Grover Algorithm

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Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the Degree of

Master of Science
in
Applied Mathematics and Computer Science

Eastern Mediterranean University
February 2011
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

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ABSTRACT

Although, the past years brought many exciting and pathfinder achievements in computer science, computer engineers still agreed on a point that the computers of the next generation should be the quantum computers. These will be computation devices to make direct use of quantum mechanical phenomena, such as superposition and entanglement, to carry out operations on data.

However, the quantum computer means we need a quantum programming language to understand and to be able to use it. Sadly, the quantum computing is improved so slowly that we can say it is still in its infancy. Even so, after the big surprise of Peter Shor in 1994[12], Lov Grover came across in 1996 [9] with another surprising algorithm that searches an unsorted database in less than linear time unlike the models of classical computation.

Keywords: Quantum Search, Search, Quantum Computing, Grover's Algorithm.

ÖZ

Geçen yıllar bilgisayar bilimi adına birçok heyecan verici ve yol gösterici gelişmelere sahne olmasına rağmen, bilgisayar mühendisleri hala gelecek nesilin bilgisayarlarının kuantum bilgisayarlar olması konusunda hemfikirler. Bu bilgisayarlar doğrudan doğruya kuantum mekanizmasını veri tabanları üzerinde uygulamak için inşa edilecekler.

Ancak, kuantum bilgisayar demek bu bilgisayarları anlayabilmek ve kullanabilmek için quantum programlama diline ihtiyaç duyduğumuz anlamına geliyor. Maalesef kuantum bilgisayar teknolojisi oldukça yavaş ilerliyor. Buna rağmen, Peter Shor'un 1994'teki büyük sürprizinden sonra, Lov Grover 1996'da bu alandaki bir başka buluşa imza attı . Bu buluş klasik tarama algoritmalarına göre daha hızlı olmasının nedeni iç içe geçmiş ve paralel işlemleme tekniğini kullanmasından kaynaklanıyor.

Anahtar Kelimeler: Kuantum Taraması, Tarama, Kuantum Hesaplaması, Grover Algoritması.

PREFACE

My thesis discusses the latest achievements about the quantum computing by taking the subject as the Grover's algorithm with, furthermore, taking care of its complexity class. And, it finalizes with an implementation of Grover's algorithm in a quantum programming language QCL, which is written by Bernhard Ömer in 1998 [14] in the frame of his master thesis.

Chapter 1 gives an early description about the quantum computer science by explaining the quantum mechanics in order to make a preparation to Grover algorithm.

Chapter 2 talks about the mean of the Grover's algorithm and the definition of it in quantum boundaries. After Grover, many people discussed and worked on improving this algorithm, which brought a very hard working on many resources [3][6][8].Although, one came into prominence between others while writing this chapter that I can't abnegate [7]. Despite the traditional computation search algorithms within linear time, Grover's quantum search algorithm searches an unsorted database with N entries in $O\left(N^{1/2}\right)$ time by using only $O(\log N)$ storage space. This carries out an idea if this algorithm is an answer to the biggest question in complexity analysis.

In writing chapter 3, the critical point was to make attention on working on an up to date resource[2]. This because, complexity analysis is still in seeking hardly by many people around the world discusses that it is away of showing $NP \subseteq BQP$, but solves an even more general problem which is the satisfiability of a circuit with n inputs.

Chapter 4 clarifies everything by using the fact that the programming languages allow the complete and constructive description of quantum algorithms including their control structure for arbitrary input sizes. This makes it much better than the quantum circuits or quantum Turing machines. While writing this chapter, although the documentations [14][15] [16][17] of Bernhard Ömer was the main supporter, I can't reckon without mention a very nice practice[5] on this programming language.

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Chapter 1

PRELIMINARIES

1.1 Introduction to Quantum Computing

In the early 1980's, Richard Feynman observed that certain quantum mechanical effects cannot be simulated efficiently on a classical computer. This observation brought about the beliefs that general computation could be done more efficiently if it made use of these quantum effects. Unfortunately, building computational machines that use such quantum effects, which we called Quantum Computers, found very tricky. Even more, no one was sure how to use the quantum effects to speed up the computation, till 1994. In that year, Peter Shor surprised the world by describing a polynomial time quantum algorithm for factoring integers. Then, the field of quantum computing came into its own. This discovery prompted a flurry of activity, both among experimentalists trying to build quantum computers and theoreticians trying to find other quantum algorithms. Just two years later, Lov Grover tapered with another quantum algorithm and surprised the experimentalists again. The algorithm provided a quadratic speedup over its classical counterpart for searching an unsorted database. Classically, the time it takes to do certain computations can be decreased by using parallel processors. This provides that achieving an exponential decrease in time requires an exponential increase in the number of processors, and hence an exponential increase in the amount of physical space needed. However, in quantum systems the amount of parallelism increases exponentially with the size of the system. In other words, an exponential increase in parallelism requires only a linear increase in the amount of physical space needed. This effect is called quantum parallelism which was mentioned by David Deutsch and Richard Jozsa in 1992.

Unexpectedly, this came along with a big handicap. While quantum system can perform massive parallel computation, accessing the results of the computation is strictly restricted. Access to the results is equivalent to making a measurement which interrupts the quantum state. This problem brings a situation which is even poor than the classical one that we can only read the result of one parallel thread. Worse, because of the measurement is probabilistic, we cannot even choose which one we get.

After all, in the past years, various people have found clever ways of addressing the measurement problem to exploit the power of quantum parallelism. This sort of manipulation has no classical similarity, and requires non-traditional programming techniques. One technique manipulates the quantum state so that a common property of all of the output values such as the symmetry or period of a function can be read off. This technique is used in Shor's factorization algorithm. Another technique transforms the quantum state to increase the probability that output of interest will be read. Such an amplification technique is used in Grover's search algorithm.

1.2 Quantum Mechanics

Since most of our everyday experiences are not applicable, quantum mechanical phenomena are difficult to understand. Hence, we will try to give some feeling as to the nature of quantum mechanics and some of the mathematical formalisms needed to work with quantum mechanics to the range needed for quantum computing. Quantum mechanics is a theory in the mathematical sense that it is governed by a set of axioms. The consequences of these axioms describe the behavior of quantum systems.

1.3 State Spaces and Bra/Ket Notation

For quantum computing, we need to deal with finite quantum systems and it suffices to consider finite dimensional complex vector spaces with inner product. We can describe the quantum state spaces and the transformations acting on them in terms of vectors and matrices, or in the more compact bra/ket notation which was introduced by Paul Dirac in 1939. Kets like $|x\rangle$ denote column vectors. These are typically used to describe quantum states. The corresponding notation bra, $\langle x|$, denotes the conjugate transpose of $|x\rangle$. For instance, the orthonormal basis $\{|0\rangle, |1\rangle\}$ can be expressed as $\left\{(1,0)^T, (0,1)^T\right\}$, which is equivalent to $\left\{\binom{1}{0}, \binom{0}{1}\right\}$. Any complex linear combination of $|0\rangle$ and $|1\rangle$, for example $a|0\rangle + b|1\rangle$, can be written as $(a,b)^T$.

Combining $\langle x|$ and $|y\rangle$ as in $\langle x||y\rangle$, simply written as $\langle x|y\rangle$, denotes the inner product of the two vectors. For example, since $|0\rangle$ is a unit vector we have $\langle 0|0\rangle = 1$, and since $|0\rangle$ and $|1\rangle$ are orthogonal we have $\langle 0|1\rangle = 0$.

The notation $|x\rangle\langle y|$ is the outer product of $|x\rangle$ and $\langle y|$. For instance, $|x\rangle\langle 1|$ is the transformation that maps $|1\rangle$ to $|0\rangle$ and $|0\rangle$ to $(0,0)^T$ since

$$|0\rangle\langle 1||1\rangle = |0\rangle\langle 1|1\rangle = |0\rangle = \binom{1}{0} \qquad \textit{due to} \quad \langle 1|1\rangle = 1$$

$$|0\rangle\langle 1||0\rangle = |0\rangle\langle 1|0\rangle = 0\,|0\rangle = \binom{0}{0} \quad \textit{due to} \quad \langle 1|0\rangle = 0.$$

In the similar way, $|0\rangle\langle 1|$ can be written in matrix form $|0\rangle = \binom{1}{0}$, and $\langle 1| = (0,1)$.

Then,

$$|0\rangle\langle 1| = \begin{pmatrix}1\\0\end{pmatrix}(0,1) = \left(\begin{array}{cc} 0 & 1 \\ \\ 0 & 0 \end{array}\right).$$

This notation gives us a comfortable way of indicating transformations on quantum states in terms of what happens to the basis vectors. To give an example, we can consider a transformation that exchanges $|0\rangle$ and $\langle 1|$ is given by the matrix

$$X = |0\rangle\langle 1| + |1\rangle\langle 0|.$$

We will use a sensitive notation

$$\mathbf{X}|0\rangle = |1\rangle$$
$$\mathbf{X}|1\rangle = |0\rangle$$

that clearly explains the result of a transformation on the basis vectors[10].

1.4 Quantum Bits

In a two dimensional complex vector space, a quantum bit, or simply qubit, is a unit vector for the fixed particular basis $\{|0\rangle, |1\rangle\}$. For the aim of quantum computation, the basis states $|0\rangle$ and $|1\rangle$ are taken to represent the classical bit values 0 and 1, respectively. However, despite classical bits, qubits can be in a superposition of $|0\rangle$ and $|1\rangle$ such as $a|0\rangle \pm b|1\rangle$ where a and b are complex numbers such that $|a|^2 + |b|^2 = 1$. If we measure this superposition with respect to the basis $\{|0\rangle, |1\rangle\}$, the probability of getting the measured value $|0\rangle$ is $|a|^2$, and the probability that the measured value is $|1\rangle$ is $|b|^2$. Although a qubit can be put in infinitely many superposition states, the only possible extract from a single qubit is equivalent to the worth of information of a single classical bit. Here, the main reason of no more information could be gained from a qubit than from a classical bit is this information can only be achieved by measurement. When we measure a qubit, the measurement changes the state to one of the basis states. Since every measurement can result in only one of two states, one of the basis vectors associated to the given measuring device. So, there are only two possible results just like in the classical case. As measurement changes the state, it is not possible to measure the state of a qubit in two different bases.

