# Type-2 Fuzzy Logic in Modeling Uncertainty

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# ABSTRACT

Thisthesis aims to analyze the type-2 fuzzy logic and sets to model uncertainty. Basic concepts of type-2 fuzzy sets are described. Operations on type-2 fuzzy sets are performed. Generalized and interval type-2 fuzzy sets are represented. Mamdani and Sugeno type-2 fuzzy systems are considered.

**Keywords:**Fuzzy logic and Sets,Uncertainty, Generalized and Interval Type-2 Fuzzy Sets, Mamdani and Sugeno Type-2 Fuzzy Systems

# ÖZ

Bu tezin amacı tip-2 bulanık mantık ve kümeler teorisini kullanarak belirsizligi modellemedir. Tip-2 bulanık kümeler ile ilgili temel kavramlar açıklanır. Tip-2 bulanık kümeler üzerinde işlemler gerçekleştirilir. Genelleştirilmişve aralık tip-2 bulanık kümeler belirtilir. Mamdani ve Sugeno tip-2 bulanık sistemler incelenir.

**Anahtar Kelimeler:**Bulanık Mantık ve Kümeler,Belirsizlik, Genelleştirilmişve Aralık Tip-2 Bulanık Kümeler, Mamdani ve Sugeno Tip-2 Bulanık Sistemler

I would like to dedicate this work to my beloved Mother who has always stood by me and light up my life with her wisdom words, guidance, spiritual support and her precious love and prays

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# Chapter 1

# INTRODUCTION

Earlier studies on the concept of ordinary fuzzy sets (the type-1 fuzzy sets) reveal the necessity of paying much consideration on the importance of general uncertainties in almost all facet of the real world. The word 'Fuzzy' is fondly regarded to be a state of unclear, unrealistic and inaccuracies in representation of information within a specific context. Several studies and opinions suggested situations where occurrences in real world are associated with uncertainties.

Professor Lotfi A. Zadeh presented the general fuzzy system (comprises of fuzzy sets and fuzzy logic systems) in the mid-60's. His idea of fuzzy is derived from the crisp nature of all aspect of life, for instance, every real number is believed to have associated element of fuzziness in it. And of course for real number 2 which is crisp and exact, it also has an associated fuzziness that could be represented as fuzzy 2 ( ) which is not a crisp number, not exact and not deterministic but with deterministic associated membership function (MF).

Comprehensive studies that followed the introduction of fuzzy by Zadeh had shown the use of fuzzy sets in representing varying degrees of uncertainties. The introduction of type-1 fuzzy sets also with their fuzzy logic systems have proved to be an effective tool in modeling some different situations in the concept of fuzzy sets. But due to the ineffectiveness in the modeling of some levels of uncertainties by the type-1 fuzzy sets and systems precisely in 1975 by Professor Lotfi A. Zadeh, extensive research confirmed the durability of new form of fuzzy sets known as the type-2 fuzzy sets and systems. The introduction of type-2 fuzzy set did not immediately gain popularity among researchers despite its known effectiveness because much time is naturally needed to carefully study the advantages and limitations of type-1 fuzzy sets in order to establish confidence in the durability of type-2 fuzzy sets over the former.

Zadeh subsequently introduced type-2 fuzzy set which is designed to make-up for the inefficiency of type-1 fuzzy sets in modeling the uncertainties. This type of fuzzy sets (type-2) is categorized of grades of membership (principal membership function) which are also fuzzy and fully represented as footprint of uncertainty of the type-2 membership function. Hence the core modeling parameter of type-2 fuzzy set is based on the fact that the membership function is of three-dimensional which therefore gives it new design values of freedom in proper representation of varying levels of uncertainties. Taking the primary variables (force, humidity) for instance, both the associated membership functions (second membership functions) with their domain primary membership functions are both located in the interval [0,1] and also their range which is known as secondary grades (which may also exist within an interval [0,1]) all make up the three dimension of the variables force and humidity.

Generally, several researchers had extensive studies on the set-theoretic operation and structures of type-2 set with its properties of membership grades which include operation of algebraic product and sum. In addition to these properties are the 'join' and 'meet' operations under minimum and product t-norm of fuzzy numbers which are also extended to fuzzy valued logic (general type-2 fuzzy logic systems).

Importantly, the popularity of type-2 fuzzy sets can be traced to its varying applicability in real world which include its use in processing survey data, decision making, solving both fuzzy relation and engineering-related equations.

# Chapter 2

# REVIEW OF EXISTING LITERATURE ON TYPE-2 FUZZY SETS AND LOGIC SYSTEMS

Fuzzy sets and logic have gained much popularity and have useful application in many areas. These applications of fuzzy sets, fuzzy logic and systems based on continuous research and studies are noticeable in real life. A review of recent studies in fuzzy type-2 and its applications is examined in this chapter.

[1] Introduces a robust fuzzy logic system which is very much related and useful in dealing with uncertainties and is applicable to a particular development that leads to a new operation called type-reduction. The study is also extended to set operations on type-2 sets, their memberships, relations and their compositions, and also defuzzification.

Implementation of type-2 fuzzy logic systems specifically involves inference, operations of fuzzification and processing of output [2]. Processing of these outputs which involve reduction and defuzzification shows that type-reduction methods are purely of type-1 defuzzification methods. The type-2 fuzzy logic system can be applied to time-varying channel equalization and hence preferably adopted for general computations done in type-1.

[3] Is about recent advances and development in type-2 fuzzy sets and systems. The main focus here is on the advances through theoretical knowledge on type-2 fuzzy sets and systems.

The relationship between type-1 and interval type-2 fuzzy logic systems is examined in [4]. The study shows the operations of input-output mappings of both types of these fuzzy logic systems where the type-1 fuzzy logic system is regarded as universal approximator. It is notated that type-1 fuzzy logic system can be discontinuous under a specific and derived condition. The interval type-2 fuzzy logic sets and systems are considered to have six types of reduction and defuzzification methods. All these methods - Karnik-Mendel method, Wu-Tan method, Nie-Tanmethod, Du-Ying method, Begian-Melek-Mendel method and the uncertainty bound method are also studied by observing the various conditions that are specifically applicable to both their continuous and discontinuous input-output mappings. The methods and procedure to obtain continuous and discontinuous interval type-2 fuzzy logic systems are also well-analyzed and detailed. With the use of this outlined procedure for interval type-2 fuzzy logic systems, favorable and continuous interval type-2 fuzzy logic system is obtained through selection of certain parameters of valued membership functions.

