determines the gravitational redshift[18]. The Maxwell

electric two-form field **F** provides the matter source which is given by

$$\mathbf{F} = E(r)dt \wedge dr, \tag{3.3}$$

and it's dual form

$$^*\mathbf{F} = -E(r)e^{\Phi}r^2 \sin\theta d\theta \wedge d\varphi, \tag{3.4}$$

where $E(r)$ is radial electric field. Therefore from $d^*\mathbf{F} = 0$ we can derive

$$E(r) = \frac{q}{r^2}e^{-\Phi}, \tag{3.5}$$

where the integration constant q is equal to the charge of black hole Q. From field equation 2.8 we obtain

$$\Box F = \Box\frac{df}{dR} = \frac{1}{\sqrt{-g}}\partial_r\left(\sqrt{-g}g^{rr}\partial_r F\right), \tag{3.6}$$

$$\nabla^t\nabla_t F = g^{\alpha t}\left[F_{,t,\alpha} - \Gamma^m_{t\alpha}F_{,m}\right], \tag{3.7}$$

because the line element 3.2 is static spherically symmetric metric so $\alpha = t$. The Ricci sacalar is the function of r and $F = F(r) = \frac{df}{dR}$ therefore $F_{,t,t} = \frac{d^2F}{dt^2} = 0$ and the only non zero Γ^m_{tt} term is Γ^r_{tt} so that

$$\nabla^t\nabla_t F = \frac{1}{2}g^{tt}g^{rr}g_{tt,r}F_{,r}. \tag{3.8}$$

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Similarly for r, θ and φ components we have

$$\nabla^r \nabla_r F = g^{rr} F_{,r,r} - g^{rr} \Gamma^r_{rr} F_{,r} = g^{rr} F_{,r,r} - \frac{1}{2} (g^{rr})^2 g_{rr,r} F_{,r}, \qquad (3.9)$$

$$\nabla^\varphi \nabla_\varphi F = \nabla^\theta \nabla_\theta F = \frac{1}{2} g^{\theta\theta} g^{rr} g_{\theta\theta,r} F_{,r}. \qquad (3.10)$$

The stress-energy tensor in 2.8 is given by

$$T^\nu_\mu = \frac{1}{4\pi} \left(\mathcal{F} \delta^\nu_\mu - F_{\mu\lambda} F^{\nu\lambda} \right), \qquad (3.11)$$

whereas from 2.3 we know that only $F_{rt} \neq 0$ therefore

$$\mathcal{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{4} (F_{tr} F^{tr} + F_{rt} F^{rt}) = \frac{1}{4} (2 g^{tt} g^{rr} (F_{tr})^2) = \frac{Q^2}{2r^4}. \qquad (3.12)$$

Consequently the stress-enegy tensor components are

$$T^t_t = \frac{1}{4\pi} \left(\mathcal{F} \delta^t_t - F_{tr} F^{tr} \right) = \frac{1}{4\pi} \left(\frac{Q^2}{2r^4} - \frac{Q^2}{r^4} \right) = -\frac{1}{8\pi} \frac{Q^2}{r^4}, \qquad (3.13)$$

$$T^r_r = \frac{1}{4\pi} \left(\mathcal{F} \delta^r_r - F_{tr} F^{tr} \right) = \frac{1}{4\pi} \left(\frac{Q^2}{2r^4} - \frac{Q^2}{r^4} \right) = -\frac{1}{8\pi} \frac{Q^2}{r^4}, \qquad (3.14)$$

$$T^\theta_\theta = \frac{1}{4\pi} \left(\mathcal{F} \delta^\theta_\theta - F_{\theta\lambda} F^{\theta\lambda} \right) = \frac{1}{4\pi} \left(\frac{Q^2}{2r^4} - 0 \right) = \frac{1}{8\pi} \frac{Q^2}{r^4}, \qquad (3.15)$$

$$T^\varphi_\varphi = \frac{1}{4\pi} \left(\mathcal{F} \delta^\varphi_\varphi - F_{\varphi\lambda} F^{\varphi\lambda} \right) = \frac{1}{4\pi} \left(\frac{Q^2}{2r^4} - 0 \right) = \frac{1}{8\pi} \frac{Q^2}{r^4}, \qquad (3.16)$$