1.5 Multiple Qubits

The state of a qubit can be represented by a vector in the two dimensional complex vector space spanned by $|0\rangle$ and $|1\rangle$ as we considered before. In classical physics, the possible states of a system of n particles, such that individual states can be described by a vector in a two dimensional vector space, form a vector space of $2n$ dimensions. But, in a quantum system, a system of n qubits has a state space of 2^n dimensions which is much larger than the classical one. Here the state space is the set of normalized vectors in this 2^n dimensional space, just as the state $a\,|0\rangle + b\,|1\rangle$ of a qubit is normalized so that $|a|^2 + |b|^2 = 1$. Actually, this is the exponential growth of the state space with the number of particles that suggests a possible exponential speed-up of computation on quantum computers over classical ones. While individual state spaces of n particles combine classically through the Cartesian product, quantum states combine through the tensor product.

Instead of talking only about the tensor product, which is already explained in Appendix A before, let's take a look on the differences of the Cartesian and the tensor product that will be fateful to understand quantum computation.

Let V and W be 2 two-dimensional complex vector spaces with bases $\{v_1, v_2\}$ for

V , and $\{w_1, w_2\}$ for W. The Cartesian product $V \times W$ can take as its basis the union of the bases of its component spaces $\{v_1, v_2, w_1, w_2\}$. Because $dim\,(X \times Y) = dim\,(X) + dim\,(Y)$, easy to see that the dimension of the state space of multiple classical particles grows linearly with the number of particles. The tensor product of V and W has basis $\{v_1 \otimes w_1, v_1 \otimes w_2, v_2 \otimes w_1, v_2 \otimes w_2\}$. So, the state space for two qubits, each with basis $\{|0\rangle\,, |1\rangle\}$, has basis $\{|0\rangle \otimes |0\rangle\,, |0\rangle \otimes |1\rangle\,, |1\rangle \otimes |0\rangle\,, |1\rangle \otimes |1\rangle\}$ which can also be noted as $\{|00\rangle\,, |01\rangle\,, |10\rangle\,, |11\rangle\}$. In a more general manner, we write $|x\rangle$ to mean $|b_n b_{n-1} ... b_0\rangle$ where b_i are the binary digits of the number x.

A basis for a three qubit system is $\{|000\rangle\,, |001\rangle\,, |010\rangle\,, |011\rangle\,, |100\rangle\,, |101\rangle\,, |110\rangle\,, |111\rangle\}$. So, in general, an n qubit system has 2^n basis vectors. Now, we can see the exponential growth of the state space with the number of quantum particles. The tensor product $X \otimes Y$ has dimension $dim\,(X) \times dim\,(Y)$.

But, as it is more clear now, we can see the recycling is not possible. The state $|00\rangle + |11\rangle$ is an example of a quantum state that cannot be described in terms of the state of each of its qubits separately. In other words, we cannot find $\{a_1, a_2, b_1, b_2\}$ such that

$$(a_1\,|0\rangle + b_1\,|1\rangle) \otimes (a_2\,|0\rangle + b_2\,|1\rangle) = |00\rangle + |11\rangle$$

since $(a_1\,|0\rangle + b_1\,|1\rangle) \otimes (a_2\,|0\rangle + b_2\,|1\rangle) = a_1 a_2\,|00\rangle + a_1 b_2\,|01\rangle + b_1 a_2\,|10\rangle + b_1 b_2\,|11\rangle$ and $a_1 b_2 = 0$ implies that either $a_1 a_2 = 0$ or $b_1 b_2 = 0$. States which cannot be decomposed in this way are called entangled states. These states represent situations which have no classical counterpart. Besides, these are the states that provide the exponential growth of quantum state spaces with the number of particles.

1.6 Measurement

The result of a measurement is probabilistic and the process of measurement changes the state to that measured. Here is an example of measurement in a two qubit system. Any two qubit state can be expressed as $a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle$, where *a, b, c,* and *d* are complex numbers such that

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1.$$

Suppose we wish to measure the first qubit with respect to the standard basis $\{|0\rangle\,, |1\rangle\}$. For convenience, we will rewrite the state $a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle$.

$$\begin{aligned} a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle &= |0\rangle \otimes (a\,|0\rangle + b\,|1\rangle) + |1\rangle \otimes (c\,|0\rangle + d\,|1\rangle) \\ &= v\,|0\rangle \otimes \left(\frac{a}{v\,|0\rangle} + \frac{b}{v\,|1\rangle}\right) + w\,|1\rangle \otimes \left(\frac{c}{w\,|0\rangle} + \frac{d}{w\,|1\rangle}\right) \end{aligned}$$

For $v = \sqrt{|a|^2 + |b|^2}$ and $w = \sqrt{|c|^2 + |d|^2}$ the vectors $a/v\,|0\rangle + b/v\,|1\rangle$ and $c/w\,|0\rangle + d/w\,|1\rangle$ are of unit length. Once we write the state as above which is a tensor product of the bit being measured and a second vector of unit length, the probabilistic result of a measurement is easy to read off. Measurement of the first bit will return $|0\rangle$ with probability $v^2 = |a|^2 + |b|^2$ projecting the state to $|0\rangle \otimes (a/v\,|0\rangle + b/u\,|1\rangle)$ or return $|1\rangle$ with probability $w = |c|^2 + |d|^2$ projecting the state to $|1\rangle \otimes (c/w\,|0\rangle + d/w\,|1\rangle)$. As $|0\rangle \otimes (a/v\,|0\rangle + b/u\,|1\rangle)$ and $|1\rangle \otimes (c/w\,|0\rangle + d/w\,|1\rangle)$ are both unit vectors, no scaling is necessary. In the same manner, measuring the second bit works similarly.

Any quantum state can be written as the sum of two vectors, one in each of the subspaces. A measurement of k qubits in the standard basis has 2^k possible outcomes. Measuring k qubits of an n-qubit system splits of the 2^n-dimensional state space $\mathcal{H}$ into a Cartesian product of orthogonal subspaces $S_1, S_2, ..., S_{2^k}$ with $\mathcal{H} = S_1 \times S_2 \times ... \times S_{2^k}$, such that the value of the k qubits being measured is m_i, and the state after measurement is in space S_i for some i. The S_i is chosen randomly with probability the square of the amplitude of the component of ψ in S_i, and projects the state into that component, scaling to give length 1. In the same manner, the probability that the result of the measurement to be a given value is the sum of the squares of the absolute values of the amplitudes of all basis vectors compatible with that value of measurement.

The way of the measurement brings out some other ideas about entangled particles. Particles are not entangled if the measurement of one has no effect on the other. For example, the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is entangled. Because the probability of the measurement of the first bit to be $|0\rangle$ is $\frac{1}{2}$, if the second bit has not been measured. However, if the second bit had also been measured, the probability of the measurement of the first bit to be $|0\rangle$ would be either 1 or 0 depending on whether the second bit was measured as $|0\rangle$ or $|1\rangle$ respectively. This why, the possible result of measuring the first bit is changed by a measurement of the second bit. Unlikely, the state $\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$ is not entangled. Because $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, any measurement of the first bit will return $|0\rangle$ regardless of whether the second bit was measured or not. Likewise, the second bit has 50% chance of being measured as $|0\rangle$ without consideration whether the first bit was measured.

1.7 Quantum Gates

Any linear transformation on a complex vector space can be described by a matrix. Let $M^\dagger$ denote the conjugate transpose of the matrix M. A matrix M is unitary if $MM^\dagger = I$. In other words, we say M is a unitary transformation. The most important property of it is that any unitary transformation is reversible. Here is the explanation of some useful simple, single-qubit, quantum state transformations, each with a little example in addition:

1.7.1 Quantum Identity Gate:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{array}{l} |0\rangle \to I \to |0\rangle \\ |1\rangle \to I \to |1\rangle \end{array} \qquad \alpha|0\rangle + \beta|1\rangle \to I \to \alpha|0\rangle + \beta|1\rangle$$

1.7.2 Quantum Not Gate (Negation):

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{array}{l} |0\rangle \to X \to |1\rangle \\ |1\rangle \to X \to |0\rangle \end{array} \qquad \alpha|0\rangle + \beta|1\rangle \to X \to \alpha|1\rangle + \beta|0\rangle$$

1.7.3 Pauli Y Gate:

$$Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{array}{l} |0\rangle \to Y \to -|1\rangle \\ |1\rangle \to Y \to |0\rangle \end{array} \qquad \alpha|0\rangle + \beta|1\rangle \to Y \to i(\alpha|1\rangle - \beta|0\rangle)$$

1.7.4 Pauli Z Gate (Phase Shift Operation):

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{array}{l} |0\rangle \to Z \to |0\rangle \\ |1\rangle \to Z \to -|1\rangle \end{array} \qquad \alpha|0\rangle + \beta|1\rangle \to Z \to \alpha|0\rangle - \beta|1\rangle$$

It can be easily verified that these gates are unitary. For instance

$$YY^\dagger = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = I.$$

1.7.5 Controlled-Not Gate:

The controlled-not gate CNOT works in a way that it changes the second bit if the first bit is 1, and leaves this bit unchanged otherwise. In order to represent transformations of this space in matrix notation we need to choose an isomorphism between this space and the space of complex four tuples. Such an isomorphism associates basis $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$ to the standard 4-tuple basis $(1,0,0,0)^T$, $(0,1,0,0)^T$, $(0,0,1,0)^T$, and $(0,0,0,1)^T$ respectively. So, we have CNOT with the representations

$$\text{CNOT:} \quad \begin{array}{c} |00\rangle \to |00\rangle \\ |01\rangle \to |01\rangle \\ |10\rangle \to |11\rangle \\ |11\rangle \to |10\rangle \end{array} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The transformation CNOT is unitary since $CNOT^{\dagger} = CNOT$ and $CNOTCNOT = I$. In addition, the CNOT gate cannot be decomposed into a tensor product of two single-bit transformations.

Also, we use some graphical representations of quantum transformations to make it easier to see the operations. The controlled-not gate CNOT is mostly represented by a circuit in the following form:

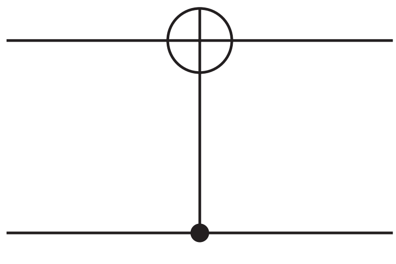


Figure 1.1: Controlled-Not Gate

Here, the black point indicates the control bit, and the solid circle indicates the target bit, which is toggled when the control bit is 0.

1.7.6 Controlled-Controlled-Not Gate (*Toffoli Gate*):

Similarly, the controlled-controlled-not gate CCNOT has the following graphical representation.

