Some Properties of Appell Polynomials

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Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the Degree of

Doctor of Philosophy
in
Mathematics

Eastern Mediterranean University
February 2014
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

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ABSTRACT

This thesis consists of five chapters. The first Chapter gives general information about the thesis. In the second Chapter, some preliminaries and auxilary results that are used throughout the thesis are given.

The original parts of the thesis are Chapters 3, 4 and 5 which are established from [35], [46] and [48]. In Chapter three, extended 2D Bernoulli and 2D Euler polynomials are introduced. Moreover, some recurrence relations are given. Differential, integro-differential and partial differential equations of the extended 2D Bernoulli and the extended 2D Euler polynomials are obtained by using the factorization method. The special cases reduces to differential equation of the usual Bernoulli and Euler polynomials. Note that the results for the usual 2D Euler polynomials are new.

In Chapter four, we consider Hermite-based Appell polynomials and give partial differential equations of them. In the special cases, we present the recurrence relation, differential, integro-differential and partial differential equations of the Hermite-based Bernoulli and Hermite-based Euler polynomials.

In Chapter five, introducing k-times shift operators, factorization method is generalized. The differential equations of the Appell polynomials are obtained. For the special case $k = 2$, differential equation of Bernoulli and Hermite polynomials are exhibited.

Keywords: 2D Bernoulli polynomial, 2D Euler polynomial, extended 2D Bernoulli polynomial, extended 2D Euler polynomial, Hermite-based Appell polynomials, factorization method

ÖZ

Bu tez beş bölümden oluşmuştur. Birinci bölümde, tez ile ilgili genel bilgiler verilmiştir. İkinci bölümde, tezde kullanılan tanım ve kavramlar hakkında temel bilgiler ve sonuçlar verilmiştir.

Bu tezin orijinal kısımları [35], [46] ve [48] nolu referanslardan ortaya çıkan üçüncü, dördüncü ve beşinci bölümlerdir. Üçüncü bölümde, iki değişkenli genişletilmiş Bernoulli ve Euler polinomları tanımlanmıştır. Buna ek olarak, iki değişkenli genişletilmiş Bernoulli ve Euler polinomlarının sağladığı rekürans bağıntıları verilmiştir. Faktorizasyon metodu kullanılarak, bu polinom ailelerinin sağladığı diferensiyel, integro-diferensiyel ve kısmi diferensiyel denklemler bulunmuştur. Özel durumlar, Bernoulli ve Euler polinomlarının diferensiyel denklemlerine düşer. Belirtelim ki, sonuçlar iki değişkenli Euler polinomları için yenidir.

Dördüncü bölümde, Hermite tabanlı Appell polinomları göz önüne alınmış ve bu polinomların sağladığı kısmi diferensiyel denklemler bulunmuştur. Özel durumlar olarak, Hermite-tabanlı Bernoulli ve Hermite-tabanlı Euler polinomlarının diferensiyel, integro-diferensiyel ve kısmi diferensiyel denklemleri verilmiştir.

Beşinci bölümde, k-defa artıran ve k-defa azaltan operatörler kullanılarak, faktorizasyon metodu genişletilmiş ve böylece Appell polinomlarının diferensiyel denklemleri bulunmuştur. Özel olarak, $k = 2$ için Bernoulli ve Hermite polinomlarının diferensiyel denklemleri verilmiştir.

Anahtar Kelimeler: İki değişkenli Bernoulli polinomu, iki değişkenli Euler poli-

nomu, genişletilmiş iki değişkenli Bernoulli polinomu, genişletilmiş iki değişkenli Euler polinomu, Hermite-tabanlı Appell polinomları, faktorizasyon metodu

ACKNOWLEDGMENT

I would like to thank to my supervisor Assoc. Prof. Dr. Mehmet Ali Özarslan for his suggestions, patient and support during the Phd process and the Phd thesis.

Also, I would like to thank to Prof. Dr. Nazım Mahmudov, Asst. Prof. Dr. Arif Akkeleş, Assoc. Prof. Dr. Sonuç Zorlu, Assoc. Prof. Dr. Hüseyin Aktuğlu , Asst. Prof. Dr. Mehmet Bozer for their support during the Phd process.

Besides, I would like to thank to my family for their support, love and patient during the Phd process.

NOTATIONS and SYMBOLS

$R_n(x)$	Appell Polynomial,
$B_n(x)$	Bernoulli Polynomial,
$E_n(x)$	Euler Polynomial,
$H_n(x)$	Hermite Polynomial,
$B_n(x,y)$	2D Bernoulli Polynomial,
$E_n(x,y)$	2D Euler Polynomial,
$B_n^{(\alpha,j)}(x,y)$	Generalized 2D Bernoulli Polynomial,
$E_n^{(\alpha,j)}(x,y)$	Generalized 2D Euler Polynomial,
L_n^-	Derivative Operator,
L_n^+	Multiplicative Operator,
$B_n^{(\alpha,j)}(x,y,c)$	Extended 2D Bernoulli Polynomial,
$E_n^{(\alpha,j)}(x,y,c)$	Extended 2D Euler Polynomial,
${}_AH_n(x,y,z)$	Hermite-Based Appell Polynomials,

$\theta_n^{-(k)}$	*k*-times Derivative Operator,
$\theta_n^{+(k)}$	*k*-times Multiplicative Operator,
$H_n^{(j)}(x,y)$	Gould-Hopper Polynomial,
H-K.F	Hermite-Kampé de Fériet Polynomial,
$P_n^{(j,c)}(x,y)$	Extended Gould-Hopper Polynomial,
${}_H B_n(x,y,z)$	Hermite-based Bernoulli Polynomial,
${}_H E_n(x,y,z)$	Hermite-based Euler Polynomial,
D_x	Derivative with respect to *x*,
D_x^{-1}	Integral.

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Chapter 1

INTRODUCTION

A polynomial set $\{P_n(x)\}_{n=0}^{\infty}$ is quasi-monomial under the action of the operators Θ_n^+ and Θ_n^-, independent of n, possess the following representation

$$\Theta_n^+(P_n(x)) = P_{n+1}(x) \text{ and } \Theta_n^-(P_n(x)) = nP_{n-1}(x)$$

and if Θ_n^+ and Θ_n^- are differential realizations then they satisfy the following differential equation

$$\Theta_{n+1}^-\Theta_n^+(P_n(x)) = P_n(x)$$

where

$$P_0(x) := 1 \textit{ and } P_{-1}(x) := 0.$$

The operators Θ_n^- and Θ_n^+ are called derivative and multiplicative operators, respectively. The following commutation relation is satisfied by the operators Θ_n^- and Θ_n^+

$$\left[\Theta_n^-, \Theta_n^+\right] = I$$

where I is the identity operator and [,] denotes Lie paranthesis. It was Y.Ben Cheikh who proved that "Every polynomial set is quasi-monomial" [9]. With the aid of monomiality principle new consequences were found for Hermite, Laguerre, Legendre and Appell polynomials in [8], [13], [15], [20], [38].

Throughout the thesis, we take into consideration of the Appell polynomials and their differential equations. First, we introduce some facts about Appell polynomials. The well known Appell polynomials are generated by

$$A(t)e^{xt} = \sum_{n=0}^{\infty} R_n(x)\frac{t^n}{n!}$$

where $A(t)$ is given via

$$A(t) = \sum_{n=0}^{\infty} \alpha_n t^n$$

which is an analytic function at $t = 0$. Considering

$$\frac{A'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!},$$

it is directly seen that for any $A(t)$, the derivatives of $R_n(x)$ satisfy

$$R'_n(x) = nR_{n-1}(x).$$

Thus, $\{R_n(x)\}_{n=0}^{\infty}$ are called an Appell polynomial set. The special choices of $A(t)$ give many well known polynomial sets. For instance,

- Taking $A(t) = 1$, we get the monomials $R_n(x) = x^n$.
- If $A(t) = e^{-\frac{t^2}{2}}$, then $R_n(x) = He_n(x)$, the Hermite polynomial (see [12]).

- Choosing $A(t)=\dfrac{t}{\lambda e^t-1}$ ($|t|<2\pi$ when $\lambda=1; |t|<\left|\log\lambda\right|$ when $\lambda\neq 1$), then $R_n(x)=\mathcal{B}_n(x)$, the Apostol-Bernoulli polynomial (see [2], [28], [40]). Note that when $\lambda=1$, we have the Bernoulli polynomial (see [45]).

- Letting $A(t)=(1-t)^{-\alpha}$ ($|t|<1$), then $R_n(x)=n!L_n^{(\alpha-n)}(x)$, the modified Laguerre polynomial (see [18]).

- By taking $A(t)=e^{ht^m}$, then $R_n(x)=g_n^m(x,h)$ the Gould-Hopper polynomial (see [19]).

- Choosing $A(t)=\dfrac{2}{\lambda e^t+1}$ ($|t|<\pi$ when $\lambda=1; |t|<\left|\log(-\lambda)\right|$ when $\lambda\neq 1$), then $R_n(x)=\mathcal{E}_n(x)$, the Apostol-Euler polynomial (see [24], [29], [36], [42]). Note that when $\lambda=1$, we have the Euler polynomial (see [45]).

- Putting $A(t)=\dfrac{2t}{\lambda e^t+1}$ ($|t|<\left|\log(-\lambda)\right|$), then $R_n(x)=\mathcal{G}_n(x)$, the Apostol-Genocchi polynomial (see [26], [27], [30], [36], [42]). The case $\lambda=1$ gives the Genocchi polynomial.

- Letting $A(t)=\prod_{i=1}^{m}\dfrac{\alpha_i t}{e^{\alpha_i t}-1}(|\alpha_i t|<2\pi)$, then $R_n(x)$ is the Bernoulli polynomial of oder m (see [5]). Note that, when $\alpha_i=1$ ($i=1,\cdots,m$) then these polynomials are called Barnes polynomials.

- Taking $A(t)=\prod_{i=1}^{m}\dfrac{2}{e^{\alpha_i t}+1}(|\alpha_i t|<\pi)$, then $R_n(x)$ is the Euler polynomial of order m (see [5]).

- Choosing $A(t)=e^{\sum_{i=0}^{d+1}\xi_i t^i}$ ($\xi_{d+1}\neq 0$), then $R_n(x)$ is the generalized Gould-Hopper polynomial (see [13]). This polynomial include the Hermite polynomial when $d=1$ and $d-$orthogonal polynomials for each positive integer d.

The differential equations of Bernoulli and Euler polynomials were found by He and Ricci (see [20]).

Two dimensional Appell polynomials

$$A(t)e^{xt+yt^j} = \sum_{n=0}^{\infty} R_n(x,y)\frac{t^n}{n!}$$

were defined by Bretti and Ricci (see [5]). Besides, recurrence relation and corresponding equations of $2D$ Appell polynomials were presented in [5]. Also, for the special case $j=2$, they obtained the corresponding recurrence and differential equations for $2D$ Bernoulli polynomials.

Afterward, the Hermite-Based Appell polynomials (H-B Appell) defined by Khan et al. (see [23]) via

$$A(t)e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} R_n(x,y,z)\frac{t^n}{n!}.$$

Moreover, H-B Apostol Bernoulli, Euler and Genocchi polynomials were introduced and investigated by Özarslan (see [34]).

In this thesis, we study how to obtain the differential equation, integro-differential equation and partial differential equation of the following Appell polynomial families:

- Extended $2D$ Bernoulli polynomial (E2DBP),
- Extended $2D$ Euler polynomial (E2DEP),

- Hermite-based Bernoulli polynomial (H-BBP),
- Hermite-based Euler polynomial (H-BEP).

To obtain differential equations of them, we present factorization method. The main idea of the factorization method is to find the derivative operator L_n^- and the multiplicative operator L_n^+ such that

$$L_{n+1}^- L_n^+ \ (A_n(x,y,z)) = \ A_n(x,y,z).$$

In order to generalize the factorization method, for each fixed $k \in \mathbb{N}_0$, we introduce $k-$times derivative operator by $\Theta_n^{-(k)}$

$$\Theta_n^{-(k)} (P_n(x)) = P_{n-k}(x)$$

and $k-$times multiplicative operator by $\Theta_n^{+(k)}$

$$\Theta_n^{+(k)} (P_n(x)) = P_{n+k}(x).$$

With the help of these operators, we introduce generalized factorization method by

$$\left(\Theta_{n+k}^{-(k)} \Theta_n^{+(k)}\right) (P_n(x)) = P_n(x).$$

This thesis is organized as follows:

In Chapter 2,

- some basic definitions and properties related with Appell polynomials,
- main definitions, some elementary properties of Bernoulli and Euler polynomials

are studied.

The original parts of the thesis are Chapters 3, 4 and 5 which are established from the papers [35], [46] and [48].

In Chapter 3, the E2DBP [48] is introduced by

$$(\frac{t}{e^t-1})^{\alpha} c^{xt+yt^j} = \sum_{n=0}^{\infty} B_n^{(\alpha,j)}(x,y,c)\frac{t^n}{n!},\ c>1$$

and the E2DEP [48] is introduced via

$$(\frac{2}{e^t+1})^{\alpha} c^{xt+yt^j} = \sum_{n=0}^{\infty} E_n^{(\alpha,j)}(x,y,c)\frac{t^n}{n!},\ c>1.$$

Notice that in the case $c=e$ and $\alpha=1$, these polynomials coincide with the usual 2DBP and 2DEP, respectively [5]. The corresponding results for the usual 2D Bernoulli polynomial were presented in [5]. However, the results for the usual 2D Euler polynomial are new. In obtaining differential equation of the E2DBP and E2DEP, we use the factorization method.