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or

$$T_\mu^\nu = \frac{1}{8\pi} \frac{Q^2}{r^4} diag[-1, -1, 1, 1].$$ (3.17)

It is clear that $T = T_\mu^\nu = 0$ so the trace of equation of motions is

$$FR - 2f + 3\Box F = 0,$$ (3.18)

and by using the trace equation 2.8 we can siplify the field equations and rewrite them

as

$$FR_\mu^\nu - \frac{1}{4}\delta_\mu^\nu(FR - \Box F) - \nabla_\mu \nabla^\nu F = \kappa T_\mu^\nu.$$ (3.19)

From metric 3.2 one can find the horizon of the black hole from $g_{tt} = 0$ or

$$r_\pm = M \pm \sqrt{M^2 - Q^2},$$ (3.20)

where r_+ is called outer horizon and r_- is inner horizon. We use the $r_+ = r_0$ as an

event horizon in the following and we replace the mass of black hole by $M = \frac{r_0^2 + Q^2}{2r_0}$

equation.

Based on the *near horizon test* introduced in [5, 6] we expand all the unknown func-

tions about the horizon. This would lead to the expansions

$$R(r) = R_0 + R_0'(r - r_0) + \frac{1}{2}R_0''(r - r_0)^2 + O\left((r - r_0)^3\right),$$ (3.21)

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$$\Phi(r) = \Phi_0 + \Phi_0'(r - r_0) + \frac{1}{2}\Phi_0''(r - r_0)^2 + O\left((r - r_0)^3\right), \qquad (3.22)$$

$$F = F_0 + F_0'(r - r_0) + \frac{1}{2}F_0''(r - r_0)^2 + O\left((r - r_0)^3\right), \qquad (3.23)$$

where sub zero shows the value of quantity at the horizon and the prime denotes derivative with respect to the coordinate r. Then, we put equations 3.21, 3.22 and 3.23 into the equation 3.19 and after some calculations for the zeroth order we obtain the three equations

$$f_0 r_0^4 - \left(E_0 R_0' + 3\Phi_0' F_0\right) r_0^3 + Q^2 \left(E_0 R_0' + 3\Phi_0' F_0\right) r_0 + 2Q^2 (F_0 - 1) = 0, \qquad (3.24)$$

$$f_0 r_0^4 - 2r_0^3 E_0 R_0' + 2Q^2 r_0 E_0 R_0' - 2Q^2 (F_0 - 1) = 0, \qquad (3.25)$$

$$R_0 = \frac{3\Phi_0' \left(r_0^2 - Q^2\right)}{r_0^3}. \qquad (3.26)$$

From the first order equations we derive

$$F_0 R_0' r_0^4 + \left[\left(2\Phi_0'^2 - 5\Phi_0''\right) F_0 - 3\Phi_0' E_0 R_0' - 3H_0 R_0'^2 + 4f_0 - 3E_0 R_0''\right] r_0^3$$

$$-2\left(3E_0 R_0' + 5\Phi_0' F_0\right) + \left[\left(-2\Phi_0'^2 + 5\Phi_0''\right) F_0 + 3\Phi_0' E_0 R_0' + 3H_0 R_0'^2\right]$$

$$+3E_0 R_0'' Q^2 r_0 + 6Q^2 E_0 R_0' + 4\Phi_0' F_0 Q^2 = 0, \qquad (3.27)$$

$$F_0 R_0' r_0^4 + 4\left(f_0 - E_0 R_0'' - H_0 R_0'^2 + \frac{1}{2}\Phi_0' E_0 R_0'\right) r_0^3 - 2\left(\Phi_0' F_0 + 3E_0 R_0'\right) r_0^2$$

$$+ \left(4E_0 R_0'' - 2\Phi_0' E_0 R_0' + 4H_0 R_0'^2\right) Q^2 r_0 + 2\Phi_0' F_0 Q^2 = 0, \tag{3.28}$$

$$R_0' = \frac{\left(5\Phi_0'' - 2\Phi_0'^2\right) r_0^3 - 2\Phi_0' r_0^2 + \left(2Q^2\Phi_0'^2 - 5Q^2\Phi_0''\right) r_0 + 8Q^2\Phi_0'}{r_0^4}. \tag{3.29}$$