One of the theoretical studies of type-1 and type-2 fuzzy sets and logic is in time series prediction [5]. The study shows both the singleton and non-singleton type-1 back propagation method which is used in designing fuzzy logic system with sixteen rules. In the analysis of the types of these singletons, the back propagation system is adjourned for a type-2 fuzzy sets that is better in forecasting compare the singleton type-1 fuzzy logic system. The types of uncertainties which include non-stationarity and stable attractors studied within this framework are shown for chaotic data. The proper use of non singleton type-1 fuzzy logic system with non-stationarity type of data gives a better output than the use of an ordinary type-2 fuzzy logic system.

The prediction of Mackey-Glass chaotic time-series is discussed in [6]. The appropriateness of using type-2 fuzzy logic system over type-1 fuzzy logic system for considering information about noisy training data is explained.

Fuzzy logic technique because of dealing with uncertainty can be used as a very effective tool for signal processing. In [7] the importance of type-2 fuzzy logic system for model-based signal processing is presented. It is underlined that type-2 fuzzy logic system outperforms type-1 fuzzy logic system for prediction of Mackey-Glass chaotic time-series.

Moving away from the theoretical and academic relevance type-2 fuzzy sets and systems, it is relevant to review various applications of type-2 fuzzy sets and systems. Measures of uncertainties like the centroid, variance, cardinality, skewness and fuzziness are studied by deriving some formulas in [8].The definitions presented in this research are useful in measuring similarities between different interval type-2 fuzzy sets.

A derivation of inner and outer-bound sets for a type-reduced set of interval type-2 fuzzy logic system is properly examined in [9]. The demonstration using simulation experiment shows that the resulting design is capable of operating with and without a type-reduced set in a way to achieve close performance. The design method analyzed is more computationally intensive in operation of the estimation of interval type-2 fuzzy logic system that is based on the bound sets.

Representation of uncertainty and fuzziness in real life problems can be better performed using and type-2 fuzzy sets compare to type-1 fuzzy sets. The interval type-2 fuzzy TOPSIS method is presented in [10] for decision making problems to better evaluate values and weights of the attributes.

Type-2 fuzzy sets are theoretically studied and shown to have a significant importance to generate a variety of truth value algebra [11]. Mainly, a finite type of algebra, in particular locally finite type of algebra generates this specific variety of type-2 fuzzy sets.

Algebraic properties of extended fuzzy S-implications and complications are studied in [12].These properties of the algebraic operations and their relationships to extended t-conorms and t-norms at varying points are considered that give significant contribution for the application of type-2 approximate reasoning applications.

In [13] the relation between uncertainty measure and type-2 fuzzy sets is established on a general platform. A generalized method for computing existing types of uncertainty measures of interval type-2 fuzzy sets with less complexity using a modernized α-plane representation is offered. Different picture of footprint of uncertainty with varying types of secondary membership functions companied with the uncertainty measures of type-2 fuzzy sets are illustrated to be good examples with proper observations in summaries made in this regard. The feasibility with the approximation of type-2 fuzzy sets using Quasi-Type-2 (QT2), a comparative analysis of uncertainty measures of QT2, IT2 and Type-2 fuzzy sets is conducted.

Recent research and investigation show the advantage of type-2 fuzzy logic system over type-1 fuzzy logic system in dynamic uncertainty [14] because of its ability to handle with negative impacts of this uncertainty. Interval type-2 fuzzy sets are used to analyze the capabilities of uncertain modeling for a specific problem by improving the accuracy of the proposed model.

In [15] the method is proposed for group decision making problem based on ranking interval type-2 fuzzy sets with fuzzy multiple attributes. The proposed method is simpler than existing methods to deal with fuzzy multiple attributes of group decision making problems.

Ranking methods, similarity measures and uncertainty measures are known to be very important concepts while using type-2 fuzzy sets [16]. The applications of these methods and measures are analyzed on real survey data to select the most suitable paradigm.

The application of interval type-2 fuzzy logic system in mobile robotics is presented based on the attributes and capability of interval type-2 fuzzy logic systems in modeling dynamic uncertainties [17]. The fuzzy controller is designed that takes the uncertain nature of interval type-2 fuzzy logic system into consideration. The antecedent and consequent uncertainty quantifiers are suggested to monitor the modeling of the uncertainty during the inference period. Design of a wall-following navigation controller used in autonomous mobile robots show more accurate results compared to methods of classical design methods.

An inverse controller based on type-2 fuzzy model is designed in [18]. An example of this implementation is in the setup of PH neutralization experiment process where the closed-loop performance to disturbance rejection is improved. The internal model control structure is posed to eliminate errors of the model. The research shows that inverse type-2 fuzzy model controller has out-powering result over both the classical control structure and the inverse type-1 fuzzy controller.

[19] Considers type-2 fuzzy sets in which the membership functions themselves are fuzzy numbers. The proposed fuzzy logic controller (FLC) is based on such kind of sets. The type-2 FLC is designed by using genetic algorithms, and then this controller is compared with three other FLCs designed by using genetic algorithm and type-1 fuzzy sets. The experimental results show the better performance of type-2 FLC in terms of accuracy and interpretability.

In engineering applications, interval type-2 fuzzy sets are confirmed to play a central role in modeling and design of systems. The characterization by both boundaries of upper and lower membership functions of these fuzzy sets is an extension of characterization by their footprints of uncertainty (FOU) [20]. In this context, symmetric interval type-2 fuzzy sets with known centroid give the measures of uncertainties is the main subjects of this analysis. It is proposed that certain geometric properties associated with FUO like the centroid of both the lower and upper membership functions are attributed with the level of uncertainty in such a specific type-2 fuzzy sets. This research demonstrates the correctness of the above proposed properties associated with the uncertainty in type-2 fuzzy sets and in the quantification of the centroid of symmetric interval type-2 fuzzy sets with their uncertainties as regards their geometric properties.

