New Exact solutions in a Model of f(R) Gravity and their Physical Properties

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ABSTRACT

Some new exact solutions in a model of $f(R)$ gravity are obtained. Three kinds of matter fields have been used to obtain exact analytic solutions. In the first solution, the Yang-Mills fields are incorporated as a matter field at constant curvature condition. In the second and third solutions, the linear and nonlinear electromagnetic fields are used as matter fields. Thermodynamic properties are explored for those solutions that admit black holes. The occurrence of naked singularities in the solutions sourced by linear electromagnetic fields is investigated within the context of quantum mechanics. The waves obeying the massless Klein-Gordon, Maxwell and Dirac fields are used to probe the singularity. It is shown that the classical curvature singularity remains singular even if it is probed with quantum waves rather than classical particles.

Keywords: Black hole solution, f(R) Gravity, Yang-Mills, Linear and Nonlinear Electromagnetism, Quantum singularities

ÖZ

$f(R)$ gravitasyon teorisinde bazı kesin çözümler elde edilmiştir. Bu çözümler üç farklı alan kullanılarak bulumuştur. Birinci çözümde, Yang-Mills alanları sabit eğrilik koşulu ile birlikte kullanılmıştır. İkinci ve üçüncü çözümlerde ise, doğrusal ve doğrusal olmayan elektromanyetik alanlar kullanılmıştır. Kara delik oluşturan çözümlerin termodinamik özellikleri incelenmiştir. Doğrusal elektromanyetik alanların kullanıldığı çözümlerde oluşan çıplak tekillikler, kuvantum mekaniksel olarak incelenmiştir. Oluşan çıplak tekillik; Klein-Gordon, Maxwell ve Dirac denklemlerini sağlayan kuvantum dalgalarıyla incelenmiştir. Klasik olarak oluşan tekilliğin kuvantum dalgalara karşı da tekil kaldığı gösterilmiştir.

Anahtar Kelimeler: Kara delik çözümleri, f(R) kütleçekim modeli, Yang-Mills alanları, Doğrusal ve doğrusal olmayan elektromanyetik alanlar, Kuvantum tekillikler

DEDICATION

To My Husband

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Chapter 1

INTRODUCTION

By the twentieth century scientists realized that Newton's theory which describes the motion of objects with speed much less than the speed of light was insufficient to describe the motion of the objects when their speed becomes close to the speed of light. As a result of this important observation, Einstein's special theory of relativity was developed and the equations governing the motion of particles close to the speed of light and that of Maxwell's equations describing electromagnetic fields were modified. The main contribution of Einstein's special theory of relativity was the concept of space and time. This concept became more important when the gravitational effects were taken into consideration, what is known today as Einstein's theory of General Relativity. More precisely, the main theme of the two theories was highlighted with the following assumptions.

1) Absolute space

2) Weak Equivalence principle which describes the equivalence of inertial and gravitational masses.

However, Einstein's theory states the following which changed our understanding of space and time forever. Einstein's theory is based on the following assumptions,

1) Principle of relativity, which means that the geometry of spacetime is not fixed and is a dynamical quantity.

2) Principle of equivalence, the inertial and gravitational masses are equal or it is impossible to distinguish between the effect of a gravitational field from those experienced in uniformly accelerated frames.

3) Principle of general relativity covariance (diffeomorphisim invariance, this principle implies that one is free to choose any set of coordinates to map spacetime and express the equations.)

4) Principle of causality, that each point of space-time should admit a universally valid notion of past, present and future. The minus sign in the metric implies causality, which means that only events in the past effect what is going on now.

Of course like any new theory, General Relativity needs experimental evidences to prove its validity. In literature, there are well known experimental tests to General Relativity such as Redshift of light (doppler effect), bending of light ray by sun (gravitational lensing), Perihelion precession of Mercury (time like trajectory) and Time delay.