In these equations $E = \frac{d^2 f}{dR^2} = \frac{dF}{dR}$ and $H = \frac{d^3 f}{dR^3} = \frac{dE}{dR}$. From zeroth, first and higher order equations we can derive the neccesary coditions for f, R and also F when we keep Φ as a known function as shown below

$$\Phi = \beta_1 \varepsilon + \beta_2 \varepsilon^2 + O\left(\varepsilon^3\right), \tag{3.30}$$

$$f = f_0 - \frac{1}{6}(f_0 r_0^4 - 6Q^2) \times$$

$$\frac{\left[2r_0\left(r_0^2 - Q^2\right)\beta_1^2 + 2\left(r_0^2 - 4Q^2\right)\beta_1 - 5r_0\left(r_0^2 - Q^2\right)\beta_2\right]}{r_0^4\left(r_0\left(r_0^2 - Q^2\right)\beta_1 - Q^2\right)}\varepsilon + O\left(\varepsilon^2\right), \tag{3.31}$$

$$F = \frac{f_0 r_0^4 - 6Q^2}{6\left(\beta_1 r_0\left(r_0^2 - Q^2\right) - Q^2\right)} + \frac{3\beta_1\left(r_0^2 - Q^2\right)\left(r_0^4 f_0 + 2Q^2\right) - 4f_0 r_0^3 Q^2}{6\left(r_0^2 - Q^2\right)\left(\beta_1 r_0\left(r_0^2 - Q^2\right) - Q^2\right)}\varepsilon + O\left(\varepsilon^2\right), \tag{3.32}$$

$$R = \frac{3\beta_1\left(r_0^2 - Q^2\right)}{r_0^3} - \frac{2r_0\left(r_0^2 - Q^2\right)\beta_1^2 + 2\left(r_0^2 - 4Q^2\right)\beta_1 - 5r_0\left(r_0^2 - Q^2\right)\beta_2}{r_0^4}\varepsilon + O\left(\varepsilon^2\right), \tag{3.33}$$

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here $\varepsilon = r - r_0$, β_1, β_2 are known constants and

$$f_0 = -\frac{6Q^2}{r_0^3} \times$$

$$\frac{8r_0\left(r_0^2 - Q^2\right)^2 \beta_1^2 - 2\left(r_0^2 - Q^2\right)\left(Q^2 + 5r_0^2\right)\beta_1 - 5r_0\left(r_0^2 - Q^2\right)^2 \beta_2}{16r_0^2\left(r_0^2 - Q^2\right)^2 \beta_1^2 + 2r_0\left(r_0^2 - Q^2\right)\left(5r_0^2 - 23Q^2\right)\beta_1 + 5r_0^2\left(r_0^2 - Q^2\right)^2 \beta_2 + 24Q^4}.$$

$$\tag{3.34}$$

The only parameter which remains unknown is Φ_0, but by redefinition of time we can absorb it to the time and consider it as $\Phi_0 = 0$.

3.2 Special Examples

In this section we shall study some special $f(R)$ gravities in RN-type black holes and derive the necessary conditions for extremal RN-type black holes.

3.2.1 Examples of $f(R)$ gravity Models

We know that for the case of $f(R) = R$ our results should satisfy in general relativity. In this case we have $f_0 = R_0$ and $F = 1$ so

$$\frac{\left(\beta_1\left(r_0^2 - Q^2\right)\right)r_0 - 2Q^2}{2\left(\beta_1 r_0\left(r_0^2 - Q^2\right) - Q^2\right)} = 1, \tag{3.35}$$

it means $\beta_1 = 0$. By using $\beta_1 = 0$ we can show that

$$f_0 = R_0 \rightarrow -\frac{6Q^2}{r_0^3}\frac{-5r_0\left(r_0^2 - Q^2\right)^2 \beta_2}{5r_0^2\left(r_0^2 - Q^2\right)^2 \beta_2 + 24Q^4} = 0, \tag{3.36}$$

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which leads us to conclude $\beta_2 = 0$. So, we proof that our general conditions with $\beta_1 = 0 = \beta_2$ are satisfied in this model.