The application of type-2 neuro-fuzzy system in modeling of predictions of stock prices with sets of specific data is analyzed in [21]. The specific dataset which can be segmented into clusters by the use of both input-similarity and output-similarity test and a unique type-2 rule is formed from each cluster to give a fuzzy rule base. Both least squares estimation and particle swarm optimization are used to refine the consequent and antecedent parameters attributed to the rules. The effectiveness of the type-2 neuro-fuzzy modeling approach in predicting the stock prices is experimentally proved.

# Chapter 3

# BASIC CONCEPTS OF TYPE-2 FUZZY SETS.OPERATIONS ON TYPE-2 FUZZY SETS

Introducing the concept of type-2 fuzzy sets by Zadeh [22] in the late seventies as an extension of fuzzy sets of type-1, he emphasized that since uncertainty is an ”attribute of information” which needed to be modeled hence it should be handled with much importance. Unlike with crisp numbers where it is difficult to determine the membership of such, fuzzy sets of type-1 are attributed with memberships function. Considering the age of a man to be thirty-two which is an exact number (crisp number), but the ages of men in a community are taken to be fuzzy sets (ordinary type-1 fuzzy sets) because an exact age is illusive. Hence the task of getting the exact value from such vagueness is difficult and then membership function of membership grade is attributed to such values in the modeling of fuzzy sets of type-1 and type-2. While completely dealing with uncertainties of any kind, specific higher grade or order like type-infinity fuzzy sets is ideally needed [23] in the perfect modeling of uncertainties. Based on the fact that type-1 fuzzy set is typically insufficient in the modeling of specific uncertainties, fuzzy set of type-2 is adjourned to model significantly better than type-1 fuzzy set.

## 3.1 Comparison of Fuzzy Logic Systems of Type-1 and Type-2

Both fuzzy logic systems of type-1 and type-2 have similar and simple structure with just minor features differentiating the two from each other. A simple structure shown in figure 1 is typical of fuzzy logic system of type-2 that models through its fuzzifier, rules base, fuzzy inference engine, to the defuzzifier (output processor) and with the integration of type-reducer to complement the output processing unit as to further automatically reduce the type-2 fuzzy set to type-1 fuzzy set [24].

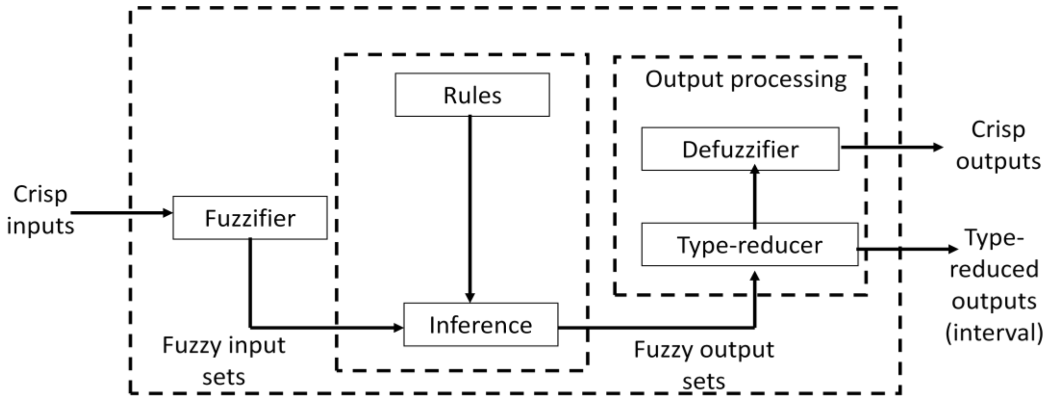


Figure 1: Type-2 Fuzzy Logic System Structure

Different sources of uncertainties are known today and the need to model them give rises to their attribution with fuzzy sets membership function. Type-1 fuzzy sets are directly unable to model such uncertainties, but type-2 fuzzy sets are known for such modeling because their membership functions are fuzzy unlike in type-1 fuzzy sets where membership functions are completely crisp. Detailed studies show that type-1 fuzzy sets are of two-dimensional membership function [25], and additional of a single dimension to these gives a type-2 fuzzy set which is known to have three dimensional membership functions. The extra single dimension added to type-1 fuzzy sets gives additional effect on the modeling of uncertainties by type-2 fuzzy sets. Although several challenges that arise from the use of type-2 fuzzy sets like the unavailability of simple collection of well-defined data, the use of complex relation formulas, difficulties in the diagrammatic representation and their computational complexities are cumbersome, but the modeling output are incomparable to that of type-1 fuzzy sets. Hence, generally type-2 fuzzy logic and sets are very useful in situations where the measurement of uncertainties exists with difficulties to determine the exact numeric membership function notwithstanding the aforementioned constrains in its computations.

In illustrating and comparing different forms of membership functions associated with fuzzy sets type-1 and type-2, we consider a small layers (hens or pullets that will be used to provide eggs) that lay and hatched for the first time a certain numbers of eggs (X) with associated membership degrees as shown in the table 1 given below.

Table 1: Membership Degree Associated with Numbers of Eggs Hatched by a Layer

| **X** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Associated Membership Degree (u)** | 0 | 0 | 0 | 0.1 | 0.6 | 1 | 0.6 | 0.1 | 0 | 0 | 0 |

From the above table, the associated membership degree attributed to the number of eggs that are hatched by the small chicken (layers) with some levels of uncertainty. These levels of uncertainty are obvious especially when a sketched diagram is used to illustrate the above information [26]. Modeling these types of uncertainties using the type-1 fuzzy sets is more of exactness of numerical values (crisp) as obviously shown in the membership function graph of type-1fuzzy sets(Figure 2)which are clearly seen to be very simple and plain as compared to the membership function graph showing the type-2 fuzzy sets (Figure 3).