Four dimensional classical Einstein's theory of relativity is described by the following action, which is known as Einstein-Hilbert action

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R + S_M, \tag{1.1}$$

where $\sqrt{-g}$ is the determinant of the metric, $\kappa = 8\pi G c^{-4}$, R is the Ricci scalar and S_M represents the action for the matter fields. From variation of the action, one can find The Einstein's equations.

The Einstein-Hilbert Lagrangian is rather singular from Hamiltonian point of view [1]. Astrophysical and cosmological observations indicate that the standard Einstein's theory does not fit exactly the observational data. Galactic, extra-galactic and cosmic scales show accelerated expansion of the Universe (low energy Universe). If General Relativity is correct, it seems that around 96% of the Universe should be in the form of dark energy and dark matter. Even if we assume that dark matter includes some particles we do not know the rest, 70% known as the dark energy. The simplest explanation for dark energy is based on a cosmological constant, which has two problems. First, there is a large discrepancy between observations and theoretical predictions on its value. Second problem concerns the coincidence problem (coincidence between the observed vacuum energy density and the current matter density)[2]. And also, searching for a unified theory that works for strong field in small scales and low energy for large scales amounts to construct a Quantum Theory of Gravity. GR is not renormalizable, therefore, it can not be conventionally quantized, so researchers go to Extended Theories of Gravity (ETG). Unfortunately a consistent (unitary and renormalizable) theory of gravity does not exist yet. We briefly review some of these Extended Theories.

1.1 Quantum Gravity

General Relativity deals with the classical geometry of spacetime, whereas quantum theory of gravity will in addition be a quantum theory of spacetime. In fact, Physicists have to look for systems under extreme conditions in which gravitational and quantum effects are on the same footing, the cases such as Black Holes and the Big Bang. The scale at which quantum gravity is necessary to describe space and time is called the Planck scale. Both string theory and loop quantum gravity are theories where space and time are effective on this tiny scale. ETGs may constitute serious approaches to a successful theory of quantum gravity.

The first attempt to quantize gravity is to use the canonical and covariant approach. Canonical formalism would be drived from Hamiltonian of GR and the canonical quantization procedure. This formalism does not need to introduce perturbative methods and hence preserves the geometric feature of GR. Covariant formalism uses quantum field theory concepts. In this approach, the metric would be separated into two parts; the flat part $\eta_{\mu\nu}$ and a dynamical part $h_{\mu\nu}$, as in using the standard techniques of perturbative quantum field theory. These formalisms lead to unstable states. Hence, their Hamiltonian does not have a ground state of energy. In particular, unitarity is violated and probability is not conserved. These two approaches do not lead to a well defined theory of quantum gravity [1].

1.2 String Theory

Late time cosmology might be predicted by some basic theory such as string theory. In this theory the concept of particle is replaced by an extended object, the fundamental string. The usual physical particle, including the spin two graviton, corresponds to excitation of the string, reproduce GR in the low energy limit. The theory has only one free parameter. This theory needs an additional fundamental field like the dilaton field, which can be interpreted as the building block of the theory.

1.3 Supergravity

The basic idea of Supergravity is the unification of the Electromagnetic and the weak interactions. This theory makes it possible to construct a consistent theory when graviton is coupled to some kind of matter fields. This theory works only at the graviton-graviton interactions (two loop level) and for matter-gravity (one-loop diagram) coupling. But, including higher order loops, destroying the renormalizablity of the theory.

1.4 f(R) Gravity

$f(R)$ gravity is a modified version of standard Einstein's gravity which incorporates an arbitrary function of the Ricci scalar R instead of the linear one.

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R) + S_M, \tag{1.2}$$

Depending only on the Ricci scalar may sound simpler at the beginning but the pertinent nonlinearity makes nothing simpler than the Einstein's gravity with sources. There are both advantages and disadvantages in adopting such a model. Curvature source constitutes its own source known as in the absence of an external matter source, S_M . Due to the nature of nonlinear structure, the identification of physical sources is not an easy task at all. The resulting complexity in the related field equations, sometimes may not admit exact analytic solutions for the function $f(R)$.