The other simple case that one can study in this method is $f(R) = R^2$ where by appling the necessary conditions we get

$$R_0^2 = -\frac{6Q^2}{r_0^3} \times$$

$$\frac{8r_0 \left(r_0^2 - Q^2\right)^2 \beta_1^2 - 2 \left(r_0^2 - Q^2\right) \left(Q^2 + 5r_0^2\right) \beta_1 - 5r_0 \left(r_0^2 - Q^2\right)^2 \beta_2}{16r_0^2 \left(r_0^2 - Q^2\right)^2 \beta_1^2 + 2r_0 \left(r_0^2 - Q^2\right) \left(5r_0^2 - 23Q^2\right) \beta_1 + 5r_0^2 \left(r_0^2 - Q^2\right)^2 \beta_2 + 24Q^4},$$

$$(3.37)$$

and

$$2R_0 = \frac{R_0^2 r_0^4 - 6Q^2}{6 \left(\beta_1 r_0 \left(r_0^2 - Q^2\right) - Q^2\right)}. \tag{3.38}$$

From 3.37 and 3.38 we can derive

$$\beta_1 = \frac{1}{6} \frac{4Q + 2\sqrt{4Q^2 - 2r_0^4}}{r_0 \left(r_0^2 - Q^2\right)}, \tag{3.39}$$

and

$$\beta_2 = \frac{-4Q[Q^2(20Q^2 - 11r_0^4) + 15r_0^2(r_0^4 - 2Q^2)]\sqrt{4Q^2 - 2r_0^4}}{45r_0^2 \left(r_0^2 - Q^2\right)^2 \left(2 \left(r_0^4 - Q^2\right) - Q\sqrt{4Q^2 - 2r_0^4}\right)} +$$

$$\frac{-4Q^2[20Q^2(2Q^2 - 3r_0^2) + r_0^4(r_0^2(2r_0 + 45) - 32Q^2)]}{45r_0^2 \left(r_0^2 - Q^2\right)^2 \left(2 \left(r_0^4 - Q^2\right) - Q\sqrt{4Q^2 - 2r_0^4}\right)}. \tag{3.40}$$

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To avoid complex results $4Q^2 - 2r_0^4 \geq 0$ must be satisfied. One of the special case is that

$$r_0^4 = 2Q^2. \tag{3.41}$$

We rewrite 3.20 as

$$r_0^2 - 2Mr_0 + Q^2 = 0, \tag{3.42}$$

having 3.41 implies

$$r_0^2 - 2Mr_0 + \frac{r_0^4}{2} = 0, \tag{3.43}$$

or

$$M = r_0 + \frac{1}{2}r_0^3. \tag{3.44}$$

As a result, for this case ($f(R) = R^2$ and $r_0^4 = 2Q^2$) we obtain the mass of RN-type black hole. Also, for this $f(R)$ model we have

$$\beta_1 = \frac{2\sqrt{2}}{3} \frac{1}{r_0(2 - r_0^2)}, \tag{3.45}$$

$$\beta_2 = \frac{8}{45} \frac{4r_0^2 - 15}{(r_0^2 - 2)^2}, \tag{3.46}$$

and $f_0 = R_0 = 1$ while $F_0 = 2$.

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3.2.2 Extremal RN-type Black Hole

Equation 3.20 shows that mass of black hole should be $M \geq Q$ to have physical meaning. One of the intresting cases is called Extremal RN-type when $M = |Q|$. In this case we have only one horizon and $r_0 = r_- = r_+ = |Q|$. By choosing $Q = b_0 \geq 0$ the line element reduces to

$$ds^2 = -e^{-2\Phi} \left(1 - \frac{b_0}{r}\right)^2 dt^2 + \frac{dr^2}{\left(1 - \frac{b_0}{r}\right)^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{3.47}$$

in which $r_0 = b_0$. As we will discuss later this black hole doesn't radiate and the $T_{BH} = 0$ but it has a specific entropy. Thus, we can define the entropy of the extremal black hole as a zero temperature entropy.