In Chapter 4, we consider H-B Appell polynomials which are defined by

$$A(t)e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} {}_{A}H_n(x,y,z)\frac{t^n}{n!},$$

where

$$A(t) = \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!}.$$

For the special case $A(t) = \dfrac{t}{e^t - 1}$ and $A(t) = \dfrac{2}{e^t + 1}$, we have H-BBP and H-BEP, respectively. Furthermore,

- recurrence relation,
- differential, integro-differential and partial differential equation of H-B Appell polynomials

are obtained.

For the special case $A(t) = \dfrac{t}{e^t - 1}$ and $A(t) = \dfrac{2}{e^t + 1}$, the corresponding equations are presented for H-BBP and H-BEP.

In Chapter 5, for a given Appell polynomial family we introduce the generalized factorization method via introducing the $k-$times shift operators $\Theta_n^{-(k)}$ and $\Theta_n^{+(k)}$ $(k \in \mathbb{N})$ (see [35]). For each $k \in \mathbb{N}$,

- recurrence relations,
- differential equations of Appell polynomials

are obtained.

For the special case $k = 2$, the differential equation of Bernoulli and Hermite polynomials are shown.

Chapter 2

PRELIMINARY AND AUXILIARY RESULTS

In this Chapter, some definitions and properties which are used throughout the thesis are presented.

2.1 Appell Polynomials and Gould-Hopper (or Hermite-Kampé de Fériet) Polynomials

In this section, we give some definitions and properties of the polynomial families which are crucial in the rest of the thesis. First, we present the Hermite–Kampé de Fériet polynomial (H-K.F), which is known as Gould-Hopper polynomial.

Definition 2.1.1 *[5] For $j \in \mathbb{N}$, the Hermite–Kampé de Fériet (H-K.F) polynomial is defined by*

$$e^{xt+yt^j} = \sum_{n=0}^{\infty} H_n^{(j)}(x,y)\frac{t^n}{n!}. \tag{2.1.1}$$

The explicit form of the H-K.F polynomial is given by

$$H_n^{(j)}(x,y) = n!\sum_{s=0}^{[\frac{n}{j}]} \frac{x^{n-js}y^s}{(n-js)!s!},\ j \in \mathbb{N} \tag{2.1.2}$$

and it is the solution of the heat equation

$$\begin{aligned}\frac{\partial^j}{\partial x^j}\mathcal{G}(x,y) &= \frac{\partial}{\partial y}\,\mathcal{G}(x,y), \\ \mathcal{G}(x,0) &= x^n.\end{aligned} \tag{2.1.3}$$

Definition 2.1.2 *[48] Extended H-K.F polynomial is defined by*

$$P_n^{(j,c)}(x,y) = n!\sum_{s=0}^{[\frac{n}{j}]}\frac{x^{n-js}y^s}{(n-js)!s!}(\ln c)^{n+s-js},\ c > 1 \tag{2.1.4}$$

where $j \geq 2$ *is an integer.*

Taking $c = e$, yields $P_n^{(j,c)}(x,y) = H_n^{(j)}(x,y)$ where $H_n^{(j)}(x,y)$ is H-K.F polynomial. The generating function of the extended H-K.F polynomial [48] is given by

$$c^{xt+yt^j} = \sum_{n=0}^{\infty}P_n^{(j,c)}(x,y)\frac{t^n}{n!}\ ;\ c > 1. \tag{2.1.5}$$

Furthermore, generalization of the extended H-K.F polynomial can be defined via

$$c^{x_1t+x_2t^2+...+x_rt^r} = \sum_{n=0}^{\infty}P_n^{(c,r)}(x_1,x_2,...,x_r)\frac{t^n}{n!}.$$

It is important to state that the generalized heat equation can be obtained in terms of the polynomial $P_n^{(j,c)}(x,y) = \mathcal{G}(x,y,c)$:

$$(\ln c)^{1-j}\frac{\partial^j}{\partial x^j}\mathcal{G}(x,y,c) = \frac{\partial}{\partial y}\,\mathcal{G}(x,y,c) \tag{2.1.6}$$

$$\mathcal{G}(x,0,c) = x^n(\ln c)^n\ .$$

Definition 2.1.3 *[5] Generalized 2D Bernoulli polynomial(G2DB) is given by*

$$(\frac{t}{e^t-1})^{\alpha} e^{xt+yt^j} = \sum_{n=0}^{\infty} B_n^{(\alpha,j)}(x,y)\frac{t^n}{n!}. \tag{2.1.7}$$

Definition 2.1.4 *[5] Generalized 2D Euler polynomial(G2DE) is given by*

$$(\frac{2}{e^t+1})^{\alpha} e^{xt+yt^j} = \sum_{n=0}^{\infty} E_n^{(\alpha,j)}(x,y)\frac{t^n}{n!}. \tag{2.1.8}$$

Definition 2.1.5 *[48] The E2DBP is defined by*

$$(\frac{t}{e^t-1})^{\alpha} c^{xt+yt^j} = \sum_{n=0}^{\infty} B_n^{(\alpha,j)}(x,y,c)\frac{t^n}{n!}, \quad c>1. \tag{2.1.9}$$

Definition 2.1.6 *[48] The E2DEP is defined by*

$$\frac{2^{\alpha}}{(e^t+1)^{\alpha}} c^{xt+yt^j} = \sum_{n=0}^{\infty} E_n^{(\alpha,j)}(x,y,c)\frac{t^n}{n!}, \quad c>1. \tag{2.1.10}$$

Definition 2.1.7 *[23] H-B Appell polynomials are defined by*

$$A(t)e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} {}_A H_n(x,y,z)\frac{t^n}{n!}, \tag{2.1.11}$$

where

$$A(t) = \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!}.$$

Taking $A(t) = \dfrac{t}{e^t - 1}$ and $A(t) = \dfrac{2}{e^t + 1}$, we get H-BBP and H-BEP which are given by

$$\frac{t}{e^t - 1} e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} {}_HB_n(x,y,z)\frac{t^n}{n!}, \tag{2.1.12}$$

$$\frac{2}{e^t + 1} e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} {}_HE_n(x,y,z)\frac{t^n}{n!}, \tag{2.1.13}$$

respectively. Besides this, Özarslan [34] defined the unification of H-B Appell polynomials via the generating relation

$$f^{\alpha}_{a,b}(x,t;k,\beta) = \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} P^{\alpha}_{n,\beta}(x;k,a,b)\frac{t^n}{n!}. \tag{2.1.14}$$

$$(k \in \mathbb{N}_0;\ a,b \in \mathbb{R}\setminus\{0\};\ \alpha,\beta \in \mathbb{C})$$

2.2 Some Properties of Bernoulli Polynomial

Bernoulli polynomial, first studied by Euler (see [3]), play an important role in the integral representation of differentiable periodic functions and in the approximation of such functions by means of polynomials (see [5]). Bernoulli polynomial, which is a special kind of Appell polynomials is given by

$$G(x,t) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}; \quad |t| < 2\pi. \tag{2.2.1}$$

First few Bernoulli polynomials are

$$B_0(x) = 1,\ B_1(x) = x - \frac{1}{2},\ B_2(x) = x^2 - x + \frac{1}{6}, \tag{2.2.2}$$
$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$$

Bernoulli numbers are defined by $B_n := B_n(0)$ and given by the following generating relation

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{2.2.3}$$

First few Bernoulli numbers are

$$B_0 = 1,\ B_1 = \frac{-1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = \frac{-1}{30} \tag{2.2.4}$$

and $B_{2k+1} = 0$ for $(k = 1, 2, ...)$. The following properties characterizes the Bernoulli polynomials:

$$\begin{aligned} B_n(x) &= \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \\ B_n(1-x) &= (-1)^n B_n(x),\ n \geq 0, \\ B_n'(x) &= nB_{n-1}(x), \\ B_n(x+1) - B_n(x) &= nx^{n-1}. \end{aligned} \tag{2.2.5}$$

Bernoulli numbers appeared in several areas of mathematics. For example,

- MacLaurin expansion of the trigonometric and hyperbolic tangent and cotangent functions

- the sums of consecutive integer powers of natural numbers

$$\sum_{k=0}^{l} k^r = \frac{B_r(l+1) - B_{r+1}}{r+1},$$

- the residual term of the Euler-Maclaurin quadrature formula (see [47]).

2.3 Euler Polynomial

Euler polynomial is given by

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{2.3.1}$$

The generating function of the Euler numbers E_n are given via:

$$\frac{2}{e^t+e^{-t}} = \sum_{n=0}^{\infty} E_n\frac{t^n}{n!}.$$

In the following formulas, the special value of Euler numbers and the relation between Euler numbers and e_k are presented

$$E_n(\frac{1}{2}) = 2^{-n}E_n$$

(see [6], [20]) and

$$e_k = (-\frac{1}{2})^k \sum_{h=0}^{k} \binom{k}{h} E_{k-h}, \tag{2.3.2}$$

respectively.

It is important to mention that some extensions of these polynomials and related polynomials were given in [16], [17], [25], [32], [33] and [44].

2.4 Generalized 2D Bernoulli Polynomial (G2DBP)

In 2004, Bretti and Ricci defined the G2DBP via [5],

$$(\frac{t}{e^t-1})^{\alpha} e^{xt+yt^j} = \sum_{n=0}^{\infty} B_n^{(\alpha,j)}(x,y)\frac{t^n}{n!} \tag{2.4.1}$$

where

$$(\frac{t}{e^t-1})^{\alpha} = \sum_{n=0}^{\infty} B_n^{(\alpha)}\frac{t^n}{n!}. \tag{2.4.2}$$

In the following theorem, the relationship between the G2DBP and H-K.F polynomials is given:

Theorem 2.4.1 *[5] The explicit form of* $B_n^{(\alpha,j)}(x,y)$ *is*

$$B_n^{(\alpha,j)}(x,y) = \sum_{k=0}^{n} \binom{n}{k} H_k^{(j)}(x,y) B_{n-k}^{\alpha} . \tag{2.4.3}$$

It was Bretti and Ricci, who found the recurrence relation and differential equations of $2D$ Bernoulli polynomial (see [5]) .

Theorem 2.4.2 *[5] For* $n \in \mathbb{N}$, *we have the following recurrence for* $2D$ *Bernoulli poly-*

nomial

$$
\begin{aligned}
B_0^{(j)}(x,y) &= 1,\\
B_{n+1}^{(j)}(x,y) &= \frac{-1}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_{n-k+1}B_k^{(j)}(x,y)\\
&\quad +(x-\frac{1}{2})B_n^{(j)}(x,y)+jy\frac{n!}{(n-j+1)!}B_{n-j+1}^{(j)}(x,y).
\end{aligned}
\tag{2.4.4}
$$

Shift operators are given by

$$L_n^- \ : \ = \frac{1}{n}D_x, \tag{2.4.5}$$

$$L_n^+ \ : \ = (x-\frac{1}{2})-\sum_{k=0}^{n-1}\frac{B_{n-k+1}}{(n-k+1)!}D_x^{n-k}+jyD_x^{j-1}, \tag{2.4.6}$$

$$\mathcal{L}_n^- \ : \ = \frac{1}{n}D_x^{-(j-1)}D_y, \tag{2.4.7}$$

$$
\begin{aligned}
\mathcal{L}_n^+ \ : \ &= (x-\frac{1}{2})+jyD_x^{-(j-1)^2}D_y^{j-1}\\
&\quad -\sum_{k=0}^{n-1}\frac{B_{n-k+1}}{(n-k+1)!}D_x^{-(j-1)(n-k)}D_y^{n-k}.
\end{aligned}
\tag{2.4.8}
$$

Corresponding equations are

$$
\begin{aligned}
&\left[\frac{B_n}{n!}D_x^n+\ldots+\frac{B_{j+1}}{(j+1)!}D_x^{j+1}+(\frac{B_j}{j!}-jy)D_x^j\right.\\
&\left.+\frac{B_{j-1}}{(j-1)!}D_x^{j-1}+\ldots+(\frac{1}{2}-x)D_x+n\right]B_n^{(j)}(x,y)=0,
\end{aligned}
\tag{2.4.9}
$$

$$
\begin{aligned}
&\left[(x-\frac{1}{2})D_y+jD_x^{-(j-1)^2}D_y^{j-1}+jyD_x^{-(j-1)^2}D_y^j\right.\\
&\left.-\sum_{k=1}^{n-1}\frac{B_{n-k+1}}{(n-k+1)!}D_x^{-(j-1)(n-k)}D_y^{n-k+1}-(n+1)D_x^{j-1}\right]B_n^{(j)}(x,y)=0,
\end{aligned}
\tag{2.4.10}
$$

$$\left[(x-\frac{1}{2})D_x^{(j-1)(n-1)}D_y+(j-1)(n-1)D_x^{(j-1)(n-1)-1}D_y+jD_x^{(j-1)(n-j)}(D_y^{j-1}+yD_y^{j})\right.$$

$$\left.-\sum_{k=1}^{n-1}\frac{B_{n-k+1}}{(n-k+1)!}D_x^{(j-1)(k-1)}D_y^{n-k+1}-(n+1)D_x^{(j-1)n}\right]B_n^{(j)}(x,y)=0;\ n\geq j \tag{2.4.11}$$

respectively where

$$D_x^{-1}=\int_0^x f(\xi)d\xi.$$

Note that, the case $j=2$ is also presented by Gabriella Bretti and Paolo E. Ricci.