In GR curvature is related through Einstein's equation to the matter/energy density. In other words, GR says that

$$\text{Gravity} = \text{Geometry}$$

and

$$\text{Geometry} = \text{Matter-Energy}$$

The question is, could the missing energy required by acceleration be an incomplete description of how matter determines geometry? Modified gravity is an alternative theory to answer this question. Modified gravity is formally equivalent to dark energy. The corresponding field equations are;

$$\begin{gathered} F\left(g_{\mu\nu}\right) + G_{\mu\nu} = \kappa T_{\mu\nu}^{M}, \\ -F\left(g_{\mu\nu}\right) = \kappa T_{\mu\nu}^{DE} \rightarrow G_{\mu\nu} = \kappa\left(T_{\mu\nu}^{M} + T_{\mu\nu}^{DE}\right), \end{gathered} \tag{1.3}$$

in which $T_{\mu\nu}^{M}$ and $T_{\mu\nu}^{DE}$ represents energy momentum tensor for matter and dark energy respectively. The Bianchi identity guarantees

$$\nabla^{\mu} T_{\mu\nu}^{DE} = 0. \tag{1.4}$$

Let us mention also that there are several models of $f\left(R\right)$ gravity and not all models are successful. R^{N} (N=an arbitrary number) model fails to produce a late time acceleration in the Universe or a matter domination era [3, 4].

1.5 Scalar - Tensor theory

If one is willing to consider some form of dark energy other than the cosmological constant by modifying gravity involves introducing new long range forces. The simplest option is a scalar field, playing the role of dark energy. It is very natural to think the scalar field might be coupled to matter, be aware that scalar does not couple to photon; photons bend in a gravitational field but not in a scalar field. Quintessence (scalar field) is dynamic whose equation of state is given by $\omega_q = \frac{p_q}{\rho_q}$ (in which p_q is a pressure and ρ_q is a density of matter), while cosmological constant is static with a fixed energy density given by $\omega_q = -1$. The theory that generalizes Einstein's theory with this new field is the Scalar - Tensor theory. If one wants to write a Lagrangian in the Scalar- Tensor theory, it can be written as,

$$S = S_{GR} + \int d^4x \left(-\frac{1}{2}\partial\varphi^2 + L_{int}(\varphi) \right) + \int d^4x \left[h_{\mu\nu}T_m^{\mu\nu} + \varphi T_m \right], \qquad (1.5)$$

where the first term is gravitational action and second integral is a scalar action including self interaction. The last integral, includes two terms, the first one is graviton-matter coupling and the second is scalar -matter coupling. Suppose the transformation between the two frames is always well defined, the results obtained should have the same physical description in both. We shall see that with a conformal transformation from $f(R)$ gravity in Jordan frame (the frame in which the form of the gravitational part of the action may be modified.) to Einstein frame (the frame in which the gravitation part of the action is the same as in standard GR), the scalar field is the first derivative of $f(R)$, i.e.

$$\varphi = \frac{df(R)}{dR}, \tag{1.6}$$

Actually, modifications of gravity will introduce new propagation degrees of freedom named as scalaron [5].

The prototypical scalar-tensor alternative to GR is the Brans-Dicke theory, which contains a scalar field. It has been shown by Hawking in 1972 that black holes which are the end point of collapse can be a solution of Brans-Dicke theory if and only if they are also solutions of GR [6].

Our main motivation in this thesis is to obtain new analytic solutions in the Extended Theory of Gravity (ETG). One of the important branches of the ETG is the $f(R)$ gravity in which the Ricci scalar R in the Einstein - Hilbert action is replaced by an arbitrary function of R. As was explained earlier, the main idea of developing this new model of gravity was to explain the accelerated expansion of our universe.