By appling the "*near horizon test*" for this metric we derived the following conditions

$$R = 6\frac{\beta}{r_0^2}\varepsilon - \frac{6\beta}{r_0^2}\left(2\beta + \frac{5}{r_0}\right)\varepsilon^2 + \frac{\beta}{r_0^2}\left(\frac{93\beta^2}{4} + \frac{71\beta}{r_0} + \frac{90}{r_0^2}\right)\varepsilon^3 + O\left(\varepsilon^4\right), \tag{3.48}$$

$$f = 6\frac{\beta}{r_0^2}\varepsilon - \frac{3\beta}{r_0^2}\left(3\beta + \frac{10}{r_0}\right)\varepsilon^2 + \frac{\beta}{r_0^2}\left(\frac{57\beta^2}{4} + \frac{49\beta}{r_0} + \frac{90}{r_0^2}\right)\varepsilon^3 + O\left(\varepsilon^4\right), \tag{3.49}$$

$$F = \left(\frac{df}{dR}\right) = 1 + \beta\varepsilon - \frac{\beta}{2}\left(\beta + \frac{2}{r_0}\right)\varepsilon^2 + \beta\left(\frac{3\beta^2}{8} + \frac{3\beta}{4r_0} + \frac{1}{r_0^2}\right)\varepsilon^3 + O\left(\varepsilon^4\right), \tag{3.50}$$

and

$$\Phi = \Phi_0 + \beta\varepsilon - \frac{\beta}{8}\left(5\beta + \frac{8}{r_0}\right)\varepsilon^2 + \beta\left(\frac{73\beta^2}{120} + \frac{73\beta}{60r_0} + \frac{1}{r_0^2}\right)\varepsilon^3 + O\left(\varepsilon^4\right), \tag{3.51}$$

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in which $\beta \neq 0$ and it's known as an arbitrary constant. In 3.51 as we did before we can absorb Φ_0 into time. It is clear that equations 3.48, 3.49 and 3.50 imply R and f are zero at the horizon but $\left(\frac{df}{dR}\right) = 1$. This leads us to write one of the possible $f(R)$ gravity model in the form of

$$f(R) = R + a_2 R^2 + a_3 R^3 + a_4 R^4 + ...,\tag{3.52}$$

where the necessary condiditions can determin the constant coefficients a_i. As an example, up to the third order one can get

$$f(R) \sim R + \frac{r_0^2}{12}R^2 + r_0^3\left(\frac{5}{72}r_0 + \frac{19}{108\beta}\right)R^3,\tag{3.53}$$

where all necessary conditions are satisfied up to the second order for this form of $f(R)$. Another $f(R)$ model that can be deduced from 3.52 is $f(R) \sim R^\nu$. For this model, necessary conditions are satisfied when we chose $\beta = 0$. It implies $\nu = 1$ or GR. Also, $f(R) = \frac{R}{1-R}$ is another $f(R)$ model which at least satisfies the above conditions up to the first order.

Chapter 4

THERMODYNAMICS OF ANALOG BLACK HOLE

4.1   Introduction to Black Hole Thermodynamics

Thermodynamics of black holes plays a key role to learn about quantum gravity and statistical mechanics. Also we can study black hole thermodynamics in modified gravity. If studying the thermodynamics of the black holes helps us to learn quantum gravity better, it will be more logical to use it in extended gravity. Gravity quantum corrections, renormalization, the low-energy limit and string theories can bring forward extra gravitational scalar fields which is coupled to curvature non-minimally and higher derivative corrections to general relativity[19].

Considering black hole as a thermodynamics system was mentioned for the first time by the J. M. Greif in 1969[20]. Then, Bardeen, Bekenstein, Carter, Penrose and Hawking tried to explain and formulate it. Bekenstein suggested that the area of the black hole can be considered as an entropy of the black hole. After that, first law af black hole thermodynamics was proved by Bardeen, Carter and Hawking. Finally, Hawking discovered black hole temperature $T_H = \frac{\frac{\partial}{\partial r} g_{tt}}{4\pi}\Big|_{r=r_0}$ by using quantum field theory in 1974[21]. In 1995 Jacobson used the local Rindler horizon and derived the entropy of the black hole as $S_{BH} = \frac{A}{4G}$ where G is Newton's constant. He showed that the field equations in GR are related to a macroscopic effective equation of state[22].

In general relativity the first law of thermodynamics can be written as

$$T_H \delta S = \delta M - \Omega_H \delta J - \varphi \delta Q, \qquad (4.1)$$

where in the left hand side S and T_H are the entropy and Hawking temperature and M, Ω_H, J, φ and Q, in the right hand side are defined as mass, angular velocity , angular momentum, electric potential and charge of the black hole, respectively. This law is akin to the M, J and Q that are measured at infinity with the S, T, A, Ω_H and φ which are local quantities and defined on the horizon.