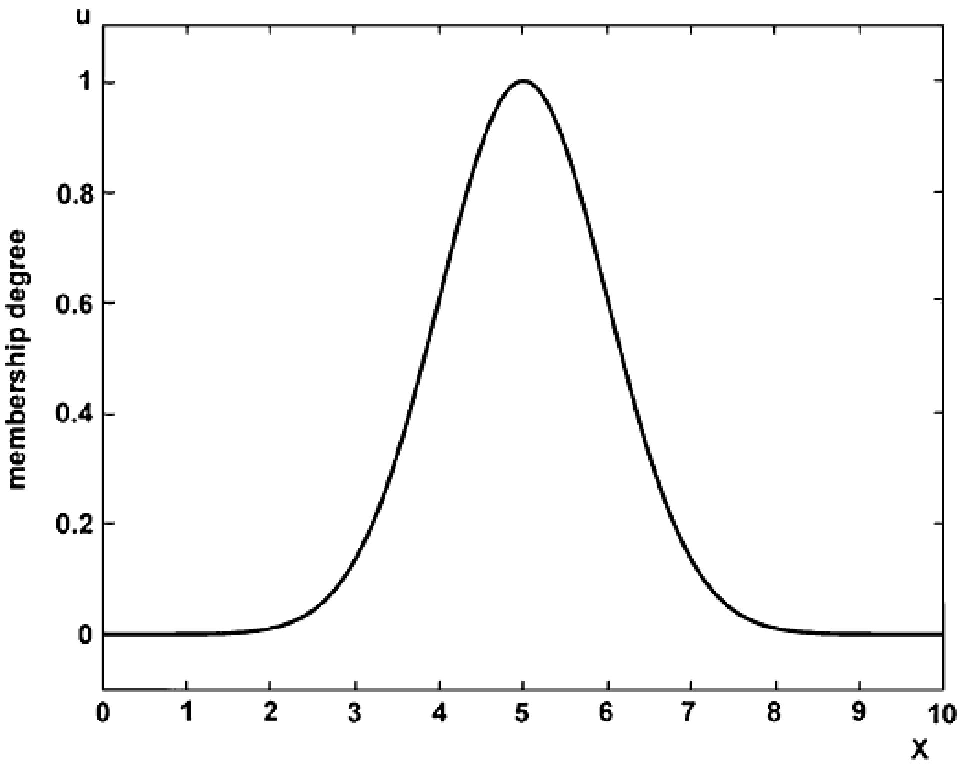


Figure 2: Membership Function Graph of Type-1 Fuzzy Sets

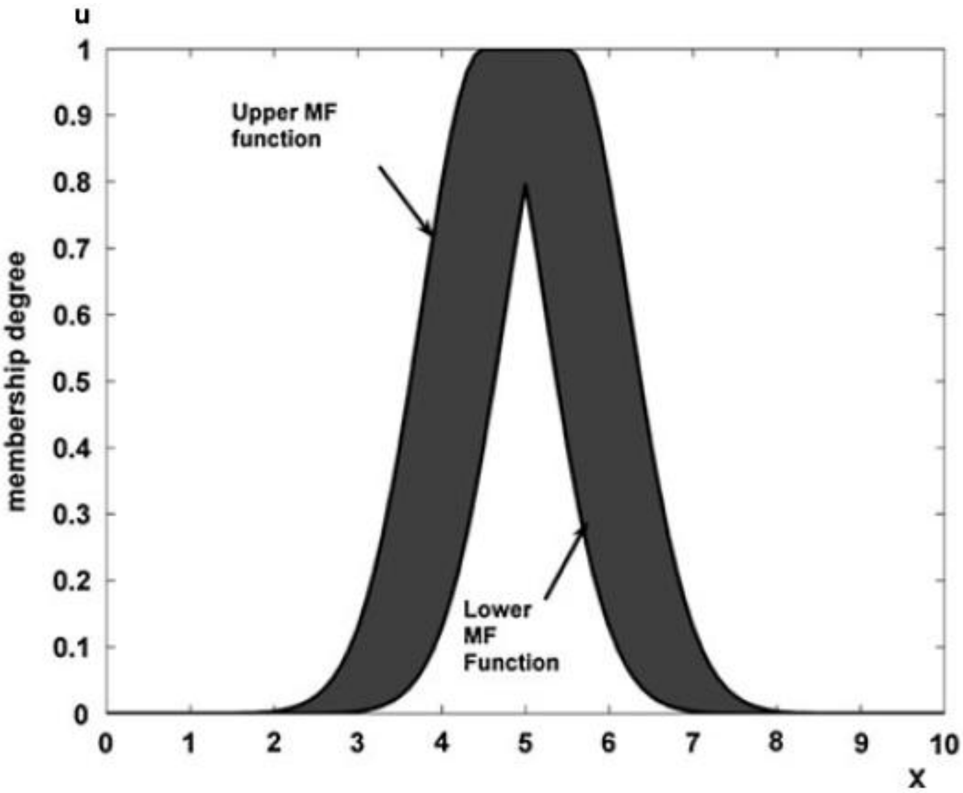

Figure 3: Membership Function Graph of Type-2 Fuzzy Sets

The thickness (level of uncertainty known as Footprint of Uncertainty-FOU) which is bounded both above and below with an upper bound and lower bound called upper membership function (UMF) and lower membership function (LMF), respectively. And the shape of the curve of the membership function of the below illustrated type-2 fuzzy sets certainly affirmed the emphases of it (type-2 fuzzy sets) in better modeling uncertainties.

Certain properties are attributed to membership grades discussed earlier for both type-1 and type-2 fuzzy sets. Generally, the properties of minimum t-norm and maximum t-conorm of membership grade are analyzed in details on the product t-norm and maximum t-conorm for convex and normal membership grades of type-2 fuzzy sets [27]. The entire reflexive, anti symmetric, transitive, idempotent, commutative, associative, absorption, distributive, involution, identity, complement and De Morgan's law properties are known to be preserved by both fuzzy sets of type-1 and type-2 in minimum t-norm. But with the exemption of idempotent, absorption, distributive and De Morgan's law properties that are not preserved under {maximum, product} t-conorm/ t-norm by type-2 fuzzy sets the rest of the properties are satisfied.

It is clearly stated that specific set-theoretic laws and properties that are unpreserved by type-1 fuzzy sets are known also not to satisfy type-2 fuzzy sets. While the converse of above may not be true, it is proven that any condition or property that is satisfied by type-1 fuzzy sets not necessarily to be satisfied by type-2 fuzzy sets [28]. Apart from the above mentioned laws and properties that define fuzzy sets of type-1 and type-2, the share five similar definitions of uncertainty measures using the centroid, cardinality, fuzziness (Yager's), variance and skewness with slightly different mathematical expressions.