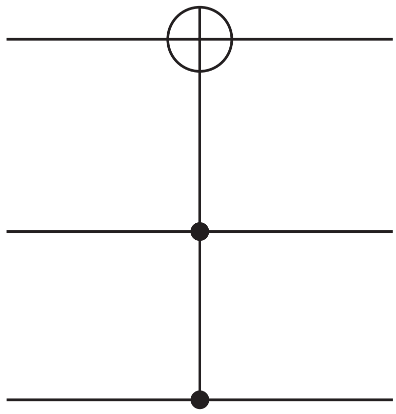


Figure 1.2: Controlled-Controlled-Not Gate (*Toffoli Gate*)

The CCNOT gate negates the last bit of three if and only if the first two are both 1.

In the same way, single bit operations are graphically represented by appropriately labeled boxes.

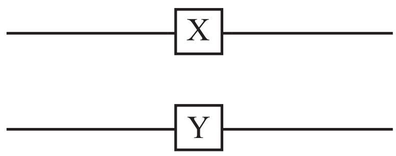


Figure 1.3: Labeled Boxes

Chapter 2

GROVER ALGORITHM

2.1 The Quantum Search Algorithm

In models of classical computation, searching through every item is optimal. In other words, searching an unsorted database cannot be done in less than linear time. Grover's algorithm clarifies that searching can be done faster than this in the quantum model. Even more, it is asymptotically the fastest possible algorithm for searching an unsorted database in the quantum model with the time complexity $O\left(N^{1/2}\right)$[1]. Unfortunately, it provides *only* a quadratic speedup, while other quantum algorithms provide exponential speedup over their classical counterparts. But still, even quadratic speedup is very substantial when N is large. The quantum search algorithm reaches to the solution with a generic search applied to a very wide range of problems. Like many other quantum algorithms, it is also probabilistic which gives the correct answer with the high probability. But yet, we can decrease the probability of failure by repeating the algorithm. In order to make it clear, let's consider an example. Suppose we know that exactly one n-bit integer satisfies a certain condition, and have a black-boxed subroutine that acts on the $N = 2^n$ different n-bit integers. It outputs 1 if the integer satisfies the condition, otherwise 0. Without any other information, we should find the special integer. With a classical computer, we can do no better than to apply the subroutine repeatedly to different random numbers till it hits on the special one. Certainly, the probability of finding the special number increases in every time we apply the subroutine. If we apply it to *M* different integers, the probability of finding the special number becomes M/N. Then, for instance, we should test $\frac{1}{2}N$ different integers to have a 50% chance of success. On the other hand, if we have a quantum computer with a subroutine that performs such a test for a very large *N*, we can find the special integer with a probability very close to 1. Even more, using a method to find the special integer calls the subroutine no greater than $(\pi/4)\sqrt{N}$ number of times.

We can consider this subroutine in various ways. It might perform a mathematical calculation to determine whether the input integer is the special one. Let's consider it on an example. Suppose an odd number *p* can be expressed as the sum of two squares, $m^2 + n^2$. Since one of *m* or *n* must be even and the other odd, *p* must be of the form $4k + 1$. This is actually an elementary theorem of the number theory: *If p is a prime number of the form* $4k + 1$*, then it can always be expressed as the sum of two squares, and in exactly one way*. For instance, $5 = 4+1$, $13 = 9+4$, $17 = 16+1$, $29 = 25+4$, $37 = 36 + 1$, $41 = 25 + 16$, $53 = 49 + 4$, $61 = 36 + 25$, etc. Given such a prime *p*, the simple-minded way to and the two squares is to take randomly selected integers x where $1 \leq x \leq N$, with $N = \left\lfloor \sqrt{p/2} \right\rfloor$. And, repeat this till we and such an x for which $\sqrt{p - x^2} = a$, where *a* is an integer. Now, imagine *p* is of the order of a trillion, then following the same procedure we would have to calculate $\sqrt{p - x^2} = a$ for nearly a million x to get a better than even chance of succeeding. Fortunately, using the quantum search algorithm with a suitably programmed quantum computer, we could find the two squares with a probability of success extremely close to 1 by calling the quantum subroutine, which evaluated $\sqrt{p - x^2}$, only fewer than a thousand times.

This very general capability of quantum computers was discovered by *Lov Grover*, and goes under the name of *Grover's search algorithm*. Shor's period-finding algorithm and Grover's search algorithm, together with their various modifications and extensions, constitute the two masterpieces of the quantum - computational software.

2.2 The Grover Iteration

Grover's algorithm actually makes operations to answer a "Yes or No" question, if the input is a solution for the search problem. In other words, mathematically, it is just a boolean function. By defining the variables *N*, the size of the search elements, *x*, the *n*-bit integer input, and *a*, the solution of the search problem, we can write down this function as:

$$f(x) \quad = \quad \begin{cases} 0 & \text{if } x \neq a \\ \\ 1 & \text{if } x = a \end{cases} \tag{2.1}$$

Despite the simplicity, of course the magic is here behind the computing of this problem. Grover, discovered a way for the case $N = 2^n$ in the sense of quantum computing, which is strictly better than the classical one. Declared a unitary operator

$$\mathbf{U}_f \left(|x\rangle_n \, |y\rangle_1 \right) = |x\rangle_n \, |y \oplus f\,(x)\rangle_1 \tag{2.2}$$

where $|x\rangle$ is the *index* register of the input x, $\oplus$ denotes the addition modulo 2, and $|y\rangle$ (oracle qubit) is a single qubit which is *flipped* if $f(x) = 1$. This operator also called as the *Oracle Operator* in some books. The aim here is to check if the single qubit $|y\rangle$ is flipped or not. In this way, it is useful to take the $|y\rangle$ in the initial state $\mathbf{H}\,|1\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. If $|y\rangle$ is flipped, then *x* is the solution of the search problem, i.e. $x = a$, and the state becomes $-\,|x\rangle\,(|0\rangle - |1\rangle)/\sqrt{2}$. Otherwise, $|y\rangle$ stays unchanged. We can write this operation as:

$$\mathbf{U}_f\left(|x\rangle_n |y\rangle_1\right) = \mathbf{U}_f\left(|x\rangle_n \mathbf{H}|1\rangle\right) = (-1)^{f(x)}|x\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) \tag{2.3}$$

In the classical way, we could do no better than to try each of the $N = 2^n$ possible inputs to check if the output register is flipped. Though by using Grover's algorithm, when the *N* is large, we can get the same answer only by performing the search subroutine no more than $\sqrt{N} = 2^{n/2}$ times (or more precisely $(\pi/4)\sqrt{N}$).

2.3 The Procedure

Now, we can define an algorithm to show that how the quantum search procedure works. Let's first define the procedure, then we can discuss it step by step.

1. Apply the Hadamard transform $H^{\otimes n}$.
2. Apply the unitary operator V.
3. Apply the Hadamard transform $H^{\otimes n}$.
4. Apply the unitary transformation W.
5. Apply the Hadamard transform $H^{\otimes n}$.

2.3.1 The Hadamard Transform $H^{\otimes n}$

The algorithm starts with the computer in the state $|0\rangle^{\otimes n}$. Next, we will use the Hadamard transform[4],

$$\mathbf{H}|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right), \tag{2.4}$$

on the *n*-Qubit input register to put the computer in the equal superposition state

$$|\phi\rangle = \mathbf{H}^{\otimes n}|0\rangle_n = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle_n = \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle_n. \tag{2.5}$$

2.3.2 The Unitary Operator V

The action of $\mathbf{U}_f$ is then to multiply the $(n+1)$-Qubit state by -1 if and only if $x = a$:

$$\mathbf{U}_f(|x\rangle \otimes \mathbf{H}\,|1\rangle) = (-1)^{f(x)}\,|x\rangle \otimes \mathbf{H}\,|1\rangle\,. \tag{2.6}$$

Obviously, $\mathbf{U}_f$ does not have any effect on the single qubit which is $\mathbf{H}\,|1\rangle$ in this case. Then, we can define an n-Qubit unitary transformation $\mathbf{V}$ on $\mathbf{U}_f$, which is also linear as $\mathbf{U}_f$ is.

$$\mathbf{V}\,|x\rangle = (-1)^{f(x)}\,|x\rangle = \begin{cases} |x\rangle\;, & x \neq a \\ \\ -\,|a\rangle\,, & x = a \end{cases} \tag{2.7}$$

Necessity use of the unitary operator $\mathbf{V}$ instead of $\mathbf{U}_f$ of course deserves an explanation here. $\mathbf{V}$ changes the sign of the component of the general superposition $|\Psi\rangle = \sum_x |x\rangle\,\langle x\,|\Psi\rangle$ of the computational basis state with $|a\rangle$, and leaves the orthogonal component to $|a\rangle$ unchanged:

$$\mathbf{V}\,|\Psi\rangle = |\Psi\rangle - 2\,|a\rangle\,\langle a\,|\Psi\rangle \tag{2.8}$$

Which can be simply implemented as

$$\mathbf{V} = 1 - 2\,|a\rangle\,\langle a|\,. \tag{2.9}$$

Here, $|a\rangle\,\langle a|$ denotes the *projection operator* (which has been described in the

Appendix A) on the state $|a\rangle$. As we could notice clearly, the unitary transformation $\mathbf{U}_f$ acts as nothing but the identity operator on the output register. The output register starts in the $\mathbf{H}\,|1\rangle$ state unentangled with the input register. In addition, $\mathbf{U}_f$ determines the output register particularly which makes the output register remain unentangled with the input register. Then, for instance, we can write the equation (2.9) again in the terms of $\mathbf{U}_f$ as:

$$\mathbf{U}_f(|\Psi\rangle \otimes \mathbf{H}\,|1\rangle) = (|\Psi\rangle - 2\,|a\rangle\,\langle a\,|\Psi\rangle\,) \otimes \mathbf{H}\,|1\rangle \tag{2.10}$$

So, obviously, we can use **V** instead of using $\mathbf{U}_f$ to implement things in a more simple way.

2.3.3 The Unitary Transformation W

We can define a second *n*-Qubit unitary **W** which is going to act on the input register with a similar manner with **V**. But this time, it'll be in a fixed form which does not depend on *a*. We will use it to simply denote the component of any state along the standard state $|\phi\rangle$, which we have defined before in (2.5), changing the sign of its component orthogonal to $|\phi\rangle$:

$$\mathbf{W} = 2\,|\phi\rangle\,\langle\phi| - 1 \tag{2.11}$$

Again, $|\phi\rangle\,\langle\phi|$ denotes the *projection operator* on the state$|\phi\rangle$. Unfortunately, the way to build **W** out of 1 and 2 Qubit unitary gates is not that obvious, and will be mentioned in another section.

