Studies on Approximations to Renewal Functions and their Applications

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ABSTRACT

Renewal equations and renewal type equations are frequently encountered in several applications when regenerative arguments are used in the modeling. These equations which are Volterra type integral equations contain the renewal function in the kernel which is a key tool in renewal processes. Analytical solutions of the renewal equations are possible only for a very few cases. Although several approximations for the renewal function are available, the use of any method depends on the characteristics of the underlying distribution function such as skewness, kurtosis, modes, and singularities. Further, all the approximations proposed so far presuppose the knowledge of the underlying distribution function. This thesis proposes non-parametric approximations to several renewal functions based only on the first few moments of the distribution function. The renewal functions considered are one-dimensional renewal function, g-renewal function, and two-dimensional renewal function. The approximations are compared with the actual values wherever available and with benchmark approximations. Examples from areas such as reliability, queuing theory, and warranty models are developed to illustrate the efficacy of the approximations.

Keywords: Renewal Function, G-renewal Function, Two-dimensional Renewal Function, Moments Based Approximation, General Repair, Renewing Warranty.

ÖZ

Modellemede tekrar üretilmiş ispatlar kullanıldığı zaman, çeşitli uygulamalarda, yenilenen denklemler ve yenilenen denklem çeşitlerine sıklıkla rastlanmaktadır. Volterra tipi tümlevsel denklemler olan bu denklemler, yenileme surecinde önemli bir araç olan kernel içerisinde yenileme fonksiyonu içermektedirler. Yenilenen denklemlerin sayısal çözümleri sadece bir kaç durumda mümkündür. Yenilenme fonksiyonları için çeşitli yaklaşımlar bulunmasına rağmen herhangi bir metodun kullanımı temel dağilim fonksiyonunun çarpıklık, basıklık, doruk ve tutarsizlık gibi özelliklerine bağlıdır. Ayrıca şimdiye kadar önerilen tüm yaklaşımlar, temel dağılım fonksiyonunun bilinmesini varsaymışlardır. Bu tez çeşitli yenileme fonksiyonuna, dağılım fonksiyonunun sadece ilk bir kaç momentine göre, parametrik olmayan yaklaşımlar öneriyor. Dikkate alınmış yenilenen fonksiyonlar tek boyutlu, g-yenilenen ve iki boyutlu yenilenen fonksiyonlardır. Yaklaşımlar, varsa gerçek değerlerle, yoksa en iyi karşılarştırmalı denektaşıyla kıyaslanmıştir. Yaklaşımların etkinliğini göstermek için örnekler güvenirlilik, kuyruk kuram ve teminat modelleri alanlarında geliştirilmiştir.

Anahtar Kelımeler: Yenileme Fonksiyonu, G-yenileme Fonksiyonu, Iki-boyutlu Yenıleme Fonksıyonu, Momentlere Bağlı Yaklaştırım, Genel Onarim, Temınat Yenilendirmesi

To My Saint Wife:

Hajieh

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INTRODUCTION

Stochastic processes deal with techniques of quantifying the dynamic relationships amongst a sequence of random variables. “Currently in the period of dynamic indeterminism in science, there is hardly a serious piece of research, which, if treated realistically, does not involve operations on stochastic processes” [Neyman (1960)]. With randomness being an integral part of the majority of phenomena around us, stochastic models play a crucial role in modeling problems of natural and engineering sciences. These models can be used to analyze the variability inherent in biological and medical sciences like the variability of neural spike trains or mutation of genes. They are very useful in modeling phenomena in diverse areas from economics and psychology to electronics and computer science. Such models provide the modeler with new perspectives and information. Newer processes continue to grow to suit the needs of the modeler. However, fundamental processes such as Markov processes, renewal processes, and branching processes score over others in terms of versatility of applications.

Renewal theory arose from the study of “self renewing aggregates” and was first introduced as a generalization of Poisson process. Renewal processes play a key role in the grand scheme of stochastic processes because of its theoretical structure as well as application in diverse areas such as man power studies, reliability, replacement and maintenance, inventory control, queuing theory, and simulation to mention a few. First, they are useful in removing the stronger distributional assumptions that are needed to build Markov models such as geometric distribution and exponential distribution of the discrete and continuous time Markov models, respectively. Secondly, renewal processes form the basis for providing a unifying theoretical framework for studying the limiting behavior of specialized stochastic processes. In this regard key renewal theorem plays an important role. Finally, renewal processes lead us to important generalizations such as renewal reward processes, regenerative processes, and Markov regenerative processes. Specific mention must be made of renewal reward processes, which play a very useful role in the computation of vital performance measures such as long run costs and revenues.

One of the key tools in the analysis of renewal processes as well as in applications is the renewal function. It simply gives us the expected number of renewals in an arbitrary time interval. This function is very important in renewal processes because it completely characterizes the process. The renewal function can at best be expressed as the solution of an integral equation known as the renewal equation. Apart from the renewal equation, renewal type equations occur in various different situations from using the renewal argument. Feller (1966) gave a number of examples of phenomena satisfying renewal type equations. Keyfitz (1968) considered use of renewal type equation in demography, Bartholomew (1973) in social processes, Bartholomew (1976), Bartholomew and Forbes (1979) in manpower studies, and Sahin (1990) in inventory models.

The renewal equations or renewal type equations, which are so important in applications, are not easy to use in practice. These integral equations contain the renewal function whose explicit evaluation is possible only in very few select cases. Hence, approximations to determine the renewal function becomes the lone option for workers in this area and the need for simple and efficient approximations have been felt more than ever. In this thesis, our endeavor has been to develop such approximations for a few important renewal functions and apply them to problems in diverse areas such as queuing theory, reliability, and warranty cost analysis. In order to understand the renewal functions under consideration and the theory behind them, we present in this sequel some basic ideas on renewal theory as well as review the existing literature on approximations to the renewal functions. We wish to mention that part of the material of the following subsections can be found in the references cited at the beginning of them.

1.1 One-dimensional Renewal Processes [Tijms (2003) and Medhi (1994)]

Renewal theory began as the study of some particular problems connected with the failure and replacements of components. However, the wealth of applications of the theory has led to a phenomenal hand in hand growth of the theory and applications.

Let $\{X_n, n = 1,2, \dots\}$ be a sequence of non-negative independent random variables. Assume that $Pr\{X_n = 0\} < 1$. Let the random variables be continuous and identically distributed with a distribution function $F(.)$. Since X_n is non-negative, it follows that $E(X_n)$ exists and let us denote

$$E(X_n) = \int_0^\infty x \, dF(x) = \mu \tag{1.1}$$

where μ may be infinite. Whenever $\mu = \infty$, $1/\mu$ shall be interpreted as 0.

Let $S_0 = 0, S_n = X_1 + X_2 + \ldots + X_n, n \geq 1$ and let $F_n(x) = Pr\{S_n \leq x\}$ be the distribution of $S_n, n \geq 1$;

$$F_0(x) = \begin{cases} 1 & if\ x \geq 0 \\ 0 & if\ x < 0 \end{cases}$$

Define the random variable $N(t) = sup\{n: S_n \leq t\}$. The process $\{N(t), t \geq 0\}$ is called a renewal process with distribution F. Among the various statistical characteristics of the random variable $N(t)$, such as mean, variance, and autocorrelation function, the most sought after measure is the mean function. In several optimization procedures wherein the objective function is the long run average measure, the mean function plays an important role. The function $M(t) = E[N(t)]$ is called the renewal function of the process with distribution F. It is clear that

$$\{N(t) \geq n\} \Leftrightarrow \{S_n \leq t\} \quad (1.2)$$

The distribution of $N(t)$ is given by

$$p_n(t) = Pr\{N(t) = n\} = F_n(t) - F_{n+1}(t) \quad (1.3)$$

where $F_n(t)$ is the n fold convolution of $F(t)$ with itself. This can be seen by observing that

$$Pr\{N(t) = n\} = Pr\{N(t) \geq n\} - Pr\{N(t) \geq n + 1\} = Pr\{S_n \leq t\} - Pr\{S_{n+1} \leq t\}$$
$$= F_n(t) - F_{n+1}(t).$$

The expected number of renewals is computed by noting that

$$M(t) = E[N(t)] = \sum_{n=0}^{\infty} n\, p_n(t) = \sum_{n=0}^{\infty} n\{F_n(t) - F_{n+1}(t)\} = \sum_{n=1}^{\infty} F_n(t) = \sum_{n=1}^{\infty} Pr\{S_n \leq t\}. \quad (1.4)$$

Taking Laplace transform on both the sides of (1.4), the above relation can be conveniently cast in the form

$$M^*(s) = \frac{f^*(s)}{s[1-f^*(s)]} \tag{1.5}$$

where $M^*(s)$ and $f^*(s)$ are the Laplace transforms of $M(t)$ and $f(t)$ respectively. From (1.5) we have the relation

$$f^*(s) = \frac{s\,M^*(s)}{1+s\,M^*(s)} \tag{1.6}$$

(1.5) and (1.6) together show that $M(t)$ and $F(t)$ can be determined uniquely one from the other. Note that $M(t) = E[N(t)]$ is a sure function and not a random function or stochastic process. The renewal density $m(t)\Delta t$ is defined as the probability of the occurrence of one or more renewals in a very small time interval $(t, t + \Delta t)$. It can be easily shown that $m(t)$ is the derivative of the renewal function $M(t)$. To see this we observe that

$$m(t) = \lim_{\Delta t \to 0} \frac{Pr\{one\ or\ more\ renewals\ in\ (t, t+\Delta t)\}}{\Delta t}$$

$$= \sum_{n=1}^{\infty} \lim_{\Delta t \to 0} \frac{Pr\{n^{th}\ renewal\ occurs\ in\ (t, t+\Delta t)\}}{\Delta t}$$

$$= \sum_{n=1}^{\infty} \lim_{\Delta t \to 0} \frac{f_n(t)\Delta t + o(\Delta t)}{\Delta t}$$

$$= \sum_{n=1}^{\infty} f_n(t) = \sum_{n=1}^{\infty} F'_n(t) = M'(t) \tag{1.7}$$

The function $m(t)$ also specifies the mean number of renewals to be expected in narrow interval near t. Note that $m(t)$ is not a probability density function. Taking Laplace transform on both sides of (1.7) and using (1.5), it is readily seen that

$$m^*(s) = \frac{f^*(s)}{1-f^*(s)} \tag{1.8}$$

An integral equation can be obtained for the renewal function $M(t)$, which is given below:

$$M(t) = F(t) + \int_0^t M(t-x)dF(x) \tag{1.9}$$

To prove (1.9) we first condition on the duration of the first renewal X_1 to get

$$M(t) = E[N(t)] = \int_0^\infty E\{N(t)|X_1 = x\}\, dF(x)$$

Consider $x > t$; given that $X_1 = x > t$, no renewal occurs in [0, t], so that $E[N(t)|X_1 = x] = 0$.

Consider $0 \leq x \leq t$; given that the first renewal occurs at x $(\leq t)$, the process starts again at epoch x, and the expected number of renewals in the remaining interval of length $(t - x)$ is $E[N(t-x)]$, so that

$$E[N(t)|X_1] = 1 + E[N(t-x)] = 1 + M(t-x)$$

Thus considering the above two equations, we get

$$M(t) = \int_0^t \{1 + M(t-x)\}dF(x) = F(t) + \int_0^t M(t-x)dF(x)$$

The equation (1.9) is called the integral equation of renewal theory (or simply renewal equation) and the argument used to derive it, is known as "renewal argument".

The renewal equation (1.9) can be generalized as follows:

$$v(t) = g(t) + \int_0^t v(t-x)dF(x), t \geq 0, \tag{1.10}$$

where g and F are known and v is unknown. The equation (1.10) is called a renewal type equation. A unique solution of $v(t)$ exists in term of g and F which will be formally stated as below.

If $v(t) = g(t) + \int_0^t v(t-x)dF(x), t \geq 0$ then

$$v(t) = g(t) + \int_0^t g(t-x)dM(x) \tag{1.11}$$

where $M(t) = \sum_{n=1}^{\infty} F_n(t)$.

To prove (1.11), we first take Laplace transform on both sides of (1.10) to get

$v^*(s) = g^*(s) + v^*(s)\, f^*(s)$

so that $v^*(s) = \frac{g^*(s)}{1-f^*(s)} = g^*(s)\left[1 + \frac{f^*(s)}{1-f^*(s)}\right] = g^*(s)[1 + s\, M^*(s)]$ (1.12)

Inverting the Laplace transform on both sides of (1.12), we get (1.11). The solution $v(t)$ is unique, since a function is uniquely determined by its Laplace transform.

The renewal equation (1.9) satisfied by the renewal function is a Volterra integral equation. The closed form solution of this equation is not available, excepting a very few cases in which the renewal process is driven by exponential or gamma distributions. In view of the importance of the renewal function in practical applications, several approximations for the same have been proposed. We will very briefly review some approximations, which in our view have contributed to the state of art.

(i) Methods of Substitution

Bartholomew (1963) was one of the earliest to provide an approximate solution of the integral equation of the renewal theory. Using the identity

$$F(T) = \int_0^T m(T-t)\bar{F}(t)dt \tag{1.13}$$

in the renewal equation, he derived an expression for the renewal density purely in terms of the distribution function $F(t)$ (where $\bar{F}(t)$ is the survivor function of T). Deligonul (1985) further improved Bartholomew's approximation that was quite robust for renewal densities with high degree of skewness. Smeitink and Dekker (1990) proposed an approximation by replacing the original distribution function $F(.)$ by another distribution function $\hat{F}(.)$ with the same mean and variance as $F(.)$. Politis and Pitts (1998) gave an approximation, which was on the lines of Smeitink and Dekker by approximating a density whose output is not known analytically by another density with easy output. They obtained explicit formulae for their approximations, which in many cases can be easily implemented on computer algebra software.

(ii) Riemann-Stieltjes methods

These methods are based on evaluating the renewal equation (1.9) using direct numerical Riemann-Stieltjes integration. Xie (1989) proposed such a method by partitioning the total interval into subintervals and using the midpoint method in numerical analysis, he recursively computed the value of the renewal function. This method is particularly useful when the probability density function has singularities. Ayhan *et al* (1999) also proposed a direct Riemann-Stieltjes integration method. However, instead of directly computing the integral they provided bounds on the renewal function by simply computing the lower and upper sums of the Riemann-Stieltjes integral. Xie *et al* (2003) obtained upper bounds and the error terms when some direct Riemann-Stieltjes integration methods are used.

(iii) Bounds

These methods investigate asymptotic behavior and bounds of the solutions to renewal equations. Bounds by themselves are interesting problems in many areas like upper bound on the reliability function, ruin probabilities, etc. Tighter and tighter bounds take us closer to the actual value. Marshall (1973) defined a sequence of "best" linear bounds which were sharpest bounds and when iterated converges monotonically to the renewal function $M(t)$ for all t. Daley (1976) in a classic paper showed that the renewal function $U(x) = \sum_0^\infty F^n(x)$ satisfies $U(x) \leq \lambda x_+ + C\,\lambda^2\,E(X^2)$ for a certain constant C independent of F, where $\lambda = 1/E(x)$. He further showed that $C \leq 1.3185649 \ldots$. Li and Luo (2005) studied upper and lower bounds for the solutions of Markov renewal equations and applied them to a shock model as well as an age dependent branching process under Markovian environment. Ran *et al* (2006) studied analytical and numerical bounds on the renewal function based on a simple iterative procedure. They also studied some interesting monotonicity properties and approximation error.

(iv) Methods of moment matching

One can approximate a general distribution function $F(x)$ by a phase type distribution and compute the renewal function for the approximated distribution. This approach leads to tangible results because the structure of phase type distribution gives rise to a Markovian state description for which the solution of the renewal equation is possible. The phase type distributions mainly used in the literature are Coxian-2 distributions (K_2) and mixture of exponentials (H_2).

Marie (1980) developed a two-moment approximation for the case $C^2 > 1$ where C^2 is the square of coefficient of variation given by μ_2/μ_1^2. He also gave formulas to approximate general distribution with $C^2 > 0.5$ by a two-stage phase type distribution with certain parameters. Whitt (1982) analyzed the general problem of fitting distribution functions by matching the moments. He empirically showed that if the coefficient of variation is small, then the impact of the third moment is not significant and hence there is no need to include the third moment in the representation of the original distribution by matching the moments. Altiok (1985) studied the problem of approximating a general distribution by a phase type distribution matching the first three moments. He showed that a three moment fit by Coxian-2 distribution is always possible when $\Phi_2 > 2$ and $\Phi_3 > \frac{3}{2}\Phi_2^2$, where $\Phi_2 = 1 + C^2$ and $\Phi_3 = \mu_3/\mu_1^3$. Lindsay *et al* (2000) showed how to approximate a univariate distribution with mixtures of known distribution functions. Cui and Xie (2003) approximated the Weibull distribution with normal distribution by equating the first two moments. Bux and Herzog (1977) developed a procedure based on a mathematical programming approach and fitted a Coxian distribution with uniform rates at all phases. They used the restriction that this distribution to be within a certain tolerance range with the original distribution as well as the equality of the first two moments in the constraint set. Their objective function was to minimize the number of phases.

Other directions in which research has extended include Pade' approximations [Garg and Kalagnanam (1998)], Power series expansions [Smith and Leadbetter (1963)], and Laplace transform methods [Abate (1995)].

Among the numerous approximations available, no one can be termed as an "all weather approximation" in the sense that it can be used for every distribution function. The use of a particular method depends on the characteristics of the underlying distribution function in terms of skewness, kurtosis, modes and singularities to provide a good approximation. Further every one of the methods in the literature assumes the explicit form of the distribution function $F(.)$ to be known a priori. This is a very restrictive assumption because in many practical situations we may not know the form of the distribution function, but make suitable assumptions on it. Typical examples are the failure distribution of components in reliability theory and arrival and service distributions in queuing theory. In such cases, we can collect data on the realization of the random variable X from which efficient estimators of the various moments of X could be computed. The objective of the present work is to provide an easy to evaluate but accurate all weather approximation for the evaluation of the renewal function based on the first three moments of X and requires no knowledge on the distribution function.

1.2 G-renewal Process [Kijima (1989) and Kijima *et al* (1988)]

G-renewal processes were first introduced by Kijima in the context of optimal repair and replacement of deteriorating systems. The progressive degradation of systems is often reflected in increased production cost and lead-times as well as lower product quality. Thus, optimality issues of maintenance such as repair and replacement of systems are of vital importance. The simplest maintenance policy is to replace, the system on failure by a new and identical one. With such a policy in place the number of failures can be modeled as a renewal process. However for systems consisting of many components, each having their own failure mode, a replacement is a luxury whereas repair or replacement of only the failed component is a more viable option. In this regard, the maintenance policy of minimal repairs is appealing to both researchers and practitioners. Minimal repairs restore the system to the condition that it was in just prior to failure, rendering the failure counting process as an non-homogeneous Poisson process. However, in practical situations, maintenance operations may not conform to either of these two extreme actions but may result in an intermediate level. Such maintenance operations are referred to as general repairs. We present below a general repair model that will lead us to g-renewal functions.

Consider a system, which is subject to failures. At each failure, a repair activity, which requires negligible amount of time, is performed on the system. Let a new system be put in operation in $t = 0$. Let $N(t)$ be the number of failures until time t. The failure process $N(t)$ is modeled as follows. Let V_n be the virtual age of the system immediately after the n^{th} repair and X_{n+1} , the time between n^{th} and $(n+1)^{th}$ failure since $t = 0$. Then the distribution of X_{n+1} given $V_n = y$ is distributed according to

$$Pr[X_{n+1} \leq x \mid V_n = y] = \frac{F(x+y)-F(y)}{1-F(y)} \tag{1.14}$$

where $F(x)$ is the lifetime distribution of a new system (new systems are assumed to have the virtual age $V_0 = 0$). We define a partial sum process by $S_n = \sum_{k=1}^{n} X_k$ with $S_0 = 0$. S_n may be referred to as the real age of the system at the n^{th} failure since it is the elapsed time since the system was put in operation. It is easy to see that

$$Pr[N(t) \geq n] = Pr[S_n \leq t] \tag{1.15}$$

It remains to specify the mechanism of the virtual age process $\{V_n; n \geq 0\}$. Let a_n represent the degree of the n^{th} repair. It is assumed that the n^{th} repair can remove only damages incurred during the n^{th} lifetime. That is, it reduces the additional age X_n to $a_n X_n$. Accordingly, the virtual age after the n^{th} repair becomes $V_n = V_{n-1} + a_n X_n$. Furthermore, we assume that each repair is of the same degree, say $a_n = \theta$. The virtual age V_n, then can be specified by

$$V_n = \theta S_n = \sum_{k=1}^{n} \theta X_k \tag{1.16}$$

Thus, the process $\{V_n; n \geq 0\}$ is a time homogeneous Markov process and so is $\{S_n; n \geq 0\}$.

We note that if $\theta \leq 1$ then $V_n \leq S_n$ meaning that the system is rejuvenated by the repair. If $\theta \geq 1$ then $V_n \geq S_n$ so that the repair damages the system more. Our interest is restricted to improving systems only, so that we assume that $0 \leq \theta \leq 1$ in what follows. If $\theta = 1$ we have $V_n = S_n$, so that the real age and virtual age coincide implying that a minimal repair is performed. Also if $\theta = 0$ then, $V_n = 0$, which implies that the system is renewed by each repair and hence the resulting failure process is an ordinary renewal process. The difference $S_n - V_n$ may be considered to represent the degree of rejuvenation by the repairs.

At time $t > 0$, the nearest time epoch of failure before t is $S_{N(t)}$ (if no failure, $S_{N(t)} = 0$). Let $r(t)$ be the failure rate of the system of age *t*. At the $N(t)^{th}$ repair (the 0^{th} repair means replacement), the virtual age of the system is $V_{N(t)}$ so that the failure rate at that time is $r(V_{N(t)})$, while the real age is $S_{N(t)}$. No rejuvenation occurs during the interval $[S_{N(t)}, t)$ (note that $t \geq S_{N(t)}$ almost surely).

Hence, at time t, the failure rate of the system becomes $r(V_{N(t)} + t - S_{N(t)})$. Since $V_n = \theta S_n$, we have

$$E[N(T)] = \int_0^T E\left[r\left(t - (1-\theta)S_{N(t)}\right)\right]dt \tag{1.17}$$

Further let $B_t(x)$ denote the probability distribution function of $S_{N(t)}$, so that, $B_t(x) = Pr[S_{N(t)} \leq x]$. Conditioning on $S_{N(t)}$, it is easily seen that

$$m(t) = r(t)Pr[N(t) = 0] + \int_0^t r(t - x + \theta x)dB_t(x) \tag{1.18}$$

Suppose that the n^{th} failure occurs at time x. This means that $S_n = x$ and $V_n = \theta x$. The conditional life distribution $G(t|V_n = \theta x)$ of X_{n+1} is then given from (1.14) by

$$G(t|\theta x) = \frac{F(t+\theta x) - F(\theta x)}{1-F(\theta x)} \tag{1.19}$$

It is well known that

$$Pr\left[x < S_{N(t)} \leq x + dx\right] = m(t)\{1 - G(t - x|\theta x)\}\, dx \tag{1.20}$$

It follows from (1.18) through (1.20) that

$$m(t) = f(t) + \int_0^t m(t)\frac{f(t-x+\theta x)}{1-F(\theta x)}\, dx, \quad t \geq 0. \tag{1.21}$$

Define

$$q(t|x) = \frac{f(t+\theta x)}{1-F(\theta x)}\,, \quad t, x \geq 0, \tag{1.22}$$

so that $f(t) = q(t|0)$ and

$$Q(t|x) = \int_0^t q(y|x)dy = G(t|\theta x). \tag{1.23}$$

Note that $Q(t|x)$ for any fixed x is a distribution function. It is readily seen that

$$m(t) = q(t|0) + \int_0^t m(x)\, q(t-x|x)dx \tag{1.24}$$

and the set of distribution functions $Q\{t|x: x \geq 0\}$ satisfies all the conditions needed for the Volterra integral equation (1.24) to have a unique solution. Equation (1.24) is called the g-renewal equation and the function $m(t)$ satisfying the integral equation is called a g-renewal density and $M(t)$ the g-renewal function.

Observation: If the degree of repair $\theta = 0$, then the function $q(t|x)$ becomes $f(t)$ and it is independent of x. In this case, the g-renewal equation (1.24) reduces to ordinary renewal equation. On the other hand if $\theta = 1$, then $q(t-x|x) = f(t)/\bar{F}(x)$ so that the solution of the renewal equation (1, 24) turns out to be $m(t) = r(t)$, the failure rate. Thus, the failure counting process $N(t)$ is a non-homogeneous Poisson process with mean function $\Lambda(t) = \int_0^t r(t)dt$.

For a general repair model, Kijima (1989) discussed various monotonicity properties of the process S_n, the time for the n^{th} failure with respect to stochastic orderings of general repair $\{\theta_k\}$. He also obtained an upper bound for $E(S_n)$ when a general repair is used. Kaminskiy and Krivtsov (2000) used a g-renewal process as a model for statistical warranty claim prediction. They showed that warranty claim prediction based on g-renewal process provided a higher accuracy as compared to ordinary process or non-homogeneous Poisson process. Kaminskiy (2004) obtained simple bounds on the g-renewal function with increasing failure rate underlying distributions and compared the new bounds with some known bounds for the renewal process. Dimitrakos and Kyriakidis (2007) considered the average cost optimal policy of a general repair model and developed an efficient special purpose policy iteration algorithm. They generated a sequence of improving control-limit policies. They also provided strong numerical evidence that the algorithm converges to the optimal policy. Matis *et al* (2008) discussed optimal price and prorate decisions for combined warranty policies when a general repair model is used. Kaminskiy and Krivtsov (2010) have considered g-renewal process as a repairable system model and discussed the properties and statistical estimation of the model parameters for the case of exponential and Weibull underlying distributions.

As already seen, when $\theta = 0$ the general repair model collapses to an ordinary replacement model so that the g-renewal function becomes an ordinary renewal function. On the other hand if $\theta = 1$ general repair model coincides with minimal repair model so that the g-renewal function is nothing but the mean function of the inhomogeneous Poisson process $\Lambda(t) = \int_0^t r(t)dt$. However, for the case $0 < \theta < 1$, in view of the structure of the g-renewal equation, it is not possible to obtain a closed form solution of the g-renewal function. The methods of Laplace transforms and power series expansions used in the case of one-dimensional renewal process are not applicable because of the kernel $q(t|x)$ appearing in g-renewal equation (1.24). Numerical solutions are unwieldy to employ since the g-renewal equation contains a recurrent infinite system. In view of the importance of the g-renewal function and its application in reliability and the non-availability of useful approximations in estimating them, this thesis proposes a couple of useful approximations. An optimal replacement problem with general repairs is developed which uses the computation of g-renewal functions to obtain optimal replacement periods.