Some of these properties and laws for both type-1 and type-2 fuzzy sets are easily often differentiated with use of mathematical expressions. Typically expressing the properties of minimum t-norm and product t-norm discussed earlier for both type-1 and type-2 fuzzy sets, the mathematical expressions , , $\subseteq$ and $\leq$ are used in place of V, * and $\leq$ (of type-1 fuzzy sets), respectively.

## 3.2 Modeling of Type-2 Fuzzy Sets. Operations on Type-2 Fuzzy Sets

As previously mentioned the introduction of type-reduction to the output processing unit of the type-1 fuzzy logic system shed more light on the significance of type-2 fuzzy logic system. In type-2 fuzzy logic system, IF-THEN rules are categorically adapted and used with their antecedent and consequent especially when the situations are obviously uncertain to determine the exact membership grades. In modeling uncertainties with type-2 of both fuzzy sets and fuzzy logic system, certain computational operations that include computing the union, intersection, complement on type-2 fuzzy set, performing type-2 reduction and defuzzification and also computing type-2 fuzzy relations and composition among others are performed.

The membership grades form some of these operations on type-2 fuzzy sets like the complement (negation), intersection and union are based on Zadeh's Extension principle are shown below respectively [22, 29].

Complement: (x) = {(1- ), ( )} (3.1)

Where is the complement of type-2 fuzzy set .

Intersection: (x) = min { (x), (x)} (3.2)

Union: (x) = min { (x), (x)} (3.3)

Where and are the membership degrees.

Above operations are performed on A and B of fuzzy sets of type-2 both in a universe X where (membership grades for A and B) of fuzzy sets type-2 belongs to [0.1].

The equations above are further examined in detailed using the operations of join, meet and negation under minimum t-norm operation as well as the expression of the relations and compositions using the same unions and intersections of type-2 fuzzy sets described above. The computations of both relations and compositions of type-2 fuzzy sets are performed on either the same product space or more than two different product spaces. A new form of representing type-2 fuzzy sets especially in a canonical form for better clarity and understanding is with the use of both vertical and wavy slices.

In illustrating the use of these major operators in type-2 fuzzy sets, we consider the secondary membership functions with associated membership grades of the average marks of two students A and B in particular science subjects x to form two type-2 fuzzy sets given below:

$(x) = 0.4/ 0.1 + 0.7/ 0.2$ and $(x) = 0.3/ 0.4 + 0.8/ 0.8$

Then, the major operations for type-2 fuzzy sets on and given above in form of the membership function of their union, intersection and complements can be computed as represented in the following way.

In computing the union of these two sets, we use the expression (3.3).Hence,

(x) = (x) (x) = (0.4/ 0.1 + 0.7/ 0.2) (0.3/ 0.4 + 0.8/ 0.8)

= (0.4 □ 0.3) / (0.1 □ 0.4) + (0.4 □ 0.8) / (0.1 □ 0.8) + (0.7 □ 0.3) / (0.2 □ 0.4) + (0.7 □ 0.8) / (0.2 □ 0.8)

= 0.3/ 0.4 + 0.4/ 0.8 + 0.3/ 0.4 + 0.7/ 0.8

= max (0.3, 0.3)/ 0.4 + max (0.4, 0.7)/ 0.8

= 0.3/ 0.4 + 0.7/ 0.8

In computing the intersection of these two type-2 fuzzy sets and , we use the expression (3.2). Hence,

(x) = (x) (x) = (0.4/ 0.1 + 0.7/ 0.2) (0.3/ 0.4 + 0.8/ 0.8) =

= (0.4 □ 0.3) / (0.1□ 0.4) + (0.4 □ 0.8) / (0.1 □ 0.8) + (0.7 □ 0.3) / (0.2 □ 0.4) + (0.7 □ 0.8) / (0.2 □ 0.8) =

= 0.3/ 0.1 + 0.4/ 0.1 + 0.3/ 0.2 + 0.7/ 0.2

= max (0.3, 0.4)/ 0.1 + max (0.3, 0.7)/ 0.2 =

= 0.4/ 0.1 + 0.7/ 0.2

Also the complement of these fuzzy sets and can likewise be computed using the expression (3.1).Hence,

(x) = 0.4/ (1- 0.1) + 0.7/ (1- 0.2)= 0.4/ 0.9 + 0.7/ 0.8

In similar manner the complement of type-2 fuzzy set is computed below:

(x) = 0.3/ (1- 0.4) + 0.8/ (1- 0.8)= 0.3/ 0.6 + 0.8/ 0.2

While the computations above involving two simple type-2 fuzzy sets are slightly easy, in computing several or numerous type-2 fuzzy sets we need to adopt more complex computational methods like using programming and other mathematical software (like MATLAB, LaTeXetc.) which can generate accurate, relatively easier and time efficient outputs.

# Chapter 4

# GENERALIZED AND INTERVAL TYPE-2 FUZZY SETS.MAMDANI AND SUGENO TYPE-2 FUZZY SYSTEMS

A brief introduction was earlier stated about the relationship between type-1 and type-2 fuzzy sets and one of them is that type-2 fuzzy sets are confirmed to be an extension of type-1 fuzzy sets. Research also showed that type-2 fuzzy sets (fuzzy logic systems) undergoes fuzzification (type-reduction) to be type-1 fuzzy logic systems. For some time now much interest in emphases has turned to more general types of type-2 fuzzy sets and systems. These types of type-2 fuzzy sets and systems are generally known interval type-2 fuzzy sets and interval type-2 fuzzy logic systems.

## 4.1 Generalized Type-2 Fuzzy Sets and Fuzzy Logic Systems

Modeling with type-2 fuzzy sets, representation theorem in terms of simplification of type-2 fuzzy sets is recently used in deriving theoretical results which of course rarely used for computational purpose because of its huge explicit an enumeration demand and hence is better approached differently. Notwithstanding, this representation theorem for type-2 fuzzy sets is effected and gives most useful results in type-2 fuzzy set theory because of its relevant in deriving useful parameters attributed with the theorem in a very simple and direct manner irrespective of either old or new. A part from widely known applications of type-2 fuzzy sets is also majorly in a rule-based fuzzy logic system known as type-2 fuzzy logic system [3]. It is not a surprise that type-2 fuzzy logic system is new and majorly calculated as type-reduction which is a description that shows the mapping of type-2 fuzzy sets into type-1 fuzzy sets that is further defuzzified to obtaining a number (of crisp value) as an output of the type-2 fuzzy logic systems.