Chapter 3

DIFFERENTIAL EQUATION OF THE EXTENDED 2D BERNOULLI AND THE EXTENDED 2D EULER POLYNOMIALS

In this Chapter, we present some results of our study [48].

3.1 Construction of the E2DB and E2DE Polynomials

The differential equation, recurrence relation, shift operators of the G2DBP and the G2DEP have not been found before. In this Chapter, first we extend the G2DBP and G2DEP and then, we find the differential equation, recurrence relation, shift operators for the E2DBP and E2DEP. In the special cases, we exhibit the results for the G2DBP and G2DEP.

3.2 The Extended 2D Bernoulli Polynomial(E2DBP)

We define the E2DBP via [48]

$$\frac{t^{\alpha}}{(e^{t}-1)^{\alpha}}c^{xt+yt^{j}} = \sum_{n=0}^{\infty} B_{n}^{(\alpha,j)}(x,y,c)\frac{t^{n}}{n!}, \quad c>1. \tag{3.2.1}$$

The following theorem states the relation between the E2DBP and the extended H–K.F polynomials:

Theorem 3.2.1 *[48] The explicit form of* $B_n^{(\alpha,j)}(x,y,c)$ *is*

$$B_n^{(\alpha,j)}(x,y,c)=\sum_{k=0}^{n}\binom{n}{k}P_k^{(j,c)}(x,y)B_{n-k}^{\alpha}\ ;\ c>1. \tag{3.2.2}$$

Proof. *Using (2.1.5) and (2.4.2) in the generating function of the E2DBP,*

$$\sum_{n=0}^{\infty}B_n^{(\alpha,j)}(x,y,c)\frac{t^n}{n!}=\frac{t^{\alpha}}{(e^t-1)^{\alpha}}c^{xt+yt^j},$$

the theorem is proved applying the Cauchy product of the series. ∎

Note that, taking $c=e$ and $j=2$, we get the explicit representation of generalized Bernoulli polynomials obtained in [5].

The following theorem includes the recurrence relation and corresponding operators and equations of the E2DBP:

Theorem 3.2.2 *[48] For* $n\in\mathbb{N}$ *the E2DBP satisfies the following recurrence relation*

$$B_0^{(\alpha,j)}(x,y,c)=1,\ B_{-k}^{(\alpha,j)}(x,y,c):=0$$

$$\begin{aligned}B_{n+1}^{(\alpha,j)}(x,y,c) &= (x\ln c-\frac{\alpha}{2})B_n^{(\alpha,j)}(x,y,c)+yj\frac{n!}{(n-j+1)!}(\ln c)B_{n-j+1}^{(\alpha,j)}(x,y,c)\\ &\quad -\frac{\alpha}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k^{(\alpha,j)}(x,y,c)B_{n+1-k}\end{aligned} \tag{3.2.3}$$

where B_n *is given by (2.2.3).*

Corresponding operators are

$$L_n^- \quad : \quad = \frac{1}{n\ln c} D_x, \tag{3.2.4}$$

$$L_n^+ \quad : \quad = x\ln c - \frac{\alpha}{2} + yj(\ln c)^{(2-j)} D_x^{(j-1)} \tag{3.2.5}$$

$$-\alpha \sum_{k=0}^{n-1} \frac{B_{n+1-k}}{(n+1-k)!} (\ln c)^{(k-n)} D_x^{n-k},$$

$$\mathcal{L}_n^- \quad : \quad = \frac{(\ln c)^{j-2}}{n} D_x^{1-j} D_y, \tag{3.2.6}$$

$$\mathcal{L}_n^+ \quad : \quad = (x\ln c - \frac{\alpha}{2}) + yj(\ln c)^{(j-1)(j-2)+1} D_x^{-(j-1)^2} D_y^{j-1} \tag{3.2.7}$$

$$-\alpha \sum_{k=0}^{n-1} \frac{B_{n+1-k}}{(n+1-k)!} (\ln c)^{(n-k)(j-2)} D_x^{-(j-1)(n-k)} D_y^{n-k},$$

where $n \geq 1,\ j \geq 2$ *is an integer and* $c > 1$.

The corresponding equations for the E2DBP are

$$\Big[(x - \frac{\alpha}{2\ln c}) D_x + yj(\ln c)^{1-j} D_x^j \tag{3.2.8}$$

$$-\alpha \sum_{k=1}^{n-1} \frac{B_{n+1-k}}{(n+1-k)!} (\ln c)^{k-n-1} D_x^{n+1-k} - n\Big] B_n^{(a,j)}(x,y,c) = 0,$$

$$\Big[(x\ln c - \frac{\alpha}{2}) D_y + j(\ln c)^{(j-1)(j-2)+1} D_x^{-(j-1)^2} D_y^{j-1} \tag{3.2.9}$$

$$+yj(\ln c)^{(j-1)(j-2)+1} D_x^{-(j-1)^2} D_y^j - \alpha \sum_{k=1}^{n-1} \frac{B_{n+1-k}}{(n+1-k)!} (\ln c)^{(j-2)(n-k)} D_x^{-(j-1)(n-k)} D_y^{n-k+1}$$

$$-(n+1)(\ln c)^{2-j} D_x^{j-1}] B_n^{(a,j)}(x,y,c) = 0,$$

$$\Big[(x\ln c-\frac{\alpha}{2})D_x^{(j-1)(n-1)}D_y+(j-1)(n-1)D_x^{(j-1)(n-1)-1}D_y \qquad (3.2.10)$$

$$+j(\ln c)^{(j-1)(j-2)+1}D_x^{(j-1)(n-j)}D_y^{j-1}(1+yD_y)$$

$$-\alpha\sum_{k=1}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}(\ln c)^{(j-2)(n-k)}D_x^{(j-1)(k-1)}D_y^{n-k+1}-(n+1)(\ln c)^{2-j}D_x^{n(j-1)}\Big]$$

$$\times B_n^{(a,j)}(x,y,c)=0.$$

It is important to note that (3.2.10) does not contain anti-derivatives for $n \geq j$.

Proof. *Taking derivative with respect to t* in *(3.2.1)*

$$\frac{t^\alpha}{(e^t-1)^\alpha}c^{xt+yt^j}=\sum_{n=0}^{\infty}B_n^{(\alpha,j)}(x,y,c)\frac{t^n}{n!}$$

then applying series manipulations and (2.2.4), we get the recurrence relation

$$B_{n+1}^{(\alpha,j)}(x,y,c)=(x\ln c-\frac{\alpha}{2})B_n^{(\alpha,j)}(x,y,c)+yj\frac{n!}{(n-j+1)!}(\ln c)B_{n-j+1}^{(\alpha,j)}(x,y,c)$$

$$-\frac{\alpha}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k^{(\alpha,j)}(x,y,c)B_{n+1-k}.$$

Differentiating generating relation (3.2.1) with respect to x and comparing coefficients of t^n yields

$$D_xB_n^{(\alpha,j)}(x,y,c)=n\ln cB_{n-1}^{(\alpha,j)}(x,y,c).$$

Thus, the following relation holds for $L_n^- := \dfrac{1}{n\ln c} D_x$ *:*

$$L_n^-(B_n^{(\alpha,j)}(x,y,c)) = B_{n-1}^{(\alpha,j)}(x,y,c).$$

Obviously

$$\begin{aligned} B_k^{(\alpha,j)}(x,y,c) &= \left[L_{k+1}^- \ L_{k+2}^- ... L_n^-\right] B_n^{(\alpha,j)}(x,y,c) \\ &= \frac{k!}{n!}(\ln c)^{k-n} D_x^{n-k} B_n^{(\alpha,j)}(x,y,c), \end{aligned} \tag{3.2.11}$$

$$\begin{aligned} B_{n-j+1}^{(\alpha,j)}(x,y,c) &= \left[L_{n-j+2}^- \ L_{n-j+3}^- ... L_n^-\right] B_n^{(\alpha,j)}(x,y,c) \\ &= \frac{(n-j+1)!}{n!}(\ln c)^{1-j} D_x^{j-1} B_n^{(\alpha,j)}(x,y,c). \end{aligned} \tag{3.2.12}$$

Taking into account (3.2.11) and (3.2.12) in (3.2.3), we get the multiplicative operator L_n^+ *by*

$$L_n^+ := x\ln c - \frac{\alpha}{2} + yj(\ln c)^{(2-j)} D_x^{(j-1)} - \alpha \sum_{k=0}^{n-1} \frac{B_{n+1-k}}{(n+1-k)!}(\ln c)^{(k-n)} D_x^{n-k}.$$

By applying the factorization method (see [22], [21]),

$$L_{n+1}^- L_n^+ B_n^{(\alpha,j)}(x,y,c) = B_n^{(\alpha,j)}(x,y,c)$$

we get

$$\begin{aligned} &\Big[(x - \frac{\alpha}{2\ln c})D_x + yj(\ln c)^{1-j} D_x^j \\ &-\alpha \sum_{k=1}^{n-1} \frac{B_{n+1-k}}{(n+1-k)!}(\ln c)^{k-n-1} D_x^{n+1-k} - n\Big] B_n^{(a,j)}(x,y,c) = 0. \end{aligned}$$

To obtain (3.2.9), first we take derivative with respect to y in (3.2.1). Thus, we have

$$(\ln c)B_{n-j}^{(a,j)}(x,y,c)n(n-1)...(n-j+1) = \frac{\partial B_n^{(a,j)}(x,y,c)}{\partial y}.$$

Consequently, we have:

$$\mathcal{L}_n^- := \frac{(\ln c)^{j-2}}{n} D_x^{1-j} D_y.$$

By using the above derivative operator in (3.2.3), we have

$$\mathcal{L}_n^+ := (x\ln c - \frac{\alpha}{2}) + yj(\ln c)^{(j-1)(j-2)+1} D_x^{-(j-1)^2} D_y^{j-1} \\ -\alpha \sum_{k=0}^{n-1} \frac{B_{n+1-k}}{(n+1-k)!} (\ln c)^{(n-k)(j-2)} D_x^{-(j-1)(n-k)} D_y^{n-k}.$$

Using the factorization relation

$$\mathcal{L}_{n+1}^- \mathcal{L}_n^+ B_n^{(\alpha,j)}(x,y,c) = B_n^{(\alpha,j)}(x,y,c),$$

we get (3.2.9). Differentiating each sides of (3.2.9) with respect to x, $(j-1)(n-1)$ times, we obtain

$$\Big[(x\ln c - \frac{\alpha}{2}) D_x^{(j-1)(n-1)} D_y + (j-1)(n-1) D_x^{(j-1)(n-1)-1} D_y \\ + j(\ln c)^{(j-1)(j-2)+1} D_x^{(j-1)(n-j)} D_y^{j-1} (1 + yD_y) \\ -\alpha \sum_{k=1}^{n-1} \frac{B_{n+1-k}}{(n+1-k)!} (\ln c)^{(j-2)(n-k)} D_x^{(j-1)(k-1)} D_y^{n-k+1} - (n+1)(\ln c)^{2-j} D_x^{n(j-1)} \Big] \\ \times B_n^{(a,j)}(x,y,c) = 0.$$

■

In the following Corollary, the important case $c = e$ is mentioned for the generalized $2D$ Bernoulli polynomials.