1.3 Two-dimensional Renewal Processes [Hunter (1974(a), 1974(b), and 1977)]

There has been a steady growth in the one-dimensional renewal theoretic models and their applications to wide ranging areas. A natural extension of the one-dimensional models to higher dimensions has led to the development in multidimensional models and in particular two-dimensional renewal processes. However, such an extension is fraught with pitfalls such as $F(x,y) + \bar{F}(x,y)$ is not equal to unity as in the one-dimensional case. Again, in the one-dimensional case the failure rate $\lambda(t)$ is defined as $\lambda(t) = f(t)/\bar{F}(t)$ which gives rise to the differential equation $d\ ln\ f(t)/dt = \lambda(t)$ with the solution $f(t) = e^{-\int_0^t \lambda(u)du}$. However, in the two-dimensional case, the bivariate failure rate $\lambda(t,u)$ is given by $\lambda(t,u) = f(t,u)/\bar{F}(t,u)$ where $f(t,u) = \delta^2 F(t,u)/\delta t\ \delta u$. Unlike the one-dimensional case, the solution of the partial differential equation $\frac{\delta^2 F(t,u)}{\delta t\ \delta u} = \lambda(t,u)\bar{F}(t,u)$ has not been found yet. Hunter (1974(a), 1974(b), and 1977) in a series of three classic papers built up the edifice of two-dimensional renewal theory. We present here the basics of two-dimensional renewal processes in a run up to the two-dimensional renewal equation, which is the focus of our study.

Let $\{(X_n, Y_n)\}, n = 1, 2, ...,$ be a sequence of independent and identically distributed non-negative bivariate random variables with common joint distribution function

$$F(x,y) = Pr\{X_n \le x, Y_n \le y\}. \tag{1.25}$$

Let $S_n = \left(S_n^{(1)}, S_n^{(2)}\right) = (\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i)$. The sequence of bivariate random variables $\{(X_n, Y_n)\}$ is known as a bivariate renewal process. Each of the marginal sequences, $\{X_n\}$ and $\{Y_n\}$ are (univariate) renewal processes. In order to distinguish between the different renewal processes, we shall say that an X-renewal occurs at the point x on the X-axis if $S_n^{(1)} = x$ for some n, a Y-renewal occurs at the point y on the Y-axis if $S_n^{(2)} = y$ for some n, and an (X, Y)-renewal occurs at the point (x, y) in (X, Y) plane if $S_n^{(1)} = x$ and $S_n^{(2)} = y$ for some n.

Define

$$N_x^{(1)} = \max\left\{n: S_n^{(1)} \leq x\right\},$$

$$N_y^{(2)} = \max\left\{n: S_n^{(2)} \leq y\right\},$$

$$N_{x,y} = \max\left\{n: S_n^{(1)} \leq x, S_n^{(2)} \leq y\right\}.$$

Thus associated with a bivariate renewal process we have various counting processes. Firstly, $N_x^{(1)}$ and $N_y^{(2)}$ are the univariate renewal counting processes for the X-renewals and the Y-renewals. We call the random pair $(N_x^{(1)}, N_y^{(2)})$ the bivariate renewal counting process. Secondly, $N_{x,y}$ records the number of (X, Y)-renewals that occur in the closed region of the positive quadrant of the (X, Y) plane bounded by the axes and the lines $X = x$ and $Y = y$. We call $N_{x,y}$ the two-dimensional renewal counting process.

1.3.1 The Distribution of $N_{x,y}$

For $x, y \geq 0$, and $n \geq 0$,

$$Pr\left\{N_{x,y} = n\right\} = F_n(x, y) - F_{n+1}(x, y). \tag{1.26}$$

Since $N_{x,y} = \min\left(N_x^{(1)}, N_y^{(2)}\right)$ we observe that

$$\left\{N_{x,y} \geq n\right\} = \left\{N_x^{(1)} \geq n\right\} \cap \left\{N_y^{(2)} \geq n\right\} = \left\{S_n^{(1)} \leq x, S_n^{(2)} \leq y\right\}.$$

Thus $Pr\{N_{x,y} \geq n\} = F_n(x,y)$ along with $Pr\{N_{x,y} = n\} = Pr\{N_{x,y} \geq n\} - Pr\{N_{x,y} \geq n+1\}$ proves the result.

1.3.2 The Two-dimensional Renewal Function and Renewal Density

In analogy with the univariate theory, we define the two-dimensional renewal function, $M(x,y) = E(N_{x,y})$. For all $x, y \geq 0$, one can easily show that

$$M(x,y) = \sum_{n=1}^{\infty} F_n(x,y). \tag{1.27}$$

There are various ways of establishing this result. In particular, from (1.26)

$$E[N_{x,y}] = \sum_{n=1}^{\infty} Pr\{N_{x,y} \geq n\} = \sum_{n=1}^{\infty} F_n(x,y) \tag{1.28}$$

Just as the probability generating function of $N_{x,y}$ provides information concerning $N_x^{(1)}$ and $N_y^{(2)}$, the univariate renewal functions can be obtained from $M(x,y)$, since from $E\left[N_x^{(1)}\right] = \sum_{i=1}^{\infty} F_i^1(x)$, $E\left[N_y^{(2)}\right] = \sum_{j=1}^{\infty} F_j^2(y)$, (for $x, y \geq 0$) and equation (1.27), $M_1(x) = M(x,\infty)$ and $M_2(y) = M(\infty, y)$. Furthermore, from (1.27), we observe that

$$M ** F(x,y) = \sum_{n=1}^{\infty} F_{n+1}(x,y) = M(x,y) - F(x,y) \tag{1.29}$$

where the operator $**$ denotes convolution with respect to x and y. From (1.29) we obtain the "integral equation of two-dimensional renewal theory",

$$M(x,y) = F(x,y) + \int_0^x \int_0^y M(x-u, y-v)\, dF(u,v). \tag{1.30}$$

This is analogous to the well-known integral equation of one-dimensional renewal theory:

$$M_1(x) = F^1(x) + \int_0^x M_1(x-u) dF^1(u)$$

From (1.27) or (1.30) we can derive an expression for $M^*(p,q)$the bivariate Laplace-Stieltjes transform of $M(x,y)$:

$$M^*(p,q) = \frac{F^*(p,q)}{1-F^*(p\,,q)} \tag{1.31}$$

A similar expression holds for the univariate Laplace-Stieltjes transform of $M_1(x)$:

$$M_1^*(p) = \frac{F^{1*}(p)}{1 - F^{1*}(p)}$$

It should be remarked that knowledge of the two-dimensional renewal function $M(x, y)$ implies complete knowledge of all aspects of the two-dimensional renewal process. This is obvious from (1.31), and expressions for the higher order moments of $N_{x,y}$ can be found in terms of $M(x, y)$. If we assume $F(x, y)$ to be absolutely continuous then we can define the two-dimensional renewal density function

$$m(x,y) = \frac{\partial^2}{\partial x\ \partial y} M(x,y) = \sum_{n=1}^{\infty} f_n(x,y) \tag{1.32}$$

Thus, from (1.30) we obtain the "two-dimensional renewal density integral equation"

$$m(x,y) = f(x,y) + \int_0^x \int_0^y m(x-u, y-v)\, f(u,v)\, du\, dv; \tag{1.33}$$

which upon taking bivariate Laplace transforms, yields

$$m^*(p,q) = \frac{f^*(p,q)}{1- f^*(p,q)} \tag{1.34}$$

Note that $m^*(p, 0) = m_1^*(p)$ and $m^*(0, q) = m_2^*(q)$, the Laplace transforms on the univariate renewal densities, $m_1(x)$ and $m_2(y)$ respectively. It can be shown that $m_1(x) = \int_0^\infty m(x,y)\, dy$ and $m_2(y) = \int_0^\infty m(x,y)\, dx$.

The theory of multidimensional stochastic processes has been developed mainly by the Russian school of probability and applied to several interesting areas like cascade showers, seismicity, and neural activity. Kotz *et al* (2000) have studied several models of continuous multivariate distributions and their applications. Bivariate exponential distributions, very much like their one-dimensional analog are simple to use but versatile in applications. Various bivariate exponential distributions have been proposed in the literature (see Kotz *et al* (2000) for details). One form of the distribution with desirable characteristics was introduced by Downton (1970) in the context of reliability theory. The bivariate exponential density function is given by

$$f(x,y) = \frac{\lambda_1 \lambda_2}{1-\rho} e^{\left[-\frac{\lambda_1 x+\lambda_2 y}{1-\rho}\right]} I_0\left[\frac{2\sqrt{\rho\, \lambda_1\, \lambda_2\, x\, y}}{1-\rho}\right] \quad (1.35)$$

where $\lambda_1 = 1/\mu_1 > 0$, $\lambda_2 = 1/\mu_2 > 0$, and $0 \le \rho < 1. I_n(.)$is the modified Bessel function of the first kind of n^{th} order.

The advantage of using this form of the bivariate exponential distribution is that this is one of the very few joint distributions whose explicit form of the double Laplace transform of the two-dimensional renewal function is known. It is given by

$$M^*(p,q) = \frac{1}{pq(\mu_1 p+\mu_2 q+(1-\rho)\mu_1\mu_2 pq)} \quad (1.36)$$

Analogous to the one-dimensional elementary renewal theorem, Hunter (1974(b)) gave the following results for two-dimensional renewal function.

$$\lim_{t\to\infty} M(t,t) = \frac{1}{\max(\mu_1,\mu_2)} \quad (1.37)$$

$$\lim_{t\to\infty} \frac{M(\mu_1 t,\mu_2 t)}{t} = 1 \quad (1.38)$$

$$\lim_{t\to\infty} M(\mu_1 t,\mu_2 t) = t - D\sqrt{\frac{t}{2\Pi}} + o\sqrt{t} \quad (1.39)$$

where $D = \sqrt{(\frac{\sigma_1}{\mu_1})^2 + (\frac{\sigma_2}{\mu_2})^2 - 2\rho\frac{\sigma_1\sigma_2}{\mu_1\mu_2}}$

Hunter (1977) proposed a collection of upper and lower bounds for the two-dimensional renewal functions. Chen *et al* (2010) in their two-dimensional renewal risk model, were interested in finite time ruin probabilities which requires the computation of two-dimensional renewal functions. However, they considered only asymptotic formula. One of the major applications of two-dimensional renewal theory has been in the area of two-dimensional warranty policies (Murthy *et al* (1995)). A two-dimensional warranty, which is the natural extension of the one-dimensional warranty, is characterized by a region in two dimensions with the two axes representing age and usage. The theory requires heavy usage of two-dimensional renewal functions.

Approximations to one-dimensional renewal function have received considerable attention in the literature. However, strangely there has been practically no attempt to provide efficient approximations to two-dimensional renewal functions although their occurrences in applications are quite frequent. Iskandar (1991) has provided a two-dimensional renewal function solver, which is a computational procedure with a computer program to solve the two-dimensional renewal integral equation. We wish to observe that even the only available approximation requires the explicit form of the joint distribution function $F(x, y)$. In our view, this is very restrictive because in practical applications, presupposing the joint distribution may lead to erroneous conclusions. However, past data records might provide us with good estimates of the statistical characteristics of the two variables. This thesis develops an efficient approximation to the two-dimensional renewal function based only on the first two moments of the variables and their correlation coefficient. The proposed approximation is checked for accuracy with the benchmark approximation of Iskandar for several bivariate distributions. We also consider two-dimensional renewal warranty models as applications to apply our approximation procedure.

The layout of the thesis is as follows. In chapter 1 we provide the basics of renewal processes needed for the approximations of the renewal functions that are developed in this thesis. In chapter 2 a moment based non-parametric approximation to the one-dimensional renewal function is presented. Numerical comparison of our approximation with the benchmark approximations of Deligonul (1985), Deligonul and Bilgen (1984), Bartholomew (1963), Xie (1989), and Giblin (1983) are made. Some illustrations are provided to show the performance of our approximation. An application of the moment based approximation developed in chapter 2 in the form of computing the performance measures in queuing systems is presented in chapter 3. This chapter also proposes an alternative procedure of approximation by matching moments for the calculation of the performance measures. Numerical illustrations are also provided. Chapter 4 provides a couple of approximations for the computation of g-renewal function of which one is based on Riemann integrals. In chapter 5 we develop an optimal system design model which uses the computation of the g-renewal function using the methods developed in chapter 4. In chapter 6 we develop an approximation for the two-dimensional renewal function. Apart from numerical examples, we discuss a two-dimensional warranty model in which the warranty costs are developed using our approximation. In conclusion, the last chapter makes some interesting observations and scope of the proposed work.

Chapter 2

MOMENTS BASED APPROXIMATION TO THE RENEWAL FUNCTION

2.1 Introduction

There are many applications like reliability, queuing theory, and inventory theory in which renewal equations are encountered. Since a closed form expression for the solution of the renewal equation is not available, in most cases bounds, numerical and approximation methods have been employed. Researches have been done in several directions to find simpler approximations to the renewal function. These include Laplace transform methods [Abate(1995)], approximations [Bartholomew (1963), Deligonul (1985), Smeitink and Dekker (1990), and Politis and Pitts (1998)], Pade' approximations [Garg and Kalagnanam (1998)], power series expansions [Smith and Leadbetter (1963)], Riemann-Stieltjes integration methods [Xie (1989) and Xie *et al* (2003)] and bounds [Daley (1976), Ayhan *et al* (1999), Li and Luo (2005), and Ran *et al* (2006)] to mention a few. Early approximations to the renewal density can be traced to Bartholomew's work (1963) which was later on improved by Deligonul (1984). Interestingly Bartholomew's approximation provided good results for small values of t while not so good matching for lager values of t whereas Deligonul's approximation provided good match for larger values of t and not so good approximation for smaller t. Smeitink and Dekker (1990) provided an interesting approximation by approximating the original distribution function F(t) by another distribution function $\hat{F}(t)$ with the same mean μ and variance σ^2 as that of F(t). Amongst renewal function approximation for distribution function F(t), Weibull distribution has attracted the attention of researchers in the recent past. This is because "the Weibull distribution describes in a relatively simple analytical manner a wide range of realistic behavior and its shape and scale parameters can be readily determined with graphical or statistical procedure. On the other hand, there are no closed form analytical solutions for the Weibull renewal function except for the special case of exponential distribution" [Cui and Xie (2003)]. Cui and Xie (2003) proposed some approximations based on normal approximation of Weibull distribution. Jiang (2008) studied a series truncation approximation for computing the Weibull renewal function by approximating the Weibull distribution function by a mixture of n-fold convolution of gamma and normal distribution functions. Jin and Gonigunta (2009, 2010) in a couple of papers used exponential approximation to Weibull distribution and generalized exponential approximation to Weibull and gamma distributions. Min Xie and his team of researchers (1989, 2003) have proposed several approximations using numerical Riemann-Stieltjes integration method. Another interesting approach in the determination of renewal functions is to provide bounds. Stone (1972) provided an upper bound for the renewal function in terms of the mean and variance, which was subsequently improved by Daley (1976). Ayhan *et al* (1999) provided tight upper and lower bounds for renewal function based on Riemann-Stieltjes integration. Upper and lower bounds for the solutions of Markov renewal equations for some special cases under specific marginal conditions and in an alternating environment were studied by Li and Luo (2005). Ran *et al* (2006) studied analytical bounds based on a simple iterative procedure. They provided some convergent analytical results and investigated bounds and approximations for a recursive algorithm for numerical computation. However all these methods make use of the distribution function *F(t)* of the renewal process which is assumed to be known. In practical applications like reliability and queuing theory, this assumption may not hold. At best, one might be in possession of the statistical characteristics of the underlying distribution. There are a number of cases where the moments of a distribution are easily obtained, but theoretical distributions are not available in closed forms [Lindsay *et al* (2000)]. Alternatively, from the observed sample data efficient estimators for the various moments of the underlying distributions could be calculated. Thus, a more appropriate problem would be the computation of the renewal function based only on the moments of the distribution without recourse to the distribution function.

In this chapter we first propose an approximation for the evaluation of the renewal function based on the first three moments of the distribution function *F(t)*. The proposed method requires no knowledge of the explicit form of *F(t)* and is applicable when the first three moments of *F(t)* are finite and known. It is applicable both for distributions with coefficient of variation smaller than one (smaller dispersion) and for distributions whose coefficient of variation is greater than one (larger dispersion) and is easy to carry out. The method produces exact results of the renewal function for certain important distributions like mixture of two exponentials and K_2(Coxian-2) used widely in applications. We also propose an iterative procedure to improve the approximation, which is shown to converge to the value of the renewal function. However the iterative procedure uses the distribution function *F(t)*. When this is not known, we propose that the unknown distribution function *F(t)* can be approximated by a K_2 distribution by fitting the first three moments [Heijden (1988) & Altiok (1985)]. In the region where a three-moment fit is not possible, a procedure for fitting the first two moments exactly and matching the third moment as close as possible has been suggested. Finally, an optimal replacement problem is used to illustrate the computations and efficacy of the proposed method.

2.2 Notations used

T: random variable denoting the inter-arrival time between successive renewals of a stationary renewal process

f(t): density function of *T*

F(t): distribution function of *T*

N(t): number of renewals occurring during the interval *(0,t)*

M(t): renewal function of the renewal process whose inter arrival time of events is specified by *T*

$\mu'_1 = \mu$, μ'_2 and μ'_3: first three raw moments (about the origin) of *T*

σ : standard deviation of T

C: coefficient of variation of T given by σ/μ

$\Phi_2 = C^2 + 1$

Φ_3: coefficient of skewness of T given by μ'_3/μ^3

$f^*(s)$: Laplace Transform of the function *f(t)*

$E_{k-1,k}$: mixture of Erlangean distributions

2.3 The Moment Based Approximation

Consider the renewal counting process $\{N(t);\ t \geq 0\}$ whose renewal function is defined as the mean number of renewals in *[0,t]* so that *M(t) = E[N(t)]*. *M(t)* can be expressed as:

$$M(t) = \sum_{n=1}^{\infty} F^{(n)}(t) \qquad ,t \geq 0 \tag{2.1}$$

where $F^{(n)}(t)$ denotes the n-fold convolution of $F(.)$ with itself, recursively defined as:

$$F^{(n)}(t) = \int_0^t F^{(n-1)}(t-u)\, dF(u), t \geq 0, n \geq 2, \tag{2.2}$$

with $F^{(1)}(t) = F(t)$. Note that $F^{(n)}(t)$ is the probability of n or more renewals occurring in the interval $[0,t]$. It is well-known that

$$M(t) = t/\mu + (\mu'_2 - 2\mu^2)/2\mu^2 + o(1) \text{ as } t \to \infty \tag{2.3}$$

While the right hand side of (2.3) is an asymptotic result, we now propose the following approximation to the renewal function:

Theorem 1. Suppose that the raw moments $\mu'_n = E(T^n)$; $n = 1, 2, 3$ of T exist and are known. Then, the following result holds for renewal function $M(t)$:

$$M(t) = t/\mu + (\mu'_2 - 2\mu^2)(1 - e^{s_0 t}) / (2\mu^2) + o(1) \tag{2.4}$$

where $s_0 = 6\, \mu(\mu'_2 - 2\mu^2) / (3\mu'^2_2 - 2\, \mu\, \mu_3')$. (2.5)

Proof:

The renewal density $m(t) = M'(t)$ satisfies the integral equation:

$$m(t) = f(t) + \int_0^t m(t-u) f(u) du \tag{2.6}$$

Applying Laplace Transform to both sides of (2.6) we have:

$$m^*(s) = \frac{f^*(s)}{1-f^*(s)} \tag{2.7}$$

Noting that $m^*(s)$ has a singularity at $s=0$, we approximate it by a rational function as follows:

$$m^*(s) \approx \frac{A}{s} + \frac{B}{s-s_0} \tag{2.8}$$

Inverting (2.8) and integrating we obtain:

$$M(t) \approx At - \frac{B(1-e^{s_0 t})}{s_0} \tag{2.9}$$

In order to obtain A, B and s_0, we proceed as follows. It is known that $f^*(s)$, the Laplace transform of a density function admits the power series expansion

$$f^*(s) = \sum_{n=0}^{\infty} \frac{(-1)^n s^n}{n!} \acute{\mu}_n \tag{2.10}$$

Using (2.10) and (2.8) in (2.7) and comparing the coefficient of s^0, s^1 and s^2 on both sides we obtain after some algebra

$A=1/\mu$

$B = -s_0(\mu'_2-2\mu^2)/2\mu^2$ and

$s_0 = -6\mu(\mu'_2 - 2\mu^2)/(2\mu\mu'_3 - 3\mu'^2_2)$

Thus an approximation for the renewal function is:

$$M_o(t) = t/\mu + (\mu'_2-2\mu^2)(1-e^{s_0 t})/(2\mu^2) \tag{2.11}$$

where s_0 is given by (2.5). The rational function approximation (2.8) clearly shows that the above approximation for the renewal function is of *o(1)*. This completes the proof. Q.E.D.

Note 1: For the applicability of the approximation, it is necessary that s_0 must be less than or equal zero. To get an intuitive meaning of this condition we proceed as follows. Let $\Phi_2 = \frac{\mu'_2}{\mu^2}$ and $\Phi_3 = \frac{\mu'_3}{\mu^3}$. Now $s_0 \leq 0$ implying either $\begin{cases} 6\mu(\mu'_2 - 2\mu^2) \geq 0 \\ and\ \ 3\mu'_2 - 2\mu\mu'_3 \leq 0 \end{cases}$ or $\begin{cases} 6\mu(\mu'_2 - 2\mu^2) \leq 0 \\ and\ \ 3\mu'_2 - 2\mu\mu'_3 \geq 0 \end{cases}$. Now it can be easily established that the first set of conditions imply $\Phi_2 \geq 2$ and $\Phi_3 \geq \frac{3}{2}\Phi_2^2$ where as the second sets of conditions lead us to $\Phi_2 \leq 2$ and $\Phi_3 \leq \frac{3}{2}\Phi_2^2$. Since Φ_2 and Φ_3 are measures of coefficient of variation and skewness respectively, we observe that given the distribution F, the applicability of the method can be determined using its coefficient of variation and skewness. Figure 2.1 plots the regions in which the condition is satisfied.

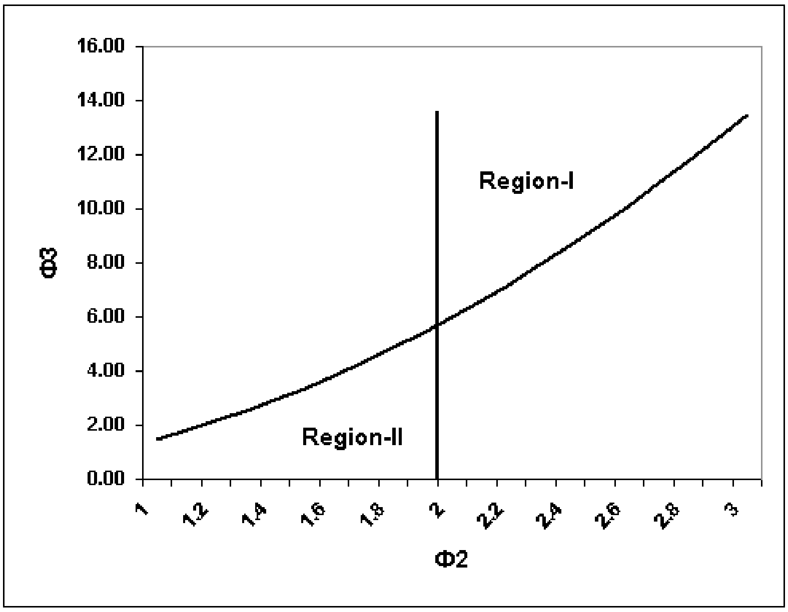

Figure 2.1: Plot of Feasible Regions for the Approximation

Note 2: When $F(t)$ has decreasing failure rate (DFR), the bounds for $M(t)$ (Ross 1996) are

$t/\mu \leq M(t) \leq t/\mu + (\mu'_2 - 2\mu^2)/(2\mu^2)$. We observe that the approximation $M_o(t)$ improves the upper bound and gives a tighter bound.

Note 3: Employing analysis similar to that of the proof of Theorem 1, the following approximations for the variance of $N(t)$ and $E[N(t)^3]$ can be derived.

$Var[N(t)] \approx$

$$\frac{\sigma^2}{\mu^3}t + [\frac{2\sigma^2}{\mu^2} + \frac{3}{4} + \frac{5\sigma^4}{4\mu^4} - \frac{2\mu'_3}{3\mu^3}] - [\frac{5\sigma^2}{2\mu^2} + \frac{1}{2} + \frac{\sigma^4}{\mu^4} - \frac{2\mu'_3}{3\mu^3}]e^{s_0 t} + 2t\upsilon\ [\frac{1}{\mu} + \upsilon s]e^{s_0 t} - \upsilon^2 e^{2s_0 t} \quad (2.12)$$

$$E[N(t)^3] \approx \upsilon + \frac{12\upsilon}{\mu s_0} + \frac{18\upsilon}{\mu^2 s_0^2} + \frac{36\upsilon^2}{\mu s_0} - 6\upsilon^2\left(\frac{1}{s_0} - \upsilon\right) + t\left[\frac{1}{\mu} + \frac{12\upsilon}{\mu} + \frac{18\upsilon}{\mu^2 s_0} + \frac{18\upsilon^2}{\mu}\right]$$
$$+ \frac{3t^2}{\mu^2}(1+3\upsilon) + \frac{t^3}{\mu^3} + \left[-\upsilon - \frac{12\upsilon}{\mu s_0} - \frac{18\upsilon}{\mu^2 s_0^2} - \frac{36\upsilon^2}{\mu s_0} + 6\upsilon^2\left(\frac{1}{s_0} - \upsilon\right)\right]e^{s_0 t}$$
$$+ t\left[\frac{18\upsilon^2}{\mu} - 6\upsilon^2(1 - \upsilon s_0)\right]e^{s_0 t} + t^2\left[3\upsilon^2\left(s_0 - \upsilon s_0^2\right)\right]e^{s_0 t}$$

(2.13)

where $\upsilon = \frac{\mu'_2 - 2\mu^2}{2\mu^2}$ and $\sigma^2 = \mu'_2 - \mu^2$.