In $f(R)$ gravity, where equation of motion are derived by using the thermodynamics of local Rindler horizon, we have to redefine the entropy expression to satisfy that property correctly. There are some attempts to define black hole entropy in extended gravity like Bekenstein-Hawking entropy in scalar-tensor and extended gravity, Wald's Noether charge, field redefinition techniques and the Euclidean path integral approaches[19].

In 1996 Kang[23] realized that the second law of thermodynamics (area law) is violated in extended gravity when he studied black hole entropy in Brans-Dicke gravity. He introduced another definition for the entropy

$$S_{BH} = \frac{1}{4} \int_\Sigma d^2x \sqrt{g^{(2)}} \phi, \qquad (4.2)$$

here ϕ is Brans-Dicke scalar and $g^{(2)}$ is the determinant of the restriction $g^{(2)}_{\mu\nu} \equiv g_{\mu\nu}|_\Sigma$ of the metric $g_{\mu\nu}$ to the horizon surface Σ.

One can write this equation by replacing G with the effective gravitational coupling $G_{eff} = \phi^{-1}$ so that

$$S_{BH} = \frac{A}{4G_{eff}}. \tag{4.3}$$

We replaced this quantity because we want to write the field equation as an effective Einstein equation and consider scalar field or geometry in $f(R)$ gravity terms as an effective form of matter. One can easily show that Einstein frame goes to the Brans-Dicke theory by conformal rescaling of the metric[19].

In the following sections we shall derive the Hawking temperature, entropy and heat capacity of RN-type/extremal RN-type black holes then Misner-Sharp (MS) energy will be calculated from the first law of thermodynamics.

4.2 Hawking Temperature, Entropy and Heat Capacity of Analog Black Holes

The Hawking temperature expression remains unchanged in modified gravity

$$T_H = \frac{\frac{\partial}{\partial r} g_{tt}}{4\pi} \bigg|_{r=r_0} = T_H^{(RN)} = \frac{1}{4\pi r_0} \left(1 - \frac{Q^2}{r_0^2}\right), \tag{4.4}$$

in which $T_H^{(RN)}$ implies RN Hawking temperature. By using the equivalence between Brans-Dicke theory and metric $f(R)$ gravity for 4.3 we derive the

$$S_{BH} = \frac{\mathcal{A}}{4G} F \bigg|_{r=r_0} = \pi r_0^2 F_0, \tag{4.5}$$

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as a form of entropy in which $\mathcal{A}|_{r=r_0} = 4\pi r_0^2$ is the surface area of the black hole at the horizon and $F|_{r=r_0} = F_0$. So we derived the exact values for T_H and S in order to find the heat capacity of the black hole

$$C_q = T \left(\frac{\partial S}{\partial T} \right)_Q = C_q^{(RN)} I, \tag{4.6}$$

in which

$$I = 12Q^2 \left(r_0^2 - Q^2 \right) \Pi, \tag{4.7}$$

where

$$\Pi = \frac{5r_0^3 \left(\left(r_0^4 - Q^4 \right) \beta_1 - 4Q^2 r_0 \right) \beta_2 + 16r_0^3 \beta_1^3 \left(r_0^4 - Q^4 \right)}{\left[r_0^2 \left(r_0^2 - Q^2 \right)^2 \left(5\beta_2 + 16\beta_1^2 \right) + 2r_0 \left(r_0^2 - Q^2 \right) \left(5r_0^2 - 23Q^2 \right) \beta_1 + 24Q^4 \right]^2} +$$

$$\frac{4Q^2 r_0^2 \beta_1^2 \left(7r_0^2 - 23Q^2 \right) + \frac{2Q^2 \left(24Q^4 - r_0 \beta_1 \left(15r_0^4 + 32r_0^2 Q^2 - 59Q^4 \right) \right)}{\left(r_0^2 - Q^2 \right)}}{\left[r_0^2 \left(r_0^2 - Q^2 \right)^2 \left(5\beta_2 + 16\beta_1^2 \right) + 2r_0 \left(r_0^2 - Q^2 \right) \left(5r_0^2 - 23Q^2 \right) \beta_1 + 24Q^4 \right]^2}. \tag{4.8}$$

From 4.6 one can easily check that in the GR limit C_q goes to $C_q^{(RN)}$ (i.e., $\beta_i \to 0$) or I becomes unit as expected.