In this thesis, we consider this model of gravity in the presence of two kinds of matter fields. First, we consider minimally coupled Yang - Mills fields, and present some exact solutions at the constant curvature condition (i. e. R=constant). Another solution is also obtained in the presence of linear and nonlinear electromagnetic fields. Each solution has its own characteristic properties such that depending on the parameters the obtained solutions may admit black holes or solutions with no horizon. holes.

The physical properties of the black hole solutions are investigated by calculating thermodynamic quantities and it was shown to satisfy the first law of thermodynamics. The solution that results with naked singularities are studied in the frame of quantum mechanics.

Organization of the thesis is as follows. We start with chapter 2, that is about $f(R)$ gravity and different approaches in this theory. In chapter 3, the new exact analytic solutions to $f(R)$ gravity which is minimally coupled with Yang-Mills field at the constant curvature conditions (i. e. $R=$constant) is presented. The energy conditions are also discussed. In chapter 4, another interesting exact solution in $f(R)$ gravity is obtained in the presence of linear / nonlinear electromagnetic fields. Thermodynamic properties of the solutions admitting black holes is also considered. Chapter 5, deals with the analysis of the singularity structure of the solutions that results with naked singularity. The classical curvature singularity is investigated within the framework of quantum mechanics. The thesis is finalized with a conclusion in chapter 6.

Chapter 2

$f(R)$ GRAVITY

The Universal gravitational interaction depends on the curvature of spacetime. Geometric theory of gravitational interactions in GR is a Riemannian manifold in which metric is a fundamental geometric entity. But as we shall see in the following, in some approaches it is possible to put metric and connections as independent geometric entities. In this theory, there are three different formalisms. In this chapter, we aim to explain these formalisms.

2.1 Metric Formalism

The action for this formalism is given by

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R) + S_M, \tag{2.1}$$

where S_M denotes the matter field and $\kappa = 8\pi Gc^{-4}$. In this formalism variation of the action with respect to metric $g_{\mu\nu}$ gives

$$f^{'}(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \left(\nabla_\mu\nabla_\upsilon - g_{\mu\nu}\Box\right)f^{'}(R) = \kappa T_{\mu\nu}, \tag{2.2}$$

in which prime denote derivative with respect R, $\Box=\nabla_\mu\nabla^\mu=\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\right)$ and the energy momentum tensor may be written as,

$$T_{\mu\nu}=\frac{-2}{\sqrt{-g}}\frac{\partial S_M}{\partial g^{\mu\nu}}. \tag{2.3}$$

Hence, the Einstein's equations can be written as

$$\begin{aligned} G_{\mu\nu} &\equiv R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R \\ &= G_{eff}\left(T_{\mu\nu}+T_{\mu\nu}^{eff}\right), \qquad G_{eff}\equiv\frac{G}{f'(R)} \end{aligned} \tag{2.4}$$

where

$$T_{\mu\nu}^{eff}=\frac{1}{\kappa}\left[\frac{f(R)-Rf'(R)}{2}g_{\mu\nu}+\left(\nabla_\mu\nabla_\upsilon-g_{\mu\nu}\Box\right)f'(R)\right] \tag{2.5}$$

$$f'(R)R-2f(R)+3\Box f'(R)=\kappa T. \tag{2.6}$$

Here $T=g^{\mu\nu}T_{\mu\nu}$ links R with T as differential relations, unlike algebraic one in GR, where $R=\kappa T$. The field equations in $f(R)$ gravity allow much wider set of solutions compared to GR. To illustrate such a statement we remark the Jebsen-Birkhoff's theorem, stating that Schwarzschild solution represents the unique spherically symmetric vacuum solution (even if the field source involves radial motions, the field beyond the region occupied by matter remains constant and is always described by the Schwarzschild solution), does not hold in the $f(R)$ theory. We note that $T=0$ does not require that R =constant [7].