Proposition 1. The condition that s_0 is non-positive which is necessary for the approximation to hold is satisfied by many standard distributions like uniform, gamma, mixture of exponentials, lognormal, $E_{k-1,k}$, $k \geq 2$ (see section 2.5) and Weibull. In addition, s_0 is non-positive for the following distributions under the stated conditions:

Truncated Normal pdf with $\alpha > 0$:

$$f(t) = [(\beta\sqrt{2\pi})(1 - \phi(-\alpha/\beta))]^{-1} e^{[-(t-\alpha)^2/2\beta^2]}, t \geq 0, \beta > 0$$

where $\phi(x) = (1/\sqrt{2\pi})\int_{-\infty}^{x} e^{(-y^2/2)}dy$.

Inverse Gaussian pdf with $1 < \lambda/\mu < \sqrt{3}$

$$f(t) = (\lambda/2\pi t^3)^{1/2} e^{[-\lambda(t-\mu)^2/2t\mu^2]}, t > 0, \mu > 0, \lambda > 0.$$

Note 4: Comparing the exact expression for the renewal function [Sahin (1990), Smeitink and Dekker (1990)], simple algebra shows that the approximation $M_o(t)$ is exact for several distributions like exponential, mixture of two exponentials, and Erlang with two phases, and K_2. We wish to mention that these distributions in general and K_2 in particular have wide applicability in modeling arrival and service distributions in queuing theory, life times in reliability analysis and approximation of probability distributions to evaluate multimedia systems. Thus, the evaluation of the exact values of the renewal function for such important distributions using simple computations assumes significance.

We give below a summary of the procedure to approximate the mean number of events *M(t)* given the sample data.

Step 1: Compute the first three moments μ_1', μ_2', and μ_3' from the given sample data.

Step 2: Calculate s_0 based on (2.5).

Step 3: If $s_0 > 0$, this method is not applicable.

Step 4: If $s_0 \leq 0$, go to step 5.

Step 5: Approximate the renewal function *M(t)* using (2.4).

To illustrate the above procedure we present an example and compare our results with benchmark approximations. In step1, set the first three moments as $\mu_1' = 2, \mu_2' = 6,$ and $\mu_3' = 3$ (the choice of the moments was motivated in order to make a comparison with Xie (1989) and others). In step2, s_0 is evaluated as -4. Since s_0 is negative we proceed to step 5. In step5, we compute $M_o(t)$ the approximation to renewal function for various values of *t* as shown in Table 2.1.

In order to make a comparison of the values obtained using ours with other benchmark approximations, we have chosen the moments in our example as the moments of the gamma distribution of order 2 with scale parameter equal to one. In this case, we have the closed form solution $M(t) = 0.25(2t - 1 + e^{-2t})$. In Table 2.1

along with $M_0(t)$ obtained using our approximation we also present the results obtained using the following benchmark approximations:

$\hat{M}_{RS}(t)$: RS method of Xie (1989) without using the known distribution function *F(t)*.

$M_{RS}(t)$: RS method of Xie (1989) using the known distribution function *F(t)*.

$M_G(t)$: Generating function algorithm of Giblin (1983).

$M_{DB}(t)$: Results obtained using the cubic spline Galerkin solution of Deligonul and Bilgen (1984).

M(t): Exact values of the renewal function.

Table 2.1: Values of Renewal Function for Gamma Distribution

GAMMA	$f(t)=te^{-t}$			$\mu_1=2, \mu_2'=6, \mu_3'=3, \sigma^2=2, s_0=-4$						
Time	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**	**2.0**	**3.0**	**4.0**	**5.0**	**8.0**
***M(t)* (Baxter et al [10])**	0.017580	0.062332	0.125299	0.200474	0.283834	0.754579	1.250620	1.750084	2.250011	3.750000
$M_0(t)$	0.017580	0.062332	0.125299	0.200474	0.283834	0.754579	1.250620	1.750084	2.250011	3.750000
$M_{DB}(t)$	0.01756	0.06233	0.12534	0.20039	0.28388	0.75457	1.25059	1.75003	2.24993	3.74982
$M_G(t)$	0.0176	0.0624	0.1254	0.2007	0.2842	0.7559	1.2531	1.7538	2.2550	3.7588
$M_{RS}(t)$	0.017580	0.062332	0.125299	0.200475	0.283834	0.754580	1.250621	1.750085	2.250012	3.750001
$\hat{M}_{RS}(t)$	0.017581	0.062335	0.125302	0.200479	0.283890	0.754590	1.250636	1.750107	2.250042	3.750061

A similar computational procedure was carried out for Weibull and Truncated Normal distributions. In Tables 2.2 and 2.3, we have compared the values of *M(t)* using our approximation $M_o(t)$ with the actual values for Weibull and Truncated Normal distributions [Baxter(1981)], and those of Smeitink and Dekker (1990) and Bartholomew (1963). Excepting for values of t very close to the origin, it can be seen that our approximation works at least as good as or better than theirs. Also it should be mentioned that our method is very simple to evaluate.

Table 2.2: Values of Renewal Function for Weibull Distribution

WEIBULL	$f(t) = 1.5t^{0.5}e^{-t^{1.5}}$ μ_1=0.9027, μ'_2=1.1906, μ'_3=2, σ^2=0.3757, s_0= -3.7066												
Time	0.2	0.3	0.4	0.5	0.6	0.8	1	1.25	1.5	1.75	2	2.5	3
M(t) (Baxter et al [6])	0.0879	0.1591	0.2408	0.3303	0.4257	0.6287	0.8416	1.1147	1.3910	1.6682	1.9455	2.4998	3.0538
$M_o(t)$	0.0805	0.1515	0.2348	0.3266	0.4243	0.6306	0.8449	1.1178	1.3931	1.6694	1.9461	2.4999	3.0537
Relative error	8.47%	4.80%	2.50%	1.12%	0.33%	0.30%	0.39%	0.28%	0.15%	0.07%	0.03%	0.00%	0.00%
Smeitink[23]	0.0864	0.1554	0.2344	0.3210	0.4139	0.6138	0.8265	1.1024	1.3828	1.6639	1.9438	2.4999	3.0531
Bartholomew[5]	0.0881	0.1600	0.2428	0.3341	0.4319	0.6412	0.8620	1.1460	1.4329	1.7194	2.0040	2.5675	3.1253

Table 2.3: Values of Renewal Function for Truncated Normal Distribution

Truncated Normal	$f(t) = e^{[-(t-1.5)^2/2]} / 0.9332\sqrt{2\pi}$ μ_1=1.6388, μ'_2=3.4582, μ'_3=8.4648, σ^2=0.7726, s_0= -2.3129													
Time	0.3	0.5	0.7	0.9	1.1	1.3	1.5	1.75	2	2.5	3	3.5	4	4.5
M(t) (Baxter et al [6])	0.0529	0.1024	0.1647	0.2404	0.3292	0.4300	0.5410	0.6904	0.8472	1.1662	1.4781	1.7822	2.0848	2.3891
$M_o(t)$	0.0049	0.0610	0.1415	0.2375	0.3430	0.4547	0.5702	0.7179	0.8677	1.1705	1.4748	1.7797	2.0847	2.3898
Relative error	90.83%	40.44%	14.06%	1.23%	4.20%	5.75%	5.41%	3.99%	2.43%	0.37%	0.22%	0.14%	0.00%	0.03%
Smeitink[23]	0.0517	0.0984	0.1566	0.2267	0.3095	0.4050	0.5125	0.6610	0.8208	1.1536	1.4800	1.7915	2.0944	2.3954
Bartholomew[5]	0.0529	0.1025	0.1651	0.2416	0.3321	0.4359	0.5516	0.7100	0.8791	1.2305	1.5761	1.9049	2.2202	2.5286

2.4 Iterative Procedure to Improve the Approximation

In this section we present an iterative procedure to improve the approximation to the renewal function specified by $M_o(t)$. However this procedure uses the distribution function $F(t)$ of the renewal process. In cases where $F(t)$ is unknown, there exist methods to approximate theoretical univariate distributions with mixtures of phase type distribution by matching the first two or three moments. We will indicate this in section 2.5.

Proposition 2.

(i)An iterative procedure based on $M_o(t)$ whose n^{th} iterate $M_n(t)$ is given by

$$M_n(t) = \sum_{i=1}^{n} F^{(i)}(t) + \upsilon F^{(n)}(t) + t/\mu - \sum_{i=1}^{n-1} F^{(i)} * Fe(t) - \upsilon F^{(n)} * g(t) \tag{2.14}$$

where $g(t) = e^{s_0 t}$ and $F_e(t)$ is the equilibrium distribution given by

$$F_e(t) = \int_0^t (1-F(u))du/\mu \tag{2.15}$$

and * is the convolution operator defined as $F*G(t) = \int_0^t F(t-u)\, dG(u)$

(ii) $M_n(t)$ converges to $M(t)$ as $n \rightarrow \infty$.

(iii) Further if $F(t)$ is NBUE (new is better than used in expectation), then the sequence of $M_n(t)$ is monotonically non-increasing in n for any fixed t and converges to $M(t)$.

Proof:

(i) Our approximation $M_o(t)$ to the renewal function is given as:

$$M_o(t) = t/\mu + \upsilon\,(1-g(t)) \tag{2.16}$$

where $\upsilon = (\mu'_2 - 2\mu^2)/2\mu^2$ and $g(t) = e^{s_0 t}$.

The renewal function $M(t)$ satisfies the renewal equation

$$M(t) = F(t) + \int_0^t M(t-u)dF(u) \tag{2.17}$$

Substituting (2.16) in (2.17) results in

$$M_1(t) = F(t) + \int_0^t [t/\mu - u/\mu + \upsilon\,(1-g(t-u))]\, dF(u)$$

$$= F(t) + \upsilon\,(1-g(t))*F(t) + t/\mu - F_e(t) \tag{2.18}$$

where $F_e(t)$ is the equilibrium distribution and $g*F(t)$ is the convolution of $g(t)$ with $F(t)$.

Substitution of $M_1(t)$ of (2.18) in the renewal equation (2.17) we obtain $M_2(t)$. Repeating this process n times we obtain

$$M_n(t) = \sum_{i=1}^{n} F^{(i)}(t) + \upsilon\,(1\text{-}g(t))*F^{(n)}(t) + t/\mu - \sum_{i=1}^{n-1} F^{(i)}*F_e(t) \qquad (2.19)$$

(ii) The convergence of $M_n(t)$ to M(t) as $n \to \infty$ for any t can be established by observing that

$$\sum_{i=1}^{n} F^{(i)}(t) \to M(t), \quad \sum_{i=1}^{n} F^{(i)}*F_e(t) = t/\mu, \quad F^{(n)}(t) \text{ and } F^{(n)}*g(t) \to 0 \text{ as } n \to \infty.$$

(iii)Monotonicity property is shown by induction as follows. The fact that $M_1(t) \geq M_o(t)$ can be seen by observing that $\upsilon\,(1\text{-}g(t)) \geq \upsilon\,(1\text{-}g(t))*F(t)$ and $F(t) - F_e(t) \leq 0$. The latter can be established from the fact that when $F(t)$ is NBUE (new is better than used in expectation) then the mean residual life time

$$E[T\text{-}t \mid T \geq t] = \frac{\int_t^{\infty}(1-F(u))du}{(1-F(t))} \leq \mu \qquad (2.20)$$

$$\Rightarrow \frac{\int_t^{\infty}(1-F(u))du}{\mu} \leq (1-F(t))$$

$$\Rightarrow 1-\frac{\int_t^{\infty}(1-F(u))du}{\mu} \geq F(t)$$

$$\Rightarrow \frac{\int_0^{\infty}(1-F(u))du-\int_t^{\infty}(1-F(u))du}{\mu} \geq F(t)$$

$$\Rightarrow \frac{\int_0^{t}(1-F(u))du}{\mu} \geq F(t)$$

$$\Rightarrow F_e(t) \geq F(t) \qquad (2.21)$$

Thus the result is true for $n=1$. Let the result be true for $n=m$. Then

$M_m(t) \geq M_{m-1}(t)$ so that

$$\sum_{i=1}^{m} F^{(i)}(t) + \upsilon F^{(m)}(t) + t/\mu - \sum_{i=1}^{m-1} F^{(i)}*F_e(t) - \upsilon F^{(m)}*g(t) \geq \sum_{i=1}^{m-1} F^{(i)}(t) + \upsilon F^{(m-1)}(t) +$$

$$t/\mu - \sum_{i=1}^{m-2} F^{(i)}*F_e(t) - \upsilon F^{(m-1)}*g(t) \tag{2.22}$$

Convoluting both sides of (2.22) with F and adding $F(t)-F_e(t)$ to both sides, we obtain

$$\sum_{i=1}^{m+1} F^{(i)}(t) + \upsilon F^{(m+1)}(t) + t/\mu - \sum_{i=1}^{m} F^{(i)}*F_e(t) - \upsilon F^{(m+1)}*g(t) \geq \sum_{i=1}^{m} F^{(i)}(t) + \upsilon$$

$$F^{(m)}(t) + t/\mu - \sum_{i=1}^{m-1} F^{(i)}*F_e(t) - \upsilon F^{(m)}*g(t)$$

This implies that $M_{m+1}(t) \geq M_m(t)$, completing the proof. Q.E.D.

At this juncture, we would like to mention that the monotonicity property is desirable from the computational aspects. The NBUE property of $F(t)$ is only a necessary condition and in practical applications the convergence of the sequence of iterates is much faster.

2.5 Approximation of Distribution Functions through Moment Matching

In this section we turn our attention to the fitting of a distribution function whose first three moments match the corresponding moments of a given distribution $F(t)$. Extensive literature is available on this aspect. However, we shall confine our attention to the problem of fitting a K_2 distribution to the given distribution $F(t)$. The probability density function of K_2 distribution is given by

$$f(t) = \begin{cases} p\lambda e^{(-\lambda t)} + (1-p)\lambda^2 t e^{(-\lambda t)}, \lambda_1 = \lambda_2 = \lambda \\ \frac{p\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2}\lambda_1 e^{(-\lambda_1 t)} + \left(1 - \frac{p\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2}\right)\lambda_2 e^{(-\lambda_2 t)}, \lambda_1 \neq \lambda_2 \end{cases} \tag{2.23}$$

where λ_1 and λ_2 are positive and $0<p\leq 1$. The renewal function of this distribution function is given by

$$M(t) = \frac{\lambda_1\lambda_2}{\lambda_1(1-p)+\lambda_2}t - \frac{\lambda_1(1-p)(\lambda_2 - p\lambda_1)}{(\lambda_1(1-p)+\lambda_2)^2}(1-e^{-(\lambda_1(1-p)+\lambda_2)t}) \tag{2.24}$$

Since the renewal function of the K_2 distribution has the same form as that of approximation $M_o(t)$, we have chosen to fit a K_2 distribution to $F(t)$. Altiok (1985) showed that a three moment fit by K_2 distribution is always possible when $\Phi_2>2$ and $\Phi_3>\frac{3}{2}\Phi_2^2$. This region is exactly the region-I in Figure 2.1 where the proposed method is applicable. Thus, we can use the fitted K_2 distribution to improve the approximation. On the other hand in region-II where $\Phi_2\leq 2$ and $\Phi_3 \leq\frac{3}{2}\Phi_2^2$, a three moment K_2 fit may not be possible and hence we propose the following procedure. We divide the region-II into two parts according as $1<\Phi_2\leq\frac{3}{2}$ and $\frac{3}{2}<\Phi_2\leq 2$. It is well known that if the coefficient of variation is small, then the impact of the third moment is not significant and hence there is no need to include the third moment in the representation of the original distribution by matching the moments [Whitt (1982)]. Thus in the region $1<\Phi_2 \leq\frac{3}{2}$ we use a two moment fit with an $E_{k-1,k}$ distribution and K_2 distribution is a particular case of $E_{k-1,k}$ distribution. In the region $\frac{3}{2}<\Phi_2\leq 2$, we adopt the following procedure in which a K_2 distribution is fitted by matching the first two moments exactly while the third moments are matched as closely as possible which is given in the following theorem.

Theorem 2. Given the first three raw moments μ_1', μ_2', and μ_3' of a distribution function *F(t)*, the parameters *p*, λ_1, and λ_2 of the K_2 distribution whose first two moments exactly match μ_1' and μ_2' and whose third moment matches μ_3' as closely as possible (in the sense of squared differences) are given by:

$$\frac{1}{\lambda_1} = \frac{1}{\lambda_2} = \frac{2+\sqrt{2(1-C^{2)}}}{\mu_1'(1+C^2)} \tag{2.25}$$

$$1-p = \frac{(1-C^2)+\sqrt{2(1-C^2)}}{1+C^2} \tag{2.26}$$

where $C^2 = \frac{\mu_2' - \mu_1'^2}{\mu_1'^2}$ is the square of the coefficient of variation of *F(t)* such that $0.5 \le C^2 < 1$.

Proof:

Equating the first three moments of the K_2 distribution given in (2.23) with the three given moments μ_1', μ_2', and μ_3' after setting $\gamma_i = 1/\lambda_i, i = 1, 2$ and $q = 1-p$ we obtain

$$\mu_1' = \gamma_1 + q\gamma_2 \tag{2.27}$$

$$\mu_2' = 2\gamma_1^2 + 2q(\gamma_1^2 + \gamma_1\gamma_2) \tag{2.28}$$

$$\mu_3' = 6\gamma_1^3 + 6q\gamma_2(\gamma_1^2 + \gamma_2^2 + \gamma_1\gamma_2) \tag{2.29}$$

From (2.27) we have $q = \frac{\mu_1' - \gamma_1}{\gamma_2}$ (2.30)

Substituting the value of q from (2.30) in the equations (2.28) and (2.29) we obtain

$$2\mu_1'(\gamma_1 + \gamma_2) - \mu_2' = 2\gamma_1\gamma_2 \tag{2.31}$$

$$\mu_3' = 6\,\mu_1' \,(\gamma_1{}^2 + \gamma_2{}^2 + \gamma_1\gamma_2) - 6\,\gamma_1\gamma_2\,(\gamma_1 + \gamma_2) \tag{2.32}$$

Thus, the proposition reduces to the nonlinear optimization problem specified as follows:

$$\text{Min } \Phi^2(\gamma_1, \gamma_2) = [\,\mu_3' - 6\,\mu_1'\,(\gamma_1{}^2 + \gamma_2{}^2 + \gamma_1\gamma_2) + 6\,\gamma_1\gamma_2\,(\gamma_1 + \gamma_2)]^2 \tag{2.33}$$

subject to the constraint (2.31) along with $\gamma_1 \geq 0$and $\gamma_2 \geq 0$. We first ignore the non-negativity constraints and apply Lagrangean multiplier method. The necessary conditions reduce to

$$\Phi(\gamma_1, \gamma_2)\,[-2\,\mu_1'\,\gamma_1 - \mu_1'\,\gamma_2 + 2\,\gamma_1\gamma_2 + \gamma_2{}^2] - \tau(\mu_1' - \gamma_2) = 0 \tag{2.34}$$

$$\Phi(\gamma_1, \gamma_2)\,[-2\,\mu_1'\,\gamma_2 - \mu_1'\,\gamma_1 + 2\,\gamma_1\gamma_2 + \gamma_1{}^2] - \tau(\mu_1' - \gamma_1) = 0 \tag{2.35}$$

where τ is the Lagrangean multiplier. Assuming that the third moments do not match exactly, (2.34) and (2.35) yield

$$(\mu_1' - \gamma_2)[-2\,\mu_1'\,\gamma_2 - \mu_1'\,\gamma_1 + 2\,\gamma_1\gamma_2 + \gamma_1{}^2] = (\mu_1' - \gamma_2)[-2\,\mu_1'\,\gamma_1 - \mu_1'\,\gamma_2 + 2\,\gamma_1\gamma_2 + \gamma_2{}^2] \tag{2.36}$$

The above equations give $\gamma_1 = \gamma_2$ or $\gamma_1 = \mu_1'$ or $\gamma_2 = \mu_2'$. The last two possibilities are ignored as they correspond to the case $C^2 = 1$. Thus, the parameters of the K_2 distribution are as given in (2.25) and (2.26).

The solution $\gamma_1 = \gamma_2 = \dfrac{2 - \sqrt{2(1 - C^2)}}{\mu_1'(1 + C^2)}$ in (2.25) is neglected because it leads to

$$p = \mu_1' - \frac{\gamma_1}{\gamma_2} \leq 0\,.$$

Clearly $0 \leq q \leq 1$ if $C^2 \geq 0.5$. Thus, the solution (2.25) and (2.26) is acceptable when $0.5 \leq C^2 \leq 1$ and the third moments cannot be matched. Q.E.D.

In Table 2.4, we continue with our illustration using Weibull distribution. In this table $M_1(t)$ shows the values of the renewal function using the iterative procedure once. However we assume here that the distribution function $F(t)$ is known to be Weibull and is used in (2.18). We observe that the iteration has taken us very close to the exact value with the relative error not exceeding 0.5%. $M_1^m(t)$ is computed using a two moment fit with an $E_{k-1,k}$ distribution for $F(t)$ since Φ_2 lies in the region $1<\Phi_2 \leq \frac{3}{2}$. In this case, the iteration improves the approximation $M_o(t)$ but this cannot be guaranteed for all distributions. However one can be sure that successive iterates will take it closer to the actual value but could prove to be computationally costly.

Table 2.4: Values of Renewal Function for Weibull Distribution and Using Moment Matched $E_{k-1,k}$ Distribution

WEIBULL	$f(t) = 1.5t^{0.5}e^{-t^{1.5}}$ μ_1=0.9027, μ'_2=1.1906, μ'_3=2, σ^2=0.3757, s_0= -3.7066 k=3, p=0.61, L=2.64																
Time	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.25	1.5	1.75	2	2.25	2.5	2.75	3
M(t) (Baxter et al [6])	0.0879	0.1591	0.2408	0.3303	0.4257	0.5256	0.6287	0.7343	0.8416	1.1147	1.3910	1.6682	1.9455	2.2228	2.4998	2.7768	3.0538
$M_o(t)$	0.0805	0.1515	0.2348	0.3266	0.4243	0.5260	0.6306	0.7370	0.8449	1.1178	1.3931	1.6694	1.9461	2.2230	2.4999	2.7768	3.0537
Relative error	8.47%	4.80%	2.50%	1.12%	0.33%	0.08%	0.30%	0.37%	0.39%	0.28%	0.15%	0.07%	0.03%	0.01%	0.00%	0.00%	0.00%
$M_I(t)$	0.0876	0.1584	0.2396	0.3287	0.4238	0.5234	0.6265	0.7322	0.8398	1.1138	1.3910	1.6689	1.9466	2.2239	2.5008	2.7776	3.0543
Relative error	0.36%	0.46%	0.50%	0.49%	0.45%	0.41%	0.34%	0.28%	0.21%	0.08%	0.00%	0.04%	0.06%	0.05%	0.04%	0.03%	0.02%
$M_I^m(t)$	0.0686	0.1389	0.2235	0.3175	0.4179	0.5223	0.6292	0.7376	0.8470	1.1221	1.3979	1.6737	1.9495	2.2254	2.5016	2.7779	3.0545
Relative error	21.98%	12.69%	7.20%	3.86%	1.84%	0.63%	0.08%	0.45%	0.64%	0.67%	0.50%	0.33%	0.21%	0.12%	0.07%	0.04%	0.02%

2.6 An Optimal Periodic Replacement Problem

We present below an optimal replacement problem to test the approximation. Consider a system subject to randomly occurring shocks. Each shock increases the operating cost by c/unit time. There is a fixed cost rate *a*/unit time of operating the system and a fixed cost of replacement c_0. The system is replaced at periodic instants of time T_0, $2T_0$, $3T_0$, etc with identical copies of the same. It has been shown [Abdel-Hameed (1986)] that the long run average cost/unit time is given as follows:

$$A(T_0) = \frac{aT_0 + c\int_0^{T_0} M(t)dt + c_0}{T_0}$$

where *M(t)* is the renewal function corresponding to the shock arrival process.

The optimal replacement time T_0^* is given as the unique solution of the integral equation: $\int_0^{T_0^*} (M(T_0^*) - M(t))dt = \frac{c_0}{c}$.

Let us consider the case when the moments of the shock arrivals distribution are $\mu_1' = 3, \mu_2' = 12,$ and $\mu_3' = 60$. Using our approximation $M_o(t)$ to *M(t)* given in (4) the optimal *Ť* and the cost optimal *A(Ť)* are computed and given in Table2.5. With a view to make a comparison with the optimal solution using the exact expression for *M(t)*, we have chosen the values for the moments as the moments of the gamma distribution of order three with scale parameter equal to one ($f(t) = t^2 e^{-t} /2, t \geq 0$), so that the renewal function *M(t)* in this case is given by

$$M(t) = \frac{e^{-1.5t}}{3}\cos(0.5\sqrt{3}t) + \frac{e^{-1.5t}\sqrt{3}}{9}\sin(0.5\sqrt{3}t) + \frac{t-1}{3} \qquad (2.37)$$

Using (2.37) the optimal T^* and the optimal cost $A(T^*)$ have been computed and presented in Table 2.5. These values along with the relative error for certain values of c_0/c are shown in Table 2.5. It is noted that the relative errors are very small. A comparison of the relative errors using our formula with those obtained using the approximation of Deligonul (1985) shows that our method works efficiently with minimal labor in such replacement problems.