Corollary 3.2.3 *[48] The recurrence relation of the G2DBP is as follows:*

$$B_{n+1}^{(\alpha,j)}(x,y) = (x-\frac{\alpha}{2})B_n^{(\alpha,j)}(x,y) + yj\frac{n!}{(n-j+1)!}B_{n-j+1}^{(\alpha,j)}(x,y)$$
$$-\frac{\alpha}{n+1}\sum_{k=0}^{n-1}\binom{n+1}{k}B_k^{(\alpha,j)}(x,y)B_{n+1-k}.$$

Corresponding operators are

$$L_n^- \; : \; = \frac{1}{n}D_x,$$
$$L_n^+ \; : \; = x - \frac{\alpha}{2} + yjD_x^{(j-1)} - \alpha\sum_{k=0}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}D_x^{n-k},$$
$$\mathcal{L}_n^- \; : \; = \frac{1}{n}D_x^{1-j}D_y,$$
$$\mathcal{L}_n^+ \; : \; = (x-\frac{\alpha}{2}) + yjD_x^{-(j-1)^2}D_y^{j-1}$$
$$-\alpha\sum_{k=0}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}D_x^{-(j-1)(n-k)}D_y^{n-k}.$$

The corresponding equations are

$$\Big[(x-\frac{\alpha}{2})D_x + yjD_x^j$$
$$-\alpha\sum_{k=1}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}D_x^{n+1-k} - n\Big]B_n^{(a,j)}(x,y) = 0,$$

$$\Big[(x-\frac{\alpha}{2})D_y + jD_x^{-(j-1)^2}D_y^{j-1} + yjD_x^{-(j-1)^2}D_y^j$$
$$-\alpha\sum_{k=1}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}D_x^{-(j-1)(n-k)}D_y^{n-k+1} - (n+1)D_x^{j-1}\Big]B_n^{(a,j)}(x,y) = 0,$$

$$\Big[(x-\frac{\alpha}{2})D_x^{(j-1)(n-1)}D_y+(j-1)(n-1)D_x^{(j-1)(n-1)-1}D_y$$
$$+jD_x^{(j-1)(n-j)}D_y^{j-1}(1+yD_y)$$
$$-\alpha\sum_{k=1}^{n-1}\frac{B_{n+1-k}}{(n+1-k)!}D_x^{(j-1)(k-1)}D_y^{n-k+1}-(n+1)D_x^{n(j-1)}\Big]B_n^{(a,j)}(x,y)=0;\ n\geq j.$$

3.3 The Extended 2D Euler Polynomial(E2DEP)

In this section, we define the E2DEP and find the differential equations of the E2DEP. The E2DEP is given by [48]

$$(\frac{2}{e^t+1})^{\alpha}c^{xt+yt^j}=\sum_{n=0}^{\infty}E_n^{(\alpha,j)}(x,y,c)\frac{t^n}{n!},\ c>1. \tag{3.3.1}$$

The next theorem states the recurrence relation, shift operators, differential, integro-differential and partial differential equations of the E2DEP. Since the proof is similar with the E2DBP, we only present the theorem.

Theorem 3.3.1 *[48] The recurrence formula of the E2DEP is given by:*

$$E_{n+1}^{(\alpha,j)}(x,y,c)=(x\ln c-\frac{\alpha}{2})E_n^{(\alpha,j)}(x,y,c)+yjE_{n-j+1}^{(\alpha,j)}(x,y,c)\frac{n!}{(n-j+1)!}(\ln c) \tag{3.3.2}$$
$$+\frac{\alpha}{2}\sum_{k=0}^{n-1}\binom{n}{k}e_{n-k}E_k^{(\alpha,j)}(x,y,c).$$

Corresponding operators are given by:

$$L_n^- \; : \; = \frac{1}{n \ln c} D_x, \tag{3.3.3}$$

$$L_n^+ \; : \; = x \ln c - \frac{\alpha}{2} + yj(\ln c)^{2-j} D_x^{j-1} \tag{3.3.4}$$

$$+\frac{\alpha}{2} \sum_{k=0}^{n-1} \frac{e_{n-k}}{(n-k)!} (\ln c)^{k-n} D_x^{n-k},$$

$$\mathcal{L}_n^- \; : \; = \frac{(\ln c)^{j-2}}{n} D_x^{1-j} D_y, \tag{3.3.5}$$

$$\mathcal{L}_n^+ \; : \; = (x \ln c - \frac{\alpha}{2}) + yj(\ln c)^{(j-1)(j-2)+1} D_x^{-(j-1)^2} D_y^{j-1} \tag{3.3.6}$$

$$+\frac{\alpha}{2} \sum_{k=0}^{n-1} \frac{e_{n-k}}{(n-k)!} (\ln c)^{(n-k)(j-2)} D_x^{-(n-k)(j-1)} D_y^{n-k}.$$

Corresponding equations are:

$$\Bigg[(x - \frac{\alpha}{2 \ln c}) D_x + yj(\ln c)^{1-j} D_x^j \tag{3.3.7}$$

$$+\frac{\alpha}{2} \sum_{k=1}^{n-1} \frac{e_{n-k}}{(n-k)!} (\ln c)^{k-n-1} D_x^{n-k+1} - n\Bigg] E_n^{(\alpha,j)}(x,y,c) = 0,$$

$$\Bigg[(x \ln c - \frac{\alpha}{2}) D_y + (\ln c)^{(j-1)(j-2)+1} jD_x^{-(j-1)^2} D_y^{j-1} \tag{3.3.8}$$

$$+yj(\ln c)^{(j-1)(j-2)+1} D_x^{-(j-1)^2} D_y^j + \frac{\alpha}{2} \sum_{k=1}^{n-1} \frac{e_{n-k}}{(n-k)!} (\ln c)^{(j-2)(n-k)} D_x^{-(j-1)(n-k)} D_y^{n-k+1}$$

$$-(n+1)(\ln c)^{2-j} D_x^{j-1}] E_n^{(\alpha,j)}(x,y,c) = 0,$$

$$
\Big[(x\ln c-\frac{\alpha}{2})D_x^{(j-1)(n-1)}D_y+(j-1)(n-1)D_x^{(j-1)(n-1)-1}D_y \tag{3.3.9}
$$

$$
+(\ln c)^{(j-1)(j-2)+1}jD_x^{(j-1)(n-j)}(D_y^{j-1}+yD_y^{j})
$$

$$
+\frac{\alpha}{2}\sum_{k=1}^{n-1}\frac{e_{n-k}}{(n-k)!}(\ln c)^{(j-2)(n-k)}D_x^{(j-1)(k-1)}D_y^{n+1-k}-(n+1)(\ln c)^{2-j}D_x^{(j-1)n}\Big]
$$

$$
\times E_n^{(\alpha,j)}(x,y,c)=0.
$$

Similarly, as in (3.2.10), we should take $n\geq j$ in (3.3.9).

Since the case $c=e$ reduces to the G2DEP, we thus have the following corollary:

Corollary 3.3.2 *[48] For the G2DEP, we have the following recurrence:*

$$
\begin{aligned}
E_{n+1}^{(\alpha,j)}(x,y) &= (x-\frac{\alpha}{2})E_n^{(\alpha,j)}(x,y)+yjE_{n-j+1}^{(\alpha,j)}(x,y)\frac{n!}{(n-j+1)!} \\
&\quad +\frac{\alpha}{2}\sum_{k=0}^{n-1}\binom{n}{k}e_{n-k}E_k^{(\alpha,j)}(x,y).
\end{aligned}
$$

Shift operators:

$$
\begin{aligned}
L_n^- &:= \frac{1}{n}D_x, \\
L_n^+ &:= x-\frac{\alpha}{2}+yjD_x^{j-1}+\frac{\alpha}{2}\sum_{k=0}^{n-1}\frac{e_{n-k}}{(n-k)!}D_x^{n-k}, \\
\mathcal{L}_n^- &:= \frac{1}{n}D_x^{1-j}D_y, \\
\mathcal{L}_n^+ &:= (x-\frac{\alpha}{2})+yjD_x^{-(j-1)^2}D_y^{j-1}+\frac{\alpha}{2}\sum_{k=0}^{n-1}\frac{e_{n-k}}{(n-k)!}D_x^{-(n-k)(j-1)}D_y^{n-k}.
\end{aligned}
$$

The corresponding equations are :

$$\Big[(x-\frac{\alpha}{2})D_x + yjD_x^j + \frac{\alpha}{2}\sum_{k=1}^{n-1}\frac{e_{n-k}}{(n-k)!}D_x^{n-k+1} - n\Big]E_n^{(\alpha,j)}(x,y) = 0,$$

$$\Big[(x-\frac{\alpha}{2})D_y + jD_x^{-(j-1)^2}D_y^{j-1} + yjD_x^{-(j-1)^2}D_y^j + \frac{\alpha}{2}\sum_{k=1}^{n-1}\frac{e_{n-k}}{(n-k)!}D_x^{-(j-1)(n-k)}D_y^{n-k+1} - (n+1)D_x^{j-1}\Big]E_n^{(\alpha,j)}(x,y) = 0,$$

$$\Big[(x-\frac{1}{2})D_x^{(j-1)(n-1)}D_y + (j-1)(n-1)D_x^{(j-1)(n-1)-1}D_y + jD_x^{(j-1)(n-j)}D_y^{j-1}(1+yD_y) + \frac{\alpha}{2}\sum_{k=1}^{n-1}\frac{e_{n-k}}{(n-k)!}D_x^{(j-1)(k-1)}D_y^{n-k+1} - (n+1)D_x^{(j-1)n}\Big]E_n^{(\alpha,j)}(x,y) = 0;\ (n \geq j).$$

Chapter 4

HERMITE-BASED APPELL POLYNOMIALS

This Chapter consists of results of our recent study [46].

4.1 Construction and Auxilary Results

It was Khan et al. [23] who defined the H-B Appell polynomials by

$$G(x,y,z;t) = A(t)\exp(\mu t) = \sum_{n=0}^{\infty} {}_HA_n(x,y,z)\frac{t^n}{n!}, \tag{4.1.1}$$

where

$$\mu = x + 2y\frac{\partial}{\partial x} + 3z\frac{\partial^2}{\partial x^2} \tag{4.1.2}$$

denotes the multiplicative operator of the 3-variable Hermite polynomials which are given by

$$\exp(xt + yt^2 + zt^3) = \sum_{n=0}^{\infty} H_n^{(3)}(x,y,z)\frac{t^n}{n!} \tag{4.1.3}$$

and

$$A(t) = \sum_{n=0}^{\infty} a_n t^n,\ a_0 \neq 0. \tag{4.1.4}$$

Using Berry decoupling identity

$$e^{A+B} = e^{\frac{m^2}{12}} e^{((\frac{-m}{2})A^{\frac{1}{2}}+A)} e^B, \quad [A,B] = mA^{\frac{1}{2}}, \tag{4.1.5}$$

they introduced H-B Appell polynomials ${}_HA_n(x,y,z)$ as

$$G(x,y,z;t) = A(t)\exp(xt+yt^2+zt^3) = \sum_{n=0}^{\infty} {}_HA_n(x,y,z)\frac{t^n}{n!}. \tag{4.1.6}$$

In this Chapter, we consider the H-BBP ${}_HB_n(x,y,z)$, H-BEP ${}_HE_n(x,y,z)$ and the H-BGP ${}_HG_n(x,y,z)$ via the following generating functions (see [23]):

$$\frac{t}{e^t-1}\exp(xt+yt^2+zt^3) = \sum_{n=0}^{\infty} {}_HB_n(x,y,z)\frac{t^n}{n!}, \ |t|<2\pi, \tag{4.1.7}$$

$$\frac{2}{e^t+1}\exp(xt+yt^2+zt^3) = \sum_{n=0}^{\infty} {}_HE_n(x,y,z)\frac{t^n}{n!}, \ |t|<\pi, \tag{4.1.8}$$

and

$$\frac{2t}{e^t+1}\exp(xt+yt^2+zt^3) = \sum_{n=0}^{\infty} {}_HG_n(x,y,z)\frac{t^n}{n!}, \ |t|<\pi, \tag{4.1.9}$$

respectively.

In the special case $z=0$, these generating functions reduce to the generating functions of 2DBP, 2DEP and 2DGP. The special cases of (4.1.7) and (4.1.8) were investigated by Bretti and Ricci, which is given in [5]. In the case $y=z=0$, we have the usual Bernoulli, Euler and Genocchi polynomials, respectively.

4.2 Recurrence Relation and Shift Operators for Hermite-Based Appell Polynomials

The following theorem gives the recurrence relation and shift operators for Hermite-based Appell polynomials:

Theorem 4.2.1 *[46] The recurrence relation of H-B Appell polynomials is:*

$$
\begin{gathered}
{}_HA_{-1}(x,y,z) := 0,\ {}_HA_{-2}(x,y,z) := 0; \\
{}_HA_{n+1}(x,y,z) = (x+\alpha_0)\ {}_HA_n(x,y,z) + \sum_{k=1}^{n} \binom{n}{k} \alpha_k\ {}_HA_{n-k}(x,y,z) \\
+2ny\ {}_HA_{n-1}(x,y,z) + 3zn(n-1)\ {}_HA_{n-2}(x,y,z)
\end{gathered}
\tag{4.2.1}
$$

where α_k $(k = 0,1,2,...)$ *are given by the expansion*

$$
\frac{A'(t)}{A(t)} = \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!}. \tag{4.2.2}
$$

Shift operators are as follows:

$$
{}_x\mathcal{L}_n^- \ :\ = \frac{1}{n} D_x, \tag{4.2.3}
$$

$$
{}_y\mathcal{L}_n^- \ :\ = \frac{1}{n} D_x^{-1} D_y, \tag{4.2.4}
$$

$$
{}_z\mathcal{L}_n^- \ :\ = \frac{1}{n} D_x^{-2} D_z, \tag{4.2.5}
$$

$$
{}_x\mathcal{L}_n^+ \ :\ = x + \alpha_0 + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^k + 2yD_x + 3zD_x^2, \tag{4.2.6}
$$

$$
{}_y\mathcal{L}_n^+ \ :\ = x + \alpha_0 + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{-k} D_y^k + 2yD_x^{-1} D_y + 3zD_x^{-2} D_y^2, \tag{4.2.7}
$$

$$
{}_z\mathcal{L}_n^+ \ :\ = x + \alpha_0 + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{-2k} D_z^k + 2yD_x^{-2} D_z + 3zD_x^{-4} D_z^2, \tag{4.2.8}
$$

where

$$D_x := \frac{\partial}{\partial x},\ D_y := \frac{\partial}{\partial y}, ..., D_x^{-1} := \int_0^x f(\xi)d\xi.$$