2.2 Palatini Formalism

The manifold that is chosen in GR, is Riemannian manifold with its own properties (like parallel transport and connections). In other words, in this manifold, metric is symmetric, non-singular, tensor and connections also are symmetric. However, for a Palatini formalism (metric-affine theory of gravity) this is not necessarily true. Since the metric and the connection are independent, the metric can be symmetric without the connection being symmetric as well. We start with the following action

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R) + S_M\left(g_{\mu\nu},\psi\right), \tag{2.7}$$

S_M, is assumed to depend only on the metric and the matter fields and not on the independent connection. So after variation the action with respect to metric gives

$$f^{'}(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{2.8}$$

and variation with respect to the connection gives

$$\nabla_\sigma\left(\sqrt{-g}f^{'}(R)g^{\sigma(\mu}\right)\delta_\lambda^{\upsilon)} - \nabla_\lambda\left(\sqrt{-g}f^{'}(R)g^{\mu\nu}\right) = 0. \tag{2.9}$$

By taking the trace of above equation, we get

$$\nabla_\lambda\left(\sqrt{-g}f^{'}(R)g^{\mu\nu}\right) = 0. \tag{2.10}$$

One can see that it is possible to introduce a new metric such as $h_{\mu\nu} = f^{'}(R)\, g_{\mu\nu}$. And $\Gamma^{\lambda}_{\mu\nu}$ becomes the Levi-Civita connection of $h_{\mu\nu}$, i.e.,

$$\Gamma^{\lambda}_{\mu\nu} = \frac{h^{\lambda\rho}}{2}\left(\partial_{\mu} h_{\rho\nu} + \partial_{\nu} h_{\rho\mu} - \partial_{\rho} h_{\mu\nu}\right). \tag{2.11}$$

Let us discuss about this formalism more in the simplest case, when we have a vacuum or Maxwell fields. In these cases trace of energy momentum is zero, i.e. $T = 0$. So 2.8 becomes

$$f^{'}(R)\, R - 2f(R) = 0, \tag{2.12}$$

this equation can have three solutions. The first one is the equation that has no real solution. For the second one, we consider special choice of $f(R)$. If

$$f(R) = \alpha R^{2}, \tag{2.13}$$

the homogeneous and isotropic cosmology in R^{N} gravity seems to show a complete disagreement with what is required to explain to show the current feature of the universe. In a paper by L. G. Jaime et al. they showed with numerical evidence that R^{N} gravity produces a late time acceleration in the universe or a matter dominant era but not both[3, 4]. Beside that this model seems to have a solution when the trace of energy momentum is zero ($T = 0$). The third possibility is that it admits a solution

R =constant, so $f^{'}(R)$ is also a constant. We can use 2.10, to have

$$\nabla_{\lambda}\left(\sqrt{-g}g^{\mu\nu}\right)=0, \tag{2.14}$$

This is the metricity condition for the affine connections, $\Gamma^{\lambda}_{\mu\nu}$ in this new frame. So the affine connections now become the Levi-civita connections of metric. With $\Gamma^{\lambda}_{\mu\nu}=\left\{ {\lambda \atop \mu\nu} \right\}$. Then 2.8 reads

$$R_{\mu\nu}-\frac{1}{4}C_{i}g_{\mu\nu}=0, \tag{2.15}$$

which is exactly Einstein field equation with a cosmological constant (maximally symmetric solution corresponds to the space of deSitter or anti-deSitter). This is not the case if one uses the metric variational principle.

2.3 Metric-Affine Formalism

In Palatini formalism we assumed that matter depended only on the metric and the matter fields. Here in this approach we suppose the matter in addition depends on connections as well. As usual we start with the action

$$S=\frac{1}{2\kappa}\int d^{4}x\sqrt{-g}f(R)+S_{M}\left(g_{\mu\nu},\psi,\Gamma\right). \tag{2.16}$$