As mentioned in the previous chapter, a general type-2 fuzzy logic system shares certain similarities with type-1 fuzzy logic system. The introduction of type-reducer within the new output processing compartment as against defuzzifier compartment in type-2 fuzzy logic systems proves to be the major difference between these two types of fuzzy logic systems (type-1 fuzzy logic system and type-2 fuzzy logic system). The processes from the inputs stage through the fuzzifier, inference, rules and the defuzzifier to outputs stage are similar in both types. But at the type-reducer in type-2 fuzzy logic system, the application of type-2 fuzzy set is enhanced which causes its (output of the inference engine) reduction to type-1 fuzzy logic system and after further defuzzification the number value obtained as an output is a crisp type.

## 4.2 Representation of Interval Type-2 Fuzzy Sets and Fuzzy Logic Systems

The representation theorem earlier discussed can be specialized to interval type-2 fuzzy sets because of its much relevance generally for fuzzy sets. And this interval type-2 fuzzy set is believed to have connections with interval-valued fuzzy sets and as such these two are also proved to be the same [22].

In representing interval type-2 fuzzy sets, both the lower and upper membership functions within the footprint of uncertainty (FOU) are used for detail description. In general type-2, fuzzy sets are always considered to be unique type of interval type-2 fuzzy sets and hence any form of design for type-2 fuzzy sets is specially adapted for interval type-2 fuzzy sets. But the interval type-2 fuzzy sets are preferably used ahead of the other types of type-2 fuzzy sets like interval value fuzzy sets basically because of their simplicity during application. The form of type-2 fuzzy sets that are of interval type-2 fuzzy sets have all secondary grades to be equal 1 which actually present it (interval type-2 fuzzy set) as special form of type-2 fuzzy sets [25].

Previously we discussed on how fuzzy logic system can be described completely in term of type-1 fuzzy sets which is referred to type-1 fuzzy logic system and also how fuzzy logic system with at least one type-2 fuzzy system is known as type-2 fuzzy logic system. We emphasized also that because of the impossibility of type-1 fuzzy logic systems to directly model associated uncertainties in this context because it uses type-1 fuzzy sets which are of certain values; hence type-2 fuzzy logic systemsare preferably used. Because of the type-reduction and difuzzification processes associated with type-2 fuzzy logic and system where the output processer of type-2 fuzzy logic systems is respectively mapped into a type-1 fuzzy set and finally to crisp output. In computing type-2 fuzzy logic systems, it is only practically possible when every type-2 fuzzy set is interval type-2 fuzzy system which is also regarded as interval type-2 fuzzy logic system. Because of the problem associated with the use of type-reduction in interval type-2 fuzzy logic systems for computation in real-time applications especially during the iterative Karnik-Mendel (KM) algorithm, recent research has shown that the use of minimax uncertainty bounds (lower and upper bounds) can effectively replace type-reduction method. The effectiveness of KM algorithm is studied and shown to be very simple, able to run in parallel because of its independency and also of their ability to converge monotonically and of high-exponentially faster [30].

## 4.3 Representation of Interval Type-2 Fuzzy Sets using Ranking Methods

An important method of representing interval type-2 fuzzy sets is with use of ranking of their composite elements. Researches had shown that there are more than 35 different methods of ranking in type-1 fuzzy sets as compared to only one known method of ranking for interval type-2 fuzzy sets. This method of ranking of type-2 fuzzy sets which is called Mitchell's method [31] remains an ideal and known method of ranking interval type-2 fuzzy sets with the use of reasonable ordering properties and new ranking method recently surfacing.

It is important to discuss the ordering properties for interval type-2 fuzzy sets which are fondly regarded by Wang and Kerre [32] as reasonable ordering properties. In the study, seven reasonable ordering properties for type-1 fuzzy sets were comprehensively observed and when extended to interval type-2 fuzzy sets these properties were observed. The following shows these reasonable ordering properties in interval type-2 fuzzy sets.

a) If ≥ and ≥ , then ~ .

b) If ≥ and ≥ , then ≥ .

c) If = and is rightwise of , then ≥ .

d) The order of and is unaffected by the IT2 FSs under comparison.

e) For any interval type-2 fuzzy sets , ≥ ( this an exceptional ordering property).

f) If ≥ , then + ≥ + .

g) If ≥ , then ≥ .

Where ≥, ~, and ∩ as shown above indicate larger than or equal to (in term of ranking), the same rank, empty rank and intersection of rank, respectively.

A part from the fifth reasonable ordering property shown above which is explicitly satisfied in the new centroid-based ranking method, also only the third property above among the others six properties is satisfied using Mitchell's method for ranking interval type-2 fuzzy sets.

## 4.4 Advance Representations of Type-2 Fuzzy Sets and Logic Systems

Earlier, it was stated that a traditional method of type-1 fuzzy sets is always relevant and applied in information modeling especially when there exists difficulty in determining the membership of an element in a set as either 0 or 1. Hence for this same reason it was adequately acknowledged that the type-1 fuzzy sets are widely insufficient categorically when the situations are deeply (highly) fuzzy such that it is very challenging and if not almost impossible to determining the membership degree especially as crisp number [0, 1]. And in such regard described lately above, a type-2 fuzzy set is widely considered appropriate in modeling such level of variance in uncertainty. But even with the generalized modeling techniques and the level of uncertainty in type-2 fuzzy sets, adequate research studies has also proved the need for more sophisticated and slightly complex approach to model efficiently some higher level of inconsistencies in information and possibly modeling of linguistic information. Illustratively, a kind of type-2 fuzzy set which is known as the interval type-2 fuzzy set is adequately represented using an advance measures and model techniques. Previously, several authors had proposed methods of solving some of these real-world problems especially with the use of aggregation operators by using different sources of information such as arriving at a precise conclusion with a numerical aggregation. While the basic aggregation operators which include both the arithmetic, geometric and harmonic mean and weighted mean, the likes of Choquet Integral, Sugeno Integral and Ordered Weighted Averaging (OWA) are regarded as more complex and pretty adequate for modeling varying forms of uncertainties.