Table 2.5: Comparison of Optimal Replacement Times and Costs

Co/C	Exact Value		Approximate Value		Relative Error	Approximate Value (Deligonul)		Relative Error
	T^*	$C(T^*)$	$\check{T}$	$A(\check{T})$		$\check{T}$	$A(\check{T})$	
1.66667	3.373	48.3546	3.371	48.2118	0.30%	3.3	49.0298	1.40%
2.5	4.044	41.5575	4.038	41.4752	0.20%	4	42.225	1.61%
3.33333	4.620	37.1889	4.611	37.1392	0.13%	4.5	37.8127	1.68%
5	5.598	31.6507	5.590	31.6307	0.06%	5.5	32.171	1.64%

2.7 Block Replacement Problem

The most commonly used replacement policy implies replacement of failed item as and when it fails by a similar new one. Besides this replacements policy there are other policies, which are used in replacement problems. Two most important such replacement policies are age and block replacement policies. Under an age replacement policy, items are replaced upon failure *or* when it reaches an age T*, whichever is earlier. Under the block replacement policy, the item is replaced upon failure and also preventively at periodic times T, 2T, etc. The cost of failure and the cost of preventively replacement are C_f and $C_p$respectively. One can easily derive the average cost as

$$g(t) = \frac{C_p + C_f\, M(t)}{t} \quad ,t > 0$$

"Sufficient, but not necessary conditions for the existence of a unique minimum of $g(t)$ are that $C_p/C_f < 0.5\,(1 - C^2)$ and that $m(t) = dM(t)/dt$ is strictly increasing in t. The latter is true for example, Weibull distributions with $C^2 < 1$, with t up to about μ" [Berg (1980)]. In Table 2.6, for select values of shape and scale parameters of Weibull distribution and the ratio of the cost C_f/C_p , we present the optimal preventive replacement times T*, and the corresponding average cost per unit time. These results which are calculated using our moments based approximation to the renewal function are compared with the results obtained using Smeitink and Dekker's (1990) recursive simple approximation and the exact values of $M(t)$. It is to be noted that our method with minimal computations efforts provides very accurate results.

Table 2.6: Optimal Preventive Replacement Times and Corresponding Average Cost

Distribution	C_f/C_p	Smeitink and Dekker (1990)		Exact Values		Moments Based Approximation	
Weibull (with $C^2 < 1$)		$\tilde{T}$	$\tilde{g}$	T^*	g^*	$\hat{T}$	$\hat{g}$
Shape parameter =1.5 Scale parameter = 1	5	0.75	5.084	0.75	5.179	0.75	5.1641
	10	0.4	8.355	0.4	8.520	0.39	8.4892
	20	0.24	13.534	0.25	13.740	0.23	13.6974
Shape parameter =2 Scale parameter = 1	5	0.53	4.284	0.5	4.308	0.51	4.2501
	10	0.34	6.166	0.35	6.220	0.34	6.1275
	20	0.23	8.831	0.25	8.920	0.23	8.7576

2.8 Conclusions

This chapter proposes a nonparametric method for the approximation of the renewal function. The method is easy to evaluate and gives exact values for some important distributions (see note 6). An iterative procedure has also been suggested to improve the approximation which requires the usage of the distribution function *F(t)*. Moment matching procedures are spelt out to approximate *F(t)*. Since our approximation includes Φ_3, the skewness coefficient, we hope that the method will be robust even for highly skewed distribution functions. We wish to observe that when there is no information on the distribution function excepting the moments one can also resort to maximum entropy distributions as an approximation. Work in this direction is in progress.

Chapter 3

MOMENTS BASED APPROXIMATIONS TO PERFORMANCE MEASURES IN QUEUING SYSTEMS

3.1 Introduction

The growth of queuing theoretic applications has been phenomenal ranging from communication and multimedia systems to inventory and reliability theory. This has led to a sustained interest in the methods of evaluation of the performance measures in queuing theory. In the case of non-Markovian queues, the computations of these measures involve the arrival and/or service distributions explicitly. However in practical applications like management, optical and communication networks, the specific forms of these distributions might not be known and at best one might be in possession of the moments of the underlying distribution only. Thus, computation of performance measures based on the first two or three moments of the arrival and/or service distributions are very useful. Whitt-I (1984) in a classic expose discussed approximations using extremal distributions giving the upper and lower bounds for the performance measures in a GI/M/1 system. Smith (2011) proposed a two-moment approximation for the probability distribution of M/G/1/K systems and extended it to the analysis of M/G/1/K queuing networks. Sohn and Lee (2004) conducted a Monte Carlo simulation based on a designed experiment in order to relate performance measures such as the mean and the expected maximum number of requests in queue, and the mean and expected maximum time in queue to the degree of LRD in arrival, traffic intensities in G/G/1 queue. Recent works on such systems with working vacations for the server have immense applications in ATM machines and internet systems such as optical nets, electric nets and communication nets [Li *et al* (2008), Chae *et al* (2009) and Baba (2005)]. In these applications, the arrival epochs could be observed or at worst simulated. Thus, our motivation in this chapter has been to obtain approximations to the performance measures of a GI/M/1 system using only the first three moments of the arrival distribution without recourse to the arrival distribution explicitly.

In this sequel, we will discuss our problem with specific reference to a GI/M/1 queuing system, even though our methods work in similar situations for other non-Markovian queues as well. Consider a GI/M/1 queue whose traffic intensity is ρ= E(service time)/E (arrival time), C^2 is the squared coefficient of variation of inter-arrival time, L is the expected equilibrium queue length and σ is the steady state probability that a customer will have to wait to commence his service. It is well known that

$$L = \rho/(1-\sigma) \tag{3.1}$$

where σ is the unique root in the open interval (0, 1) of the equation

$$\Phi(\mu(1-\sigma)) = \sigma \tag{3.2}$$

with μ=1/E(service time) and $\Phi(s)$, the Laplace-Stieltjes Transform of the inter-arrival distribution function say F given by:

$$\Phi(s)=\int_0^\infty \exp(-st)dF(t) \tag{3.3}$$

We note that the evaluation of the performance measures σ and L require the prior knowledge of the inter-arrival distribution function F and not just the moments of F. As remarked in the beginning of this section, many queuing applications are likely to produce the moments of F and not the distribution itself. Thus, the problem is to find σ and L on the basis of the first few moments of F only. Whitt-I (1984) showed that there is a considerable reduction in the range of possible values of σ and L when the third moment is also used as compared to two moments of F. Our endeavor in this chapter is to propose simple and accurate methods to evaluate σ and L based on the first three moments of the distribution function F in the absence of any knowledge on the form of F. In section 3.2 we propose a non-parametric method based only on the first three moments of F without recourse to approximating F by another distribution function. Numerical illustrations are provided to compare the values of σ and L using the present method with their exact values. The method is seen to provide exact results for certain important arrival distributions like Erlang of order 2, Coxian (K_2), mixture of two exponentials, and exponential distribution. We also provide two optimization illustrations to obtain economic performance measures in the application of GI/M/1 queuing systems.

Approximations of probability distributions by phase type distributions by matching moments up to a certain order has attracted the attention of researchers because of the necessity and their wide applicability. Among the various members of the family of phase type distributions that has been studied, mixtures of two exponential distributions known as H_2 distributions play a key role in many approximations used in queuing theory. These distributions are log convex in nature and can approximate highly skewed distributions reasonably accurately. It is well known that an arbitrary probability distribution F can be approximated by H_2 (mixture of two exponentials) distribution whose first three moments match the corresponding moments of F if and only if m_1, m_2, and m_3, the first three moments of F satisfy the conditions $m_1 \geq 0$,

$\Phi_2 = m_2/m_1^2 = C^2 + 1 \geq 2$, and $\Phi_3 = m_3/m_1^3 \geq (3/2)\ \Phi_2^2$. However, Karlin and Studden (1966) have shown that m_1, m_2, and m_3 are the moments of some probability distributions on the positive real line if and only if $m_1 \geq 0$, $\Phi_2 \geq 1$, and $\Phi_3 \geq \Phi_2^2$. Thus in the region where exact three moment matching is not possible researchers have used adhoc methods to find the approximating H_2 distribution (See Figure 3.1).

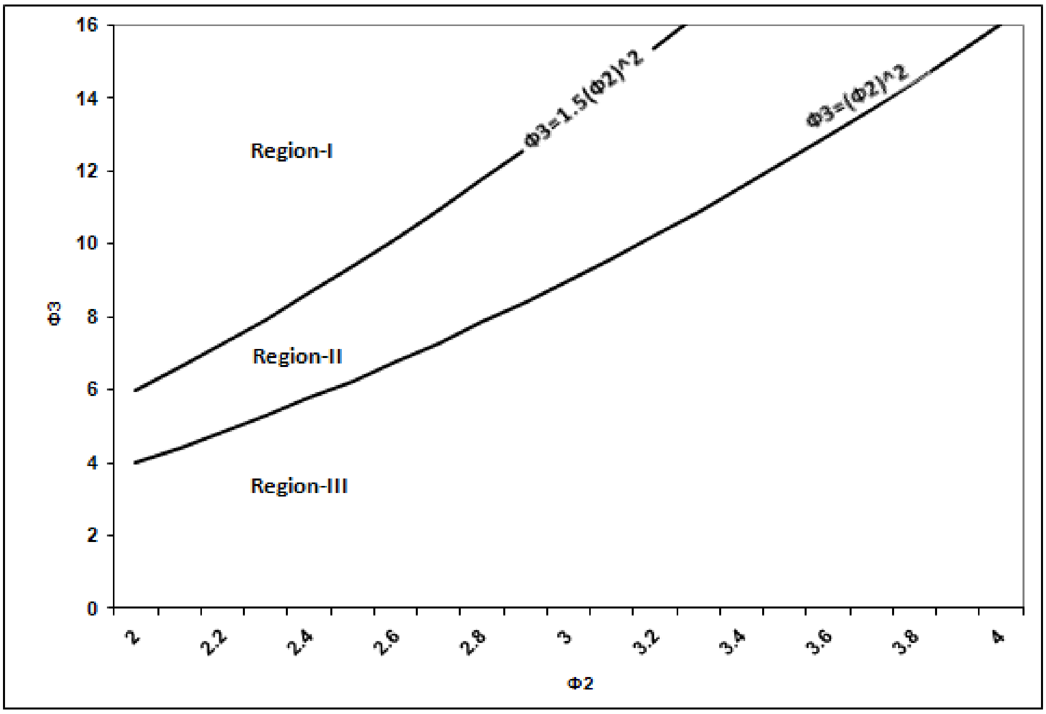

Figure 3.1: Region of Three Moment Matching (Region-I: Exact Three Moment Match Possible, Region-II: Exact Three Moment Match Not Possible, Region-III: Infeasible Region for Three Moments of a Distribution Function)

Lopez-Herrero (2002) in the absence of information on the service distribution used the maximum entropy principle approach to estimate the true distribution of the number of customers served during the busy period in an M/G/1 retrial system. Whitt (1982) suggests "a procedure to replace m_3 by something slightly larger than (3/2) m_2^2/m_1". In section 3.3 we suggest a simple nonlinear programming method in which the first two moments are matched exactly while the third moments are matched as close as possible. This method works in the entire region of possible values of (m_1, m_2, m_3) and provides exact three-moment match wherever possible. The approximated H_2 distribution with the given three moments of the inter arrival distribution is then used to calculate σ and L. Numerical illustrations are provided to validate the approximation and computation of the performance measures.

3.2 A Non-parametric Method

We observe from (3.2), that the computation of the performance measures σ and L in the GI/M/1 system requires the use of $\Phi(s)$, the Laplace Transform of the density function f corresponding to the distribution function F. However, without the prior knowledge of F and armed with only the first three moments of F, an approximation to $\Phi(s)$ is obtained from the following proposition.

Proposition 3. Suppose that the first three moments m_1, m_2, and m_3 of the distribution function F exist and are known. Then the following approximation to the Laplace transform of the distribution function F holds.

$$\Phi(s) \approx \frac{A(s-s_o)+Bs}{s(s-s_o)+ A(s-s_o)+Bs} \tag{3.4}$$

where $s_0 = \dfrac{6m_1\,(m_2-2m_1^2)}{3m_2^2-2m_1m_3}$, $A=\dfrac{1}{m_1}$, and $B=\dfrac{-s_0(m_2-2m_1^2)}{2m_1^2}$

Proof:

In the classical renewal theory, the renewal density $m(t)$ of a renewal process with interval density $f(t)$ satisfies the integral equation:

$$m(t) = f(t) + \int_0^t m(t-u)f(u)du \tag{3.5}$$

Applying Laplace transform to both sides of (3.5) we obtain the Laplace Transform of $m(t)$ as:

$$m^*(s) = \Phi(s) /(1- \Phi(s)) \tag{3.6}$$

where $\Phi(s)$ is the Laplace transform of the density function f.

It is well known that [Ross (1996)] $m^*(s)$ can be approximated by a rational function and has a singularity at s=0. Thus, we approximate $m^*(s)$ by:

$$m^*(s) \approx A/s + B/(s-s_0) \tag{3.7}$$

In order to obtain A, B and s_0, we proceed as follows. It is known that $\Phi(s)$, the Laplace transform of the density function $f(t)$ admits the power series expansion

$$\Phi(s) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1} s^n}{n!} m_n \tag{3.8}$$

where m_n is the n^{th} order moment about the origin of f. Using (3.8) in (3.6) and (3.7) and comparing the coefficient of s^0, s^1 and s^2 on both sides we obtain after some algebra

$A=1/m_1$

$B = -s_0(m_2-2m_1^2)/2m_1^2$ and

$$s_0 = -6m_1(m_2 - 2m_1^2)/(2m_1m_3 - 3m_2^2) \tag{3.9}$$

Substituting these values in (3.7) and using (3.6) we obtain (3.4). We note that for (3.4) to hold s_0 must be non-positive. Q.E.D.

Note1: For the approximation to hold, it is necessary that s_0 must be less than or equal zero. Simple calculations show that this condition implies that $\Phi_2>2$ and $\Phi_3 \geq (3/2)\Phi_2^2$ or $\Phi_2<2$ and $\Phi_3 \leq (3/2)\Phi_2^2$ (see Figure 3.1).

Note2: It is to be observed that the error in the approximation (3.4) is of $o(1)$ as $s \to 0$.

Note3: The condition that s_0 is non-positive which is necessary for the approximation to hold is satisfied by many standard arrival distributions like uniform, gamma, mixed exponential, lognormal, Coxian (K_2), mixture of Erlangean, and Weibull. Also s_0 is non-positive for truncated normal and inverse Gaussian probability density functions under certain conditions.

Using (3.4) in equations (3.2) and (3.1) with the values of m_1, m_2, and m_3, the performance measures σ and L could straightaway be obtained. To illustrate the efficiency of the proposed method we present in Tables 3.1 and 3.2 the values of σ and L computed using (3.4) for certain choice of the set $\{m_1, m_2, m_3\}$. The triplet (m_1, m_2, m_3) is chosen from the regions I and II where three-moment match is possible and where it is not. In order to compare the approximations with exact values we have considered the values of the moments of commonly used arrival distributions namely gamma and *PH4* distributions. The values of σ and L are calculated for various values of traffic intensity ρ. Whitt-I (1984) calculated the maximum relative error in the computation of the performance measure L using the formula

$$MRE\ (in\ L) = (L_u - L_l)/L_l \qquad (3.5)$$

where L_u and L_l are the upper and lower bounds of L. In order to calculate L_u and L_l he used extremal two point distributions approximating the distribution function F. Tables 3.1 and 3.2 also present the upper and lower bounds for σ and L given two and three moments and the corresponding maximum relative errors specified in (3.5). It can be seen that our method captures the values of σ and L with low relative errors.

Table 3.1: Approximations for *σ* and *L* with *PH4* Inter-arrival Distribution.

PH4(Hyper exponential)		***ρ=0.5***		***ρ=0.7***		***ρ=0.9***	
Case-I	Exact	σ	0.54207	σ	0.73870	σ	0.91790
		L	1.09187	L	2.67891	L	10.96224
	Nonparametric method	σ	0.54157	σ	0.73583	σ	0.91792
		L	1.09068	L	2.67717	L	10.96491
	Relative Error	0.11%		0.06%		0.02%	
C^2= 1.517	Matching moments with H2 distribution	σ	0.54157	σ	0.73853	σ	0.91792
		L	1.09068	L	2.67717	L	10.96491
m1= 19.997	Relative Error	0.11%		0.06%		0.02%	
m2= 1006.48	Upper (2 moments)	σ	0.68342	σ	0.78823	σ	0.92330
m3= 101021.195		L	1.57938	L	3.30547	L	11.73403
Region- I	Lower (2 moments)	σ	0.20319	σ	0.46700	σ	0.80690
		L	0.62750	L	1.31332	L	4.66080
	MRE (in L)	151.69%		151.69%		151.76%	
	Upper (3 moments)	σ	0.68342	σ	0.78823	σ	0.92330
		L	0.57938	L	3.30547	L	11.73400
	Lower (3 moments)	σ	0.42194	σ	0.71530	σ	0.91730
		L	0.86496	L	2.45873	L	10.88270
	MRE (in L)	82.60%		34.44%		7.82%	
Case-II	Exact	σ	0.77892	σ	0.89250	σ	0.96900
		L	2.261625	L	6.51163	L	29.03226
	Nonparametric method	σ	0.76586	σ	0.89113	σ	0.96901
		L	2.135475	L	6.42699	L	29.04163
	Relative Error	5.58%		1.30%		0.03%	
C^2= 5.821	Matching moments with H2 distribution	σ	0.76587	σ	0.89110	σ	0.96900
		L	2.13557	L	6.42792	L	29.03226
m1= 20.004	Relative Error	5.57%		1.29%		0.00%	
m2= 2729.4164	Upper (2 moments)	σ	0.88318	σ	0.92186	σ	0.97170
m3= 836216.8692		L	4.28009	L	8.95828	L	31.80212
Region- I	Lower (2 moments)	σ	0.20319	σ	0.46700	σ	0.80690
		L	0.62750	L	1.31332	L	4.66080
	MRE(in L)	582.08%		582.11%		582.33%	
	Upper (3 moments)	σ	0.88318	σ	0.19286	σ	0.97170
		L	4.28009	L	8.95828	L	31.80212
	Lower (3 moments)	σ	0.58466	σ	0.87540	σ	0.96880
		L	1.20383	L	5.61798	L	28.84615
	MRE(in L)	255.54%		59.46%		10.25%	

Table 3.2: Approximations for σ and L with Gamma Inter-arrival Distribution (Note that in Case II, a 3 Moment Fit of *H2* Distribution is Not Carried Out as $C^2 < 1$).

GAMMA		ρ=0.5		ρ=0.7		ρ=0.9	
Case-I	Exact	σ	0.73911	σ	0.85242	σ	0.95297
		L	1.91653	L	4.74319	L	19.13469
	Nonparametric method	σ	0.72789	σ	0.85082	σ	0.95292
		L	1.83749	L	4.69232	L	19.11640
	Relative Error	4.12%		1.07%		0.10%	
C^2= 3.3333	Matching moments with H2 distribution	σ	0.72793	σ	0.85086	σ	0.95295
		L	1.83776	L	4.69358	L	19.12859
	Relative Error	4.11%		1.05%		0.03%	
m1= 0.45	Upper (2 moments)	σ	0.81612	σ	0.87700	σ	0.95540
m2= 0.8775		L	2.71916	L	5.69106	L	20.17937
m3= 3.0273	Lower (2 moments)	σ	0.20319	σ	0.46700	σ	0.80690
Region - I		L	0.62750	L	1.31332	L	4.66080
	MRE(in L)	333.33%		333.33%		332.96%	
	Upper (3 moments)	σ	0.81612	σ	0.87700	σ	0.95540
		L	2.71916	L	5.69106	L	20.17937
	Lower (3 moments)	σ	0.63907	σ	0.84200	σ	0.95270
		L	1.38531	L	4.43038	L	19.02748
	MRE(in L)	96.29%		28.46%		6.05%	
Case-II	Exact	σ	0.49619	σ	0.69739	σ	0.89904
		L	0.99243	L	2.31317	L	8.91442
	Nonparametric method	σ	0.49629	σ	0.69740	σ	0.89905
		L	0.99264	L	2.31332	L	8.91530
	Relative Error	0.02%		0.01%		0.01%	
C^2= 0.9804	Upper (2 moments)	σ	0.59765	σ	0.73086	σ	0.90250
		L	1.24270	L	2.60088	L	9.23077
m1= 0.51	Lower (2 moments)	σ	0.20319	σ	0.46700	σ	0.80690
m2= 0.5151		L	0.62750	L	1.31332	L	4.66080
m3= 0.77801	MRE(in L)	98.04%		98.04%		98.05%	
Region - II	Upper (3 moments)	σ	0.59765	σ	0.73086	σ	0.90250
		L	1.24270	L	2.60088	L	9.23077
	Lower (3 moments)	σ	0.45049	σ	0.69026	σ	0.89885
		L	0.90990	L	2.25996	L	8.89768
	MRE(in L)	36.58%		15.09%		3.74%	

Eckberg (1977) obtained sharp bounds on the Laplace Stieltjes transform of the distribution function of a random variable in terms of various partial characterization of the random variable. The random variable itself was assumed to be defined over the interval (0, b] where b possibly could be infinite. In Table 3.3 we present the upper and lower bounds of σ [see equation (3.2)] obtained using Eckberg's sharp bounds as well as the value of σ using our approximation. We wish to observe that our approximation falls within the bounds of Eckberg (1977) and hence is an improvement over his results.

Table 3.3: Comparison of the values of σ using Eckberg's (1977) bounds and our approximation

Distribution moments				ρ = 0.667					ρ = 0.9		
				b = ∞		b = 10		Non-parametric method	b= ∞ b = 10		Non-parametric method
m1	**m2**	**m3**	**C^2**	σ_u	σ_l	σ_u	σ_l	**σ**	σ_u	σ_l	**σ**
0.4500	0.4275	0.6199	1.1111	0.7242	0.6710	0.7118	0.6710	0.68270	0.9085	0.9049	0.90530
0.3500	0.4720	1.1053	2.8543	0.8490	0.8000	0.8489	0.8000	0.81386	0.9499	0.9470	0.94726
0.5703	1.7556	8.6139	4.3986	0.8921	0.8646	0.8921	0.8646	0.87276	0.9642	0.9626	0.96272

3.3 Inter-arrival Distribution Approximation by Matching Moments

Approximating general distributions by phase type distributions are important in queuing theory because their structure leads to Markovian state description and consequently analytical tractability. Although several phase type distributions have been used in the literature, two distributions that are prime candidates for such approximations because of their simplicity and suitability are mixtures of two exponentials (H_2) and Coxian (K_2) distributions. In this section we take up the former for analysis as these distributions provide a fairly accurate match when the C^2 is large which is true for arrival distributions in a queuing system. Further, in some of the examples discussed by Whitt-III (1984), the maximum relative error (MRE) when two moments are fitted was found to be 200 percent while working with mixtures of exponentials reduced the MRE to 50 percent and also specifying the third moment reduced the MRE to 5 percent. Thus, a three moment match using H_2 distributions for inter-arrival distributions seem to provide useful results.

Using empirical study, Bere (1981) showed that when the service time distribution is approximated using the first two of its moments, the third moment has a considerable effect on the probability distribution of the number of customers in an M/G/1 queue if $C^2>1$. He also showed that the probability distribution of the number of customers and the average number of customers in $\lambda(n)$/G/1/N system are highly sensitive to the third moment of the service time distribution if $C^2>1$. Altiok (1985), in justifying the inclusion of third moment in matching refers to Bere's empirical work. Whitt (1982) empirically showed that the effect of the third moment on the average number in the system in a GI/G/1 queues becomes considerable as C^2 increases. If the service time distribution $C^2<1$, the impact of third moment is not significant. Since the present work deals with matching an H_2 distribution with three moments, we confine our attention to the range $C^2>1$ only.

It is well known that a necessary condition that three numbers m_1, m_2, and m_3 can be the first three moments of a distribution function F provided $m_1\geq 0$, $m_2/m_1^2\geq 1$, and $m_3/m_1^3\geq m_2^2/m_1^4$. Further, if the first three moments exist for a distribution F, then an H_2 distribution exists with these three moments if and only if in addition to these, the condition $m_3/m_1^3 \geq (3/2)\, m_2^2/m_1^4$ holds. These regions are clearly exhibited in Figure 3.1. The parameters p, μ_1, and μ_2 of H_2 [see (3.6)] when (m_1, m_2, m_3) falls in region-I where the three moments can be matched exactly are well documented. However, when these moments fall in region-II so that the three moments cannot be matched exactly, methods suggested in the literature are recipes in nature. Whitt (1982) suggests "if m_3 turns out to be too small when attempting an H_2-fit, one procedure is to replace m_3 by something slightly larger than $(3/2)\, m_2^2/m_1$". Altiok (1985) suggests the use of $m_3 = 3m_2^2/2m_1 + \varepsilon m_1^3$ but does not indicate how to calculate the perturbation parameter ε. In the following algorithm, we propose a goal programming procedure of matching the first two moments exactly and matching the third moment as close as possible in region-II. This procedure subsumes region-I as a particular case. It is to be noted that region-III is an infeasible region for the existence of (m_1, m_2, m_3).

Given the first three moments of a distribution function F, say m_1, m_2, and m_3 which are assumed to be finite, the parameters p, μ_1 and μ_2 of the H_2 distribution whose first three moments either match m_1, m_2 and m_3 exactly or match the first two moments exactly and matches the third moment as closely as possible in the sense of squared differences are given by the following algorithm.

Step 1: Find optimal p using the Golden section method [Rao (2009) and Kambo (1991)] to solve the following one dimension problem:

$$\underset{\frac{C^2-1}{C^2+1}<p<1}{MinZ =} \left\{ m_3 - 6m_1^3 \left[1 + \frac{3}{2}\left(C^2-1\right) + \left(\frac{C^2-1}{2}\right)^{3/2} \frac{2p-1}{\sqrt{p(1-p)}} \right] \right\}^2$$

Step 2: Using the value of p found in step1 compute:

$$\mu_1 = m_1 \left[1 - \sqrt{\frac{1-p}{2p}\left(C^2-1\right)} \right]$$

Step 3: Using the value of p found in step1 compute:

$$\mu_2 = m_1\left[1+\sqrt{\frac{p}{2(1-p)}\left(C^2-1\right)}\right]$$

The steps of the algorithm are justified using the following arguments.