As we discussed before for extremal case the Hawking temperature is

$$T_H = \frac{\frac{\partial}{\partial r} g_{tt}}{4\pi} \bigg|_{r=r_0} = \frac{2}{4\pi} e^{-2\phi} (1 - \frac{b_0}{r}) \frac{b_0}{r^2} \bigg|_{r=b_0} = 0, \tag{4.9}$$

however $S = \frac{\mathcal{A}}{4G} F \big|_{r=r_0} \neq 0$ and $C_q = 0$.

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4.3 Misner-Sharp Energy

From general relativity we know that gravitational field is totally intertwined with the energy. Although defining the energy in this case is one of the most challenging parts, Arnowitt-Deser-Misner (ADM) energy and Bondi-Sachs (BS) energy are two well-known expressions for the energy in GR at spatial and null infinity, respectively, which are described in an asymptotic flat spacetime for the isolated system[24].

Eenergy-momentum pseudotensor of the gravitational field which is related to metrics and its first derivative, in a locally flat coordinate will die in any point of the space-time therefore its local energy density cannot help us to define the total energy in other cases. Consequently, it leads us to define the quasilocal energy. There are some well-known definition for the quasilocal energy like Brown-York energy, Misner-Sharp energy, Hawking-Hayward energy and Chen-Nester energy[24].

Among all, we can only define Misner-Sharp energy in the spherically symmetric spacetime and also it has a nice connection between the first law of thermodynamics and Einstein equation in the Friedmann-Robertson-Walker (FRW) cosmological and black hole metrics[24].

In this section in order to derive the Misner-Sharp energy in non-asymptotic flat spacetime, we will use the equation of motion with the previous section results and first law of thermodynamics as shown below

$$G^{\nu}_{\mu} = \kappa \left[\frac{1}{F} T^{\nu}_{\mu} + \frac{1}{\kappa} \check{T}^{\nu}_{\mu} \right]. \tag{4.10}$$

Here G_μ^ν is the Einstein tensor,

$$\check{T}_\mu^\nu = \frac{1}{f_R} \left[\nabla^\nu \nabla_\mu F - \left(\Box F - \frac{1}{2} f + \frac{1}{2} RF \right) \delta_\mu^\nu \right],$$ (4.11)

and in this case we consider general form of the metric

$$ds^2 = -e^{-2\Phi} U d^2 t + \frac{1}{U} d^2 r + r^2 d\Omega^2.$$ (4.12)

Since we want to derive the Misner-Sharp energy from equations of motion, from the tt component of field equation we have

$$G_0^0 = \kappa \left[\frac{1}{F} T_0^0 + \frac{1}{\kappa} \frac{1}{F} \left[\nabla^0 \nabla_0 F - \left(\Box F - \frac{1}{2} f + \frac{1}{2} RF \right) \right] \right],$$ (4.13)

where

$$G_0^0 = \frac{U' r - 1 + U}{r^2},$$ (4.14)

and

$$\nabla^0 \nabla_0 F = \frac{1}{2} \left(-2\Phi' U + U' \right) F',$$ (4.15)

and

$$\Box F = \frac{2}{3} f - \frac{1}{3} RF.$$ (4.16)

Because MS energy is introduced at the horizon we have to write the equations of

motion at the horizon (where $U(r_0) = 0$) which yields

$$G_0^0 = \frac{U_0' r_0 - 1}{r_0^2}, \nabla^0 \nabla_0 F = \frac{1}{2} U_0' F_0', \qquad (4.17)$$

Thus, field equation 4.13 can be written as

$$\frac{F_0 U_0'}{r_0} - \frac{F_0}{r_0^2} = \kappa T_0^0 + \left(\frac{1}{2} U_0' F_0' - \frac{1}{6} (f_0 + R_0 F_0) \right). \qquad (4.18)$$

Now, we have to derive the first law of thermodynamics from the field equation therefore we multiply both sides by the spherical volume element at the horizon $dV_0 = \mathcal{A} dr_0$ to get

$$\frac{F_0 U_0'}{r_0} \mathcal{A} dr_0 = \left(\frac{F_0}{r_0^2} + \frac{1}{2} U_0' F_0' - \frac{1}{6} (f_0 + R_0 F_0) \right) \mathcal{A} dr_0 + \kappa T_0^0 dV_0. \qquad (4.19)$$