2.3.4 Application of V and W

As we have constructed the unitary operators **V** and **W** successfully, now we can use them on the initial state $|\phi\rangle$. To see what we get by repeatedly applying **WV** to the $|\phi\rangle$, remember that both **V** and **W** acting on either $|\phi\rangle$ or $|a\rangle$ results in linear combinations of these two states. Since $\langle a\,|\phi\rangle = \langle\phi\,|a\rangle\; = 1/2^{n/2}$ is true for any value of a, the linear combinations have real coefficients such that

$$\begin{aligned} \mathbf{V}\,|a\rangle &= -\,|a\rangle\ \ , \quad \mathbf{V}\,|\phi\rangle = |\phi\rangle - \tfrac{2}{2^{n/2}}\,|a\rangle \\ \mathbf{W}\,|\phi\rangle &= -\,|\phi\rangle\,, \quad \mathbf{W}\,|a\rangle = \tfrac{2}{2^{n/2}}\,|\phi\rangle - |a\rangle \end{aligned} \tag{2.12}$$

Let's start with the state $|\phi\rangle$ and suppose any sequence of **V** and **W** act successively, then the resulting states will always remain in the two - dimensional plane spanned by real linear combinations of $|\phi\rangle$ and $|a\rangle$. Consequently, finding the result of repeated applications of **WV** to the initial state $|\phi\rangle$ lessens to an exercise in plane geometry.

We continue with the form (2.3) of $|\phi\rangle$. If we consider$|\phi\rangle$ and $|a\rangle$ as vectors in the plane of their real linear combinations, they are very nearly perpendicular because of the cosine of the angle γ between them is given by

$$\cos\gamma = \langle a\,|\phi\rangle\; = 2^{-n/2} = 1/\sqrt{N} \tag{2.13}$$

which gets smaller when *N* gets larger. Appropriately, we can define $|a_{\perp}\rangle$ to be the normalized real linear combination of $|\phi\rangle$ and $|a\rangle$ that is strictly orthogonal to $|a\rangle$ and makes the small angle $\theta = \pi/2 - \gamma$ with $|\phi\rangle$, as illustrated in 2.1 and 2.2. Since

$$\sin\theta = \cos\gamma = 2^{-n/2} = 1/\sqrt{N} \tag{2.14}$$

θ is very accurately give by

$$\theta \approx 2^{-n/2} \tag{2.15}$$

when $\sqrt{N}$ is large.

The geometrical action of **W** on any vector in two-dimensional plane containing $|\phi\rangle$, $|a\rangle$, and $|a_\perp\rangle$ is simply to substitute the vector with its reflection at the mirror line through the origin along $|\phi\rangle$. That's because **W** leaves $|\phi\rangle$ unchanged and reverses the direction of any vector orthogonal to $|\phi\rangle$. On the other side, **V** reverses the direction of $|a\rangle$ while leaving any vector orthogonal to $|a\rangle$ unchanged. Thus, it acts on a general vector in the two-dimensional plane by substituting it with its reflection in the mirror line through the origin along $|a_\perp\rangle$. In the same manner, the product **WV**, which is a product of two two-dimensional reflections, is a two dimensional rotation. The way of achieving a two-dimensional reflection is adding a third dimension perpendicular to the plane and performing a 180^o rotation with the mirror line axis. This reverses the irrelevant direction orthogonal to the plane. The product of two such three-dimensional rotations is also a rotation, takes the plane into itself, and does not reverse the third orthogonal direction, so it is a two-dimensional rotation in the plane. The angle of the rotation **WV** is easily seen by regarding the effect of **WV** on $|a_\perp\rangle$ 2.2. While the application of **V** leaves $|a_\perp\rangle$ unchanged, the subsequent action of **W** on $|a_\perp\rangle$ reflects it in the line through the origin along the direction of $|\phi\rangle$. Thus, the resulting effect of the rotation **WV** on $|a_\perp\rangle$ is to rotate $|a_\perp\rangle$ past $|\phi\rangle$ through a total angle which is twice

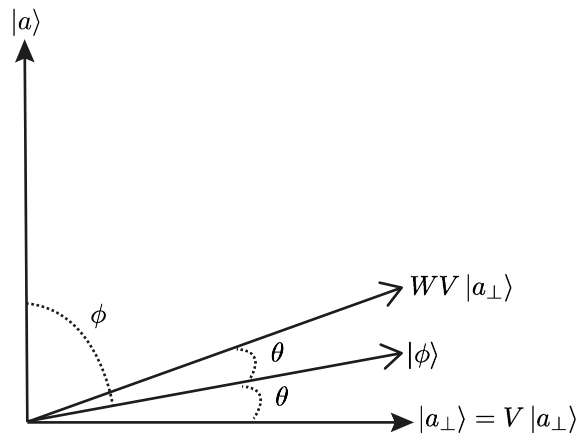


Figure 2.1: Rotation of $|\phi\rangle$

the angle θ between $|a_\perp\rangle$ and $|\phi\rangle$ [13]. Obviously **WV** is a rotation with no difference from other rotations. The result of applying it to any other vector in the plane is also to rotate that vector through the angle 2θ with the direction from $|a_\perp\rangle$ to $|\phi\rangle$. Thus, we can apply it to the initial state $|\phi\rangle$ which gives a vector rotated away from $|a_\perp\rangle$ by 3θ, since $|\phi\rangle$ is already rotated away from $|a_\perp\rangle$ by θ (2.2). If we apply **WV** twice, it results in a vector rotated away from $|a_\perp\rangle$ by 5θ. So that the subsequent application of **WV** increases the angle between the final state and $|a_\perp\rangle$by another 2θ. Because of the angle θ is very close to $2^{-n/2}$, after a certain number of applications as near as possible to $(\pi/4)2^{n/2}$, the resulting state will be very nearly orthogonal to $|a_\perp\rangle$ in the plane spanned by $|\phi\rangle$ and $|a_\perp\rangle$. In other words, it will be nearly equal to $|a\rangle$ itself.

As the matters stand, a measurement of the input register in the computational basis will give a with a very close probability to 1. To see that whether we have been successful or not, we can query the oracle. We can confirm if we have found the desired a by checking the value of $f\,(a)$ is 1 or not. And, we will see it is true with very high probability. If we were really unlucky, we might have to repeat the whole procedure a few more times to get the right answer.

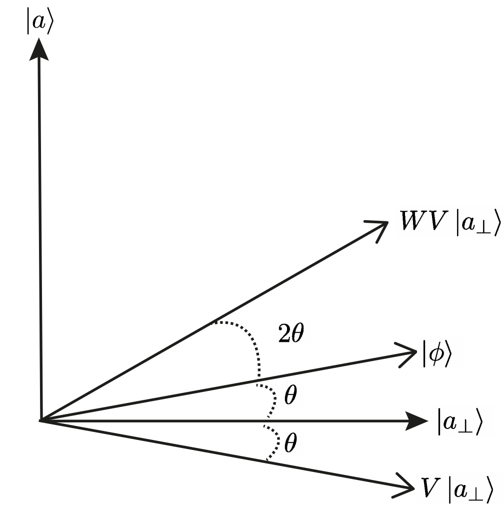


Figure 2.2: Rotation by 3θ

Chapter 3

COMPLEXITY

While talking about one of the most basic and quite useful algorithm of quantum computers, of course we should discuss about the complexity of this algorithm. At the rest, I have used a lot of terms belong to the complexity theory. For any unknown notation or name, you can check the Appendix A.

Let's begin with a NP-complete problem SAT of finding; given n-variable Boolean formula φ, whether there is an assignment $a \in \{0,1\}^n$ such that $\varphi(a) = 1$. Customary, we do not know how to solve this problem better than the trivial poly$(n)\,2^n$-time algorithm by using "classical" deterministic or probabilistic TM's. But, Grover's algorithm solves SAT in poly$(n)\,2^{n/2}$-time on a quantum computer. Even if it is away of showing that $NP \subseteq BQP$, this is still a significant improvement over the classical case. On the other hand, Grover's algorithm actually solves an even more general problem, namely, satisfiability of a circuit with n inputs[12].

Theorem 1. *(Grover's Algorithm): There is a quantum algorithm that given as input every polynomial-time computable function* $f : \{0,1\}^n \to \{0,1\}$ *(i.e., represented as a circuit computing f) finds in* $poly(n)2^{n/2}$ *time a string "a" such that* $f(a) = 1$ *(if such a string exists).*

It is best to describe Grover's algorithm geometrically. Assume that the function f has a *single* satisfying assignment a. Consider an n-qubit register, and let u be the *uniform state vector* of this register, which means

$$u = \frac{1}{2^{n/2}} \sum_{x} |x\rangle \;\; where \; x \in \{0,1\}^n .$$

In this way, the angle θ between u and the basis state $|a\rangle$ is equal to the inverse cosine of their inner product

$$\langle u, |a\rangle\rangle = \frac{1}{2^{n/2}}.$$

Because of this inner product is a positive number,

$$cos^{-1}\left(\langle u, |a\rangle\rangle\right) = cos^{-1}\left(\frac{1}{2^{n/2}}\right) = \theta < \frac{\pi}{2} = 90^{\circ} \;\; since \; \frac{1}{2^{n/2}} > 0.$$