We apply variation with respect to metric and connection for matter, respectively. We obtain

$$T_{\mu\nu}=-\frac{2}{\sqrt{-g}}\frac{\partial S_{M}}{\partial g^{\mu\nu}},\ \text{and}\ \Delta^{\mu\nu}_{\lambda}=-\frac{2}{\sqrt{-g}}\frac{\partial S_{M}}{\partial\Gamma^{\lambda}_{\mu\nu}}, \tag{2.17}$$

for some matters like scalar field and electromagnetic field $\Delta_{\lambda}^{\mu\nu}=0$. One example for non-zero torsion is massive vector field or Dirac field [8, 9]. Generally, if we make variation with respect to metric and connection, respectively, for our action, it gives us

$$f^{'}(R)R_{\mu\nu}-\frac{1}{2}f(R)g_{\mu\nu}=\kappa T_{\mu\nu}, \tag{2.18}$$

$$\begin{aligned}\frac{1}{\sqrt{-g}}\left[\nabla_{\sigma}\left(\sqrt{-g}f^{'}(R)g^{\sigma(\mu}\right)\delta_{\lambda}^{\upsilon)}-\nabla_{\lambda}\left(\sqrt{-g}f^{'}(R)g^{\mu\nu}\right)\right]\\ +2f^{'}(R)g^{\mu\sigma}S_{\sigma\lambda}^{\nu}=\kappa\left(\Delta_{\lambda}^{\mu\nu}-\frac{2}{3}\Delta_{\sigma}^{\sigma[\nu}\delta_{\lambda}^{\mu]}\right),\end{aligned} \tag{2.19}$$

$$S_{\mu\sigma}^{\sigma}=0, \tag{2.20}$$

in which

$$\begin{aligned} S_{\mu\nu}^{\lambda} &\equiv \Gamma_{[\mu\nu]}^{\lambda} \quad \text{Cartan torsion tensor,}\\ Q_{\mu\nu\lambda} &\equiv -\nabla_{\mu}g_{\nu\lambda} \quad \text{Non-metricity tensor,}\\ Q_{\mu} &\equiv \frac{1}{4}Q_{\mu\nu}^{\nu} \quad \text{Weyl tensor.}\end{aligned}$$

[8, 9].

2.4 Stability Issues

For the sake of the stability analysis we prefer to consider modified gravity from Einstein gravity as

$$f(R)=R+\Delta(R), \tag{2.21}$$

so the trace of 2.6 from metric formalism can be written as

$$\Box\Delta_R = \frac{1}{3}\left[R + 2\Delta - \Delta_R R\right] + \frac{\kappa}{3}T \equiv \frac{\partial V_{eff}}{\partial \Delta_R}, \tag{2.22}$$

which is a kinetic term and an effective potential $V_{eff}\left(\Delta_R\right)$. We must have $|\Delta \ll R|$ and $|\Delta_R| \ll 1$ at high curvatures to be consistent with our knowledge of the high redshift universe or in other words

$$\lim_{R\to\infty} \Delta_R = 0 \;\text{ and }\; \lim_{R\to\infty} \frac{\Delta}{R} = 0. \tag{2.23}$$

In this limit, the extremum of the effective potential lies at the GR value $R = \kappa T$ [10]. Whether this extremum is a minimum or a maximum is determined by the second derivative of V_{eff} at the extremum, which is also the squared mass of the scalaron:

$$m^2_{\Delta_R} \equiv \frac{\partial^2 V_{eff}}{\partial \Delta_R^2} = \frac{1}{3}\left[\frac{1+\Delta_R}{\Delta_{RR}} - R\right]. \tag{2.24}$$

At high curvature, when $|R\Delta_{RR}| \ll 1$ and $\Delta_R \to 0$ it approximates to

$$m^2_{\Delta_R} \approx \frac{1+\Delta_R}{3\Delta_{RR}} \approx \frac{1}{3\Delta_{RR}}. \tag{2.25}$$

It then follows that in order for the scalaron not to be tachyonic one must require $\Delta_{RR} > 0$. Classically, $\Delta_{RR} > 0$ is required to keep the evolution in the high-curvature regime stable against small perturbations.