Consider the following probability density function of mixture of two exponentials (H_2 distribution):

$$f(x) = p\frac{\exp\left(-x/\mu_1\right)}{\mu_1} + (1-p)\frac{\exp\left(-x/\mu_2\right)}{\mu_2}, x \geq 0,\ 0 \leq p \leq 1 \tag{3.6}$$

The first three moments of the above distribution function are:

$$m_1 = p\mu_1 + (1-p)\mu_2 \tag{3.7}$$

$$m_2 = 2p\mu_1^2 + 2(1-p)\mu_2^2 \tag{3.8}$$

$$m_3 = 6\left[p\mu_1^3 + (1-p)\mu_2^3\right] \tag{3.9}$$

From (3.7) we have

$$\mu_2 = \frac{m_1 - p\mu_1}{1-p} \tag{3.10}$$

Substituting (3.10) in (3.8) and after simple algebra

$$\mu_1 = m_1\left[1 \pm \sqrt{\frac{1-p}{2p}\left(C^2-1\right)}\right] \tag{3.11}$$

We have two cases. First we consider

$$\mu_1 = m_1\left[1 - \sqrt{\frac{1-p}{2p}\left(C^2-1\right)}\right] \tag{3.12}$$

We know that

$\frac{1-p}{2p}\left(C^2-1\right)\geq 0$ which implies that $C^2 \geq 1$

Also $\mu_1>0$ implies that $p>\frac{C^2-1}{C^2+1}$ (3.13)

By substituting (3.12) and (3.13) in (3.10) results in

$$\mu_2 = m_1\left[1+\sqrt{\frac{p}{2(1-p)}\left(C^2-1\right)}\right] \tag{3.14}$$

The third parameter *p* is obtained by matching the third moment as close as possible. Thus, we minimize

$$\left\{m_3 - 6m_1^3\left[1+\frac{3}{2}\left(C^2-1\right)+\left(\frac{C^2-1}{2}\right)^{3/2}\frac{2p-1}{\sqrt{p(1-p)}}\right]\right\}^2 \tag{3.15}$$

subject to: $1>p>\frac{C^2-1}{C^2+1}$

In the second case, we set

$$\mu_1 = m_1\left[1+\sqrt{\frac{1-p}{2p}\left(C^2-1\right)}\right] \tag{3.16}$$

Using algebra similar to the first case results in

$$\mu_2 = m_1\left[1-\sqrt{\frac{p}{2(1-p)}\left(C^2-1\right)}\right] \tag{3.17}$$

and the third parameter p is determined by solving

$$\underset{\frac{2}{C^2+1}>p>0}{MinZ=} \left\{m_3 - 6m_1^3\left[1+\frac{3}{2}\left(C^2-1\right)+\left(\frac{C^2-1}{2}\right)^{3/2}\frac{2p-1}{\sqrt{p(1-p)}}\right]\right\}^2 \tag{3.18}$$

It is easily seen that cases 1 and 2 lead to the same result but with the roles of p and $1-p$ interchanged.

In Tables 3.1 and 3.2, we continue with our *PH4* and gamma inter-arrival distributions introduced in section 3.2. As illustrations, we fit H_2 distributions for these two distributions by matching the moments specified by the algorithm. The performance measures σ and L obtained by using the fitted H_2 distribution in (3.1), (3.2), and (3.3) are given in Tables 3.1 and 3.2. Attention is drawn to the MRE for these measures when three moments are matched [Whitt-I (1984)] and the actual relative error in using the approximated H_2 distributions. Extensive calculations show that when relative error is used to compare, fitted H_2 distributions provide slightly better approximations than the non-parametric method for increasing C^2. We also wish to point out that these two methods improve in accuracy for increasing ρ as they should be.

3.4 Two Optimization Illustrations

3.4.1 Illustration 1

In practice, queuing managers are generally interested in optimizing for the model parameters under their control by minimizing the operating cost or maximizing the business profit. In the first illustration, we will be interested in obtaining the optimal service rate in a cost minimization problem for a GI/M/1 queuing system. The objective cost function consists of two components, which are the cost due to customers waiting in line known as the delay cost, and the service cost rate. Thus, the cost function to be minimized is given by:

$$C(\mu) = c_1 (\lambda W) + c_2 \mu \quad (3.19)$$

where λ and μ are the arrival and service rates respectively, W is the expected waiting time of a customer in the system, c_1 is the expected cost per unit time of a customer's wait and c_2 is the service cost rate. Using Little's formula, (3.19) reduces to

$$C(\mu) = c_1 L + c_2 \mu \tag{3.20}$$

The optimal μ* of the above objective function was computed using our non-parametric method introduced in section 3.2 by assuming only the first three moments of the arrival distribution and the cost rates. However in order to compare our results with the exact values, the moments were chosen so as to correspond to Coxian (K_2) and Inverse Gaussian distributions commonly used in queuing theory. The results are presented in Tables 3.4 and 3.5. When the Coxian arrival distribution was used, our method provided the exact values of μ* whereas in the case of Inverse Gaussian distribution the relative errors were significantly small.

Table 3.4: The Optimal Service Rate μ* with Coxian Inter-arrival Distribution (The Optimal μ* Computed Using Our Method and Using the Coxian Distribution Exactly Match).

$f(t)=\frac{p\lambda_1-\lambda_2}{\lambda_1-\lambda_2}\lambda_1 e^{-\lambda_1 t}+\left(1-\frac{p\lambda_1-\lambda_2}{\lambda_1-\lambda_2}\right)\lambda_2 e^{-\lambda_2 t}$					P=0.8, λ_1=2, λ_2=0.2 (m_1=1.5, m_2=11.5, m_3=167.25 and s_0=-0.6, A=0.6667, B=0.9333)			
C(μ)								
Approximation (Non-parametric)	μ=2.22	μ=1.667	μ=1.333	μ=1.111	μ=0.952	μ=0.833	μ=0.741	
	22.9	17.87	15.33	14.35	14.86	17.91	29.74	c_1=1, c_2=10
	11.79	9.54	8.667	8.793	10.1	13.74	26.04	c_1=1, c_2=5
	5.127	4.54	4.667	5.459	7.242	11.24	23.81	c_1=1, c_2=2

Table 3.5: The Optimal Service Rate μ* with Inverse Gaussian Inter-arrival Distribution

$f(t)=\left[\frac{L}{2\pi t^{3}}\right]^{1/2} e^{\frac{-L(t-M)^{2}}{2M^{2}t}}$	L=1, M=2 (m₁=2, m₂=12, m₃=152 and s₀=-0.273, A=0.5, B=0.136)							
C(μ)								
	μ=1.667	**μ=1.25**	**μ=1.00**	**μ=0.833**	**μ=0.7143**	**μ=0.625**	**μ=0.5556**	
Actual	17.126	13.255	11.193	10.215	***10.213***	11.76	18.505	$c_1=1, c_2=10$
	8.793	7.005	6.193	***6.048***	6.642	8.635	15.727	$c_1=1, c_2=5$
	3.793	3.255	***3.193***	3.548	4.499	6.76	14.061	$c_1=1, c_2=2$
Approximation (Non-parametric)	17.139	13.265	11.198	10.213	***10.206***	11.75	18.505	$c_1=1, c_2=10$
	8.806	7.015	6.198	***6.046***	6.635	8.625	15.727	$c_1=1, c_2=5$
	3.806	3.265	***3.198***	3.546	4.492	6.75	14.061	$c_1=1, c_2=2$
	Relative Error =		0.16%	0.03%	0.07%			

3.4.2 Illustration 2

In the second illustration, we consider an optimization problem in a GI/M/1 queue with working vacation for the server. These problems have wide applications in internet systems such as optical, electrical and communication nets [Li *et al* (2008)]. We consider a single server queue system, which has the general arrival process. The working vacation and vacation interruption are connected and the server enters into vacation when there are no customers and he can take service at the lower rate during the vacation period. If there are customers in the system at the instant of a service completion during the vacation period, the server will come back to the normal working level no matter whether the vacation ends. Otherwise, he continues the vacation. The performance measures L, the mean queue length and $P(J=0)$ and $P(J=1)$ which are the state probabilities of a server in the steady state have been derived by Li *et al* (2008). We refer the reader to their paper for the relevant expressions. Li et al (2008) considered the problem of optimizing the service rate η during the server's vacation period for a given cost structure. Let c_w represent the unit time cost of every waiting customer, and c_1 and c_2 are the service costs per unit time during the normal working level and vacation period, respectively. The expected net cost function to be optimized can be seen to be

min: $Z = c_w L + c_1 \mu P(J=1) + c_2 \eta P(J=0)$ (3.21)

where μ is the service rate during the service period. The optimal service rate η^* was computed using our non-parametric method of section 3.2 for certain values of the model parameters and cost parameters. We have used the Coxian arrival distribution and its moments used in Illustration 1 to obtain the optimal service rate η^*. However, the Inverse Gaussian distribution could not be used as the objective function (3.21) loses its convexity and becomes monotonic. We have used Erlangean of order 2 [used by Li *et al* (2008)] in its place. Figures 3.2 and 3.3 present the values of η versus the associates cost. The optimal η^* and the corresponding cost obtained using our method are very close to the exact values.

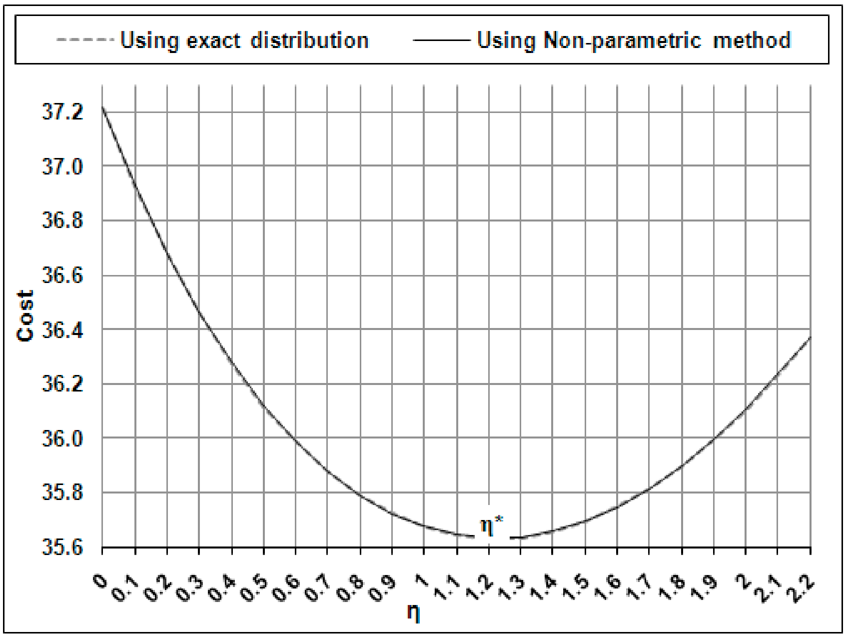

Figure 3.2: (c_w=4, c_1=15, c_2=10, Θ=1, ρ=0.65, and Coxian Distribution Parameters are p=0.8, λ_1=2, λ_2=0.2) Optimal Service Rate η^* during Servers Vacation Period with Coxian Inter-Arrival Distribution.

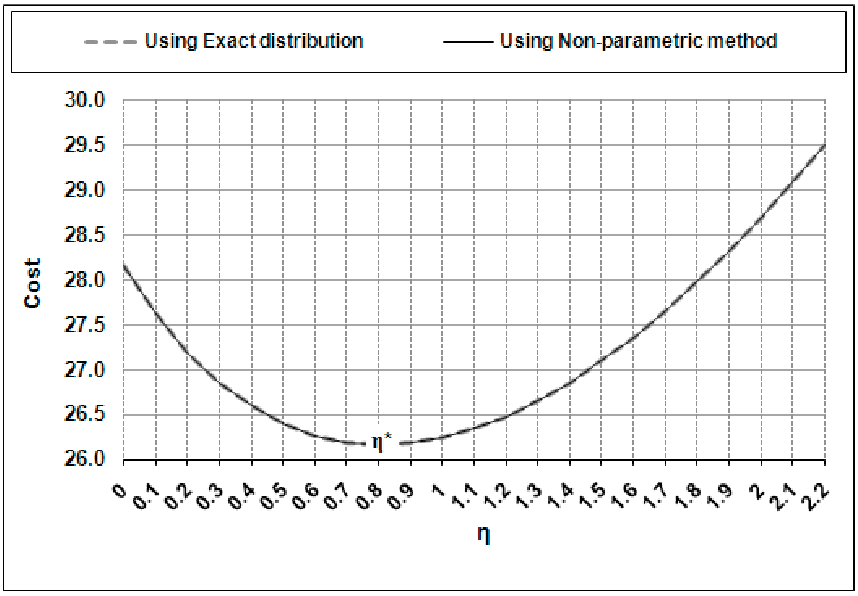

Figure 3.3: (c_w=4, c_1=15, c_2=10, Θ=1, ρ=0.65, and Erlangean Distribution Parameters are K=2, L=2.5) Optimal Service Rate η* during Servers Vacation Period with Erlangean of Order 2 Inter-Arrival Distribution.

3.5 Conclusions

This chapter introduces two methods for the evaluation of performance measures in a GI/M/1 queuing system in the absence of information on the arrival distribution and when only the first three moments are known. The first method is non-parametric as it does not use the distribution function whereas the second method uses an H_2 distribution obtained by moment matching procedure. This procedure involves the computationally economical Golden section method. We wish to note that a Coxian (K_2) distribution is also a good phase type distribution for consideration. It is worth pursuing the regions (Φ_2, Φ_3) in which each of these approximations score over the others in terms of relative errors. The usefulness of the methods in optimization procedures has been illustrated with examples.

Chapter 4

APPROXIMATIONS TO G-RENEWAL FUNCTIONS

4.1 Introduction

Renewal function is a key tool used in many applications like reliability, queuing theory, inventory theory and financial engineering. However, researchers have been hampered by the non-availability of a closed form expression for the solution of the renewal equation. In this regard the literature has extended in several directions which includes Laplace transform methods, approximations, power series methods, Riemann-Stieltjes integration methods, Pade' approximations and bounds. A similar function known as g-renewal function, first introduced by Kijima and Sumita (1986) has been found to be extremely useful in reliability analysis. Maintenance of systems on failure can be broadly classified into one of the following: Replace on failure in which case the failures can be modeled as a renewal process. Minimally repair the system so that it is restored to the condition that it was in just prior to failure. In this case, failures can be represented by a non-homogeneous Poisson process whose intensity function is the failure rate. The most interesting possibility is to perform a 'general repair'. The system after a general repair is returned neither to a good as new state nor bad as old state but to an intermediate state, which is better than old but worse than new. General repairs encompass replacement and minimal repair as special cases.

Let the distribution function and density function of the failure time of a new system be $F(t)$ and $f(t)$ respectively and the failure counting process be $\{N(t); t \geq 0\}$. The expected number of failures observed in the interval $(0,t)$ is denoted by $M(t) = E[N(t)]$. It is well known that when systems are replaced on failures, the renewal function can be obtained as the solution of the renewal equation

$$M(t) = F(t) + \int_0^t M(t-x)dF(x) \tag{4.1}$$

Analytical solutions of the above integral equation are known only for exponential and gamma distributions whereas various approximations are available in the literature (Deligonul (1985), Xie (1989), Marshall (1973), and Kambo *et al* (2011)). When systems on failure are maintained by a general repair policy, Kijima *et al* (1988) and Kijima (1989) showed that the corresponding renewal function known as g-renewal function is given as the solution of the g-renewal equation

$$M_g(t) = Q(t \mid 0) + \int_0^t Q(t-x \mid x) m_g(x) dx \tag{4.2}$$

where $Q(t \mid x) = \int_0^t q(y \mid x) dy$ (4.3)

and $q(t \mid x) = \dfrac{f(t+\theta x)}{1 - F(\theta x)}, t, x \geq 0$ (4.4)

The function $m_g(t)$ is the corresponding g-renewal density, which satisfies the equation

$$m_g(t) = q(t \mid 0) + \int_0^t m_g(x) q(t-x \mid x) dx \tag{4.5}$$

θ is known as the parameter of degree of repair (parameter of rejuvenation). It is readily seen that if $\theta = 0$ the rejuvenation is perfect so that the general repair model coincides with the replacement model. On the other hand if $\theta = 1$ the present model collapses to the minimal repair model. The closed form solution of the g-renewal equation given by (4.2) is not available for the case $0 < \theta < 1$. The methods of Laplace transform and power series expansion fail because of the kernel $q(t \mid x)$. Numerical solutions are difficult to obtain since the equation contains a recurrent infinite system. In view of the importance of the g-renewal functions and the unavailability of useful methods in obtaining them, it seems worthwhile to look for approximations to g-renewal functions, which are simple to execute and yield fairly accurate results. Matis *et al* (2008) have considered a repairable product under a non-renewing combined warranty policy that is subject to a displaced log linear demand function of the product price and pro-rata period length. They derived expressions for the manufacturer's long-run average profit per unit time under replacement, minimal and general repairs that involves numerical computation of the g-renewal function. Dampener (1997) has considered renewal type equations for a general repair process by generalizing Kijima's general repair model. Dimitrakos and Kyriakidis (2007) proposed an improved algorithm for the computation of the optimal repair/replacement policy under general repairs based on the embedding technique of Tijms (1994). Kaminiskiy and Krivstov (2000 and 2010) have discussed the properties and statistical procedures for the estimation of the process parameters of a g-renewal process. Simple bounds on the cumulative intensity functions of g-renewal processes were provided by Kaminiskiy (2004). In this chapter we propose two methods for obtaining approximations to the g-renewal function. The first method is based on a simple approximation and is very easy to execute, but provides results with acceptable accuracy. The second is a method of successive approximation based on direct numerical Riemann integration requiring some computational efforts.

We also use an optimal replacement problem of a system subject to failures and general repairs as an illustration of the proposed methods. Optimal replacement time and the optimal cost are computed numerically with the proposed methods and compared with the exact values.

4.2 Approximations to the G-renewal Function

The g-renewal densities and the corresponding renewal functions were first encountered in the analysis of systems, which are maintained on failure by general repairs. More specifically it was assumed that if a system of age t fails, then a general repair reduces the age of the system to θt. If $0 \leq \theta \leq 1$, the system is rejuvenated by the repair whereas if $\theta > 1$ the repair damages the system more. If $m_g(t)$ is the renewal density of system failures then it can be established that

$$m_g(t) = f(t) + \int_0^t m_g(x) \frac{f(t - x + \theta x)}{1 - F(\theta x)} dx \tag{4.6}$$

where $f(t)$ is the failure time density of a new system. Equation (4.6) can be interpreted in the following way: the failure at time t is the first failure (which corresponds to the first term) or a subsequent failure (in this case, the last failure before t occurs at x and the general repair rejuvenation reduces the age to θx followed by a failure at time t) given by the second term.

4.2.1 A Simple Approximation

By integrating the g-renewal equation, we obtain equation (4.2) satisfied by the renewal function $M_g(t)$. The following identity then holds

$$\frac{Q(t \mid 0)}{\int_0^t \overline{Q}(t-x \mid x) m_g(x) dx} = 1 \tag{4.7}$$

where $Q(t|x)$ was given in (4.3) and $\bar{Q}(t|x) = 1 - Q(t|x)$. Using (4.7) in the g-renewal density equation (4.5) we obtain

$$m_g(t) = q(t \mid 0) + \frac{Q(t \mid 0) \int_0^t q(t-x \mid x) m_g(x) dx}{\int_0^t \overline{Q}(t-x \mid x) m_g(x) dx} \tag{4.8}$$

Note that $m_g(t)$ tends to $1/\mu$ as t tends to infinity, where μ is the mean of the distribution function $F(t)$. By ignoring the path $m_g(t)$ for finite values of t so that $m_g(t) = 1/\mu$ for all t, we obtain our approximation for $m_g(t)$ as

$$m_g(t) = q(t \mid 0) + \frac{Q(t \mid 0) \int_0^t q(t-x \mid x) dx}{\int_0^t \overline{Q}(t-x \mid x) dx} \tag{4.9}$$

Integrating both sides

$$M_g(t) = Q(t \mid 0) + \int_0^t \frac{Q(u \mid 0) \int_0^u q(u-x \mid x) dx}{\int_0^u \overline{Q}(u-x \mid x) dx} du \tag{4.10}$$

The right hand side of (4.10) gives us an approximation for the renewal function as it contains only terms of the distribution function $F(t)$ and density function $f(t)$. It can be easily seen that if the failure time distribution is exponential so that $q(t-x \mid x) = \lambda e^{-\lambda(t-x)}$ then it is easy to see from (4.10) that $M_g(t) = \lambda t$.

In order to check the accuracy of approximation (4.10), we use the following conditional density function as an illustration:

$$q(t-x \mid x) = ae^{-a(t+x)} + b\left(1-e^{-ax}\right)e^{-bt}, \ a,b>0, \ t,x \geq 0 \tag{4.11}$$

This choice of the conditional density function is motivated by the fact that for such a choice of the kernel, the exact solution of the g-renewal equation is known.

In Table 4.1 we present the values of the renewal function for various values of t using our approximation (4.10) and compare it with the exact values of the same. The parameters a and b of (4.11) have been chosen to be 1 and 2 respectively. It is to be noted that the relative errors of the approximation do not exceed 1.32%.

4.2.2 Method of Successive Approximation Based on Riemann Integrals

We start with equation (4.2) for the renewal function. Partition the interval $(0,t)$ into n equal sub intervals (t_{i-1},t_i), $1 \leq i \leq n$ of length Δ. Then equation (4.2) can be written as

$$M_g(t_n) = Q(t_n \mid 0) + \int_0^{t_n} Q(t_n - x \mid x) m_g(x) dx$$

$$= Q(t_n \mid 0) + \int_0^{t_{n-1}} Q(t_n - x \mid x) m_g(x) dx + \int_{t_{n-1}}^{t_n} Q(t_n - x \mid x) m_g(x) dx \tag{4.12}$$

Using the approximations

$$Q(t_n - x \mid x) = Q(t_{n-1} + \Delta - x \mid x) = Q(t_{n-1} - x \mid x) + \Delta \frac{\partial Q(t \mid x)}{\partial t}\Big|_{t=t_{n-1}-x} + o(\Delta)$$

and $\int_a^{a+\Delta} f(t)dt \approx \Delta f(a)$ in (4.12), we obtain

$$M_g(t_n) = Q(t_n \mid 0) + \int_0^{t_{n-1}} \left[Q(t_{n-1} - x \mid x) + \Delta \frac{\partial Q(t_{n-1} - x \mid x)}{\partial t} \right] m_g(x) dx + \int_{t_{n-1}}^{t_n} Q(t_n - x \mid x) m_g(x) dx$$

$$= Q(t_n \mid 0) + \int_0^{t_{n-1}} \left[Q(t_{n-1} - x \mid x) + \Delta q(t_{n-1} - x \mid x) \right] m_g(x) dx + \Delta Q(\Delta \mid t_{n-1}) m_g(t_{n-1})$$

(4.13)

The renewal function $M_g(t)$ at $t = t_{n-1}$ is obtained from (4.2) as

$$M_g(t_{n-1}) = Q(t_{n-1} \mid 0) + \int_0^{t_{n-1}} Q(t_{n-1} - x \mid x) m_g(x) dx \tag{4.14}$$

Subtracting (4.14) from (4.13)

$$M_g(t_n) = M_g(t_{n-1}) + [Q(t_n \mid 0) - Q(t_{n-1} \mid 0)] + \int_0^{t_{n-1}} \Delta q(t_{n-1} - x \mid x) m_g(x) dx + \Delta Q(\Delta \mid t_{n-1}) m_g(t_{n-1})$$

$$= M_g(t_{n-1}) + [Q(t_n \mid 0) - Q(t_{n-1} \mid 0)] + \sum_{r=1}^{n-1} \int_{t_{r-1}}^{t_r} \Delta q(t_{n-1} - x \mid x) m_g(x) dx + \Delta Q(\Delta \mid t_{n-1}) m_g(t_{n-1})$$

$$= M_g(t_{n-1}) + [Q(t_n \mid 0) - Q(t_{n-1} \mid 0)] + \sum_{r=1}^{n-1} \Delta^2 m_g(t_{r-1}) q(t_{n-1} - t_{r-1} \mid t_{r-1}) + \Delta Q(\Delta \mid t_{n-1}) m_g(t_{n-1})$$

(4.15)

Substituting for $m_g(t_{n-1}) = \dfrac{M_g(t_n) - M_g(t_{n-1})}{\Delta}$ in (4.15) and rearranging the terms, we obtain the required approximation as

$$M_g(t_n) = M_g(t_{n-1}) + \frac{[Q(t_n \mid 0) - Q(t_{n-1} \mid 0)] + \Delta \sum_{r=1}^{n-1} q(t_{n-1} - t_{r-1} \mid t_{r-1})[M_g(t_r) - M_g(t_{r-1})]}{1 - Q(\Delta \mid t_{n-1})}$$

(4.16)

We wish to observe that the error in the computation of the renewal function in each of the subintervals is of $o(\Delta)$. However, for large values of t, the approximation sums over more subintervals leading to the accumulation of the errors. As an illustration, we use the conditional density function given in (4.11) to compute the values of the g-renewal function using the method of successive approximation given in (4.16).

We observe that the relative error is an increasing function of time and that for small and medium values of time the accuracy of the approximation is good.

Table 4.1: Values of the Renewal Function Using the Two Approximations

Time	Exact Value	Simple approximation		Successive approximation	
		Approximate Value	Relative Error	Approximate Value	Relative Error
0.1	0.1002	0.1002	0.00%	0.1001	0.10%
0.3	0.3037	0.3036	0.03%	0.3034	0.10%
0.5	0.515	0.5149	0.02%	0.5145	0.10%
0.7	0.7368	0.7362	0.08%	0.736	0.11%
0.9	0.9704	0.9689	0.15%	0.9692	0.12%
1.1	1.2165	1.2134	0.25%	1.2149	0.13%
1.5	1.7465	1.7377	0.50%	1.7438	0.15%
2.5	3.2722	3.2363	1.10%	3.2646	0.23%
3.5	5.0135	4.9471	1.32%	4.9975	0.32%
4.5	6.884	6.7967	1.27%	6.8558	0.41%
5.5	8.8229	8.7249	1.11%	8.7787	0.50%
7.5	12.7834	12.6787	0.82%	12.695	0.69%
10	17.7756	17.6698	0.60%	17.6097	0.93%

4.3 An Illustration

Consider a system, which is subject to failures. When the system fails at an age say x, a general repair activity, which reduces the age of the system to θx, $0<\theta<1$ is carried out and this activity requires negligible time. The system is periodically replaced by a new one after every T time units. Let the replacement and general repair costs be C_0 and C_1 respectively. It is easy to see that the long-run expected cost $C(T)$ per unit time is given by

$$C(T)=\frac{C_0+C_1M_g(T)}{T},T>0 \tag{4.17}$$

and the optimal T^* is such that $m_g(T^*)*T^*-M_g(T^*)=\frac{C_0}{C_1}$.

Let us use conditional failure time density function given by (4.11) so that $q(t\,|\,0) = ae^{-at}$ and $Q(t\,|\,0) = 1 - e^{-at}$.