Hence, we denote it by

$$\frac{\pi}{2} - \theta \;\; where \; sin\theta = \frac{1}{2^{n/2}}.$$

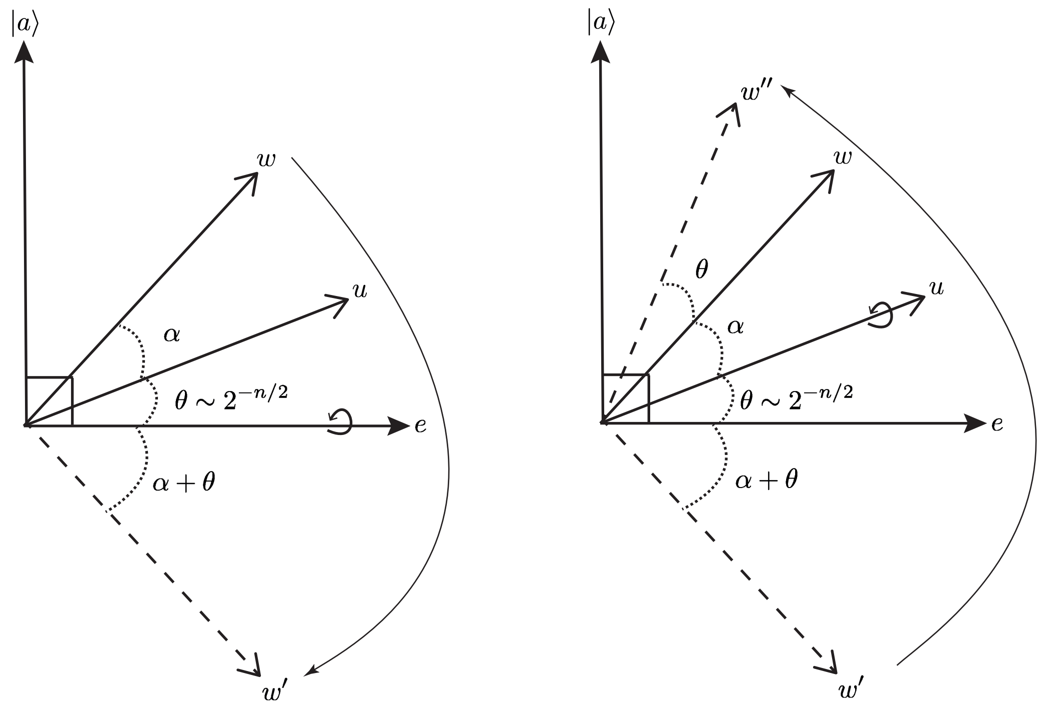
Thus, by using the inequality $\theta \geq sin\theta$ for every $\theta > 0$, we get

$$\theta \geq 2^{-n/2}.$$

At the beginning, the algorithm is in the state u, and at each step it gets closer to the state $|a\rangle$. If its current state makes an angle $\frac{\pi}{2} - \alpha$ with $|a\rangle$, then at the end of the step it makes an angle $\frac{\pi}{2} - \alpha - 2\theta$. Therefore, it will get to a state v whose inner product with $|a\rangle$ is larger than $\frac{1}{2}$ in

$$O\left(\frac{1}{\theta}\right) = O\left(2^{n/2}\right)$$

steps. This implies that a measurement of the register will yield a with probability at least $\frac{1}{4}$.



Step 1: Reflecting around *e* Step 2: Reflecting around *u*

Figure 3.1: Reflection of w

Here the aim is to rotate a vector w toward the unknown vector $|a\rangle$ by an angle of θ. To do this, it is enough to take two reflections around the vector u, and the vector$\theta \sim 2^{-n/2}$

$$e = \sum_{x \neq a} |x\rangle \, .$$

The vector e is the vector orthogonal to $|a\rangle$ on the plane spanned by u and $|a\rangle$.

We need to show how we can take the reflections around the vectors u and e to complete the description of algorithm. This implies that we need to show how we can transform a state w of the register into the state in polynomial time. This state is the reflection of w' around e where w' is the reflection of w around u. Actually, we are not going to work with an n-qubit register, instead we will work with an m-qubit register where m is a polynomial in n. However, the extra qubits will only serve as "scratch workspace" and will always contain zero except during intermediate computations. Thus, it can be accepted to ignore these.

3.1 Reflecting Around *e*

To reflect a vector w around a vector v, we indicate w as $\alpha v + v^{\perp}$ and output $\alpha v - v^{\perp}$. Note that $v^{\perp}$ is orthogonal to v. Therefore, the reflection of w around e is equal to

$$\sum_{x \neq a} w_x \left| x \right\rangle - w_a \left| a \right\rangle .$$

Unexpectedly, it is actually easy to represent this transformation.

3.1.1 Steps of Reflecting Around *e*

1. Because of f is computable in polynomial time, we can compute the transformation

$$\left| x\sigma \right\rangle \rightarrow \left| x \left(\sigma \oplus f\left(x \right) \right) \right\rangle$$

in polynomial time (This is actually same with the operator U_f that we have defined before in 2.2). This transformation implies $\left| x_0 \right\rangle \rightarrow \left| x_0 \right\rangle$ for all $x \neq a$ and $\left| a_0 \right\rangle \rightarrow \left| a_1 \right\rangle$.

2. Follows that, we apply the Pauli-Z gate (1.7.4) on the qubit σ, which maps $\left| 0 \right\rangle$ to $\left| 0 \right\rangle$ and $\left| 1 \right\rangle$ to $- \left| 1 \right\rangle$. It infers $\left| x_0 \right\rangle \rightarrow \left| x_0 \right\rangle$ for all $x \neq a$, and $\left| a_1 \right\rangle \rightarrow - \left| a_1 \right\rangle$.

3. After that, we apply the transformation we have used in step 1 again. This refers $\left| x_0 \right\rangle \rightarrow \left| x_0 \right\rangle$ for all $x \neq a$ and $- \left| a_1 \right\rangle \rightarrow - \left| a_0 \right\rangle$.

Consequently, the vector $\left| \alpha_0 \right\rangle$ is mapped to itself for all $x \neq a$, yet $\left| a_0 \right\rangle$ is mapped to $- \left| a_0 \right\rangle$. Ignoring the last qubit, this is exactly a reflection around $\left| a \right\rangle$.

3.2 Reflecting Around *u*

Instead of directly reflecting around u, we follow a different way to make this process easier. Firstly, we apply the Hadamard operation to each qubit which maps u to $\left| 0 \right\rangle$.

Then, we make the reflection this time around $|0\rangle$. To do this, we can use the same way as reflecting around $|a\rangle$, but this time instead of the function f using a function $g : \{0,1\}^n \rightarrow \{0,1\}$ which outputs 1 if and only if its input is all 0s. At last, we apply the Hadamard operation again, mapping $|0\rangle$ back to u.

In the gleam of these operations, we can take a vector in the plane spanned by $|a\rangle$ and u, and rotate it 2θ radians closer to $|a\rangle$. Hence, if we start with the vector u, we will only need to repeat them $O(1/\theta) = O\left(2^{n/2}\right)$ times to obtain a vector that, when measured, yields $|a\rangle$ with constant probability.

3.3 Grover Search Algorithm

Eventually, let's try to gather all of the process into one piece.

Target: Given a polynomial - time computable $f : \{0,1\}^n \rightarrow \{0,1\}$ with a unique $a \in \{0,1\}^n$ such that $f(a) = 1$, find a.

Quantum register: We use an $(n+1+m)$-qubit register, where m is large enough so we can compute the transformation $|x\sigma 0^m\rangle \rightarrow |x(\sigma \oplus f(x))\, 0^m\rangle$.

Operation	**State**
	Initial state: $\left\|0^{n+m+1}\right\rangle$
Apply Hadamard operation to first n qubits.	$u\left\|0^{m+1}\right\rangle$ (where $u=\frac{1}{2^{n/2}}\sum_{x\in\{0,1\}^n}\|x\rangle$)
For $i=1,\ldots,2^{n/2}$ do:	$v^i\left\|0^{m+1}\right\rangle$
	We let $v^1=u$ and maintain the invariant that $\left\langle v^i,\|a\rangle\right\rangle=sin\left(i\theta\right)$, where $\theta\sim 2^{-n/2}$ is such that $\langle u,\|a\rangle\rangle=sin\left(\theta\right)$
Step 1: *Reflect around* $e=\sum_{x\neq a}\|x\rangle$:	
1.1 Compute $\|x\sigma 0^m\rangle\to\|x\left(\sigma\oplus f\left(x\right)\right)0^m\rangle$	$\sum_{x\neq a}v_x^i\|x\rangle\left\|0^{m+1}\right\rangle+v_a^i\|a\rangle\|10^m\rangle$
1.2 If $\alpha=1$ then multiply vector by -1, otherwise do nothing.	$\sum_{x\neq a}v_x^i\|x\rangle\left\|0^{m+1}\right\rangle-v_a^i\|a\rangle\|10^m\rangle$
1.3 Compute $\|x\sigma 0^m\rangle\to\|x\left(\sigma\oplus f\left(x\right)\right)0^m\rangle$	$w^i\left\|0^{m+1}\right\rangle=\sum_{x\neq a}v_x^i\|x\rangle\left\|0^{m+1}\right\rangle-v_a^i\|a\rangle\|00^m\rangle$
	(w^i is v^i reflected around $\sum_{x\neq a}\|x\rangle$)
Step 2: *Reflect around* $u=\frac{1}{2^{n/2}}\sum_{x\in\{0,1\}^n}\|x\rangle$:	
2.1 Apply Hadamard operation to first n qubits.	$\left\langle w^i,u\right\rangle\|0^n\rangle\left\|0^{m+1}\right\rangle+\sum_{x\neq 0^n}\alpha_x\|x\rangle\left\|0^{m+1}\right\rangle$
	$\alpha_x=\sum_z\left(-1\right)^{x\odot z}w_z^i\|z\rangle$
2.2 Reflect around $\|0\rangle$:	
2.2.1 If first n-qubits are all zero, then flip $(n+1)$th.	$\left\langle w^i,u\right\rangle\|0^n\rangle\|10^m\rangle+\sum_{x\neq 0^n}\alpha_x\|x\rangle\left\|0^{m+1}\right\rangle$
2.2.2 If first $(n+1)$th qubit is 1, then multiply by -1.	$-\left\langle w^i,u\right\rangle\|0^n\rangle\|10^m\rangle+\sum_{x\neq 0^n}\alpha_x\|x\rangle\left\|0^{m+1}\right\rangle$
2.2.3 If first n-qubits are all zero, then flip $(n+1)$th.	$-\left\langle w^i,u\right\rangle\|0^n\rangle\left\|0^{m+1}\right\rangle+\sum_{x\neq 0^n}\alpha_x\|x\rangle\left\|0^{m+1}\right\rangle$
2.3 Apply Hadamard operation to first n qubits	$v^{i+1}\left\|0^{m+1}\right\rangle$ (here v^{i+1} is w^i reflected around u)
Measure register and let a' be the obtained value in the first n qubits. If $f\left(a'\right)=1$, then output a'. Otherwise, repeat.	

Chapter 4

QUANTUM COMPUTING LANGUAGE

Past years induced to separate many common concepts of classical computer science from theoretical physics. However, the quantum computer science was not that lucky enough. Even now, quantum computing is widely considered as a special discipline within the very wide field of theoretical physics. This slow adaption of quantum computing by the computer science community was mainly caused by the confusing variety of formalisms, like gates, operators, matrices, Dirac notation, and so on. These, in fact, have no similarity with classical programming languages, or with physical terminology in most of the available literature.

One of the best options to fill this breach is the Quantum Computation Language (QCL). This language is implemented by Bernhard mer in 1998 with his master thesis in the Technical University of Vienna. It is a high level and architecture independent programming language for quantum computers . Syntax of it is established by deduction of classical programming languages like C or Pascal. It will be very useful of course to implement and simulate the quantum algorithms including classical components in one consistent formalism.