At the end, the requirement that the graviton is not a ghost, massive of negative norm that cause apparent lack of unitarity, imposes that

$$1+\Delta_R > 0, \tag{2.26}$$

The most direct interpretation of this condition is that the effective Newton constant, $G_{eff} = \frac{G}{1+\Delta_R}$, is not allowed to change sign.

For Palatini $f(R)$ gravity the trace equation of 2.8 is

$$f^{'}(R)R - 2f(R) = \kappa T, \tag{2.27}$$

In contrast to the metric case, 2.27 is not an evolution equation for R; it is not even a differential equation but rather an algebraic equation in R once the function $f(R)$ is specified. This is also the case in GR, in which the Einstein field equations are of second order and taking their trace yields $R = \kappa T$. In analogy with Brans-Dicke theory the scalar field φ of the equivalent $\omega_0 = -3/2$ (for Palatini approach $\omega_0 = -3/2$ and metric formalism $\omega_0 = 0$) is not dynamical; which reduces to GR in vacuum. Therefore the instability cannot occur in Palatini f(R) gravity[5, 8, 11, 12, 13].

Chapter 3

CONSTANT CURVATURE $f(R)$ GRAVITY MINIMALLY COUPLED WITH YANG-MILLS FIELD

In this chapter, we consider a particular class within minimally coupled YM field in $f(R)$ gravity with the conditions that the scalar curvature $R = R_0 =$ constant and the trace of the YM energy-momentum tensor is zero. Contrary to our expectations this turns out to be a non-trivial class with far-reaching consequences. Our spacetime is chosen spherically symmetric to be in accord with the spherically symmetric Wu-Yang ansatz for the YM field. The field equations admit exact solutions in all dimensions $d \geq 4$ with the physical parameters; mass (m) of the black hole, YM charge (Q) and the scalar curvature (R_0) of the space time. In this picture we note that cosmological constant arises automatically as proportional to R_0.

3.1 The Formalism and Solution for $R =$Constant

We choose the action as (Our unit convention is chosen such that $c = G = 1$ so that $\kappa = 8\pi$)

$$S = \int d^d x \sqrt{-g} \left[\frac{f(R)}{2\kappa} + \mathcal{L}(F) \right], \tag{3.1}$$

where $f(R)$ is a real function of Ricci scalar R and $L(F)$ is the nonlinear YM Lagrangian with $F = \frac{1}{4} tr \left(F_{\mu\nu}^{(a)} F^{(a)\mu\nu} \right)$. Obviously the particular choice $\mathcal{L}(F) = -\frac{1}{4\pi} F$

will reduce to the case of standard YM theory. The YM field $2-$form components are given by

$$\mathbf{F}^{(a)} = \frac{1}{2} F_{\mu\nu}^{(a)} dx^{\mu} \wedge dx^{\nu} \tag{3.2}$$

with the internal index (a) running over the degrees of freedom of the nonabelian YM gauge field. Variation of the action with respect to the metric $g_{\mu\nu}$ gives the EYM field equations as

$$f_R R_{\mu}^{\nu} + \left(\Box f_R - \frac{1}{2} f \right) \delta_{\mu}^{\nu} - \nabla^{\nu} \nabla_{\mu} f_R = \kappa T_{\mu}^{\nu} \tag{3.3}$$

in which

$$T_{\mu}^{\nu} = \mathcal{L}(F) \delta_{\mu}^{\nu} - tr \left(F_{\mu\alpha}^{(a)} F^{(a)\nu\alpha} \right) \mathcal{L}_F(F), \tag{3.4}$$
$$\mathcal{L}_F(F) = \frac{d\mathcal{L}(F)}{dF}.$$

Our notation here is as follows: $f_R = \frac{df(R)}{dR}$, $\Box f_R = \nabla_{\mu} \nabla^{\mu} f_R = \frac{1}{\sqrt{-g}} \partial_{\mu} \left(\sqrt{-g} \partial^{\mu} \right) f_R$, R_{μ}^{ν} is the Ricci tensor and

$$\nabla^{\nu} \nabla_{\mu} f_R = g^{\alpha\nu} \left(f_R \right)_{,\mu;\alpha} = g^{\alpha\nu} \left[\left(f_R \right)_{,\mu,\alpha} - \Gamma_{\mu\alpha}^{m} \left(f_R \right)_{,m} \right]. \tag{3.5}$$