In Table 4.2, we provide the optimal solution T^* and $C(T^*)$ corresponding to (4.17) for various values of the ratio C_0/C_1. In the computation of the optimal values, the renewal function values $M_g(t)$ were obtained using the two proposed approximations. While the cost of replacement C_0 was fixed at 2, the cost of general repair C_1 was allowed to increase. Such a choice has an effect of decreasing the optimal replacement time T^* and accordingly increase the long-run expected cost $C(T^*)$. We observe that both the methods perform satisfactorily in the determination of optimal T^* and $C(T^*)$. In particular the method of successive approximation works well for smaller values of C_0/C_1.

Table 4.2: Optimal Replacement Time T^* and Optimal Cost $C(T^*)$ (a=1, b=2, C_0=2)

Cost Ratio	Exact Values		Simple Approximation			Successive Approximation		
C_0/C_1	T^*	$C(T^*)$	$\check{T}^*$	$C(\check{T}^*)$	Relative error $\Delta C(T)$	$\check{T}^*$	$C(\check{T}^*)$	Relative error $\Delta C(T)$
2.00	5.77	1.9676	5.43	1.9679	0.015%	6.01	1.9677	0.005%
1.80	4.78	2.1436	4.68	2.1437	0.005%	4.86	2.1436	0.002%
1.60	4.13	2.3551	4.13	2.3551	0.000%	4.17	2.3551	0.001%
1.33	3.46	2.7200	3.51	2.7201	0.004%	3.48	2.7200	0.000%
1.00	2.78	3.4116	2.86	3.4120	0.012%	2.79	3.4116	0.000%

4.4 Conclusions

In this chapter, we have proposed two approximations for the evaluation of the g-renewal function. In view of the fact that the g-renewal equation is not amenable for analytical solution and the relevance of the g-renewal function in the maintenance of deteriorating systems, these methods assume significance. The first approximation is simple to execute and the accuracy increases for increasing value of t. The second method based on the successive approximation is especially useful for smaller values of t and applicable when the underlying distribution is highly skewed. It is worth investigating the type of the distribution functions for which each of the two methods can be usefully applied.

Chapter 5

OPTIMAL SYSTEM DESIGN BASED ON BURN-IN, WARRANTY, AND MAINTENANCE

5.1 Introduction

Modern technology and industrial growth have resulted in the introduction of newer products at an increasing pace. Consumers focus on product characteristics such as functionality, reliability, and maintainability. Manufacturers on their part have responded in terms of product development and testing as well as warranty services. Thus a true picture of the design and manufacturing decisions emerges by considering not only the design and manufacturing costs but also the costs of operation and maintenance support for the product.

Failure rates of many systems, which play an important role in system design, exhibit a bathtub shape. Bathtub failure curves based on failure rate functions show three distinct phases namely, infant mortality period, useful period, and wear out period, lasting till the end of the product life cycle. In the first phase where the failure rate is monotonically decreasing, failures occur mainly due to defects in design and material or poor manufacturing quality. This period is known as infant mortality or burn-in period. From a customer satisfaction point of view, infant mortalities are unacceptable. To reduce possible damage from such early failures, manufacturers use a burn-in procedure, which is carried out under electrical or thermal conditions that approximates the working conditions in field conditions (Cha, (2001)). This is because with an appropriate debugging process, early failures could be reduced/ avoided. Kwon *et al* (2010) in a recent paper have considered the optimal burn-in of systems with random minimal repair cost. The costs associated with this period are borne by the manufacturer and include the replacement cost due to burn-in failure and operating cost of burn-in.

The second phase is known as the useful period during which the failure rate remains approximately a constant. Failures in this phase are usually associated with operator and/or fluctuations in operating environment. In order to gain the maximum out of this period where the failure rate is nearly a constant, the manufacturer while launching the sale of the product offers a warranty period, which overlaps with the second phase. Warranty contracts are an integral part of product sales and play a crucial role in the manufacturer's profit and customer satisfaction and have become a critical segment of the industrial environment. A warranty is a contractual obligation offered by the manufacturer in connection with the sale of a product under which the manufacturer is required to repair/replace failed items during the warranty period. Blischke and Murthy (1994) have analyzed various warranty strategies and their cost implications. Warranty policies can be broadly classified as renewing or non-renewing. In a renewing warranty policy, whenever a product under warranty fails, it is replaced by a new item along with a new warranty replacing the old warranty. Simple examples of renewing warranty are batteries and tires. Mi (1999) made a comparison of different renewable warranty policies. However in the case of non-renewing policies, replacement of failed items does not alter the original warranty. The study of product warranty cost is important to both manufacturers and customers, although their cost perspectives are different. The main cost to the manufacturer during this period is the cost of replacement of failed items. Warranty servicing with imperfect repairs of failed items have been considered by Yun *et al* (2008) and Yeo and Yuan (2009).Models based on selecting warranty conditions with the objective of the profit maximization have been investigated by Glickman and Berger (1976).

The third phase in which the failure rate is monotonically increasing is also known as the post warranty period. During this phase, the maintenance of the system on failures wrests with the consumer who has the option of several maintenance policies. Prominent amongst these policies are replacement on failures, minimal repairs, and general repairs. Replacement on failures means that the product is as good as new after the maintenance action, rendering the failure counting process as a renewal process. Although replacements reduce the number of failures in the long run, these actions many a times may be neither warranted nor possible. For instance in a multi component series system, it may not be necessary to replace the system on the failure of a component whereas it may be reasonable to replace only the failed component. Minimal repair maintenance action restores the system to its working condition just prior to failure. Mathematically, consider a system with failure time density $f(t)$. Let the system fail at time point s, and a minimal repair action is performed on failure. The revised failure time density of the system will be $f(t+s)/\bar{F}(s)$ where $\bar{F}(s)$ is the survivor function. While replacement and minimal repair are two extremes of maintenance policies, a more realistic maintenance policy in the form of general repairs was introduced by Kijima (1989). After a general repair maintenance action of a system with age s, the age of the repaired system is neither zero (replacement) nor s (minimal repair) but an age in between zero and s. The consumer may employ any one of the three maintenance policies or a combination thereof. Monga and Zuo (1998) considered optimal system design using a combination of minimal repairs on failures and general repairs as preventive maintenance actions. The main cost for the consumer in this phase is the cost of corrective maintenance actions and preventive maintenance actions if there are any.

Existing models on burn-in, warranty and maintenance have been developed purely from the manufacturer or consumer's perspective. In other words, models have been proposed which take in to account the burn-in and/or warranty period (manufacturer) or the post warranty period dealing with maintenance policies (consumer). Thus, there is a gap in the literature for models using the entire domain of the failure curve. If an analysis is to be performed for the whole bathtub failure rate curve then the periods of burn-in and warranty have a significant effect on the system cost during the post warranty period. Thus in this chapter we propose a system design optimization model which minimizes the total system cost incurred by the manufacturer as well as the consumer over the entire lifecycle of the system. In section 5.2, the model is presented and the costs under the three phases of the failure rate curve are derived. In section 5.3, numerical examples of the optimization procedures for a specific bathtub failure rate curve are explained and interesting observations made. Concluding remarks are made in section 5.4.

5.2 The Model

We consider a parallel system consisting of m independently and identically distributed components with failure density $f(t)$ and bathtub failure rate curve $h(t)$. The system is put through a burn-in process by the manufacturer lasting a burn-in period b. Burn-in is a method, which is used to filter out defective systems before they reach the market. During the procedure, the systems are tested under conditions identical to that of field operation. Whenever the system fails during the burn-in period, it is discarded and a new system is taken for testing. Only those systems, which successfully survive the burn-in procedure, are considered to be of quality to reach the consumers.

5.2.1 Cost during the Burn-in Period $(0, b]$

The relevant costs during this period are the purchasing cost of each component c_0, the cost of installation of the component c_1, and the operating cost rate of each component c_2 and system replacement cost c_3 during the burn-in period. Let $N(b)-1$ be the number of system replacements before the successful completion of the burn-in process by a system. Then the total expected cost during b is given by

$$C_b = m(c_0 + c_1)N(b) + mc_2\left(\sum_{i=1}^{N(b)-1} x_i + b\right) + c_3\left(N(b)-1\right) \tag{5.1}$$

where x_i is the life time of i^{th} unsuccessful system which fails before completing burn-in period. From the definition of the burn-in procedure, we note that $N(b)$ has a geometric distribution given by $P\left(N(b)=k\right) = \overline{F}s(b)Fs(b)^{k-1}$ (5.2)

where $Fs(b)$ and $\overline{F}s(b)$ are the distribution and survivor functions of the parallel system. Thus $E[N(b)]$ is given by $E[N(b)] = \dfrac{1}{\overline{F}s(b)}$ (5.3)

Also using Wald's identity we have

$$E\left(\sum_{i=1}^{N(b)-1} x_i\right) = E[N(b)]E(x_1) - E\left(x_{N(b)}\right) = \frac{\int_0^b \overline{F}s(t)dt}{\overline{F}s(b)} - b \tag{5.4}$$

$$\text{Thus, } E(C_b) = \frac{m(c_0 + c_1)}{\overline{F}s(b)} + m\, c_2 \frac{\int_0^b \overline{F}s(t)dt}{\overline{F}s(b)} + c_3 \frac{Fs(b)}{\overline{F}s(b)} \tag{5.5}$$

The successful system, which survives the burn-in, is sold in the market with a renewing warranty offer of period W. This implies that systems which fail during the warranty period W will be replaced with burnt-in systems with a new warranty of W. Thus for a system to survive during the renewing warranty, should survive a period of length W. The reader's attention is drawn to the analogy between renewing warranty and Type-II counters in counter models.

5.2.2 Cost during the Warranty Period $(b, \ b+W\,]$

The relevant costs during the warranty period are C_b which is the cost of a burnt-in system and c_4 the cost of installation of the burnt-in system in the customer's place during the warranty period. Let $N_b(W)-1$ be the number of burnt-in system replacements during the renewing warranty period before the successful completion of the warranty. Then the total expected warranty cost is given by

$$C_W = (c_4 + C_b)E[N_b(W) - 1] \tag{5.6}$$

Using an argument similar to the one used during the burn-in period, we see that

$$E[N_b(W) - 1] = \frac{Fs_b(W)}{\overline{F}s_b(W)} \tag{5.7}$$

where $\overline{F}s_b(W)$ is the survivor function of a burnt-in system with

$$Fs_b(W) = \int_0^W fs_b(t)dt = \int_0^W \frac{fs(b+t)}{\overline{F}s(b)}dt = \frac{\int_b^{b+W} fs(t)dt}{\overline{F}s(b)} \tag{5.8}$$

and $$\overline{F}s_b(W) = \frac{\overline{F}s(b+W)}{\overline{F}s(b)} = \frac{\int_{b+W}^{\infty} fs(t)dt}{\overline{F}s(b)} \tag{5.9}$$

We finally obtain

$$E(C_W) = (c_4 + C_b)\frac{Fs_b(W)}{\overline{F}s_b(W)} \tag{5.10}$$

On successful completion of the burn-in and warranty period, the system whose age is $b+W$, is operated during its useful life T until it is withdrawn from the market. During the post warranty period, the system maintenance and its costs are borne by the consumer.

5.2.3 Cost during Post Warranty Period ($b+W$, $b+W+T$]

The relevant cost during the post warranty period is the cost of a maintenance action on system failure. We observe that the consumer has a choice as regards the corrective maintenance action that can be in the form of replacements, minimal repair and general repair or a combination of them. To enable the consumer to make the optimal choice in terms of cost considerations, we consider each of the above three maintenance actions separately.

Case-I: Minimal repair

In this case, the consumer opts to minimally repair failed systems. Minimal repairs restore the failed system to the condition that it was in just prior to failure. Mathematically if a system with failure distribution $F(t)$ is minimally repaired at time t, then its failure rate after repair is given by $h(t) = f(t)/\overline{F}(t)$. Also it is well known that if system failures are maintained by minimal repairs only, then the number of failures is governed by a non-homogeneous Poisson process with intensity function $h(t)$. The expected number of failures in an arbitrary time $(0,t]$ in this case is given by

$$M(t) = \int_0^t h(x)dx \tag{5.11}$$

Thus, the total expected post warranty cost under minimal repair strategy is given by

$$c_5\left[M(b+W+T)-M(b+W)\right] \tag{5.12}$$

Case-II: Replacement

In this case, the consumer opts to replace the failed system with a new burnt-in system. We know from classical renewal theory that if systems are replaced on failure by new systems then $N(t)$, the number of failures in an arbitrary interval $(0,t]$ forms a renewal counting process. In our case, the system in use at time $b+W$ is of age $b+W$ whereas the burnt-in systems used for replacement are of age b. Thus, the number of failures in $(b+W, b+W+T]$ denoted by $N_D(T)$ forms a modified renewal counting process (See Ross (1996)). It is well known that the renewal function $M_D(T)=E[N_D(T)]$ satisfies the renewal equation

$$M_D(T)=G(T)+\int_0^T M(T-x)dG(x) \tag{5.13}$$

In our case $G(t)$ is specified by

$$G(t)=F_{b+W}(t)=\frac{\left[F(b+W+t)-F(b+W)\right]}{\overline{F}_{b+W}(t)} \tag{5.14}$$

Explicit expression for the analytical solution of equation (5.13) is not possible except in very few cases like exponential and gamma distribution functions. However, good approximations are available for the evaluation of $M_D(t)$ (Deligonul (1985), Xie (1989), Kambo *et al* (2011)). Thus, the total expected post warranty cost under replacement strategy is given by

$$(c_6+C_b)[M_D(T)] \tag{5.15}$$

Case-III: General repair

The most interesting maintenance action which subsumes both replacements and minimal repairs is the general repair in which each maintenance action reduces the age of the system by a factor $\theta, 0 \le \theta \le 1$. Thus, if a system is maintained by general repairs and if a maintenance action is performed on a system with age x, then this action reduces the age of the system to $\theta\ x$. θ is called the rejuvenation factor. It can be seen that $\theta = 0$ corresponds to replacement by a new system of age 0 whereas $\theta = 1$ corresponds to a minimal repair which leaves the age of the system unaltered. A typical sample path of the general repairs in the post warranty period is exhibited in Figure 5.1.

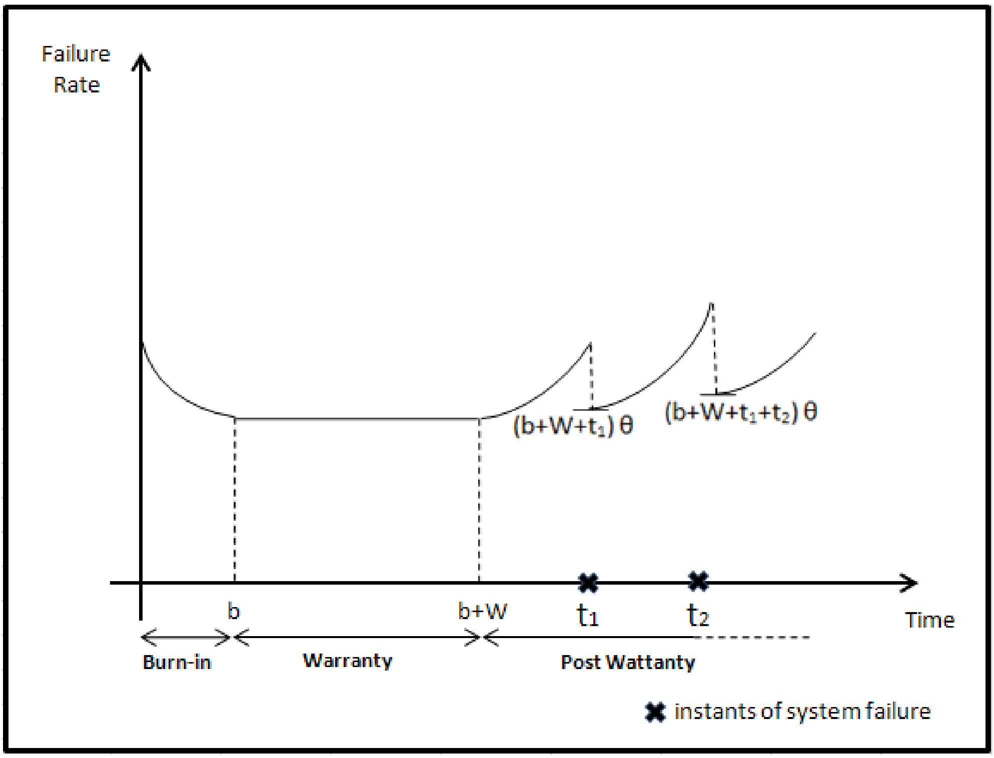

Figure 5.1: Typical Bathtub Failure Rate Curve

The expected number of failures $M_g(t)$ in an arbitrary time interval $(0,t]$ when the system is maintained under general repair policy is known as g-renewal function which was first introduced by Kijima (1989). The g-renewal functions are known to satisfy the g-renewal equation

$$M_g(t) = Q(t \mid 0) + \int_0^t Q(t - x \mid x)\ m(x) dx \tag{5.16}$$

where $Q(t \mid x) = \int_0^t q(y \mid x) dy = \int_0^t \frac{f(y + \theta\, x)}{\overline{F}(\theta\, x)} dy$ (5.17)

Thus, the total expected post warranty cost under general repair strategy is given by

$$[c_5 + (c_6 - c_5)(1 - \theta)] M_g(T) \tag{5.18}$$

where $c_6 - c_5$ is the cost proportional to the degree of rejuvenations.

Explicit expression for the solution of equation (5.16) is not possible. However, unlike their counterpart ordinary renewal functions, there are no approximations available in the literature for g-renewal functions. In order to evaluate $M_g(t)$ numerically, the computational procedure based on Riemann integral sum developed in chapter 4 was used.

The average total cost of the system until its useful age is the sum of the burn-in cost given in (5.5), the warranty cost (5.10) and the post warranty maintenance cost appropriate to the maintenance policy given by (5.12), (5.15), or (5.18). Thus, the average total costs per unit time under minimal repair, replacement, and general repair maintenance policies in the post warranty period respectively are given by

$$ATC_m = \frac{\left\{ \frac{m(c_0+c_1)}{\overline{Fs}(b)} + m\,c_2 \frac{\int_0^b \overline{Fs}(t)dt}{\overline{Fs}(b)} + c_3 \frac{Fs(b)}{\overline{Fs}(b)} + (c_4 + C_b)\frac{Fs_b(W)}{\overline{Fs}_b(W)} + c_5[M(b+W+T) - M(b+W)] \right\}}{W+T}$$

(5.19)

$$ATC_r = \frac{\left\{ \frac{m(c_0+c_1)}{\overline{Fs}(b)} + m\,c_2 \frac{\int_0^b \overline{Fs}(t)dt}{\overline{Fs}(b)} + c_3 \frac{Fs(b)}{\overline{Fs}(b)} + (c_4 + C_b)\frac{Fs_b(W)}{\overline{Fs}_b(W)} + (c_6 + C_b)[M_D(T)] \right\}}{W+T}$$

(5.20)

$$ATC_g = \frac{\left\{ \frac{m(c_0+c_1)}{\overline{Fs}(b)} + m\,c_2 \frac{\int_0^b \overline{Fs}(t)dt}{\overline{Fs}(b)} + c_3 \frac{Fs(b)}{\overline{Fs}(b)} + (c_4 + C_b)\frac{Fs_b(W)}{\overline{Fs}_b(W)} + [c_5 + (c_6 - c_5)(1-\theta)]M_g(T) \right\}}{W+T}$$

(5.21)

We wish to minimize the above cost functions to determine the optimal system design decision variables namely burn-in period b^*, warranty period W^*, the number of components in the system m^*, and the degree of general repair θ^*. In the optimization procedure, we include a few possible constraints on the system like volume, and reliability. More specifically, we consider the following two constraints:

Constraint 1: $\sum_{i=1}^{m} v_i \le V$

where v_i is the volume of the i^{th} component and V is the available volume.

Constraint 2: $Rs \ge R$

Where Rs is the reliability of the burnt-in system and R is the minimum required reliability.

In what follows, we illustrate the optimization procedure and present useful discussions of the above constrained non-linear optimization problem using a typical bathtub failure rate curve.

5.3 Illustration and Discussion

Let each component of the parallel system have the bathtub failure rate curve given by

$$h(t) = kc\lambda t^{c-1} + (1-k)bt^{b-1}\beta e^{\beta t^b} \tag{5.22}$$

This five-parameter failure rate function was introduced by Dhillon (1979). The motivation for such a choice comes from the fact that for a suitable choice of the shape parameters b and c, one can generate different shapes of the failure rate curve that can be used to model electronic or mechanical devices. In our example we fix the values of the five parameters as follows: $\lambda = 1$, $\beta = 1$, $k = 0.5$, $b = 1.5$, $c = 0.3$. Such a choice results in a failure rate curve with a steep slope of decreasing failure rate over a short period. Typical examples of systems with such a failure rate curve are electronic items. The cost parameters are chosen to be $c_0 = 100$, $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, $c_4 = 10$. In the post warranty period the cost $c_5 = 3$ if the system is maintained by minimal repairs, $c_6 = 20$ if the system is maintained by replacements so that the cost of a general repair is $c_5 + (c_6 - c_5)(1-\theta) = 3 + 17(1-\theta)$ if the system is maintained by the general repairs. We wish to observe that the case $\theta = 0$ corresponds to replacements with cost of replacement 3 and similarly the case $\theta = 1$ corresponds to minimal repairs with the cost of minimal repair being 20. The volume of each component is assumed to be 100 units while the total available volume for the system is assumed to be 350. In the discussions we consider two cases for the minimum required reliability namely 0.90 and 0.95. We present below the discussion on optimality for each of the post warranty maintenance operation.

Case-I: Minimal repair

The optimal burn-in period b^*, optimal warranty period W^*, and the optimal number of components m^* when the useful life of the system was fixed at 0.7 years and 1.4 years are given in Table 5.1. We note that when the required reliability of the system increases from 0.9 to 0.95, it is required that the system design becomes more reliable which increases the number of parallel components from 2 to 3. Also when the warranty period increases, in order to decrease the number of failures during this period, the model tries to make the system more reliable by increasing the number of components. The burn-in period is not affected by the changes in useful period. This can be attributed to the shortness of the region when the failure rate is decreasing. Finally, the average total cost is an increasing function of the minimum required reliability, warranty and the number of components.

Table 5.1: Optimal Burn-in Period, Optimal Warranty Period, Number of Components and Average Total Cost for Case-I

Burn-in	Warranty	# of components	Required reliability	Average total cost
T = 0.7				
0.1	0.5	2	0.90	478.294
0.1	0.6	3	0.95	582.542
T = 1.4				
0.1	0.3	2	0.90	303.013
0.1	0.5	3	0.95	384.785

Case-II: Replacement

In this case during the post warranty period the consumer opts to replace a failed system with a new and identical system. As expected when the minimum required reliability of the system is increased from 0.9 to 0.95 the number of parallel components in the system is increased. However, when the useful life of the system is assumed to be short (T=0.7), because of the steeply increasing nature of the failure rate curve there is no change in the warranty period. The same case is not true when the useful life is longer (T=1.4). In this case, the optimal warranty period jumps from 0.2 to 0.4. The average total cost is an increasing function of the number of components as well as the warranty period. Finally, the burn-in period is not affected by either an increase in the reliability or increase in the warranty period. This again can be attributed to steep slope of the decreasing failure rate over a short period. The results are shown in Table 5.2.

Table 5.2: Optimal Burn-in Period, Optimal Warranty Period, Number of Components and Average Total Cost for Case-II.

Burn-in	Warranty	# of components	Required reliability	Average total cost
T = 0.7				
0.1	0.4	2	0.90	659.233
0.1	0.4	3	0.95	820.222
T = 1.4				
0.1	0.2	2	0.90	533.337
0.1	0.4	3	0.95	685.359

Case-III: General repair

In this case the consumer neither employs minimal repair nor replacement of the system on failures but prefers to use a general repair mechanism with degree of repair θ. This means that when a system of age x fails and a general repair is employed, then its age on repair reduces to θx, $0 \le \theta \le 1$. Apart from the optimal design parameters b^*, W^*, and m^* we also determine the optimal degree of repair θ^*. The results are presented in Table 5.3. With the increase in the required reliability, the warranty period and the number of components increase while θ remains static.

However, when the useful life increases from 0.7 to 1.4 there is a noticeable drop in θ^* from 0.9 to 0.3. This is because with a longer post warranty period, in order to minimize the number of failures, the general repairs must be of a better quality. It is to be remembered that smaller values of θ implies more rejuvenation of the repaired system.

Table 5.3: Optimal Burn-in Period, Optimal Warranty Period, Number of Components and Average Total Cost for Case-III.

Burn-in	Warranty	# of components	Degree of repair	Required reliability	Average total cost
T = 0.7					
0.1	0.4	2	0.9	0.90	633.609
0.1	0.5	3	0.9	0.95	783.199
T = 1.4					
0.1	0.4	2	0.3	0.90	520.618
0.1	0.6	3	0.3	0.95	639.721

5.4 Conclusions

The present work deals with an optimal system design problem wherein the objective function is the total cost of the system over its entire life cycle. This approach is in contrast with the common approach of discussing optimality issues based on specific phases of the failure rate curve. The present approach is especially useful in explaining the system design parameters for varying shapes of the bathtub failure rate curve. Given the failure rate of the system and the cost parameters, the manufacturer can decide apriori his system design parameters. Similarly, the present analysis aids the consumer to choose his optimal post warranty strategy. We indicate some possible generalizations of present work. During the post warranty period the customer may opt to have periodic preventive maintenances done to enhance the system lifetime. Next, our objective function in this study is based on the total average cost per unit item sold. However, one can estimate the sales volume function and use the total average cost for all the items sold as the true objective function.

Chapter 6

TWO-DINESIONAL RENEWAL FUNCTION APPROXIMATION

6.1 Introduction

Amongst the many stochastic processes available as a tool for modeling physical phenomena, the theory of renewal processes has found favor with researchers. This is because of the applicability and simplicity of the process and analytical tractability of the underlying statistical characteristics. However, a closed form solution of the renewal equation satisfied by the all important renewal function is not available excepting in the case of exponential and gamma distributions. Realizing the importance of the renewal function, researchers have moved in several directions to find simpler and yet accurate approximations.