Simulating a quantum computer on a classical computer is a big problem. The amount of quantum memory under simulation increases exponentially with the resources required. This, distressfully, points that simulating a quantum computer with even a few dozen qubits is beyond the capability of any computer made today. Even this language simulates very small quantum computers, it is still powerful enough to demonstrate the concept behind some useful quantum algorithms.

It is most inevitable that the first quantum computers will probably consist of some exotic hardware at the core which stores and manipulates the quantum state machine just like happened by the supercomputers of last years. Inclosing this will be the hardware that supports it and presents the user with a reasonable programming environment. This language makes it available to simulate such an environment by providing a classical program structure with quantum data types and special functions to perform operations on them.

4.1 Description of QCL in Classic Words

To describe this very powerful language, let's start with demonstrating some familiar operations of classical computing languages with QCL.

4.1.1 Starting the QCL and Qubit Dump

To make the qubits implemented as Dirac notation, we use the command "-fb" which is actually "formant in binary" in simple words.

```
# qcl -fb
QCL Quantum Computation Language (32 qubits, seed 1294595263)
[0/32] 1 |0>
qcl> qureg q[1];
qcl> dump q;
1 |0>
: SPECTRUM q: <0>
```

Figure 4.1: Dumping a Qubit

Here, we assigned a one qubit variable, which is also Boolean variable, from the QCL quantum stack. The quantum state of the machine is initialized to all zeros. The notification $|\rangle$ signifies this is a quantum state, which is actually the Dirac notation for quantum mechanics. The main advantage of it comparing to traditional mathematical notation for linear algebra is that it's much easier to type.

The command "*qureg q[1]*" assigns a one qubit variable from the quantum stack. "*dump*" is a command that gives us some information about q. The line with *SPECTRUM* shows us where the qubits for q have been assigned in the quantum stack. Here, the 0-bit of q is the right most qubit in the stack. Finally the last line tells us that if we measure q, we get "0" with probability "1".

4.1.2 Implementation of A Boolean Function

Many of the quantum operators described in QCL are familiar from classical computing. For instance, the *Not()* function flips the value of a bit.

```
qcl> Not(q);
[1/32] 1 |1>
```

Figure 4.2: A Boolean Function in QCL

If we apply the operator "*Not()*" to the same qubit again, it will undo the effect of the first. This is actually same with classical computing.

4.1.3 Execution of Controlled-Not Gate

The operator *CNot(x,y)* tests the value of y and if it is "1", it flips the value of x. This is same with the statement *x^=y* in C.

```
qcl> qureg t[2];
qcl> Not(t[1]);
[2/32] 1 |10>
qcl> CNot(t[0],t[1]);
[2/32] 1 |11>
qcl> dump t[0];
: SPECTRUM t[0]: <0>
1 |1>
qcl> dump t[1];
: SPECTRUM t[1]: <1>
1 |1>
```

Figure 4.3: Controlled-Not Gate in QCL

The operator *CNot()* is its own inverse just like the operator *Not()*. If we apply it again to the same qubits, it will reverse the effect of the first, making the state same with the starting state.

In fact, the idea of reversibility is very important to understand the quantum computing. Theoretical physics reveals that every operation on quantum bits must be reversible, except for measurement. That's why, it is important to always keep enough information to work any operation backwards. Hence, the classical operations like assignment $(x = y)$, AND $(x\& = y)$, and OR $(x| = y)$ have to be modified to use in quantum computing.

4.1.4 Way of Reversing a Boolean Operator

Luckily, there is a straight forward formula to convert irreversible classical operations into quantum operations. First of all, never overwrite a quantum bit except to initialize it to "0". So, if we need to make an assignment $(x = y)$, instead of this, we initialize the target $(x = 0)$, and use exclusive or $(x\hat{} = y)$ as in the above example.

```
# qcl -fb
[0/32] 1 |0>
qcl> qureg a[3];
qcl> Not(a[1]);
[3/32] 1 |010>
qcl> Not(a[2]);
[3/32] 1 |110>
qcl> dump a;
: SPECTRUM a: <0,1,2>
1 |110>
qcl> CNot(a[0], a[1] & a[2]);
[3/32]
1 |111>
qcl> dump a;
: SPECTRUM a: <0,1,2>
1 |111>
```

Figure 4.4: Reversing in QCL

Here, the command *CNot*$(x,\ y\,\&\,z)$ will flip the value of x, if y and z are both "1". Thus, if x is initialized to "0" at the start, this operation will be the same thing of calculating y & z, and storing the value in x. Even it seems so innocent and simple, this operation is very critical in quantum computing.

4.1.5 Getting Superposition of a State

There are some operations in QCL that have no classical analogues. One of the most stunning and useful is the Hadamard operator which has been assigned as *Mix()* in QCL. It takes a computational basis state like $|0\rangle$ or $|1\rangle$, and turns the state into a quantum superposition state.

```
qcl> qureg x[1];
qcl> dump x;
: SPECTRUM x: <0>
1 |0>
qcl> Mix(x);
[1/32] 0.70711 |0> + 0.70711 |1>
qcl> dump x;
: SPECTRUM x: <0>
0.5 |0>, 0.5 |1>
```

Figure 4.5: Superposition Operator Mix()

Here, we took advantage of the quantum mechanics principle of superposition. The command *dump x*, says that if we measure *x*, we will get "0" or "1" with an equal property of 0.5 where $0.70711 = \sqrt{0.5}$. Later on, Bernhard mer added a new function named *H()*, which makes exactly the same thing with the function *Mix()*, referring to the Hadamard operator.

4.1.6 Setting a Vector State

Quantum mechanics tells us that small particles, such as electrons, can be in two places at once. Similarly, a qubit can have two different values at the same time.

Despite a classical computer where the state of the machine is merely a single string of ones and zeros, the state of a quantum computer is a vector with components for every possible string of 1s and 0s. On the other hand, the strings of ones and zeros form the basis for a vector space where our machine state lives. So, we can write down the state of a quantum computer by writing out a sum like:

```
a|X>+ b|Y\> + ...
```

Figure 4.6: Quantum Computer with a Vector State

X and Y are strings of 1s and 0s, and a and b are the amplitudes for the respective components X and Y. The notation $|X\rangle$ is just the way physicists denote a vector or a state called X.

When the Hadamard operator *Mix()* applied to a bit in the state $|0\rangle$, it will transform the state into $sqrt\,(0.5)\,(|0\rangle + |1\rangle)$ as we had in the above example (4.5). But, if we apply *Mix()* to a bit which is in the state $|1\rangle$, we get $sqrt\,(0.5)\,(|0\rangle - |1\rangle)$. Thus, if we apply *Mix()* twice to any qubit in any state, we get back to where we started. In other words, the operator *Mix()* is it's own inverse.

4.1.7 Setting Superposition of a Qubit Unrelated to Other

If we have two qubits *q* and *p* which are initialized to 0, and we perform a sequence of quantum operations on *q* and then, do the same thing to *p*, we will come across with *q* and *p* having the same value.

```
qcl> qureg q[1];
qcl> Not(q);
[1/32] 1 |1>
qcl> Mix(q);
[1/32] 0.70711 |0> - 0.70711 |1>
qcl> qureg p[1];
qcl> Not(p);
[2/32] 0.70711 |0,1> - 0.70711 |1,1>
qcl> Mix(p);
[2/32] 0.5 |0,0> - 0.5 |1,0> - 0.5 |0,1> + 0.5 |1,1>
qcl> dump q;
: SPECTRUM q: <0>
0.5 |0>, 0.5 |1>
qcl> dump p;
: SPECTRUM p: <1>
0.5 |0>, 0.5 |1>
```

Figure 4.7: Unrelated Qubits in Superposition States

4.1.8 Measuring Unrelated Qubits

In the figure 4.7, q and p are completely independent. So that if we measure one, it should have no effect on the other one.

```
qcl> measure q;
[2/8] -0.70711 |0> + 0.70711 |1>
qcl> dump p
: SPECTRUM b: |0>
0.5 |0>, 0.5 |1>
```

Figure 4.8: Measuring Unrelated Qubits

As we see, the spectrum of *p* was unchanged by measuring *q*. On the other hand, if the operations were more complicated than simple operators like *Not()* or *Mix()*, we might be allured to perform them only once on *a* and then, copy the value from *q* to *p*. In fact, we cannot copy it because it's not a reversible operation, however we can initialize *p* to 0 and *CNot(p,q)* which performs the same goal.

4.1.9 Possible to Copy a Qubit?

Unfortunately, we will not get what we have expected this time:

```
qcl> qureg q[1];

qcl> Not(q);

[1/32] 1 |1>

qcl> Mix(q);

[1/32] 0.70711 |0> - 0.70711 |1>

qcl> qureg p[1];

qcl> CNot(p,q);

[2/32] 0.70711 |0,0> - 0.70711 |1,1>

qcl> dump q;

: SPECTRUM q: <0>

0.5 |0>, 0.5 |1>

qcl> dump p;

: SPECTRUM p: <1>

0.5 |0>, 0.5 |1>
```

Figure 4.9: Preparing State for Copy

The spectrum of p and q seem alright, and of courseif we measure p or q alone, we will get the same result as above. But the difference lies in what happens when we measure both p and q.

4.1.10 Superposition of Qubits is Crushed

Note that the outcome of a measurement is random. Thus, if we repeat this operation, the result may change.

```
qcl> measure q;
[2/32] -1 |1,1>
qcl> dump p;
: SPECTRUM p: <1>
1 |1>
```

Figure 4.10: Measurement Disappointments

As we see, measuring *p* crashes the superposition of *q*. This is because *p* and *q* were entangled in what physicists call an EPR pair. It is named after Einstein, Podolsky, and Rosen where all used this in an attempt to show that quantum mechanics was an incomplete theory in 1935. Later, John Bell demonstrated that entanglement in real particles by experimental projection of the Bell's Theorem, which formalized the EPR thought experiment.

As a result, if we try to copy one quantum variable onto another one, we will end up with an entanglement rather than a real copy.

4.1.11 Implementation of Angle Rotation

The main target of quantum computing was to simulate the behavior of arbitrary quantum systems using a small set of classical components. To allow for universal quantum computation, we need the Controlled Phase function, *CPhase()*.