Proof. *Taking derivative with respect to t on both sides of (4.1.6), we have*

$$\frac{\partial}{\partial t}\mathcal{G}(x,y,z;t) = \mathcal{G}(x,y,z;t)\left(\frac{A'(t)}{A(t)} + x + 2yt + 3zt^2\right). \tag{4.2.9}$$

Inserting the corresponding series forms for $G(x,y,z;t)$ from (4.1.6) and for $\frac{A'(t)}{A(t)}$ from (4.2.2) and equating the coefficients of t^n in the equation resulting from (4.2.9), we obtain (4.2.1). Next, we take into account (4.2.1) to find the multiplicative operators ${}_x\mathcal{L}_n^+$, ${}_y\mathcal{L}_n^+$ and ${}_z\mathcal{L}_n^+$ with respect to x, y and z. First of all, in order to obtain the derivative operator ${}_x\mathcal{L}_n^-$, we differentiate both sides of the generating relation (4.1.6) with respect to x and equate the coefficients of t^n, so that we have

$$\frac{\partial}{\partial x}\{{}_HA_n(x,y,z)\} = n\ {}_HA_{n-1}(x,y,z).$$

Thus, clearly, the operator given by (4.2.3) satisfies the following relation:

$${}_x\mathcal{L}_n^-\ {}_HA_n(x,y,z) = {}_HA_{n-1}(x,y,z).$$

Differentiating the generating relation (4.1.6) with respect to y, we have

$$\frac{\partial}{\partial y}\{{}_HA_n(x,y,z)\} = n(n-1)\ {}_HA_{n-2}(x,y,z) = n\frac{\partial}{\partial x}\{{}_HA_{n-1}(x,y,z)\},$$

so that

$$D_x^{-1} D_y \, {}_H A_n(x,y,z) = n \, {}_H A_{n-1}(x,y,z), \tag{4.2.10}$$

and therefore, we get ${}_y\mathcal{L}_n^- = \frac{1}{n} D_x^{-1} D_y.$

Upon differentiating both sides of the generating relation (4.1.6) with respect to z, we have

$$\frac{\partial}{\partial z}\{\, {}_H A_n(x,y,z)\} = n(n-1)(n-2)\, {}_H A_{n-3}(x,y,z) = n\frac{\partial^2}{\partial x^2}\{\, {}_H A_{n-1}(x,y,z)\},$$

so that

$$D_x^{-2} D_z \, {}_H A_n(x,y,z) = n \, {}_H A_{n-1}(x,y,z), \tag{4.2.11}$$

which yields to ${}_z\mathcal{L}_n^- = \frac{1}{n} D_x^{-2} D_z.$

Next, in order to obtain the multiplicative operator ${}_x\mathcal{L}_n^+$, *we use the following relations:*

$$\begin{aligned} {}_H A_{n-k}(x,y,z) &= \left({}_x\mathcal{L}_{n-k+1}^- \, {}_x\mathcal{L}_{n-k+2}^- \cdots \, {}_x\mathcal{L}_{n-1}^- \, {}_x\mathcal{L}_n^- \right) {}_H A_n(x,y,z) \\ &= \frac{(n-k)!}{n!} D_x^k \; {}_H A_n(x,y,z), \end{aligned} \tag{4.2.12}$$

$$\begin{aligned} {}_H A_{n-1}(x,y,z) &= {}_x\mathcal{L}_n^- \, {}_H A_n(x,y,z) \\ &= \frac{1}{n} D_x \; {}_H A_n(x,y,z) \end{aligned} \tag{4.2.13}$$

and

$$
\begin{aligned}
{}_HA_{n-2}(x,y,z) &= \left({}_x\mathcal{L}^-_{n-1}\ {}_x\mathcal{L}^-_n\right)\ {}_HA_n(x,y,z) \\
&= \frac{1}{n(n-1)}D_x^2\ \ {}_HA_n(x,y,z).
\end{aligned} \tag{4.2.14}
$$

By substituting (4.2.12), (4.2.13) and (4.2.14) into the recurrence relation (4.2.1), we have

$$
{}_HA_{n+1}(x,y,z) = \left(x+\alpha_0+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^k+2yD_x+3zD_x^2\right)\ {}_HA_n(x,y,z)
$$

which yields the multiplicative operator ${}_x\mathcal{L}^+_n$.

To obtain the multiplicative operator ${}_y\mathcal{L}^+_n$, we use the following relations:

$$
\begin{aligned}
{}_HA_{n-k}(x,y,z) &= \left({}_y\mathcal{L}^-_{n-k+1}\ {}_y\mathcal{L}^-_{n-k+2}\cdots\ {}_y\mathcal{L}^-_{n-1}\ {}_y\mathcal{L}^-_n\right)\ {}_HA_n(x,y,z) \\
&= \frac{(n-k)!}{n!}D_x^{-k}D_y^k\ \ {}_HA_n(x,y,z),
\end{aligned} \tag{4.2.15}
$$

$$
\begin{aligned}
{}_HA_{n-1}(x,y,z) &= {}_y\mathcal{L}^-_n\ \ {}_HA_n(x,y,z) \\
&= \frac{1}{n}D_x^{-1}D_y\ \ {}_HA_n(x,y,z)
\end{aligned} \tag{4.2.16}
$$

and

$$
\begin{aligned}
{}_HA_{n-2}(x,y,z) &= \left({}_y\mathcal{L}^-_{n-1}\ {}_y\mathcal{L}^-_n\right)\ {}_HA_n(x,y,z) \\
&= \frac{1}{n(n-1)}D_x^{-2}D_y^2\ \ {}_HA_n(x,y,z).
\end{aligned} \tag{4.2.17}
$$

Inserting (4.2.15), (4.2.16) and (4.2.17) into the recurrence relation (4.2.1), we get

$$
{}_HA_{n+1}(x,y,z) = \left(x+\alpha_0+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{-k}D_y^{k}+2yD_x^{-1}D_y+3zD_x^{-2}D_y^{2}\right) \times {}_HA_n(x,y,z) \tag{4.2.18}
$$

which leads us to the multiplicative operator ${}_y\mathcal{L}_n^+$.

The derivation of the multiplicative operator ${}_z\mathcal{L}_n^+$ *would similarly make use of the following relations:*

$$
\begin{aligned}
{}_HA_{n-k}(x,y,z) &= \left({}_z\mathcal{L}_{n-k+1}^-\ {}_z\mathcal{L}_{n-k+2}^-\cdots\ {}_z\mathcal{L}_{n-1}^-\ {}_z\mathcal{L}_n^-\right)\ {}_HA_n(x,y,z) \\
&= \frac{(n-k)!}{n!}D_x^{-2k}D_z^{k}\ {}_HA_n(x,y,z),
\end{aligned}
$$

$$
\begin{aligned}
{}_HA_{n-1}(x,y,z) &= {}_z\mathcal{L}_n^-\ \ {}_HA_n(x,y,z) \\
&= \frac{1}{n}D_x^{-2}D_z\ \ {}_HA_n(x,y,z)
\end{aligned}
$$

and

$$
\begin{aligned}
{}_HA_{n-2}(x,y,z) &= \left({}_z\mathcal{L}_{n-1}^-\ {}_z\mathcal{L}_n^-\right)\ {}_HA_n(x,y,z) \\
&= \frac{1}{n(n-1)}D_x^{-4}D_z^{2}\ \ {}_HA_n(x,y,z),
\end{aligned}
$$

which, in conjunction with the recurrence relation (4.2.1), yields

$$
\begin{aligned}
&{}_HA_{n+1}(x,y,z) \qquad (4.2.19)\\
&= \left(x+\alpha_0+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{-2k}D_z^{k}+2yD_x^{-2}D_z+3zD_x^{-4}D_z^{2}\right) {}_HA_n(x,y,z)
\end{aligned}
$$

and consequently, the multiplicative operator ${}_z\mathcal{L}_n^{+}$. ■

Taking $A(t)=\dfrac{t}{e^t-1}$ in above Theorem , we get the following Corollary for Hermite-based Bernoulli polynomial:

Corollary 4.2.2 *[46] The recurrence formula of the H-BBP is given by:*

$$
\begin{aligned}
&{}_HB_{n+1}(x,y,z)\\
&= (x-\tfrac{1}{2})\ {}_HB_n(x,y,z)+2ny\ {}_HB_{n-1}(x,y,z)\\
&+3zn(n-1)\ {}_HB_{n-2}(x,y,z)-\tfrac{1}{n+1}\sum_{k=2}^{n+1}\tbinom{n+1}{k}\ {}_HB_{n-k+1}(x,y,z)B_k
\end{aligned}
$$

where B_k denotes the Bernoulli numbers and

$$
{}_HB_{-n}(x,y,z):=0,\quad n\in\mathbb{N}.
$$

Corresponding operators are:

$$
\begin{aligned}
{}_xL_n^- &: = \frac{1}{n}D_x,\\
{}_y\mathcal{L}_n^- &: = \frac{1}{n}D_x^{-1}D_y,\\
{}_z\mathcal{L}_n^- &: = \frac{1}{n}D_x^{-2}D_z,\\
{}_xL_n^+ &: = x-\frac{1}{2}+2yD_x+3zD_x^2-\sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{k-1},\\
{}_y\mathcal{L}_n^+ &: = x-\frac{1}{2}+2yD_x^{-1}D_y+3zD_x^{-2}D_y^2-\sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{1-k}D_y^{k-1},\\
{}_z\mathcal{L}_n^+ &: = x-\frac{1}{2}+2yD_x^{-2}D_z+3zD_x^{-4}D_z^2-\sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{2-2k}D_z^{k-1}.
\end{aligned}
$$

Taking $A(t)=\frac{2}{e^t+1}$ in above Theorem, we get the following Corollary for Hermite-based Euler polynomial:

Corollary 4.2.3 *[46] The recurrence relation of the H-BEP is:*

$$
\begin{aligned}
{}_HE_{n+1}(x,y,z) = (x-\tfrac{1}{2})\ {}_HE_n(x,y,z)+\tfrac{1}{2}\sum_{k=1}^{n}\binom{n}{k}e_k\ {}_HE_{n-k}(x,y,z)\\
+2ny\ {}_HE_{n-1}(x,y,z)+3zn(n-1)\ {}_HE_{n-2}(x,y,z).
\end{aligned}
$$

Shift operators are:

$$
\begin{aligned}
{}_xL_n^- &: = \frac{1}{n}D_x,\\
{}_y\mathcal{L}_n^- &: = \frac{1}{n}D_x^{-1}D_y,\\
{}_z\mathcal{L}_n^- &: = \frac{1}{n}D_x^{-2}D_z,\\
{}_xL_n^+ &: = x-\frac{1}{2}+2yD_x+3zD_x^2+\frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^k,\\
{}_y\mathcal{L}_n^+ &: = x-\frac{1}{2}+2yD_x^{-1}D_y+3zD_x^{-2}D_y^2+\frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^{-k}D_y^k,\\
{}_z\mathcal{L}_n^+ &: = x-\frac{1}{2}+2yD_x^{-2}D_z+3zD_x^{-4}D_z^2+\frac{1}{2}\sum_{k=0}^{n}\frac{e_k}{k!}D_x^{-2k}D_z^k,
\end{aligned}
$$

where e_k are the numerical coefficients that are given by (2.3.2).

4.3 Differential, Integro-differential and Partial Differential Equations of Hermite-Based Appell Polynomials

In this section, we obtain differential, integro-differential and partial differential equations for the H-B Appell polynomials via factorization method. Furthermore, we arrange the corresponding equations for H-B Bernoulli and H-B Euler polynomials.