Despite the huge usefulness and application of the so-called complex aggregation operators discussed above in the area of biometrics, intelligent techniques and in fuzzy logic, there are really largely known to be also insufficient in modeling uncertainties in information sources and its relevance within this concept of type-2 fuzzy sets especially by combining fuzzy measures and type-2 fuzzy logic using the same operator. Hence, it is necessary to briefly highlight and discuss the modification and pattern of completely new operators of representing the situation identified above which was studied recently using the approaches and concepts of Sugeno and Mamdani in both interval type-2 fuzzy logic and fuzzy logic controllers, respectively.

### 4.4.1 Mamdani Concept of Fuzzy Sets and Logic Systems

Also noted previously was the lapses of using the database of a rule-based system as a result of the presence of significant level of imprecision which tend to appear in the description of the rules given by the expert. Hence, using the inference rule-based known as the 'compositional rule of inference' which uses non-classical two valued logic led to discovery of several methods for fuzzy reasoning using the principles put forward by the duo of Zadeh and Mamdani in 1975 and 1977, respectively. Although earlier research showed that the modification of Sugeno type fuzzy controllers gives universal approximators but this has not gone a long way because many real fuzzy logic controllers do not share the same characteristics of either the membership function, type of rule-based used or the inference mechanism implored. Apart from the fact that the fuzzy logic controller (FLC) put forward by Mamdani is the type that work with crisp data as inputs, previous studies have shown the extension of it (Mamdani model) to be compatible with interval inputs where representation of the fuzzy sets are in form of triangular fuzzy number [33]. Even lately the most frequently used fuzzy inference still remain the commonly known Mamdani method [34] especially because of its applicability in modeling fuzzy interval inputs and linguistic terms which are continuing to be mans' most important facet of living. The incorporation of this advantageous nature of the Mamdani method with the uncertainty nature of membership functions of fuzzy sets associated with these linguistics terms especially of interval type-2 fuzzy sets as currently being studying will be of huge interest to researchers.

### 4.4.2 Concept of Sugeno Integral in (Interval) Type-2 Fuzzy Set

As earlier stated that certain aggregation operator tools like the Sugeno and Choquet integral with the introduction of both integrals in 1974 and 1954 respectively are becoming more and more relevant in fuzzy systems [35],and the introduction of these integrals and fuzzy measures in aggregation operators continue to extend the application of fuzzy sets and systems to numerous fields of life.

Generally, Sugeno integral uses fuzzy measures to represent the uniqueness of vectors by expressing with weighted minimum and weighted maximum. Sugeno integral is applicable to solving wide varieties of problems but of finite sets of *n* elements *X,* i.e. { , ,..., }where the element type could be criteria in multi-criteria decision problem, situations under uncertainty, multi-opinion systems and in cooperative game (of game theory). Given a function *f: X* with respect to fuzzy measure $\mu$, Sugeno integral is simply expressed as

) (4.1)

Where is equivalent to $\{x_{\sigma(j)}$ *such that* $j \leq k\}$ and $\sigma$ is a permutation given that *f* ( ) $\geq$ *f*( ) for all $i \geq 1$.

The FOU (footprint of uncertainty) earlier discussed as a parametric representation in Type-2 Fuzzy Sets is conveyed by the union of all the primary memberships of a certain Type-2 element. While the FOU of typical type-2 fuzzy sets is comprised of the lower and upper bound membership functions, the interval type-2 fuzzy set is similarly bounded by an upper membership function and lower membership function which are both type-1 fuzzy sets.

If is a type-2 fuzzy set and characterized by membership function (*x*, *u*) where *x* is an element of *X* then the upper membership function (UMF) (x) ≡ and lower membership function (LMF) _ (x) ≡ _ and subsequently the interval type-2 fuzzy sets could be simply represented as

= 1 / FOU ( ) (4.2)

especially when consideration one as the only secondary membership degree for all existing points of FOU ( ).

In a situation where there is uncertainty in many information sources with large range of possible values associated with each as stated earlier, extension of Sugeno integral to interval type-2 fuzzy logic could be applicable.

# Chapter 5

# CONCLUSION

In this thesis the detailed study of type-2 fuzzy sets and logic system is analyzed. Dating from the older studies of modeling uncertainties using different methods, comprehensive studies over the time and most recently led to the affirmation of modeling higher degree of uncertainties with the use of type-2 fuzzy sets and logic systems. Several facts about type-2 fuzzy logic system like its three-dimensional nature, possessing membership grades that are fuzzy in nature and its modeling ability to outperform type-1 fuzzy logic system are also analyzed.

The interval type-2 fuzzy logic system which is a form of generalized type-2 fuzzy sets and logic system is described according to the modeling pattern with the use of Mamdani concept and Sugeno integral. Hence, it is recommended for future study that detail representation and computation of this subject with the above concepts is considerably studied and which could also unravel other new and to enhancing better modeling concept of type-2 fuzzy sets and logic systems.

# REFERENCES


[1] Karnik, N. N., Mendel, J. M.(1998). Introduction to Type-2 Fuzzy Logic System.*IEEE International Conference on Fuzzy Systems Proceedings, IEEE World Congress on Computational Intelligence, Volume 2*, pp. 915-920.

[2] Karnik, N. N., Mendel,J. M.,and Liang, Q.(1999). Type-2 Fuzzy Logic Systems.*IEEE Transactions on Fuzzy Systems, Volume 7,Issue 6*, pp. 643-658.

[3] Mendel, J. M. (2007). Advances in type-2 fuzzy sets and systems.*Information Sciences, vol. 177*, pp. 84-110.

[4] Dongrui Wu, Mendel, J.M. (2011). On the Continuity of Type-1 and Interval Type-2 Fuzzy Logic Systems. *IEEE Transactions on Fuzzy Systems, Volume 19, Issue 1*,pp. 179-192.

[5] Yasmin Zahra Jafri, Lalarukh Kamal, Amir Waseem and S. MasoodRaza. (2012). Rule based fuzzy logic (FL) time series prediction. *International Journal of Physical Sciences, Vol. 7(11)*, pp. 1862-1873.

[6] Nilesh N. Karnik, Jerry M. Mendel. (1999). Applications of type-2 fuzzy logic systems to forecasting of time-series.*Information Sciences, Volume 120, Issues 1–4*,pp. 89-111.

[7] Mendel, J. M. (2000). Uncertainty, Fuzzy Logic, and Signal Processing.*Signal Processing Journal, Volume 80,Issue 6,*pp. 913-933.