A natural extension of the one-dimensional renewal processes to higher dimensions is possible and has been dealt with by several authors. Hoshiya and Chiba (1980) as well as Platen and Rendek (2009) used simulation methods to analyze multidimensional stochastic processes. Shurenkov (1975) proposed a method of reducing multidimensional renewal equation to an ordinary renewal equation. Sgibnev (2006) derived exact asymptotic expansion for solution of multidimensional renewal equations. Application of multidimensional stochastic processes to several interesting areas like seismicity [Delic and Radojicic (2005)] and neuronal activity (Vaillant and Lansky (2000)) have also been made. However, the study of renewal theory in two dimensions in particular has been effectively used in several applications. Realizing that the analysis of point patterns often arises in ecological studies, Newton and Campbell (1975) studied nest locations for a species of duck using two-dimensional renewal processes. Morgan and Welsh (1965) studied a two-dimensional Poisson growth process. Chen *et al* (2009) considered asymptotics for the ruin probabilities of a two-dimensional renewal risk model. The study of product warranty using two-dimensional renewal processes has received considerable attention in the literature [Murthy *et al* (1995), Corbu *et al* (2008), and Manna *et al* (2008)] to cite a few. Developing the theory of two-dimensional renewal processes purely as an extension of one-dimensional renewal process is fraught with conceptual difficulties. For instance, if $F(x,y)$ and $\bar{F}(x,y)$ are the distribution and survivor function of (X,Y) then $F(x,y) + \bar{F}(x,y) \neq 1$, as in the one-dimensional case. The seminal expose on two-dimensional renewal theory was presented by Hunter (1974(a), 1974(b), 1977) who presented a unified theory for studying renewal processes in two dimensions through a series of papers.

If $N_{x,y}$ is the two-dimensional renewal counting process, then $M(x,y) = E(N_{x,y})$ is called the two-dimensional renewal function (Please refer to the discussion on two-dimensional renewal processes in Chapter 1). It is well known that $M(x,y)$ satisfies the two-dimensional renewal equation

$$M(x,y) = F(x,y) + \int_0^x \int_0^y M(x-u, y-v)\, dF(u,v) \tag{6.1}$$

From the above integral equation, it is nearly impossible to obtain $M(x,y)$ analytically even for the simplest form of $F(x,y)$ excepting in the special case when $F(x,y)$ is specified by bivariate exponential distribution. However strangely, unlike its one-dimensional counterpart, there has been no attempt to provide efficient approximations to the two-dimensional renewal function although the use of this function in practical applications occurs often. To the best of our knowledge only Iskandar (1991) has provided a two-dimensional renewal function solver by obtaining values of $M(x,y)$ on a two-dimensional grid equispaced along the X and Y axis. The results are analogous to the one-dimensional renewal function approximation of Xie (1989) and Xie *et al* (2003). It is to be noted that even the only available approximation to the renewal function by Iskandar (1991) requires the apriori information on the joint distribution function $F(x,y)$. However, in many practical applications of two-dimensional renewal processes like ecology and warranty models, one may not be in possession of the explicit form of $F(x,y)$ but could obtain only efficient estimators of the first two moments of X and Y as well as the correlation coefficient from observed sample data. Knowledge of the distribution function is a severe restriction whereas the first few moments could easily be obtained or estimated. Thus, the need for an approximate method to evaluate the renewal function based only on the first two moments and correlation coefficient of the variables is felt more than ever.

In this chapter, we propose such an approximate method assuming the first two moments of $F(x,y)$ to exist. The method is computationally inexpensive as it provides the value of $M(x,y)$ in one go and is not an iterative procedure like that of Iskandar (1991). As we will show later, the method provides exact results for $M(x,y)$ when the joint distribution function $F(x,y)$ is bivariate exponential and a good approximation for other distributions. The layout of the present chapter is as follows. In section 6.2 we give some preliminaries of two-dimensional renewal processes and derive the approximation. In section 6.3 we compare the values of renewal function obtained using our approximation with the benchmark approximation of Iskandar for the bivariate exponential, Beta Stacy, and bivariate gamma distributions. Some useful observations are also made. Section 6.4 presents an application of the two-dimensional renewal function. For this purpose, we have chosen two-dimensional warranty cost analysis as the vehicle of illustration. The last section contains some concluding remarks.

6.2 Moments Based Approximation for Two-dimensional Renewal Function

Bivariate renewal process is a sequence of independent and identically distributed non-negative bivariate random variables $(X_n, Y_n), n = 1, 2, ...$ Let

$$S_n^{(1)} = \sum_{i=1}^{n} X_i \tag{6.2}$$

$$S_n^{(2)} = \sum_{i=1}^{n} Y_i \tag{6.3}$$

$$N_x^{(1)} = max\left\{n;\ S_n^{(1)} \le x\right\} \tag{6.4}$$

$$N_y^{(2)} = max\left\{n;\ S_n^{(2)} \le y\right\} \tag{6.5}$$

$$N_{x,y} = max\left\{n;\ S_n^{(1)} \le x, S_n^{(2)} \le y\right\} = min\left\{N_x^{(1)}, N_y^{(2)}\right\} \tag{6.6}$$

Form the above definitions we see that $N_{x,y}$ represents the number of renewals over the rectangle $[0, x) \times [0, y)$ for a two-dimensional renewal process with the origin $(0, 0)$ being a renewal point. The renewal process is characterized by $(X_i, Y_i), i = 1, 2, ...$ with each pair having the distribution function $F(x_i, y_i)$. $N_x^{(1)}$ and $N_y^{(2)}$ are each univariate renewal counting process associated with distribution functions $F_1(x)$ and $F_2(y)$ respectively which are the marginal distributions of $F(x, y)$. Thus $F_1(x) = F(x, \infty)$ and $F_2(y) = F(\infty, y)$. The renewal functions associated with the one-dimensional and two-dimensional renewal processes are defined as $M_1(x) = E(N_x^{(1)})$ and $M_2(y) = E(N_y^{(2)})$ and $M(x,y) = E(N_{x,y})$. The corresponding renewal densities are given by

$m_1(x) = \frac{d}{dx} M_1(x), m_2(y) = \frac{d}{dy} M_2(y)$ and $m(x,y) = \frac{\delta^2 M(x,y)}{\delta x\, \delta y}$. From one-dimensional renewal theory we have

$$M_1(x) = F_1(x) + \int_0^x M_1(x-u)\, dF_1(u) \quad (6.7)$$

$$M_2(y) = F_2(y) + \int_0^y M_2(y-v)\, dF_2(v) \quad (6.8)$$

To derive the corresponding renewal equation for the two-dimensional renewal function we first note that

$$M(x,y) = \sum_{n=1}^{\infty} F_n(x,y) \quad (6.9)$$

where $F_n(x,y)$ is the n fold convolution of $F(x,y)$ with itself. In (6.9) we convolute $F(x,y)$ both sides to obtain

$$M ** F(x,y) = \sum_{n=1}^{\infty} F_{n+1}(x,y) = M(x,y) - F(x,y) \quad (6.10)$$

From (6.10) we obtain the integral equation of the two-dimensional renewal theory as

$$M(x,y) = F(x,y) + \int_0^x \int_0^y M(x-u, y-v)\, dF(u,v).$$

The above renewal equation could also be derived using an extension of the one-dimensional case. Since every failure over the rectangle $[0,x) \times [0,y)$ is a renewal point, conditioning on the first renewing point at $X = u$ and $Y = v$ we obtain

$$E[N_{x,y} |\, X = u, Y = v] = \begin{cases} 1 + M(x-u, y-v), u \le x, v \le y \\ o, \qquad otherwise \end{cases} \quad (6.11)$$

Unconditioning, we have

$$M(x,y) = F(x,y) + \int_0^x \int_0^y M(x-u, y-v)\, dF(u,v) \quad (6.12)$$

Define the double Laplace transform of $f(x,y)$as

$$f^*(p,q) = \int_0^\infty \int_0^\infty e^{-px} e^{-qy} dF(x,y) \tag{6.13}$$

Taking double Laplace transform on both sides of (6.12) and simplifying, we obtain

$$M^*(p,q) = \frac{F^*(p,q)}{1-F^*(p,q)} \tag{6.14}$$

In a similar manner, we can derive the double Laplace transform of the renewal density function as

$$m^*(p,q) = \frac{f^*(p,q)}{1-f^*(p,q)} \tag{6.15}$$

Noting that $m^*(p,q)$ has a singularity at $(p,q) = (0,0)$ we approximate it by a rational function of the form

$$m^*(p,q) \approx \frac{A}{p+Bq+C} + \frac{D}{p+Eq+Fpq} \tag{6.16}$$

Using $M^*(p,q) = m^*(p,q)/pq$ and successively inverting with respect to p and then q, we obtain

$$M(x,y) \approx$$

$$\frac{A}{C}(1-e^{-Cx})Dx + \frac{A}{C}\left(e^{-Cx} - e^{-C\frac{y}{B}}\right) U_{\frac{y}{B}}(x) -$$

$$De^{-bx}e^{-\frac{y}{F}}\left[\sum_{v=1}^{N(\varepsilon)} vb^{v-1}\left(\frac{x}{a}\right)^{\frac{v}{2}} I_v\left(2\sqrt{ax}\right)\right] \tag{6.17}$$

where $a = yE/F^2$, $b = E/F$, $U_a(t) = \begin{cases} 1 & ,t \geq a \\ 0 & ,otherwise \end{cases}$ and $I_v\left(2\sqrt{ax}\right)$ is the modified Bessel function of order ν such that $I_v\left(2\sqrt{ax}\right) = \sum_{k=1}^{N(\varepsilon)} \frac{\left(\sqrt{ax}\right)^{v+2(k-1)}}{(k-1)!\Gamma(v+k)}$. The number of terms $N(\varepsilon)$ in the summation is such that the difference between two successive summands is less than the pre assigned ε. In order to obtain the constants

$A, B, C, D, E,$ and F we proceed as follows. It is well known that the double Laplace transform of the distribution function $f^*(p,q)$ admits the power series expansion

$$f^*(p,q) \approx 1 - p\mu_x - q\mu_y + \frac{p^2}{2}\mu_{x2} + \frac{q^2}{2}\mu_{y2} + pq\mu_{xy} \tag{6.18}$$

where μ_x, μ_y, μ_{x2} and μ_{y2} are the first two moments about the origin of X and Y and $\mu_{xy} = E(XY)$. Using (6.18) in (6.15) and (6.16) and comparing the coefficients of p, q, pq, p^2, p^3 and q^3 on both sides we obtain after some algebra

$$A = \frac{4\mu_x}{\mu_{x2}}\left(1 - \frac{\mu_x^2}{\mu_{x2}}\right) - \frac{1}{\mu_x} \tag{6.19}$$

$$B = \frac{\mu_y\mu_{y2}(2\mu_x^2 - \mu_{x2})}{\mu_x\mu_{x2}(2\mu_y^2 - \mu_{y2})} \tag{6.20}$$

$$C = \frac{2\mu_x(2\mu_x^2 - \mu_{x2})}{\mu_{x2}^2} \tag{6.21}$$

$$D = \frac{1}{\mu_x} \tag{6.22}$$

$$E = \frac{\mu_y}{\mu_x} \tag{6.23}$$

$$F = \frac{\mu_y\mu_{x2} - \mu_x\mu_{xy}}{\mu_x^2} \tag{6.24}$$

Thus the approximation to $M(x,y)$ given in (6.17) is completely determined. At this stage, we wish to make the following observations.

Observation 1: For the approximation to hold it is necessary that $b = E/F$ must be greater than 0 which implies that F must be positive. This leads us to the condition

$$\rho < \sqrt{\frac{\phi_x^{(2)} - 1}{\phi_y^{(2)} - 1}}$$

where ρ is correlation coefficient between the variables X and Y.

Further conditions for the approximation to hold are $B > 0$ and $C \geq 0$. These two conditions imply that the region over which the approximations hold are given by $\phi_x^{(2)} > 2$ and $\phi_y^{(2)} > 2$ where $\phi_x^{(2)}$ and $\phi_y^{(2)}$ are respectively given by

$$\phi_x^{(2)} = C_x^2 + 1 = (\frac{\sigma_x}{\mu_x})^2 + 1$$

$$\phi_y^{(2)} = C_y^2 + 1 = (\frac{\sigma_y}{\mu_y})^2 + 1$$

where C_x and C_y are coefficient of variation of X and Y, respectively.

Observation 2: We note that in the asymptotic case we obtain $M(t,t)/t \rightarrow 1/\mu_x$ as $t \rightarrow \infty$.

6.3 Illustrations and Comparison with Benchmark Approximation

In order to check the efficacy and accuracy of the proposed method, we compute the renewal function for certain special joint probability distributions $F(x,y)$ and compare it with the values obtained using the two-dimensional renewal function solver proposed by Iskandar. It is to be noted that we are hampered by the fact that exact values of the renewal function for any distribution other than bivariate exponential is not available in order to make a comparison. We choose bivariate exponential distribution, Beta Stacy distribution, and bivariate gamma distribution as illustrations.

6.3.1 Bivariate Exponential Distribution

Several bivariate exponential distributions are available in the literature [Kotz *et al* (2000)]. Some of them do not have an explicit form for the joint probability distribution while many of them impose severe restrictions on the correlation coefficient between the two variables. We choose the bivariate exponential density

[Downton (1970)] which does not have these restrictions and possess other desirable properties whose joint probability density is given by

$$f(x,y) = \frac{\lambda_x\lambda_y}{1-\rho} e^{-\frac{\lambda_x x+\lambda_y y}{1-\rho}} I_0\left[\frac{2\sqrt{\rho\lambda_x\lambda_y xy}}{1-\rho}\right] \tag{6.25}$$

where $\lambda_x = 1/\mu_x$, $\lambda_y = 1/\mu_y$ and $I_0(.)$ is the modified Bessel function of the first kind of order zero. It can be verified that the two variables X and Y have the marginal distributions which are exponential distributions with means μ_x and μ_y. Also the correlation coefficient between the variables X and Y is ρ. The double Laplace transform of the above joint probability distribution is given by

$$f^*(p,q) = \frac{1}{(1+\mu_x p)(1+\mu_y q)-\rho\mu_x\mu_y pq} \tag{6.26}$$

Substituting the values of the moments of the above distribution in (6.19) to (6.24) we note that $A = C = 0$. Thus, the approximation reduces to

$$m^*(p,q) = \frac{D}{p+Eq+Fpq} \tag{6.27}$$

with

$$D = \frac{1}{\mu_x} \tag{6.28}$$

$$E = \frac{\mu_y}{\mu_x} \tag{6.29}$$

$$F = \mu_y(1-\rho) \tag{6.30}$$

Substituting (6.28) to (6.30) in (6.27), we obtain

$$m^*(p,q) = \frac{1}{\mu_x p+\mu_y q+\mu_x\mu_y(1-\rho)pq} \tag{6.31}$$

From (6.15) we have

$$f^*(p,q) = \frac{m^*(p,q)}{1+m^*(p,q)} \tag{6.32}$$

Using (6.31) in (6.32) we obtain $f^*(p,q) = \frac{1}{(1+\mu_x p)(1+\mu_y q)-\rho\mu_x\mu_y pq}$ .This is precisely the same one given in (6.26). This shows that our approximation produces exact results for the renewal function governed by bivariate exponential distribution. In Table 6.1, we present the values of renewal function for bivariate exponential distribution with $\lambda_x = 1$ and $\lambda_y = 1$ computed using our method and Iskandar's method as well as the exact values.

Table 6.1: Comparison between the Values of the Renewal Function Computed Using our Method, Iskandar's and Exact Values

x	y	$\rho = 0.0$			$\rho = 0.5$		
		Exact value	Iskandar's	Proposed approximation	Exact value	Iskandar's	Proposed approximation
0.5	0.5	0.1632	0.1636	0.1632	0.2381	0.2393	0.2381
1.0	1.0	0.4762	0.4774	0.4762	0.6143	0.6168	0.6143
1.5	1.5	0.8403	0.8422	0.8403	1.0219	1.0257	1.0219
2.0	2.0	1.2285	1.2311	1.2285	1.4448	1.4496	1.4448

The following few observations with regard to Table 6.1 are in order. First, we note that our method gives the exact values of the renewal function. Secondly, Iskandar's method always over estimates the renewal function (although by small quantity). This is due to the fact that the Riemannian upper and lower sums to approximate the integrals essentially give the upper and lower bounds.

6.3.2 Beta Stacy Distribution

As the next illustration, we choose bivariate Beta Stacy distribution, which is a slightly modified version of the Beta Stacy distribution proposed by Mirham and Hultquist [See Johnson and Kotz (1972)]. The density function $f(x,y)$ is given by

$$f(x,\ y) = \frac{1}{\Gamma(\alpha)\beta(\theta_1,\theta_2)}\frac{c}{a^{\alpha c}\phi}x^{(\alpha c-\theta_1-\theta_2)}(y/\phi)^{(\theta_1-1)}(x-y/\phi)^{\theta_1-1}e^{[-(x/a)^c]} \quad (6.33)$$

where $x > 0, 0 < y < \phi x, \alpha, c, a, \phi, \theta_1, \theta_2 > 0$ and $\beta(.,.)$ is the Beta function.

The statistical characteristics of the above density function are given by

$$E(X) = \frac{a\Gamma\left(\alpha+\frac{1}{c}\right)}{\Gamma(\alpha)} \tag{6.34}$$

$$E(Y) = \frac{a\phi\theta_1\Gamma\left(\alpha+\frac{1}{c}\right)}{(\theta_1+\theta_2)\Gamma(\alpha)} \tag{6.35}$$

$$E(XY) = \frac{a^2\phi\theta_1\Gamma\left(\alpha+\frac{2}{c}\right)}{(\theta_1+\theta_2)\Gamma(\alpha)} \tag{6.36}$$

$$E(X^2) = \frac{a^2\Gamma\left(\alpha+\frac{2}{c}\right)}{\Gamma(\alpha)} \tag{6.37}$$

$$E(Y^2) = \frac{a^2\phi^2(\theta_1+1)\theta_1\Gamma\left(\alpha+\frac{2}{c}\right)}{(\theta_1+\theta_2+1)(\theta_1+\theta_2)\Gamma(\alpha)} \tag{6.38}$$

Substituting these values in (6.19) to (6.24), we note that F turns out to be zero, so that we use the approximation

$$m^*(p,q) \approx \frac{A}{p+Bq+C} + \frac{D}{p+Eq} \tag{6.39}$$

where the constants $A, B, C, D,$ and E are specified by

$$A = \frac{4\mu_x^2\mu_{x2} - 4\mu_x^4 - \mu_{x2}^2}{\mu_{x2}^2\mu_x} \tag{6.40}$$

$$B = \frac{\mu_y\mu_{y2}\left(2\mu_x^2 - \mu_{x2}\right)}{\mu_x\mu_{x2}\left(2\mu_y^2 - \mu_{y2}\right)} \tag{6.41}$$

$$C = \frac{2\mu_x\left(2\mu_x^2 - \mu_{x2}\right)}{\mu_{x2}^2} \tag{6.42}$$

$$D = \frac{1}{\mu_x} \tag{6.43}$$

$$E = \frac{\mu_y}{\mu_x} \tag{6.44}$$

Double inversion of (6.39) with respect to p and q yields

$$M(x,y) \approx \frac{A}{C}\left(1 - e^{-Cx}\right) + \frac{D}{E}\left[y - U_{Ex}(y)(y - Ex)\right] + \frac{A}{C}\left(e^{-Cx} - e^{-C\frac{y}{B}}\right)U_{\frac{y}{B}}(x) \tag{6.45}$$

where $U_a(t) = \begin{cases} 1 & ,t \geq a \\ 0 & ,otherwise \end{cases}$

The following observation with regard to the asymptotic behavior of $M(x, y)$ given in (6.45) is worth mentioning. Hunter (1974, Renewal theory in two-dimensions: asymptotic results, page 555) proved that

$$\lim_{t\to\infty} \frac{M(t,t)}{t} = \frac{1}{max(\mu_x, \mu_y)} \quad (6.46)$$

It can immediately be seen from (6.45) that (6.46) holds good for our approximation as well.

Iskandar considered three sets of numerical values for the parameters $a, \phi, \alpha, c, \theta_1$, and θ_2 in his two-dimensional renewal function solver which are given in Table 6.2.

Table 6.2: Parameters $a, \phi, \alpha, c, \theta_1$, and θ_2 for Three Sets of Beta Stacy Distribution

Parameters	Set		
	I	II	III
a	0.2	0.0846	0.0550
Φ	1.1	2.6	4.0
α	1.9		
c	2.5		
θ_1	1.1		
θ_2	1.1		

With the choice of the parameters as in set-II, μ_x turns out to be 0.1026 and μ_y = 0.1334, so that $\frac{1}{max(\mu_x, \mu_y)} = \frac{1}{0.1334} = 7.4963$. Figure6.2 plots the values of renewal function from which it can be seen that for large values of X $(= Y)$ the line $\frac{M(t,t)}{t} = 7.4963$ is an asymptote to the renewal function. A similar plot of $M(t, t)/t$ for sets I and III are given in Figures 6.1 and 6.3.

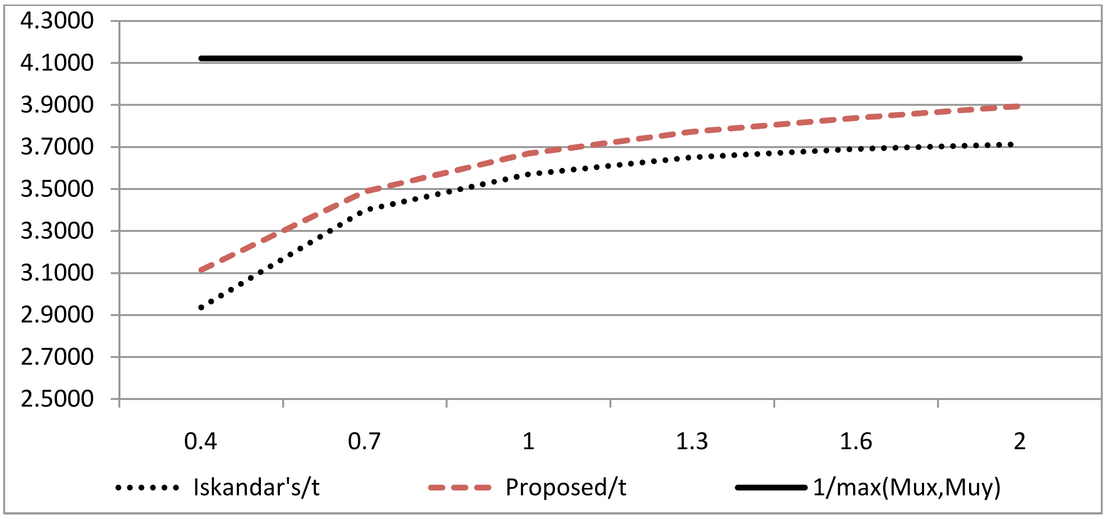

Figure 6.1: Asymptotic Nature of $M(t,t)/t$ for Set-I

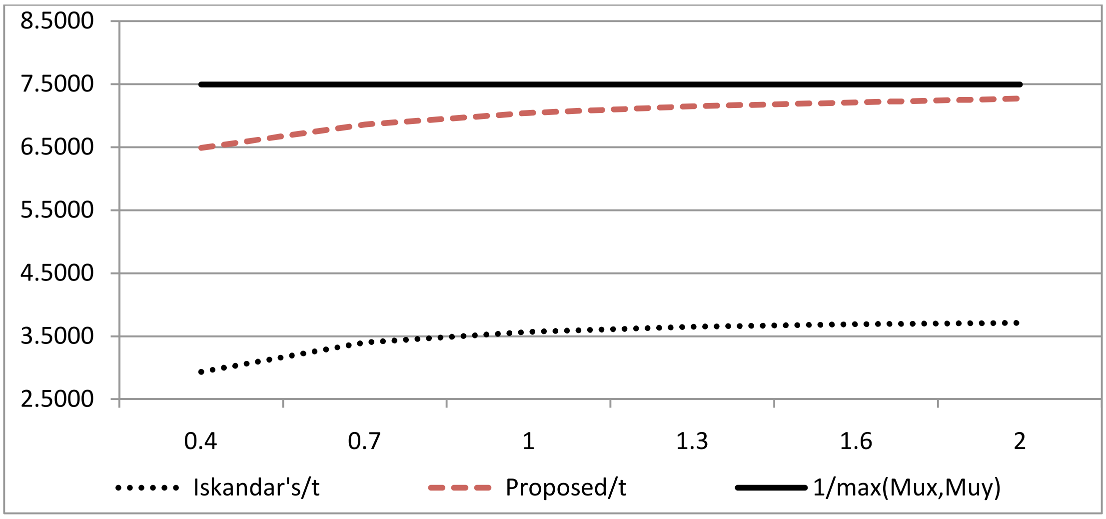

Figure 6.2: Asymptotic Nature of $M(t,t)/t$ for Set-II

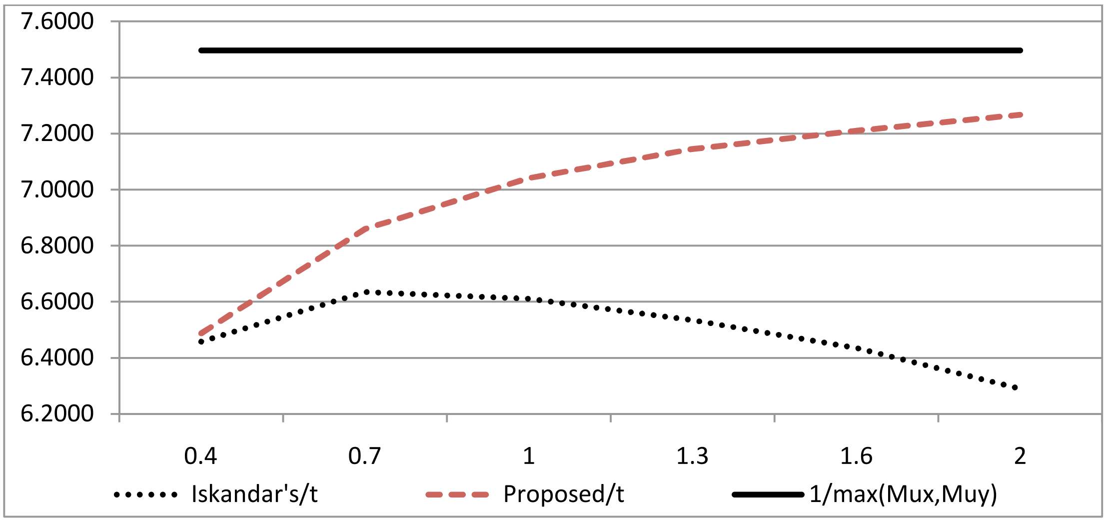

Figure 6.3: Asymptotic Nature of $M(t,t)/t$ for Set-III

As remarked earlier it is not possible to obtain the exact values of the two-dimensional renewal function $M(x,y)$. Thus, we are confronted with the question "Which of the two methods, Iskandar's or the present method provides better results?" From Figures 6.1 to 6.3 it is clear that Iskandar's approximation does not provide accurate results for large values of . In fact for values of set-III, Iskandar's rate of change in the renewal function has a negative trend for increasing values of t, whereas it should asymptotically approach the line $1/\max(\mu_x, \mu_y)$.