Theorem 4.3.1 *[46] H-B Appell polynomials satisfy the following differential equation:*

$$
\left[(x+\alpha_0)D_x+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{k+1}+2yD_x^2+3zD_x^3-n\right]{}_HA_n(x,y,z)=0 \tag{4.3.1}
$$

where

$$
\frac{A'(t)}{A(t)}=\sum_{k=0}^{\infty}\alpha_k\frac{t^k}{k!}.
$$

Proof. *Using factorization relation*

$$ {}_x\mathcal{L}^{-}_{n+1}\ {}_x\mathcal{L}^{+}_{n}\ {}_HA_n(x,y,z) = {}_H A_n(x,y,z) $$

and shift operators (4.2.3) and (4.2.6), we get the desired result. ■

Theorem 4.3.2 *[46] H-B Appell polynomials satify the following integro-differential equations:*

$$ \Bigg[(x+\alpha_0)D_y + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{-k} D_y^{k+1} + 2D_x^{-1} D_y \qquad (4.3.2) $$

$$ +2yD_x^{-1}D_y^2 + 3zD_x^{-2}D_y^3 - (n+1)D_x\Big]\ {}_HA_n(x,y,z) = 0, $$

$$ \Bigg[(x+\alpha_0)D_z + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{-2k} D_z^{k+1} + 2yD_x^{-2} D_z^2 \qquad (4.3.3) $$

$$ 3D_x^{-4}D_z^2 + 3zD_x^{-4}D_z^3 - (n+1)D_x^2\Big]\ {}_HA_n(x,y,z) = 0, $$

$$ \Bigg[(x+\alpha_0)D_y + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_x^{-2k} D_z^{k} D_y + 2D_x^{-2} D_z \qquad (4.3.4) $$

$$ +2yD_x^{-2}D_zD_y + 3zD_x^{-4}D_z^2D_y - (n+1)D_x\Big]\ {}_HA_n(x,y,z) = 0, $$

$$ \Bigg[(x+\alpha_0)D_z + \sum_{k=1}^{n} \frac{\alpha_k}{k!} D_zD_x^{-k} D_y^{k} + 2yD_x^{-1} D_yD_z \qquad (4.3.5) $$

$$ +3D_x^{-2}D_y^2 + 3zD_x^{-2}D_y^2D_z - (n+1)D_x^2\Big]\ {}_HA_n(x,y,z) = 0. $$

Proof. *Using factorization relation*

$$\mathcal{L}_{n+1}^{-}\mathcal{L}_{n}^{+}\ {}_{H}A_n(x,y,z) =_{H} A_n(x,y,z)$$

with the derivative operators (4.2.4)

$$_{y}\mathcal{L}_{n}^{-} := \frac{1}{n}D_x^{-1}D_y,$$

and the multiplicative operator (4.2.7)

$$_{y}\mathcal{L}_{n}^{+} := x+\alpha_0+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{-k}D_y^{k}+2yD_x^{-1}D_y+3zD_x^{-2}D_y^{2},$$

we get the following integro-differential equation

$$\Bigg[(x+\alpha_0)D_y+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{-k}D_y^{k+1}+2D_x^{-1}D_y$$
$$+2yD_x^{-1}D_y^{2}+3zD_x^{-2}D_y^{3}-(n+1)D_x\Big]\ {}_{H}A_n(x,y,z)=0.$$

Considering the shift operators (4.2.5)

$$_{z}\mathcal{L}_{n}^{-} := \frac{1}{n}D_x^{-2}D_z,$$

and (4.2.8)

$$_{z}\mathcal{L}_{n}^{+} := x+\alpha_0+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{-2k}D_z^{k}+2yD_x^{-2}D_z+3zD_x^{-4}D_z^{2}$$

we get corresponding equation

$$\left[(x+\alpha_0)D_z+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{-2k}D_z^{k+1}+2yD_x^{-2}D_z^2\right.$$

$$\left.3D_x^{-4}D_z^2+3zD_x^{-4}D_z^3-(n+1)D_x^2\right]\ {}_HA_n(x,y,z)=0.$$

Again using above factorization relation with shift operators (4.2.4) and (4.2.8), (4.2.5) and (4.2.7), we get the corresponding equations (4.3.4) and (4.3.5). ■

Theorem 4.3.3 *[46] H-B Appell polynomials satisfy the following partial differential equations:*

$$\left[(x+\alpha_o)D_x^{2n}D_z+2nD_x^{2n-1}D_z+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{2n-2k}D_z^{k+1}+2yD_x^{2n-2}D_z^2\right. \tag{4.3.6}$$

$$\left.+3D_x^{2n-4}D_z^2+3zD_x^{2n-4}D_z^3-(n+1)D_x^{2n+2}\right]\ {}_HA_n(x,y,z)=0,$$

$$\left[(x+\alpha_0)D_x^nD_y+nD_x^{n-1}D_y+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{n-k}D_y^{k+1}\right. \tag{4.3.7}$$

$$\left.+2D_x^{n-1}D_y+2yD_x^{n-1}D_y^2+3zD_x^{n-2}D_y^3-(n+1)D_x^{n+1}\right]\ {}_HA_n(x,y,z)=0,$$

$$\left[(x+\alpha_0)D_x^{2n}D_y+2nD_x^{2n-1}D_y+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_yD_x^{2n-2k}D_z^k\right. \tag{4.3.8}$$

$$\left.+2D_x^{2n-2}D_z+2yD_x^{2n-2}D_zD_y+3zD_x^{2n-4}D_z^2D_y-(n+1)D_x^{2n+1}\right]\ {}_HA_n(x,y,z)=0,$$

$$\Big[(x+\alpha_0)D_x^nD_z+nD_x^{n-1}D_z+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_zD_x^{n-k}D_y^k+2yD_x^{n-1}D_yD_z \tag{4.3.9}$$

$$+3D_x^{n-2}D_y^2+3zD_x^{n-2}D_y^2D_z-(n+1)D_x^{n+2}\Big]\ {}_HA_n(x,y,z)=0.$$

Proof. *Taking derivative with respect to* x, $2n-$*times in the integro-differential equation (4.3.3)*

$$\Big[(x+\alpha_0)D_z+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{-2k}D_z^{k+1}+2yD_x^{-2}D_z^2$$

$$3D_x^{-4}D_z^2+3zD_x^{-4}D_z^3-(n+1)D_x^2\Big]\ {}_HA_n(x,y,z)=0,$$

we get the partial differential equation (4.3.6)

$$\Big[(x+\alpha_o)D_x^{2n}D_z+2nD_x^{2n-1}D_z+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_x^{2n-2k}D_z^{k+1}+2yD_x^{2n-2}D_z^2$$

$$+3D_x^{2n-4}D_z^2+3zD_x^{2n-4}D_z^3-(n+1)D_x^{2n+2}\Big]\ {}_HA_n(x,y,z)=0.$$

Taking derivative with respect to x, $n-$ *times in the integro-differential equation (4.3.2), we get the partial differential equation (4.3.7). To obtain (4.3.8), we take derivatives with respect to* x, $2n-$*times in the corresponding equation (4.3.4). To obtain (4.3.9), we take derivatives with respect to* x, $n-$*times in (4.3.5).* ■

Repeating the methods and shift operators that are used in previous theorems, we obtain the following corollaries for Hermite-based Bernoulli and Hermite-based Euler polynomials:

Corollary 4.3.4 *[46] H-BBP satisfies the following differential equation:*

$$\left[(x-\frac{1}{2})D_x+2yD_x^2+3zD_x^3-\sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^k-n\right]{}_HB_n(x,y,z)=0$$

where B_k denotes the Bernoulli numbers.

Corollary 4.3.5 *[46] H-BBP satisfies the following integro-differential equations:*

$$\left[(x-\frac{1}{2})D_y+2D_x^{-1}D_y+2yD_x^{-1}D_y^2\right.$$
$$\left.+3zD_x^{-2}D_y^3-\sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{1-k}D_y^k-(n+1)D_x\right]{}_HB_n(x,y,z)=0,$$

$$\left[(x-\frac{1}{2})D_z+2yD_x^{-2}D_z^2+3D_x^{-4}D_z^2+3zD_x^{-4}D_z^3\right.$$
$$\left.-\sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{2-2k}D_z^k-(n+1)D_x^2\right]{}_HB_n(x,y,z)=0,$$

$$\left[(x-\frac{1}{2})D_y+2D_x^{-2}D_z+2yD_x^{-2}D_zD_y\right.$$
$$\left.+3zD_x^{-4}D_z^2D_y-\sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{2-2k}D_z^{k-1}D_y-(n+1)D_x\right]{}_HB_n(x,y,z)=0,$$

$$\left[(x-\frac{1}{2})D_z+2yD_x^{-1}D_yD_z+3D_x^{-2}D_y^2+3zD_x^{-2}D_y^2D_z\right.$$
$$\left.-\sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{1-k}D_y^{k-1}D_z-(n+1)D_x^2\right]{}_HB_n(x,y,z)=0,$$

where B_k denotes Bernoulli numbers.

Corollary 4.3.6 *[46] H-BBP satisfies the following partial differential equations:*

$$\left[(x-\frac{1}{2})D_x^{2n}D_z + 2nD_x^{2n-1}D_z + 2yD_x^{2n-2}D_z^2 + 3D_x^{2n-4}D_z^2 + 3zD_x^{2n-4}D_z^3 - \sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{2n-2k+2}D_z^k - (n+1)D_x^{2n+2}\right]{}_HB_n(x,y,z) = 0,$$

$$\left[(x-\frac{1}{2})D_x^nD_y + nD_x^{n-1}D_y + 2D_x^{n-1}D_y + 2yD_x^{n-1}D_y^2 + 3zD_x^{n-2}D_y^3 - \sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{n-k+1}D_y^k - (n+1)D_x^{n+1}\right]{}_HB_n(x,y,z) = 0,$$

$$\left[(x-\frac{1}{2})D_x^{2n}D_y + 2nD_x^{2n-1}D_y + 2D_x^{2n-2}D_z + 2yD_x^{2n-2}D_zD_y + 3zD_x^{2n-4}D_z^2D_y - \sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{2n-2k+2}D_z^{k-1}D_y - (n+1)D_x^{2n+1}\right]{}_HB_n(x,y,z) = 0,$$

$$\left[(x-\frac{1}{2})D_x^nD_z + nD_x^{n-1}D_z + 2yD_x^{n-1}D_yD_z + 3D_x^{n-2}D_y^2 + 3zD_x^{n-2}D_y^2D_z - \sum_{k=2}^{n+1}\frac{B_k}{k!}D_x^{n-k+1}D_y^{k-1}D_z - (n+1)D_x^{n+2}\right]{}_HB_n(x,y,z) = 0,$$

where B_k denotes Bernoulli numbers.

Corollary 4.3.7 *[46] The differential equation that is satisfied by H-BEP is given by:*

$$\left[(x-\frac{1}{2})D_x + 2yD_x^2 + 3zD_x^3 + \frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^{k+1} - n\right]{}_HE_n(x,y,z) = 0$$

where

$$e_k = (-\frac{1}{2})^k \sum_{h=0}^{k} \binom{k}{h} E_{k-h}.$$

Corollary 4.3.8 *[46] H-BEP satisfies the following integro-differential equations:*

$$\Bigg[(x-\frac{1}{2})D_y + 2D_x^{-1}D_y + 2yD_x^{-1}D_y^2 + 3zD_x^{-2}D_y^3 + \frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^{-k}D_y^{k+1} - (n+1)D_x\Bigg] {}_HE_n(x,y,z) = 0,$$

$$\Bigg[(x-\frac{1}{2})D_z + 2yD_x^{-2}D_z^2 + 3D_x^{-4}D_z^2 + 3zD_x^{-4}D_z^3 + \frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^{-2k}D_z^{k+1} - (n+1)D_x^2\Bigg] {}_HE_n(x,y,z) = 0,$$

$$\Bigg[(x-\frac{1}{2})D_y + 2D_x^{-2}D_z + 2yD_x^{-2}D_zD_y + 3zD_x^{-4}D_z^2D_y + \frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^{-2k}D_z^{k}D_y - (n+1)D_x\Bigg] {}_HE_n(x,y,z) = 0,$$

$$\Bigg[(x-\frac{1}{2})D_z + 2yD_x^{-1}D_yD_z + 3D_x^{-2}D_y^2 + 3zD_x^{-2}D_y^2D_z + \frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^{-k}D_y^{k}D_z - (n+1)D_x^2\Bigg] {}_HE_n(x,y,z) = 0,$$

where

$$e_k = (-\frac{1}{2})^k \sum_{h=0}^{k} \binom{k}{h} E_{k-h}.$$

Corollary 4.3.9 *[46] H-BEP satisfies the following partial differential equations:*

$$\Big[(x-\frac{1}{2})D_x^nD_y+nD_x^{n-1}D_y+2D_x^{n-1}D_y+2yD_x^{n-1}D_y^2$$
$$+3zD_x^{n-2}D_y^3+\frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^{n-k}D_y^{k+1}-(n+1)D_x^{n+1}\Big]\ {}_HE_n(x,y,z)=0,$$

$$\Big[(x-\frac{1}{2})D_x^{2n}D_z+2nD_x^{2n-1}D_z+2yD_x^{2n-2}D_z^2+3D_x^{2n-4}D_z^2$$
$$+3zD_x^{2n-4}D_z^3+\frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^{2n-2k}D_z^{k+1}-(n+1)D_x^{2n+2}\Big]\ {}_HE_n(x,y,z)=0,$$

$$\Big[(x-\frac{1}{2})D_x^{2n}D_y+2nD_x^{2n-1}D_y+2D_x^{2n-2}D_z+2yD_x^{2n-2}D_zD_y$$
$$+3zD_x^{2n-4}D_z^2D_y+\frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^{2n-2k}D_z^kD_y-(n+1)D_x^{2n+1}\Big]\ {}_HE_n(x,y,z)=0,$$

$$\Big[(x-\frac{1}{2})D_x^nD_z+nD_x^{n-1}D_z+2yD_x^{n-1}D_yD_z+3D_x^{n-2}D_y^2$$
$$+3zD_x^{n-2}D_y^2D_z+\frac{1}{2}\sum_{k=1}^{n}\frac{e_k}{k!}D_x^{n-k}D_y^kD_z-(n+1)D_x^{n+2}\Big]\ {}_HE_n(x,y,z)=0$$

where

$$e_k=(-\frac{1}{2})^k\sum_{h=0}^{k}\binom{k}{h}E_{k-h}.$$

Chapter 5

GENERALIZED FACTORIZATION METHOD FOR APPELL POLYNOMIALS

This Chapter is devoted to exhibition of results of our work [35].