[8] Dongrui Wu, Jerry M. Mendel. (2007). Uncertainty measures for interval type-2 fuzzy sets. *Information Sciences, 177,* pp. 5378-5393.

[9] Wu, H., J. M. Mendel. (2002). Uncertainty Bounds and Their Use in the Design of Interval Type-2 Fuzzy Logic Systems.*IEEE Transactions on Fuzzy Systems. Volume 10, No.5,* pp. 622-639.

[10] Shyi-Ming Chen, Yu-Chuan Chang. (2011). Fuzzy rule interpolation based on the ratio of fuzziness of interval type-2 fuzzy sets. *Expert Systems with Applications, Volume 38, Issue 10,* pp. 12202-12213.

[11] John Harding, Carol Walker, Elbert Walker. (2010).The variety generated by the truth value algebra of type-2 fuzzy sets.*Fuzzy Sets and Systems, Volume 161, Issue 5,* pp. 735-749.

[12] Zsolt Gera,JózsefDombi. (2008). Type-2 implications on non-interactive fuzzy truth values.*Fuzzy Sets and Systems, Volume 159, Issue 22,* pp. 3014-3032.

[13] DaoyuanZhai. (2011). Uncertainty measures for general Type-2 fuzzy sets.*Information Sciences. Volume 181, Issue 3,* pp. 503-518.

[14] Linda, O., Manic, M. (2011). Uncertainty modeling for interval Type-2 Fuzzy Logic Systems based on sensor characteristics. *IEEE Symposium on Advances in Type-2 Fuzzy Logic Systems (T2FUZZ),*pp. 31-37.

[15] Shyi-Ming Chen, Ming-Wey Yang, Li-Wei Lee, Szu-Wei Yang. (2012). Fuzzy multiple attributes group decision-making based on ranking interval type-2 fuzzy sets. *Expert Systems with Applications. Volume 39, Issue 5,* pp. 5295-5308.

[16] Dongrui Wu, Jerry M. Mendel. (2009). A comparative study of ranking methods, similarity measures and uncertainty measures for interval type-2 fuzzy sets.*Information Sciences, Volume 179, Issue 8,* pp. 1169-1192.

[17] Linda,O., Manic, M.(2011). Uncertainty modeling with Interval Type-2 Fuzzy Logic Systems in mobile robotics.*IECON 2011 - 37th Annual Conference on IEEE Industrial Electronics Society,* pp. 2441-2446.

[18]Kumbasar, T., Eksin, I., Guzelkaya, M., Yesil, E. (2012). Type-2 fuzzy model based controller design for neutralization processes.*ISA Transactions, Volume 51, Issue 2,*pp. 277-287.

[19]DongruiWu, Woei Wan Tan.(2006). Genetic learning and performance evaluation of interval type-2 fuzzy logic controllers.*Engineering Applications of Artificial Intelligence. Volume 19, Issue 8,*pp. 829-841.

[20] Mendel, J. M., H. Wu.(2006). Type-2 Fuzzistics for Symmetric Interval Type-2 Fuzzy Sets: Part 1, Forward Problems. *IEEE Transactions on Fuzzy Systems, Volume 14,* pp. 781-792.

[21]Chih-Feng Liu, Chi-Yuan Yeh, Shie-Jue Lee.(2012). Application of type-2 neuro-fuzzy modeling in stock price prediction.*Applied Soft Computing. Volume 12, Issue 4,*pp. 1348-1358.

[22] L. A. Zadeh. (1975). The Concept of a Linguistic Variable and its ApplicationtoApproximate Reasoning -I. *Information Sciences 8,* pp. 199-249.

[23] Nilesh N. Karnik, Jerry M. Mendel. (1998). Introduction to type-2 fuzzy logic systems.*IEEE International Conference on Fuzzy Systems, Volume 2,* pp. 915-920.

[24]Oscar Castillo, Patricia Melin (2012). Recent Advances in Interval Type-2 Fuzzy Systems.Type-2 fuzzy logic systems (book chapter). *Springer*, pp. 7-12.

[25] Jerry M. Mendel, Robert I. Bob John. (2002). Type-2 Fuzzy Sets Made Simple. *IEEE Transactions on Fuzzy Systems, Volume 10, No. 2,* pp. 117-127.

[26] Castillo, O., Melin, P. (2008). Type-2 Fuzzy Logic: Theory and Applications. *Springer*.

[27] M. Mizumoto, K. Tanaka. (1976). Some properties of fuzzy sets oftype-2. *Information and Computation, Volume 31, No. 4,* pp. 312–340.

[28] Nilesh N. Karnik, Jerry M. Mendel.(2001).Operations on type-2 fuzzy sets. *Fuzzy Sets and Systems, Volume 122,*pp. 327–348.

[29] D. Dubois, H. Prade.(1980). Fuzzy Sets and Systems: Theory and Applications. *Academic Press, Inc., NY.*

[30]Jerry M. Mendel. (2007). Type-2 Fuzzy Sets and Systems: An Overview. *IEEE Computational Intelligence Magazine, vol. 2, no. 1,*pp. 20-29.

[31] H.B. Mitchell. (2006). Ranking type-2 fuzzy numbers. *IEEE Transactions on Fuzzy Systems 14(2),*pp. 287–294.

[32] X. Wang, E.E. Kerre. (2001). Reasonable properties for the ordering of fuzzy quantities (II).*Fuzzy Sets and Systems 118,*pp. 387–405.

[33] Liu, F., Geng, H., Zhang, Y. Q. (2005). Interactive fuzzy interval reasoning for smart web shopping.*Applied Soft Computing. Volume 5, Issue 4,*pp. 433–439.

[34] Mamdani, E. H. &Assilian, S. (1975). An experiment in linguistic synthesis with a fuzzy logic controller.*International Journal of Man-Machine Studies. Volume 7, Issue 1,*pp. 1–13.

[35]Patricia Melin, Olivia Mendoza and Oscar Castillo (2011). Face Recognition With an Improved Interval Type-2 Fuzzy Logic Sugeno Integral and Modular Neural

Networks. *IEEE Transactions on Systems, Man and Cybernetics, Part A: Systems and Humans, Volume 41, Issue 5,* pp. 1001 - 1012.