CPhase() takes a classical rational number as its first argument, and a qubit as its second argument. *CPhase(c,q)* changes the component amplitudes of the machine's base state as the amplitudes for base states where *q* is $|0\rangle$ are unchanged, while the amplitudes for base states where *x* is $|1\rangle$ are multiplied by $exp\,(i * c) = cos\,(c) + i * sin\,(c)$.

```
# qcl -fb
QCL Quantum Computation Language (32 qubits, seed 1294652447)
[0/32] 1 |0>
qcl> qureg q[1];
qcl> Mix(q);
[1/32] 0.70711 |0> + 0.70711 |1>
qcl> CPhase(3.14159265,q);
[1/32] 0.70711 |0> - 0.70711 |1>
qcl> reset
[1/32] 1 |0>
qcl> Mix(q);
[1/32] 0.70711 |0> + 0.70711 |1>
qcl> CPhase(0.01,q);
[1/32] 0.70711 |0> + (0.70707+0.0070709i) |1>
qcl> dump q;
: SPECTRUM q: <0>
0.5 |0>, 0.5 |1>
```

Figure 4.11: Controlled Phase Rotation

The coefficients for the machine states where $q = 1$ are rotated in the complex plain by c-radians. Since $exp\,(i * pi) = -1$, $CPhase\,(pi, x)$ will flip the sign of the $|1\rangle$ component. $CPhase\,(0.01, x)$ rotates the phase of the $|1\rangle$ component by one 100 of a radian in the complex plane. The parenthesized tuple $(0.70707, 0.0070709i)$ is the QCL representation of the complex number $exp\,(0.01 * i) = 0.70707 + i * 0.0070709i$.

4.1.12 Rotation About Y-Axis

Likewise, there is an operator $RotY\,(c, q)$, where *c* is a real number and *q* is a qubit, which rotates qubit *q* by *c*.

```
qcl> qureg q[1];
qcl> RotY(pi,q);
[1/32] 1 |1>
qcl> dump q;
: SPECTRUM q: <0>
1 |1>
```

Figure 4.12: Rotation About Y-Axis

This example shows that when we rotate $q = |0\rangle$ by pi which is equivalent to 180 in degrees, we get $q = |1\rangle$.

```
qcl> qureg q[1];

qcl> RotY(pi/2,q);

[1/32] 0.70711 |0> + 0.70711 |1>

qcl> dump q;

: SPECTRUM q: <0>

0.5 |0>, 0.5 |1>

qcl> reset

[1/32] 1 |0>

qcl> Mix(q);

[1/32] 0.70711 |0> + 0.70711 |1>

qcl> dump q;

: SPECTRUM q: <0>

0.5 |0>, 0.5 |1>
```

Figure 4.13: Making Hadamard Using Rotation

Also, we can see that in this example it makes the same effect on the qubit with the Hadamard operator *Mix()*, if we take the real number argument as $pi/2$ which is equivalent to 90 in degrees.

4.2 Implementation of a Toffoli Gate in QCL

We are going to test a function which is written for the following circuit.

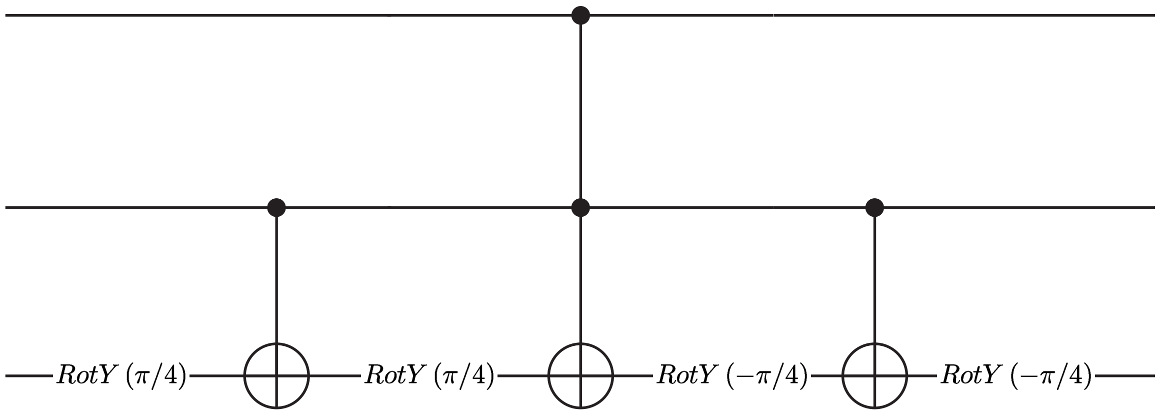


Figure 4.14: Toffoli Gate in Other Words

I have named it as PToffoli operator, because this circuit does the same job with Toffoli gate by using Pauli operators.

First of all, we write down our new operator into a text file and change its extension to ".qcl", then store it into the "lib" folder which is in the qcl directory. So, here is the function which implements the above circuit as an operator in qcl.

```
operator ptoffoli(qureg q)
{
   const n=#q;
   int i;
   int m;
   m=2*n-3;
   RotY(-pi/4,q[n-1]);
   for i=1 to m
   { if n-(1+i)<=0
      { CNot(q[n-1],q[(1+i)-n]);
         RotY(pi/4,q[n-1]); }
      else
      { CNot(q[n-1],q[n-(1+i)]);
         RotY(-pi/4,q[n-1]); }
   }
}
```

Figure 4.15: PToffoli Operator

Now, let's consider if this function works or not on examples with an input q with three qubits by using qcl. To demonstrate that this operator works, we will do the same steps, that the operator does, manually on qcl.

4.2.1 Example with the Input $q[3] = |000\rangle$

```
qcl> qureg q[3];

qcl> RotY(-pi/4,q[2]);

[3/32] 0.92388 |000> - 0.38268 |100>

qcl> CNot(q[2],q[1]);

[3/32] 0.92388 |000> - 0.38268 |100>

qcl> RotY(-pi/4,q[2]);

[3/32] 0.70711 |000> - 0.70711 |100>

qcl> CNot(q[2],q[0]);

[3/32] 0.70711 |000> - 0.70711 |100>

qcl> RotY(pi/4,q[2]);

[3/32] 0.92388 |000> - 0.38268 |100>

qcl> CNot(q[2],q[1]);

[3/32] 0.92388 |000> - 0.38268 |100>

qcl> RotY(pi/4,q[2]);

[3/32] 1 |000>
```

Figure 4.16: Input $q[3] = |000\rangle$

As we see, the CNot operators don't do anything because the control qubits are 0. So, at the end, the qubit becomes same as previous, just as our function says:

```
qcl> include "ptoffoli.qcl";
qcl> qureg q[3];
qcl> ptoffoli(q);
[3/32] 1 |000>
```

Figure 4.17: PToffoli with $q[3] = |000\rangle$

4.2.2 Example with the Input $q[3] = |010\rangle$

```
qcl> Not(q[1]);
[3/32] 1 |010>
qcl> RotY(-pi/4,q[2]);
[3/32] 0.92388 |010> - 0.38268 |110>
qcl> CNot(q[2],q[1]);
[3/32] -0.38268 |010> + 0.92388 |110>
qcl> RotY(-pi/4,q[2]);
[3/32] 1 |110>
qcl> CNot(q[2],q[0]);
[3/32] 1 |110>
qcl> RotY(pi/4,q[2]);
[3/32] -0.38268 |010> + 0.92388 |110>
qcl> CNot(q[2],q[1]);
[3/32] 0.92388 |010> - 0.38268 |110>
qcl> RotY(pi/4,q[2]);
[3/32] 1 |010>
```

Figure 4.18: Input $q[3] = |010\rangle$

This time two of the CNot gates worked. But, unfortunately, they flip between same two qubits twice. So that, at the end, the qubit became same as the beginning again.

As we will see, when we apply our function, we will get the same result.

```
qcl> qureg q[3];

qcl> Not(q[1]);

[3/32] 1 |010>

qcl> include "ptoffoli.qcl";

qcl> ptoffoli(q);

[3/32] 1 |010>
```

Figure 4.19: PToffoli with $q[3] = |010\rangle$

4.2.3 Example with the Input $q[3] = |110\rangle$

```
qcl> qureg q[3];

qcl> Not(q[0]);

[3/32] 1 |001>

qcl> Not(q[1]);

[3/32] 1 |011>

qcl> RotY(-pi/4,q[2]);

[3/32] 0.92388 |011> - 0.38268 |111>

qcl> CNot(q[2],q[1]);

[3/32] -0.38268 |011> + 0.92388 |111>

qcl> RotY(-pi/4,q[2]);

[3/32] 1 |111>

qcl> CNot(q[2],q[0]);

[3/32] 1 |011>

qcl> RotY(pi/4,q[2]);

[3/32] 0.92388 |011> + 0.38268 |111>

qcl> CNot(q[2],q[1]);

[3/32] 0.38268 |011> + 0.92388 |111>

qcl> RotY(pi/4,q[2]);

[3/32] 1 |111>
```

Figure 4.20: Input $q[3] = |110\rangle$

Now, finally our qubit is changed at the end, as we have three CNot gates all fully worked at this time. Similarly, we can see the same thing at the end of our function.

```
qcl> include "ptoffoli.qcl";
qcl> qureg q[3];
qcl> Not(q[0]);
[3/32] 1 |001>
qcl> Not(q[1]);
[3/32] 1 |011>
qcl> ptoffoli(q);
[3/32] 1 |111>
```

Figure 4.21: PToffoli with $q[3] = |110\rangle$

4.3 Implementation of the Grover's Algorithm in QCL

After all of these achievements, we are finally going to implement the Grover's algorithm in QCL. First of all, I would like to represent special thanks for this nice program and documentations [15]. In the wake of these, let's define the Grover's database search in qcl.

4.3.1 Way of Defining the Quantum Function "Query"

We will use the following quantum function to implement the search condition.

$$\text{query: } |x, 0\rangle \to |x, C(x)\rangle \text{ with } x \in B^n \text{ and } C : B^n \to B$$

Here, "*C*" is the search-condition. By using this function, we can formulate the problem within the realms of classical boolean logic. Actually, the thing we try to do here is to reduce the Grover's algorithm to simple words. If $C(x) = 1$, this means that a solution exists and it is unique. So, no extra information about $C(x)$ is required.

The following is the implementation of a test query with n solution.

```
qufunct query (qureg x, quvoid f, int n)
{
   int i;
   for i=0 to #x-1   // x -> NOT (x XOR n)
   {
      if not bit(n,i) { Not(x[i]); }
   }
   CNot(f,x);   // flip f if x=1111..
   for i=0 to #x-1   // x <- NOT (x XOR n)
   {
      if not bit(n,i) { !Not(x[i]); }
   }
}
```

Figure 4.22: Function Query

However, practically we better talk about a search for an encryption key in known-plaintext attack. We can write down the qcl code in the following way, by taking *p* the known plaintext to the ciphertext *c:*

```
qufunct encrypt (int p, quconst key, quvoid c) { ... }

qufunct query (int c, int p, quconst key, quvoid f)
{
   int i;
   quscratch s[blocklength];
   encrypt(p, key, s);
   for i=0 to #s-1   // x <- NOT (x XOR n)
   {
      if not bit(p, i) { Not(x[i]); }
   }
   CNot(f, x);   //flip f if s=1111...
}
```

Figure 4.23: Query for Text

Despite the previous example, this function utilizes a local scratch register which makes it unnecessary to uncompute *s*.