In order to make a comparison of the values obtained using our approximation with that of Iskandar's; we use the same set of values. Tables 6.3, 6.4, and 6.5 present the values of the renewal function using the two methods for certain values of x and y for the three sets of the parameters values chosen.

Table 6.3: The Values of the Renewal Function Using the Two Methods for Set-I

Y	x	0.4	0.7	1.0	1.3	1.6	2.0
0.4	Proposed approximation	1.2459	2.4824	2.5948	2.5948	2.5948	2.5948
	Iskandar's	1.1745	2.1060	2.4975	2.5994	2.6181	2.6209
0.7	Proposed approximation	1.2311	2.4410	3.6775	4.8016	4.8016	4.8016
	Iskandar's	1.1760	2.3793	3.4890	4.2407	4.5903	4.7230
1.0	Proposed approximation	1.2311	2.4365	3.6694	4.9059	6.1424	7.0417
	Iskandar's	1.1760	2.3796	3.5707	4.7236	5.6980	6.4704
1.3	Proposed approximation	1.2311	2.4365	3.6682	4.9043	6.1408	7.7895
	Iskandar's	1.1760	2.3796	3.5708	4.7454	5.8992	7.2902
1.6	Proposed approximation	1.2311	2.4365	3.6682	4.9040	6.1405	7.7892
	Iskandar's	1.1760	2.3796	3.5708	4.7454	5.9046	7.4228
2.0	Proposed approximation	1.2311	2.4365	3.6682	4.9040	6.1404	7.7891
	Iskandar's	1.1760	2.3796	3.5708	4.7454	5.9046	7.4270

Table 6.4: The Values of the Renewal Function Using the Two Methods for Set-II

y	x	0.4	0.7	1.0	1.3	1.6	2.0
0.4	Proposed approximation	2.5954	2.5954	2.5954	2.5954	2.5954	2.5954
	Iskandar's	2.4509	2.6126	2.6146	2.6146	2.6146	2.6146
0.7	Proposed approximation	3.4533	4.8025	4.8025	4.8025	4.8025	4.8025
	Iskandar's	3.2941	4.6033	4.7219	4.7247	4.7248	4.7248
1.0	Proposed approximation	3.4453	6.3684	7.0430	7.0430	7.0430	7.0430
	Iskandar's	3.3417	5.8054	6.6895	6.7753	6.7783	6.7783
1.3	Proposed approximation	3.4445	6.3668	9.2900	9.2900	9.2900	9.2900
	Iskandar's	3.3417	6.0620	8.1053	8.7117	8.7737	8.7764
1.6	Proposed approximation	3.4445	6.3665	9.2897	11.5384	11.5384	11.5384
	Iskandar's	3.3417	6.0715	8.6336	10.2520	10.6735	10.7199
2.0	Proposed approximation	3.4445	6.3665	9.2897	12.2129	14.5364	14.5364
	Iskandar's	3.3417	6.0715	8.7047	11.1412	12.6928	13.2006

Table 6.5: The values of the Renewal Function Using the Two Methods for Set-III

y	x	0.4	0.7	1.0	1.3	1.6	2.0
0.4	Proposed approximation	2.5948	2.5948	2.5948	2.5948	2.5948	2.5948
	Iskandar's	2.5833	2.5912	2.5912	2.5912	2.5912	2.5912
0.7	Proposed approximation	4.8016	4.8016	4.8016	4.8016	4.8016	4.8016
	Iskandar's	4.3662	4.6440	4.6449	4.6449	4.6449	4.6449
1.0	Proposed approximation	5.5428	7.0417	7.0417	7.0417	7.0417	7.0417
	Iskandar's	5.1098	6.5715	6.6112	6.6114	6.6114	6.6114
1.3	Proposed approximation	5.5413	9.2883	9.2883	9.2883	9.2883	9.2883
	Iskandar's	5.1822	8.1056	8.4885	8.4939	8.4939	8.4939
1.6	Proposed approximation	5.5410	10.0374	11.5362	11.5362	11.5362	11.5362
	Iskandar's	5.1826	8.8419	10.2190	10.2955	10.2962	10.2962
2.0	Proposed approximation	5.5409	10.0373	14.5337	14.5337	14.5337	14.5337
	Iskandar's	5.1826	8.9961	11.8766	12.5507	12.5799	12.5802

We note that the difference between the two approximations increase for increasing values of x and y. This aspect was mentioned in the discussions on the asymptotic nature of $\frac{M(t,t)}{t}$. One possible argument could be that with increasing values of the arguments x and y the number of grids used in Iskandar's method increase and consequently the truncation error has a cascading effect on the approximation for a fixed grid size.

The surface plots and the contour maps of the renewal function $M(x, y)$ for the three sets of data are given in Figures 6.4 and 6.5.

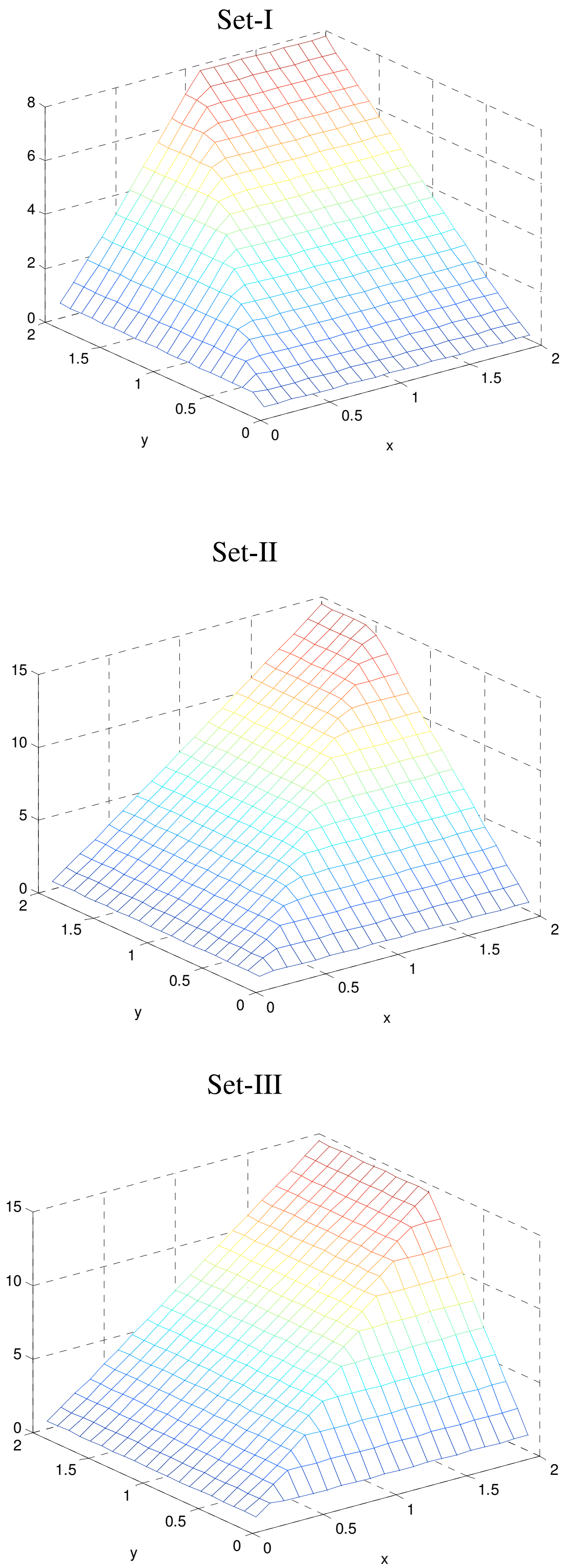

Figure 6.4: Surface Plots of the Renewal Function $M(x, y)$ for the three Sets of Data

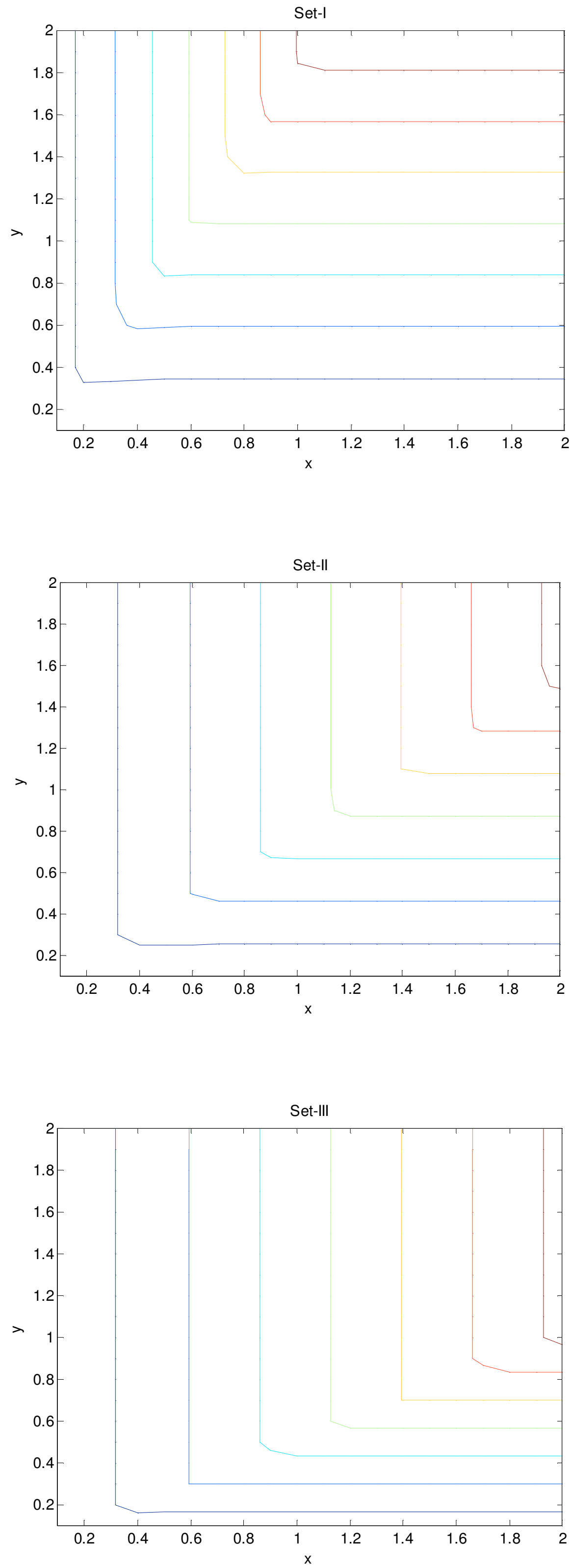

Figure 6.5: Contour Maps of the Renewal Function $M(x, y)$ for the three Sets of Data

6.3.3 McKay's Bivariate Gamma Distribution

As the third illustration, we choose one of the earliest forms of the bivariate gamma distribution, which is due to McKay defined by the following joint distribution function

$$F(x,y) = \frac{c^{a+b}}{\Gamma(a)\,\Gamma(b)} x^{a-1}(y-x)^{b-1}e^{-cy} \qquad , y> x> 0, a, b, c> 0$$

Plots of the above joint density function for a few cases have been given by Kellogg and Barnes (1987). The marginal distributions of X and Y are gamma distributed with shape parameters a and $a+b$ respectively. The correlation coefficient $\rho(X,Y) = \sqrt{\frac{a}{a+b}}$ and the conditional density functions are beta distributed. An interesting application of this distribution in hydrology was made by Clark (1980) who studied the joint distribution of annual stream flow and areal precipitation. The complete statistical characteristics of the above joint distribution needed for our approximation are given by

$$E(x) = \frac{a}{c}$$

$$E(x^2) = \frac{a(a+1)}{c^2}$$

$$E(y) = \frac{a+b}{c}$$

$$E(y^2) = \frac{(a+b)(a+b+1)}{c^2}$$

$$E(xy) = \frac{a(a+b+1)}{c^2}$$

All the conditions needed for the use of (6.17) to obtain the two-dimensional renewal function were met. Table 6.6 presents the values of the renewal function governed by the bivariate gamma distribution for a specific choice of the parameters. The corresponding surface plot and the contour map are respectively given in Figures 6.6 and 6.7.

Table 6.6: The Values of the Renewal Function using the proposed method $a = 0.4, b = 0.4, and\ c = 1$

y/x	0.5	1.0	1.5	2.0	3.0
0.6	0.3626	0.5688	0.7097	0.8016	0.8847
1.1	0.5516	0.9213	1.2171	1.4481	1.7528
1.6	0.6396	1.1098	1.5191	1.8679	2.3968
2.1	0.6801	1.2079	1.6922	2.1285	2.8498
3.1	0.7068	1.2828	1.8409	2.3754	3.3536

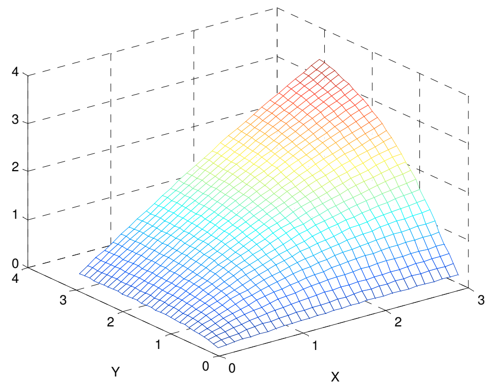

Figure 6.6: Surface Plots of the Renewal Function $M(x, y)$ for Bivariate Gamma Distribution Function

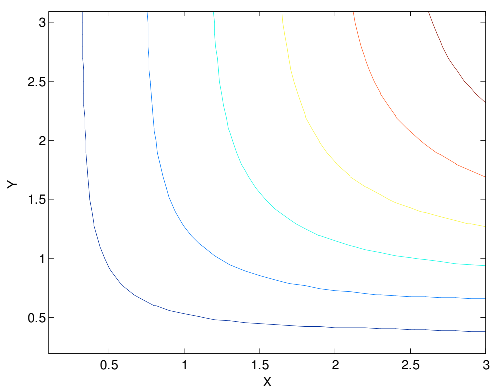

Figure 6.7: Contour Maps of the Renewal Function $M(x, y)$ for Bivariate Gamma Distribution Function

6.4 An Application: Two-dimensional Warranty Model

In the present days of consumer renaissance, most of the consumer goods are sold with a warranty. A warranty is the expression of willingness of business to stand behind its products and services. Thus from the manufacturer's point of view the warranty (i) reassures the consumer of the manufacturer's resolve in terms of product support (ii) serves as a beacon of product reliability and (iii) provides a cutting edge to the product over its competitors. However, warranty servicing is a major component of the manufacturer's cost. Reducing warranty cost is an issue of importance to the manufacturer. Blischke and Murthy (1992) have given a survey for various types of warranty policies. These policies can be broadly grouped in to one-dimensional and two-dimensional policies depending on the number of variables that characterize the failure distribution of the product. The one-dimensional policy is generally characterized by age as the variable with time interval as the warranty period. There has been a plethora of research papers on one-dimensional warranty.

In the case of two-dimensional warranties, the variables are generally labeled as age and usage and the warranty period is characterized by a region in the two-dimensional plane. The age could be the real time while the usage can be the output (copies produced for a photocopier), distance travelled (kilometers for an automobile), the number of times or hours the product has been used (takeoffs and landing of the total hours flown for an aircraft) [Jack *et al* (2009)]. The product failures are characterized by a two-dimensional distribution function. More specifically if X_i and Y_i respectively represent the time interval and the usage interval between i^{th} and $(i-1)^{th}$ failure then (X_i, Y_i) is modeled through a bivariate distribution function $F_i(x, y)$. However, if we consider (as in the present model) non repairable products, so that products are replaced on failure, then the pair (X_i, Y_i) can be thought of as a sequence of independent and identically distributed random variables with common distribution function $F(x, y)$. In this chapter we consider three different two-dimensional warranty policies proposed by Murthy *et al* (1995, page 357). In computing the expected warranty cost for each of these policies, we encounter two-dimensional renewal functions. These have been evaluated using the proposed approximation.

Policy A. The warranty region is characterized by the rectangle $[0, K) \times [0, L)$, which is given in Figure 6-8. The warranty ceases whenever a failure occurs for the first time outside the rectangle. The consumer is assured a maximum coverage for K units of time and/or L units of usage. If the average usage rate is approximately L/K, then the age and total usage are very close to K and L respectively, when the warranty ceases. If the average usage rate is low, then the warranty expires at K time units from the sale with the total usage at expiry well below L. On the other hand, if the average usage rate is high, then the warranty expires much before K time units due to the total usage exceeding the limit L. Thus, for consumers with either high or low average usage rate, the warranty policy is not very attractive as it favors the manufacturer.

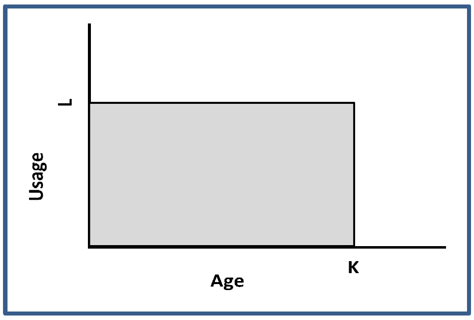

Figure 6.8: Warranty Region for Policy A

Let $N^A(K, L)$ be the number of failures under warranty and c_1, the cost of replacement of failed item. The expected warranty cost per item sold under policy A is given by

$c_1 E[N^A(K, L)] = c_1 M(K, L)$

where $M(K, L)$ is the two-dimensional renewal function. In Table 6.7, we present the expected warranty cost for certain values of K and L when the joint distribution function $F(x, y)$ is chosen to be Beta Stacy distribution.

Table 6.7: Expected Warranty Cost under Policy A with Beta Stacy Distribution. ($a = 0.0846, \Phi = 2.6, \alpha = 1.9, c = 2.5, and\ \theta_1 = \theta_2 = 1.1$) and Replacement Cost, $c_1 = 10$

L/K	0.5	1.0	1.5	2.0
0.5	33.2327	44.1964	44.1795	44.1795
1.0	33.2327	70.4302	92.8978	92.8966
1.5	33.2327	70.4302	107.8885	141.6164
2.0	33.2327	70.4302	107.8885	145.3641

Policy B. The region of warranty under this policy is characterized by two infinite dimensional strips as shown in Figure 6.9. Under this policy, the consumer is guaranteed a coverage of at least K units of time after sale and for L units of usage. The warranty comes to an end at the first time when both time and usage exceed the limits K and L, respectively. Under such a policy, a consumer with a low average usage rate gets warranty coverage for well beyond K units of time and a consumer with a heavy usage rate is covered for a time period K with total usage well in excess of L when the warranty ends. As a result, this policy is favorable to low and high usage consumers, as opposed to the manufacturer.

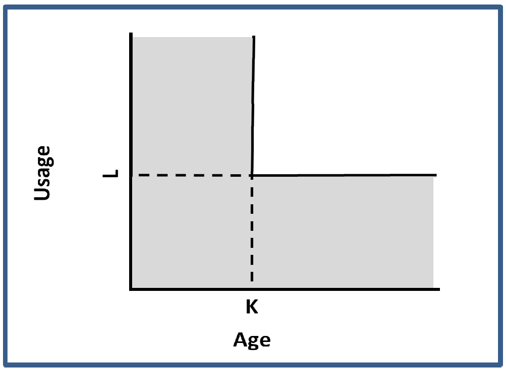

Figure 6.9: Warranty Region for Policy B

Let $N^B(K,L)$ be the number of failures under warranty. The expected warranty cost in this case is given by

$$c_1E[N^B(K,L)] = c_1[M_1(K) + M_2(L) - M(K,L)]$$

where $M_1(K)$ and $M_2(L)$ are the one-dimensional renewal functions with the marginal distributions of X and Y respectively, as the distribution functions. In Table 6.8, we present the expected warranty cost for certain values of K and L when the joint distribution function $F(x,y)$ is chosen to be Beta Stacy distribution.

Table 6.8: Expected Warranty Cost under Policy B with Beta Stacy Distribution. ($a = 0.0846, \Phi = 2.6, \alpha = 1.9, c = 2.5, and\ \theta_1 = \theta_2 = 1.1$) and Replacement Cost, $c_1 = 10$

L/K	0.5	1.0	1.5	2.0
0.5	68.6302	95.4896	133.3519	171.2372
1.0	117.3503	117.9756	133.3532	171.2398
1.5	166.0701	166.6954	167.0823	171.2398
2.0	214.7899	215.4152	215.8021	216.2119

Policy C. This policy is characterized by four parameters: $K_1, K_2, L_1,$ and L_2. The warranty region is as shown in Figure 6.10. We observe that a consumer is assured of a minimum warranty coverage of K_1 time period and a usage L_1. On the other hand, the maximum warranty coverage is for a time period of K_2 and a usage L_2. The rectangle $[0, K_1) \times [0, L_2)$ brings in better coverage to a heavy user, whereas the rectangle $[0, K_2) \times [0, L_1)$ results in a better coverage to a light user. As contrasted to policy B, the upper limits on age and usage bring in more protection to the manufacturer from excessive warranty costs.

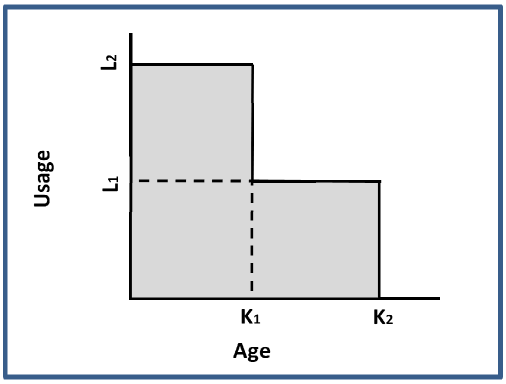

Figure 6.10: Warranty Region for Policy C

Let $N^C(K_1, L_1, K_2, L_2)$ be the number of failures under warranty. The expected warranty cost in this case is given by

$$c_1 E[N^C(K_1, L_1, K_2, L_2)] = c_1[M(K_1, L_2) + M(K_2, L_1) - M(K_1, L_1)]$$

In Table 6.9, we present the expected warranty cost for certain values of $K_1, K_2, L_1,$ and L_2 when the joint distribution function $F(x, y)$ is chosen to be Beta Stacy distribution.

Table 6.9: Expected Warranty Cost under Policy C with Beta Stacy Distribution. $(a = 0.0846, \Phi = 2.6, \alpha = 1.9, c = 2.5, and\ \theta_1 = \theta_2 = 1.1)$, Replacement Cost, $c_1 = 10$, and $K_1 = 0.5K_2, L_1 = 0.5L_2$

L_2/K_2	0.5	1.0	1.5	2.0
0.5	20.1158	33.0443	44.2535	44.1964
1.0	33.2327	44.1964	51.6748	70.4133
1.5	33.2327	68.5563	68.5379	70.4106
2.0	33.2327	70.4302	92.8978	92.8966

6.5 Conclusions

This chapter proposes a new non-parametric method based only on the moments to evaluate the two-dimensional renewal function. The method assumes significance in the light of the facts that (i) the two-dimensional renewal equation has no analytic solution (ii) there has been no useful approximation available in the literature with the possible exception of Iskandar (iii) it does not require the knowledge of the form of the joint distribution $F(x, y)$ but requires only the first two statistical characteristics of X and Y and finally (iv) it is easy to evaluate and does not require iterative computation like Iskandar's. Specific examples and an application in the form of two-dimensional warranty modeling have been presented to highlight the efficiency and efficacy of the proposed method.

Chapter 7

CONCLUDING REMARKS

The renewal equations and renewal type equations are encountered frequently in stochastic modeling whenever renewal theoretic arguments are used. The solution of these integral equations yields the renewal function, which is a key tool in optimizing the expected value criteria. However, these integral equations are not amenable for explicit solution and approximations are the only method in dealing with them. This thesis proposes several approximations to important renewal functions such as one-dimensional and two-dimensional renewal functions and g-renewal functions. The notable contribution in this regard is that the proposed methods use only the moments of the underlying distributions and do not require the explicit form of the distribution. This aspect gives the approximations a big leap over the existing literature. Several numerical examples are provided to compare the proposed methods with benchmark approximation available in the literature. We wish to make specific mention of the optimal system design model developed in chapter 5. This is a new model and enables the manufacturer to choose his optimal system design parameters while giving the consumer an optimal choice of post warranty maintenance.

We conclude this thesis with the question “where do we go from here?” Firstly our moment matching procedures have been restricted to H_2 distributions. A natural alternative to H_2 distributions are Coxian distributions. This is because in principle Coxian distribution may be used to approximate any general distribution. A detailed analysis on the class of general distributions which match each of these two phase type distributions (Coxian and H_2 distributions) will provide answers to moment matching problems. In the same vein it is also worth pursuing the region (ϕ_2, ϕ_3) in which each of these approximations score over the others when computing the renewal function in terms of relative errors. Finally, moment based approximation and moment matching procedures are used when there is no knowledge on the form of the distribution function $F(x)$. In such a scenario, another interesting method could be the resort to maximum entropy distribution as an approximation for the unknown distribution function. Given some partial information about the random variable, maximum entropy principle chooses that probability distribution for the random variable which is consistent with the given information but has otherwise maximum uncertainty associated with it.

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VISIBLE RESEARCH OUTPUTS

1- Kambo, N.S., Rangan, A., and MoghimiHadji, E. (2011). Moments based approximation to the renewal function. *Communication in Statistics – Theory and Methods*, To appear.

2- Rangan, A. and MoghimiHadji, E. (2011). Approximations to the g-renewal Function. *International Journal of Quality and Reliability Management* 28 (7).

3- MoghimiHadji, E. and Rangan, A. Optimal system design based on burn-in, warranty and maintenance". 7th international Statistics Congress, 2011, Antalya, Turkey.

4- Kambo, N.S., Rangan, A, and MoghimiHadji, E. Approximations to the Performance Measures in Queuing Systems. Journal of Computers and Operations research. In the process of revision.

5- MoghimiHadji, E., Rangan, A., and Kambo, N.S. (2011).Approximation to two dimensional renewal function and applications. Manuscript in progress.