Using $\frac{\mathcal{A}}{r_0} = \frac{1}{2} \frac{d}{dr_0} \mathcal{A}$ and some calculation we obtain

$$\frac{U_0'}{4\pi} \frac{d}{dr_0} \left(\frac{2\pi \mathcal{A}}{\kappa} F_0 \right) dr_0 = \frac{1}{\kappa} \left(\frac{F_0}{r_0^2} + U_0' F_0' - \frac{1}{6} (f_0 + R_0 F_0) \right) \mathcal{A} dr_0 + T_0^0 dV_0. \qquad (4.20)$$

By comparing this equation with the first law of thermodynamics $T ds = dE + P dV$ where $T_H = \frac{U_0'}{4\pi}$, $S_{BH} = \frac{2\pi \mathcal{A}}{\kappa} F_0$ and $P = T_r^r = T_0^0$ we can write the Misner-Sharp energy as the fallowing expression

$$E = \frac{1}{\kappa} \int \left(\frac{F_0}{r_0^2} + U_0' F_0' - \frac{1}{6} (f_0 + R_0 F_0) \right) \mathcal{A} dr_0, \qquad (4.21)$$

in which the integration constant is set to zero[25] (also for a BH-like solutions see [26]).

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Chapter 5

CONCLUSION

Extending R-theory to $f(R)$ theory of gravity is a big change and entails much novelties in general relativity. All these, however, are not free from mathematical complexity. Existence of exact solutions in $f(R)$ gravity has already been extensively studied in the literature. For this reason we concentrate on a particular type of solution, namely the Rissner-Nordström (RN) type solution. This is the static, spherically symmetric (SSS) black hole solution that carries a static electric charge. In Einstein's theory RN is the unique solution of its kind, but in the extended theories the uniqueness property is no more a valid argument. Owing to it's utmost importance, we consider RN type black hole solutions thoroughly in $f(R)$ gravity. Specifically, we apply the *"near horizon test"* to this kind of black hole solutions and derive the underlying equations/conditions. For this purpose, we expand analytically all the involved functions in the vicinity of the event horizon. Herein, by functions it is implied all metric functions plus $f(R)$ and its higher derivatives. Analyticity conditions/regularity, which amounts to admitting such expressions in the afore mentioned region, their continuity etc., all determine the necessary conditions for the existence of a RN-type black hole solutions. The equations obtained in this manner are labelled as zeroth, first, second and higher order constraint conditions. Given the intrinsic non-linearity of the theory the obtained equations are far from being solved analytically in higher orders. At least for the zeroth

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and first order equations we were able to handle them consistently and construct in this manner the necessary $f(R)$ function. This has been achieved as an infinite expansion in $(r-r_0)$, which is a small quantity around the horizon located at $r=r_0$. Next, expansion of scalar curvature R in terms of $(r-r_0)$ helps us to establish a relation between $f(R)$ and R, albeit in an infinite series form. Our analysis shows that a closed form of $f(R)$, unless the obtained infinite series are summable, is not possible. This is not an unexpected result as a matter of fact. Depending on the given physical source the first few dominating terms serve our purpose well. This is the prevailing strategy that has been adopted so far. Determining $f(R)$ alone may not suffice : additional conditions such as $\frac{df}{dR} > 0$ and $\frac{d^2 f}{dR^2} > 0$ must also be satisfied.

These are simply the conditions to avoid non-physical ghosts and instabilities[27, 28]. Again to the leading orders of expansions these can be tested. The RN-type black hole solutions that have been obtained have been studied thermodynamically. Definition of energy has been adapted from the Misner-Sharp (MS)[23]formalism (which is suitable for our formalism) and the first law of thermodynamics has been verified accordingly. The same MS definition has been used consistently before[29].

In conclusion, we have based our arguments entirely on the necessary conditions obtained from the *"near horizon test"* of RN-type and extremal RN-type black holes. It would be much desirable to obtain sufficient conditions as well, unfortunately this aspect has not been discussed in this thesis.

We admit also that since our necessary conditions for the existence of RN-type black holes are entirely local they don't involve the requirements for asymptotic flatness. Stability of such black holes must also be considered separately when one considers exact solutions.

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