4.3.2 How to Set up the Search Space

In a quantum computer, search space B^n, which is the solution space of a *n* bit query condition C, can be executed as a superposition of all eigenstates of an n qubit register.

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N} |i\rangle \text{ with } N = 2^n$$

Note that, we had talked about the implementation of the Hadamard operator in qcl in 4.1.5 which we use to prepare such a state.

4.3.3 The Principle Loop of the Algorithm

This consists of two steps.

1. Execute a conditional phase shift (4.1.11) to rotate the phase of all eigenvectors which match the condition C by π radians.

$$Q\,:\,|i\rangle \to \begin{cases} -\,|i\rangle & \text{if} \quad C\,(i) \\ |i\rangle & \text{if not} \quad C\,(i) \end{cases}$$

2. Apply the diffusion operator

$$D = \sum_{ij} |i\rangle\, d_{ij}\, \langle j| \text{ with } d_{ij} = \begin{cases} \frac{2}{N} - 1 & \text{if} \quad i = j \\ \frac{2}{N} & \text{if not} \quad i = j \end{cases}$$

Since only one eigenvector $|i_0\rangle$ is going to match the search condition C, the conditional phase shift operator will turn the initial superposition into

$$|\psi'\rangle = -\frac{1}{\sqrt{N}}\,|i_0\rangle + \frac{1}{\sqrt{N}} \sum_{i \neq i_0} |i\rangle\,.$$

Then, the effect of the diffusion operator on an arbitrary eigenvector $|i\rangle$ is

$$D\,|i\rangle = -\,|i\rangle + \frac{2}{N} \sum_{j=0}^{N-1} |j\rangle$$

and an iteration on a state of the form

$$|\psi\,(k,l)\rangle = k\,|i_0\rangle + \sum_{i \neq i_0} l\,|i\rangle$$

refers to

$$|\psi\,(k,l)\rangle \xrightarrow{Q} |\psi\,(-k,l)\rangle \xrightarrow{D} \left|\psi\left(\frac{N-2}{N}k + \frac{2\,(N-1)}{N}l, \frac{N-2}{N}l - \frac{2}{N}k\right)\right\rangle$$

4.3.4 The Number of Iterations Needed

If we apply the above main loop repeatedly to the initial superposition

$$|\psi\rangle = \left|\psi\left(\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}\right)\right\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle$$

the resulting state will be in the form $|\psi(k,l)\rangle$ and the complex amplitudes k and l are described by the following system of recursions:

$$k_{j+1} = \frac{N-2}{N} k_j + \frac{2(N-1)}{N} l_j$$

$$l_{j+1} = \frac{N-2}{N} l_j - \frac{2}{N} k_j$$

By substituting $sin^2\theta = \frac{1}{N}$, we can rewrite this system in the following closed form.

$$k_j = sin\left((2j+1)\theta\right)$$

$$l_j = \frac{1}{\sqrt{N-1}} cos\left((2j+1)\theta\right)$$

The probability p of measuring i_0 is given as $p = k^2$, and it's maximum state is $\theta = \frac{\pi}{2(2j+1)}$. Thus, for large lists $\frac{1}{\sqrt{N}} \ll 1$ we can assume $sin\theta \approx \theta$ and $\pi \gg 2\theta$ where the number of iterations m for at most p is about $m = \left\lfloor \frac{\pi}{4}\sqrt{N} \right\rfloor$ with $p > \frac{N-1}{N}$. Also, if we have $p > \frac{1}{2}$, then $m = \left\lceil \frac{\pi}{8}\sqrt{N} \right\rceil$ iterations will be enough.

4.3.5 The Implementation of the Query Operator

The operator Q can be constructed as in the following way, if we formulate the query as we have mentioned in 4.3.1 before with a flag qubit f to allow for a strictly classical implementation.

$$Q = query^{\dagger}(x,f)\, V(\pi)(f)\, query(x,f)$$

where $V(\phi)$ is the conditional phase gate.

4.3.6 Implementation of the Diffusion Operator

We can write the diffusion operator

$$D = \sum_{ij} |i\rangle \left(\frac{2}{N} - \delta_{ij} \right) \langle j|$$

also as $D = HRH$ by using the Hadamard Transform H (4.3.1) and a conditional phase rotation $R : |i\rangle = -(-1)^{\delta_{i0}} |i\rangle$ where

$$HRH = -\frac{1}{N} \sum_{i,k,j} |i\rangle (-1)^{(i,k)} (-1)^{\delta_{k0}} (-1)^{(k,j)} \langle j| \text{ while}$$

$$\sum_{k=0}^{N-1} (-1)^{(i,k)} (-1)^{\delta_{k0}} (-1)^{(k,j)} = -2 + \sum_{k=0}^{N-1} (-1)^{(i,k)(k,j)} = N\delta_{ij} - 2$$

So that, we can implement the diffusion operator by using the function *Not()* (4.3.1) and a controlled phase gate (4.1.11):

```
operator diffuse (qureg q)
{
   Mix(q); // Hadamard Transform
   Not(q); // Invert q
   CPhase(pi,q); // Rotate if q=1111..
   !Not(q); // undo inversion
   !Mix(q); // undo Hadamard Transform
}
```

Figure 4.24: Diffusion Operator

4.3.7 Implementation of the Grover's Procedure

Now, by using the previous information in this chapter, we can give a qcl implementation of the complete algorithm:

```
procedure grover(int n)
{
   int l=floor(log(n,2))+1; // no. of qubits
   int m=ceil(pi/8*sqrt(2^l)); // no. of iterations
   int x;
   int i;
   qureg q[l];
   qureg f[1];
   print l,"qubits, using",m,"iterations";
   {
      reset;
      H(q); // prepare superposition
      for i= 1 to m   // main loop
      {
         query(q,f,n); // calculate C(q)
         CPhase(pi,f); // negate |n>
         !query(q,f,n); // undo C(q)
         diffuse(q); // diffusion operator
      }
      measure q,x; // measurement
      print "measured",x;
   } until x==n;
   reset; // clean up local registers
}
```

Figure 4.25: Grover Procedure

Here, the procedure argument n is the number to be found. So that the size of the quantum registers as well as the numbers of iterations are set in the following way:

```
qcl> include "grover.qcl"; qcl> grover(500);

: 9 qubits, using 9 iterations

: measured 500


[0/32] 1 |0> qcl> grover(123);

: 7 qubits, using 5 iterations

: measured 123


[0/32] 1 |0> qcl> grover(444);

: 9 qubits, using 9 iterations

: measured 444
```

Figure 4.26: Application of Grover

CONCLUSION

Among this thesis, we have talked about one of the best algorithms of the quantum computing, which was Grover's algorithm. Also, we have taken a look with the eye of complexity to this algorithm, and finalized with the implementation of it in an abstract quantum computer with the help of a quantum programming language, which was QCL. Even now, while you read this thesis, computer engineers and programmers are probably finding new achievements about quantum computer and its programming. But, this still doesn't change the fact that it will take many years to construct a quantum computer. These achievements certainly includes also some papers or theses, where many might have no effect in the improvement of quantum computers. Even though, the main idea should be here to go ahead with trustworthy steps, and to be a part of one of the most amazing invention of next generation.

Appendix

Appendix A: Investigating Complexity

A.1 Nondeterministic

Recall that in computer science nondeterministic algorithm is an algorithm that defines multiple ways of processing the same input, without any specification of which one will be taken.

A.2 Polynomial Time

An algorithm is said to be polynomial time if its running time is upper bounded by a polynomial in the size of the input for the algorithm.

A.3 Turing Machine

It is a theoretical device that manipulates symbols on a strip of tape according to a table of rules. Despite its simplicity, a Turing machine can be adapted to simulate the logic of any computer algorithm, and is particularly useful in explaining the functions of a CPU inside a computer[11].

A.4 Decision Problems

In computational complexity theory, a decision problem is a question in some formal system with a yes-or-no answer, depending on the values of some input parameters.

A.5 Polynomial Time Reduction

A polynomial-time reduction or a polynomial-time Turing reduction is a reduction which is computable by a deterministic Turing machine in polynomial time.

A.6 Boolean Logic

It is the part of logic with the elements (or “bits”) each contain only two possible values, called various names, such as “true” and “false”, “yes” and “no”, “on” and “off”, or “1” and “0”.

A.7 SAT Problem

In full name, it is the Boolean satisfiability problem. Satisfiability is the problem of determining if the variables of a given Boolean formula can be assigned in such a way as to make the formula evaluate to TRUE. Another important thing is to determine whether no such assignments exist, which would imply that the function expressed by the formula is identically FALSE for all possible variable assignments. In this case, we call the function is unsatisfiable; otherwise it is satisfiable.

A.8 Complexity Class

It is a set of problems of related resource-based complexity. For instances, the class NP is the set of decision problems that can be solved by a non-deterministic Turing machine in polynomial time, and the class PSPACE is the set of decision problems that can be solved by a deterministic Turing machine in polynomial space.

A.9 Class NP

In computational complexity theory, NP refers to nondeterministic polynomial time. It is the set of all decision problems for which the instances where the answer is “yes” have efficiently verifiable proofs of the fact that the answer is indeed “yes”. Particularly, these proofs have to be verifiable in polynomial time by a deterministic Turing machine. In more formal words, NP is the set of decision problems where the “yes” instances can be recognized in polynomial time by a non-deterministic Turing machine.

A.10 Class NP-complete

It is a class of decision problems. A problem L is called NP-complete if it has following two properties:

1. It is in the set of NP problems: Any given solution to L can be verified quickly that is in polynomial time.

2. It is also in the set of NP-hard problems: Any NP problem can be converted into L by a transformation of the inputs in polynomial time.

A.11 Class NP-Hard

It is a class of problems that are “at least as hard as the hardest problems in NP”. A problem K is NP-hard if and only if there is an NP-complete problem L that is polynomial time Turing reducible to H. In other words, L can be solved in polynomial time by an oracle machine with an oracle for H. Informally, we can think of an algorithm that can call such an oracle machine as a subroutine for solving H, and solves L in polynomial time, if the subroutine call takes only one step to compute.

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