5.1 Construction and Auxilary Results

A polynomial set $\{P_n(x)\}_{n=0}^{\infty}$ is called quasi-monomial if and only if there exists a derivative operator Θ_n^- and a multiplicative operator Θ_n^+ such that

$$\Theta_n^-(P_n(x)) = P_{n-1}(x),\ \Theta_n^+(P_n(x)) = P_{n+1}(x). \tag{5.1.1}$$

It was Youssèf Ben Cheikh who proved that for a given polynomial sequence $\{P_n(x)\}_{n=0}^{\infty}$, there exists derivative and multiplicative operators Θ_n^- and Θ_n^+ .Therefore, he gave an affirmative answer to the Dattoli's question "May all polynomial families be viewed as quasi-monomial" [13]. More precisely, he has shown that "every polynomial set is quasi-monomial" [9]. Using the monomiality principle, several results were obtained for Laguerre, Laguerre–Konhauser, Legendre, Bernoulli and Appell polynomials (see [1], [2], [5], [8], [12], [14], [41]). On the other hand, orthogonality of some polynomial sets via quasi-monomiality was given in [41]. Obtaining the derivative and multiplicative operator of a given family of polynomials give rises some useful properties such as

$$\left(\Theta_{n+1}^{-}\Theta_{n}^{+}\right)(P_n(x)) = P_n(x), \tag{5.1.2}$$
$$\left(\Theta_{n-1}^{+}\cdots\Theta_{2}^{+}\Theta_{1}^{+}\Theta_{0}^{+}\right)(P_0(x)) = P_n(x).$$

Note that, if Θ_n^- and Θ_n^+ are differential realizations, then (5.1.2) gives the differential equation satisfied by $P_n(x)$. The technique in obtaining differential equations via (5.1.2), is known as the factorization method.

In 1935, Sheffer [37] found the infinite order differential equations for the Appell polynomials and he showed that a necessary and sufficient condition that an Appell set $\{P_n\}$ with generating function $A(t)$ satisfy a finite order equation is that $A(t)$ should be exponential type. Then, in 2002 He and Ricci [20] found the finite order differential equations of the one variable Appell polynomials. Finally, in 2013, we found all finite order differential equations for Appell polynomials [35]. In this chapter, for each $k \in \mathbb{N}$ we focus on constructing two operators $\Theta_n^{-(k)}$ and $\Theta_n^{+(k)}$ which satisfies the following

$$\Theta_n^{-(k)}\,[P_n(x)] = P_{n-k}(x) \tag{5.1.3}$$

and

$$\Theta_n^{+(k)}\,[P_n(x)] = P_{n+k}(x), \tag{5.1.4}$$

where we call them the $k-$times derivative and $k-$times multiplicative operators, respectively. Obtaining these operators for a given polynomial set will provide us several advantageous relations for that polynomial set. For instance, when $\Theta_n^{-(k)}$ and $\Theta_n^{+(k)}$ are differential operators then, for each $k \in \mathbb{N}$, the relation

$$\left(\Theta_{n+k}^{-(k)}\Theta_{n}^{+(k)}\right)(P_n(x)) = P_n(x) \tag{5.1.5}$$

gives us differential equations for this polynomial set. In this case we call such a method which is stated by (5.1.5) as generalized factorization method. This method leads us to obtain a set of differential equations for $P_n(x)$, because for each $k \in \mathbb{N}$ we have one differential equation for this polynomial. On the other hand, if $n = mk + r$, then by using few number of operators, the second relation in (5.1.2) can be given as

$$\left(\Theta_{n-1}^{+} \cdots \Theta_{mk}^{+}\Theta_{(m-1)k}^{+(k)} \cdots \Theta_{k}^{+(k)}\Theta_{0}^{+(k)}\right)(P_0(x)) = P_n(x).$$

5.2 A set of finite order differential equations for the Appell polynomials via generalized factorization method

In this section by obtaining a recurrence relation for the Appell polynomials, we determine the operators $\Theta_{n}^{-(k)}$ and $\Theta_{n}^{+(k)}$ for each $k \in \mathbb{N}$. Then using generalized factorization method, we give a set of finite order differential equations for the Appell polynomials. We exhibit the special cases of our results for $k = 1$ (the known results) and $k = 2$. We start with the following theorem:

Theorem 5.2.1 *[35] Let*

$$\frac{A^{(m)}(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_{n}^{(m)} \frac{t^n}{n!}. \tag{5.2.1}$$

Then, the recurrence is as follows

$$R_{n+k}(x) = R_n(x)\sum_{m=0}^{k}\binom{k}{m}\alpha_0^{(m)}x^{k-m} + \sum_{m=1}^{k}\binom{k}{m}x^{k-m}\sum_{l=0}^{n-1}\binom{n}{l}\alpha_{n-l}^{(m)}R_l(x). \tag{5.2.2}$$

Furthermore, the corresponding $k-$times operators are

$$\Theta_n^{-(k)} := \prod_{m=n-k+1}^{n}\Phi_m^{-} = \prod_{m=n-k+1}^{n}\frac{1}{m}D_x = \frac{(n-k)!}{n!}D_x^k$$

and

$$\Theta_n^{+(k)} := \sum_{m=0}^{k}\binom{k}{m}x^{k-m}\alpha_0^{(m)} + \sum_{m=1}^{k}\binom{k}{m}x^{k-m}\sum_{l=0}^{n-1}\frac{1}{(n-l)!}\alpha_{n-l}^{(m)}D_x^{n-l}. \tag{5.2.3}$$

Proof. *Let*

$$G(x,t) := A(t)e^{xt} = \sum_{l=0}^{\infty}R_l(x)\frac{t^l}{l!}. \tag{5.2.4}$$

Differentiating both sides of (5.2.4) $k-$times with respect to x, we get

$$\frac{\partial^k G}{\partial x^k} = t^k G(x,t).$$

Using series expansion from (5.2.4) in the above relation and equating the coefficients of $\frac{t^n}{n!}$, we get

$$R_n^{(k)}(x) = \frac{n!}{(n-k)!}R_{n-k}(x). \tag{5.2.5}$$

Introducing the familiar derivative operator by

$$\Phi_n^- = \frac{1}{n} D_x,$$

we see from (5.2.5) that

$$\Theta_n^{-(k)} [R_n(x)] := \prod_{m=n-k+1}^{n} \Phi_m^- [R_n(x)] = R_{n-k}(x).$$

Then differentiating both sides of (5.2.4) $k-$times with respect to t, we get

$$\begin{aligned} \frac{\partial^k G}{\partial t^k} &= \sum_{m=0}^{k} \binom{k}{m} D_t^m \{A(t)\} D_t^{k-m} \left\{e^{xt}\right\} \\ &= e^{xt} \sum_{m=0}^{k} \binom{k}{m} x^{k-m} \frac{\partial^m A}{\partial t^m} = G(x,t) \sum_{m=0}^{k} \binom{k}{m} x^{k-m} \frac{A^{(m)}(t)}{A(t)}. \end{aligned} \tag{5.2.6}$$

Upon using (5.2.1) and (5.2.4) in (5.2.6), we get

$$\begin{aligned} \sum_{n=0}^{\infty} R_{n+k}(x) \frac{t^n}{n!} &= \sum_{m=0}^{k} \binom{k}{m} x^{k-m} \sum_{n=0}^{\infty} \alpha_n^{(m)} \frac{t^n}{n!} \sum_{l=0}^{\infty} R_l(x) \frac{t^l}{l!} \\ &= \sum_{m=0}^{k} \binom{k}{m} x^{k-m} \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} \alpha_{n-l}^{(m)} R_l(x) \frac{t^n}{n!}. \end{aligned} \tag{5.2.7}$$

Comparing coefficients of $\frac{t^n}{n!}$ on both sides of (5.2.7), we get

$$R_{n+k}(x) = \sum_{m=0}^{k} \binom{k}{m} x^{k-m} \sum_{l=0}^{n} \binom{n}{l} \alpha_{n-l}^{(m)} R_l(x)$$

or equivalently

$$R_{n+k}(x) = R_n(x) \sum_{m=0}^{k} \binom{k}{m} x^{k-m} \alpha_0^{(m)} + \sum_{m=0}^{k} \binom{k}{m} x^{k-m} \sum_{l=0}^{n-1} \binom{n}{l} \alpha_{n-l}^{(m)} R_l(x). \tag{5.2.8}$$

Since

$$\alpha_n^{(0)} = \delta_{n,0} := \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases} \tag{5.2.9}$$

we get

$$R_{n+k}(x) = R_n(x)\sum_{m=0}^{k}\binom{k}{m}x^{k-m}\alpha_0^{(m)} + \sum_{m=1}^{k}\binom{k}{m}x^{k-m}\sum_{l=0}^{n-1}\binom{n}{l}\alpha_{n-l}^{(m)}R_l(x),$$

which is (5.2.2).

On the other hand, since

$$R_l(x) = \prod_{m=l+1}^{n}\Phi_m^-[R_n(x)] = \frac{l!}{n!}D_x^{n-l}[R_n(x)], \tag{5.2.10}$$

we get from (5.2.2) that

$$R_{n+k}(x) = \left[\sum_{m=0}^{k}\binom{k}{m}x^{k-m}\alpha_0^{(m)} + \sum_{m=1}^{k}\binom{k}{m}x^{k-m}\sum_{l=0}^{n-1}\frac{1}{(n-l)!}\alpha_{n-l}^{(m)}D_x^{n-l}\right]R_n(x).$$

Hence, the $k-$times multiplicative operator is given by (5.2.3). ■

The next Theorem gives a set of differential equations for Appell polynomials.

Theorem 5.2.2 *[35] For each $k \in \mathbb{N}$ and for all $n \in \mathbb{N}$, the Appell polynomials $R_n(x)$ satisfy the following set of differential equations:*

$$L_{n,k}^{(x)}(R_n(x)) = \left(\frac{(n+k)!}{n!} - k!\right)R_n(x), \tag{5.2.11}$$

where the differential operator $\{L_{n,k}^{(x)}\}_{n=0}^{\infty}$ *is given by*

$$\begin{aligned} L_{n,k}^{(x)} &= \sum_{j=1}^{k}\binom{k}{j}\frac{k!}{j!}x^{j}D_{x}^{j} \\ &+\sum_{m=1}^{k}\binom{k}{m}\alpha_{0}^{(m)}\sum_{j=m}^{k}\binom{k}{j}\frac{(k-m)!}{(j-m)!}x^{j-m}D_{x}^{j} \\ &+\sum_{m=1}^{k}\binom{k}{m}\sum_{l=0}^{n-1}\frac{1}{(n-l)!}\alpha_{n-l}^{(m)}\sum_{j=m}^{k}\binom{k}{j}\frac{(k-m)!}{(j-m)!}x^{j-m}D_{x}^{n-l+j}. \end{aligned} \tag{5.2.12}$$

Proof. *Taking into account the corresponding* $k-$*times shift operators from Theorem 5.2.1 and applying the generalized factorization method given by (5.1.5) to* $R_n(x)$, *we get*

$$\begin{aligned} &\frac{n!}{(n+k)!}D_{x}^{k}\left[\sum_{m=0}^{k}\binom{k}{m}\alpha_{0}^{(m)}x^{k-m}+\sum_{m=1}^{k}\binom{k}{m}x^{k-m}\sum_{l=0}^{n-1}\frac{1}{(n-l)!}\alpha_{n-l}^{(m)}D_{x}^{n-l}\right]R_{n}(x) \\ =\;& \frac{n!}{(n+k)!}\left[\sum_{m=0}^{k}\binom{k}{m}\alpha_{0}^{(m)}\sum_{j=m}^{k}\binom{k}{j}D_{x}^{k-j}\left(x^{k-m}\right)D_{x}^{j}(R_{n}(x))\right. \\ &\left.+\sum_{m=1}^{k}\binom{k}{m}\sum_{l=0}^{n-1}\frac{1}{(n-l)!}\alpha_{n-l}^{(m)}\sum_{j=m}^{k}\binom{k}{j}D_{x}^{k-j}\left(x^{k-m}\right)D_{x}^{n-l+j}(R_{n}(x))\right] \\ =\;& \frac{n!}{(n+k)!}\left[k!R_{n}(x)+\sum_{j=1}^{k}\binom{k}{j}D_{x}^{k-j}\left(x^{k}\right)D_{x}^{j}(R_{n}(x))\right. \\ &+\sum_{m=1}^{k}\binom{k}{m}\alpha_{0}^{(m)}\sum_{j=m}^{k}\binom{k}{j}D_{x}^{k-j}\left(x^{k-m}\right)D_{x}^{j}(R_{n}(x)) \\ &\left.+\sum_{m=1}^{k}\binom{k}{m}\sum_{l=0}^{n-1}\frac{1}{(n-l)!}\alpha_{n-l}^{(m)}\sum_{j=m}^{k}\binom{k}{j}D_{x}^{k-j}\left(x^{k-m}\right)D_{x}^{n-l+j}(R_{n}(x))\right] \qquad (5.2.13) \\ =\;& R_{n}(x). \end{aligned}$$

Thus, we have

$$
\begin{aligned}
&\frac{n!}{(n+k)!}(k!R_n(x)+\sum_{j=1}^{k}\binom{k}{j}\frac{k!}{j!}x^jD_x^j(R_n(x))\\
&+\sum_{m=1}^{k}\binom{k}{m}\alpha_0^{(m)}\sum_{j=m}^{k}\binom{k}{j}\frac{(k-m)!}{(j-m)!}x^{j-m}D_x^j(R_n(x))\\
&+\sum_{m=1}^{k}\binom{k}{m}\sum_{l=0}^{n-1}\frac{1}{(n-l)!}\alpha_{n-l}^{(m)}\sum_{j=m}^{k}\binom{k}{j}\frac{(k-m)!}{(j-m)!}x^{j-m}D_x^{n-l+j}(R_n(x))\\
=\ &R_n(x).
\end{aligned}
$$

This gives the desired result. ■

The cases $k = 1$ and $k = 2$ are presented in the following Corollaries:

Corollary 5.2.3 *[20] Letting $k = 1$ in Theorems 5.2.1 and 5.2.2, then taking*

$$
\frac{A'(t)}{A(t)} = \sum_{n=0}^{\infty}\alpha_n^{(1)}\frac{t^n}{n!};\ \ \alpha_n^{(1)} := \alpha_n;\ \ \alpha_n^{(0)} = \delta_{n,0} := \begin{cases} 1, & n=0 \\ 0, & otherwise \end{cases},
$$

we get the recurrence relation

$$
R_{n+1}(x) = (x+\alpha_0)R_n(x) + \sum_{l=0}^{n-1}\binom{n}{l}\alpha_{n-l}R_l(x).
$$

On the other hand, $1-$*times shift operators (or simply the shift operators) are given by*

$$
\Theta_n^- := \Phi_n^- = \frac{1}{n}D_x
$$

and

$$\Theta_n^+ := (x+\alpha_0) + \sum_{l=0}^{n-1} \frac{1}{(n-l)!} \alpha_{n-l} D_x^{n-l}.$$

Finally, for $\lambda_{n,1} = n$, the differential equation is given by

$$L_{n,1}^{(x)} (R_n(x)) = nR_n(x),$$

where the differential operator is given by

$$L_{n,1}^{(x)} := (x+\alpha_0) D_x + \sum_{l=0}^{n-1} \frac{1}{(n-l)!} \alpha_{n-l} D_x^{n-l+1}.$$

Note that these results are same with the results obtained in [20].

Corollary 5.2.4 *[35] Letting $k = 2$ in Theorems 5.2.1 and 5.2.2, by setting*

$$\frac{A'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n^{(1)} \frac{t^n}{n!} \quad and \quad \frac{A''(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n^{(2)} \frac{t^n}{n!},$$

the recurrence is as follows

$$R_{n+2}(x) = \left(x^2 + 2\alpha_0^{(1)} x + \alpha_0^{(2)}\right) R_n(x) + \sum_{l=0}^{n-1} \binom{n}{l} \left(2x\alpha_{n-l}^{(1)} + \alpha_{n-l}^{(2)}\right) R_l(x).$$

$2-$*times shift operators are*

$$\Theta_n^{-(2)} := \Phi_{n-1}^- \Phi_n^- = \frac{1}{(n-1)n} D_x^2$$

and

$$\Theta_n^{+(2)} := \left(x^2 + 2\alpha_0^{(1)} x + \alpha_0^{(2)}\right) + \sum_{l=0}^{n-1} \binom{n}{l} \left(2x\alpha_{n-l}^{(1)} + \alpha_{n-l}^{(2)}\right) D_x^{n-l}.$$

Finally, for $\lambda_{n,2} = n^2 + 3n$, *the differential equation is given by*

$$L_{n,2}^{(x)} \left(R_n(x)\right) = \left(n^2 + 3n\right) R_n(x),$$

where

$$\begin{aligned} L_{n,2}^{(x)} \quad : \quad &= \Big[\left(4xD_x + x^2 D_x^2\right) + 4\alpha_0^{(1)} D_x + 2\alpha_0^{(1)} x D_x^2 + \alpha_0^{(2)} D_x^2 \\ &\quad + \sum_{l=0}^{n-1} \frac{2\alpha_{n-l}^{(1)}(2D_x^{n-l+1} + xD_x^{n-l+2}) + \alpha_{n-l}^{(2)} D_x^{n-l+2}}{(n-l)!}\Big]. \end{aligned}$$

5.3 Applications of Main Theorems

In this section, we apply the results of Section 5.2 to the two famous representatives of the Appell polynomials: the Hermite and the Bernoulli polynomials. Since the case $k = 1$ gives the usual results for these polynomial sets, we exhibit the case $k = 2$.

5.3.1 Hermite Polynomial

Hermite polynomial is generated by the following relation

$$e^{2xt - \frac{t^2}{2}} = \sum_{n=0}^{\infty} He_n(x) \frac{t^n}{n!}. \tag{5.3.1}$$

Taking $A(t) = e^{-\frac{t^2}{2}}$ we get

$$\frac{A'(t)}{A(t)} = -t = \sum_{n=0}^{\infty} \alpha_n^{(1)} \frac{t^n}{n!} \tag{5.3.2}$$

and hence

$$\alpha_1^{(1)} = -1; \quad \alpha_0^{(1)} = \alpha_2^{(1)} = \alpha_3^{(1)} = ... = 0. \qquad (5.3.3)$$

On the other hand

$$\frac{A''(t)}{A(t)} = -1 + t^2 = \sum_{n=0}^{\infty} \alpha_n^{(2)} \frac{t^n}{n!}, \qquad (5.3.4)$$

so

$$\alpha_0^{(2)} = -1, \alpha_2^{(2)} = 2; \ \alpha_1^{(2)} = \alpha_3^{(2)} = \alpha_4^{(2)} = ... = 0. \qquad (5.3.5)$$

Corollary 5.3.1 *[35] Using the above results for k = 2 in Theorems 5.2.1 and 5.2.2, we get*

$$He_{n+2}(x) = \left(x^2 - 1\right) He_n(x) - 2nxHe_{n-1}(x) + n(n-1)He_{n-2}(x), \qquad (5.3.6)$$

the shift operators are

$$\Theta_n^{-(2)} = \frac{1}{(n-1)n} D_x^2, \ \Theta_n^{+(2)} := \left(x^2 - 1\right) - 2nxD_x + n(n-1)D_x^2 \qquad (5.3.7)$$

and the fourth order differential equation is given by

$$D_x^2(x^2 - 1 - 2xD_x + D_x^2)He_n(x) = (n+2)(n+1)He_n(x). \qquad (5.3.8)$$

5.3.2 Bernoulli Polynomial

With the aid of the generalized factorization method, which is mentioned in Section 5, we apply the procedure for the case $k = 2$ to obtain the differential operator $L_{m,2}^{(x)}$ such that

$$L_{m,2}^{(x)}(B_m(x)) = \left(m^2 + 3m\right) B_m(x). \tag{5.3.9}$$

Here $B_m(x)$ denotes the Bernoulli polynomial which has the generating function

$$\frac{t}{e^t - 1} e^{xt} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}.$$

Taking derivatives with respect to t in the above generating function, we get

$$\frac{\partial^2 G(x,t)}{\partial t^2} = G(x,t) \left(\frac{A''(t)}{A(t)} + \frac{2xA'(t)}{A(t)} + x^2 \right) \tag{5.3.10}$$

where $A(t) = \dfrac{t}{e^t - 1}$. Therefore we obtain

$$\begin{aligned}
&\frac{\partial^2 G(x,t)}{\partial t^2} \\
= \;& G(x,t)\left(-\frac{e^t}{e^t - 1} - \frac{2e^t(e^t - 1 - te^t)}{t(e^t - 1)^2} + 2x\left(\frac{e^t - 1 - te^t}{t(e^t - 1)}\right) + x^2\right) \\
= \;& G(x,t)\left(-\frac{1 - e^t - 1}{1 - e^t} - 2\frac{e^t}{t}\left(\frac{1}{e^t - 1} - \frac{te^t}{(e^t - 1)^2}\right) + 2x\left(\frac{1}{t} - \frac{e^t}{e^t - 1}\right) + x^2\right) \\
= \;& \left[\left(\frac{1}{1 - e^t} - 1\right) - 2\frac{e^t}{t^2}\frac{t}{e^t - 1}\right. \\
& \left. 2\frac{e^t}{t(e^t - 1)}\frac{te^t}{e^t - 1} + 2\frac{x}{t} - 2x\frac{e^t}{t}\frac{t}{e^t - 1} + x^2\right] G(x,t).
\end{aligned} \tag{5.3.11}$$

Substituting the series relations, we obtain

$$
\begin{aligned}
&\sum_{m=0}^{\infty} B_{m+2}(x)\frac{t^m}{m!} \\
= \;& \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}\Bigg[(-\frac{1}{t}\sum_{k=0}^{\infty} B_k\frac{t^k}{k!} - 1 - \frac{2}{t^2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}B_k\frac{t^n}{n!} \\
&+ \frac{2}{t}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}B_k\frac{t^n}{n!} + \frac{2}{t^2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}B_lB_{k-l}\frac{t^n}{n!} \\
&+ (\frac{2x}{t} + x^2) - \frac{2x}{t}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}B_k\frac{t^n}{n!}\Bigg]. \qquad (5.3.12)
\end{aligned}
$$

Using (2.2.2) and (2.2.4) and comparing the coefficients of $\frac{t^m}{m!}$, we have the following recurrence:

$$
\begin{aligned}
B_{m+2}(x) &= (x^2 - x - \frac{2}{m+2} - \frac{2}{m+1} + \frac{7}{6})B_m(x) \\
&-\sum_{k=0}^{m-1}\frac{m!}{(m-k-1)!(k+2)!}B_{m-k-1}(x)B_{k+2} \\
&-2\sum_{n=0}^{m-1}\sum_{k=0}^{n+3}\left[\binom{m+2}{n+3}\binom{n+3}{k}B_k - \sum_{l=0}^{k}\binom{m+2}{k}\binom{n+3}{k}\binom{k}{l}B_lB_{k-l}\right] \\
&\times\frac{B_{m-n-1}(x)}{(m+1)(m+2)} + 2(1-x)\sum_{n=0}^{m-1}\sum_{k=0}^{n+2}\binom{m+1}{n+2}\binom{n+2}{k}\frac{B_kB_{m-n-1}(x)}{m+1}. \qquad (5.3.13)
\end{aligned}
$$

Since

$$B_{m-k-1}(x) = \frac{(m-k-1)!}{m!}D_x^{k+1}B_m(x) \qquad (5.3.14)$$

and

$$B_{m-n-1}(x) = \frac{(m-n-1)!}{m!} D_x^{n+1} B_m(x) \tag{5.3.15}$$

the multiplicative operator can be written as

$$\begin{aligned}\Theta_m^{+(2)} = x^2 - x - \frac{2}{m+2} - \frac{2}{m+1} + \frac{7}{6} - \sum_{k=0}^{m-1} \frac{B_{k+2}}{(k+2)!} D_x^{k+1} \\ -2\sum_{n=0}^{m-1}\sum_{k=0}^{n+3}\left[\frac{B_k}{(n+3-k)!k!} - \sum_{l=0}^{k}\binom{n+3}{k}\binom{k}{l}\frac{(m-n-1)!}{(m+2-k)!k!}B_l B_{k-l}\right] D_x^{n+1} \\ +2(1-x)\sum_{n=0}^{m-1}\sum_{k=0}^{n+2}\frac{B_k}{(n+2-k)!k!} D_x^{n+1}.\end{aligned} \tag{5.3.16}$$

On the other hand, using the fact that the 2- times derivative operators for all Appell polynomials is

$$\Theta_n^{-(2)} = \frac{1}{(n-1)n} D_x^2 \tag{5.3.17}$$

and by using the generalized factorization method with $k = 2$

$$\Theta_{m+2}^{-(2)} \Theta_m^{+(2)} B_m(x) = B_m(x). \tag{5.3.18}$$

After some manipulations we obtain that the differential equation for the Bernoulli polynomial for the case $k = 2$ as

$$L_{m,2}^{(x)}(B_m(x)) = \left(m^2 + 3m\right) B_m(x), \tag{5.3.19}$$

where

$$
\begin{aligned}
L^{(x)}_{m,2} \quad : \quad &= x^2 D_x + 2xD_x - xD_x^2 + 2 - \frac{2}{m+2} D_x^2 + \frac{7}{6} D_x^2 - \sum_{k=0}^{m-1} \frac{B_{k+2}}{(k+2)!} D_x^{k+3} \\
&-2 \sum_{n=0}^{m-1} \sum_{k=0}^{n+3} \Big(\frac{B_k}{(n+3-k)!k!} D_x^{n+3} - \sum_{l=0}^{k} \binom{n+3}{k} \binom{k}{l} \frac{(m-n-1)!}{(m+2-k)!k!} B_l B_{k-l} D_x^{n+3} \\
&+2(1-x) \sum_{n=0}^{m-1} \sum_{k=0}^{n+2} \frac{B_k}{(n+2-k)!k!} D_x^{n+3}. \qquad (5.3.20)
\end{aligned}
$$

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