The trace of the EYM equation 3.3 yields

$$f_R R + (d-1) \Box f_R - \frac{d}{2} f = \kappa T \tag{3.6}$$

in which $T=T_{\mu}^{\mu}$. The $SO(d-1)$ gauge group YM potentials are given by

$$\mathbf{A}^{(a)}=\frac{Q}{r^2}C_{(i)(j)}^{(a)}\,x^i dx^j,\quad Q=\text{YM magnetic charge},\;\; r^2=\sum_{i=1}^{d-1}x_i^2, \tag{3.7}$$

$$2\leq j+1\leq i\leq d-1,\;\;\text{and}\;\;1\leq a\leq(d-2)(d-1)/2,$$

$$x_1=r\cos\theta_{d-3}\sin\theta_{d-4}...\sin\theta_1,\; x_2=r\sin\theta_{d-3}\sin\theta_{d-4}...\sin\theta_1,$$

$$x_3=r\cos\theta_{d-4}\sin\theta_{d-5}...\sin\theta_1,\; x_4=r\sin\theta_{d-4}\sin\theta_{d-5}...\sin\theta_1,$$

$$...$$

$$x_{d-2}=r\cos\theta_1,$$

in which $C_{(b)(c)}^{(a)}$ are the non-zero structure constants of $\frac{(d-1)(d-2)}{2}-$parameter Lie group $\mathcal{G}$ [14, 15, 16]. The metric ansatz is spherically symmetric which reads

$$ds^2=-A(r)\,dt^2+\frac{dr^2}{A(r)}+r^2d\Omega_{d-2}^2, \tag{3.8}$$

with the only unknown function $A(r)$ and the solid angle element

$$d\Omega_{d-2}^2=d\theta_1^2+\sum_{i=2}^{d-2}\prod_{j=1}^{i-1}\sin^2\theta_j\,d\theta_i^2, \tag{3.9}$$

with

$$0\leq\theta_{d-2}\leq 2\pi, 0\leq\theta_i\leq\pi,\quad 1\leq i\leq d-3.$$

Variation of the action with respect to $\mathbf{A}^{(a)}$ implies the YM equations

$$\mathbf{d}\left[{}^{\star}\mathbf{F}^{(a)}L_F(F)\right]+\frac{1}{\sigma}C_{(b)(c)}^{(a)}L_F(F)\,\mathbf{A}^{(b)}\wedge{}^{\star}\mathbf{F}^{(c)}=0, \tag{3.10}$$

in which σ is a coupling constant and * means duality. One may show that the YM invariant satisfies

$$F=\frac{1}{4}tr\left(F_{\mu\nu}^{(a)}F^{(a)\mu\nu}\right)=\frac{(d-2)(d-3)Q^{2}}{4r^{4}} \tag{3.11}$$

and

$$tr\left(F_{t\alpha}^{(a)}F^{(a)t\alpha}\right)=tr\left(F_{r\alpha}^{(a)}F^{(a)r\alpha}\right)=0, \tag{3.12}$$

while

$$tr\left(F_{\theta_{i}\alpha}^{(a)}F^{(a)\theta_{i}\alpha}\right)=\frac{(d-3)Q^{2}}{r^{4}}, \tag{3.13}$$

which leads us to the exact form of the energy momentum tensor

$$T_{\mu}^{\nu}=\mathrm{diag}\left[\mathcal{L},\mathcal{L},\mathcal{L}-\frac{(d-3)Q^{2}}{r^{4}}\mathcal{L}_{F},\mathcal{L}-\frac{(d-3)Q^{2}}{r^{4}}\mathcal{L}_{F},...,\mathcal{L}-\frac{(d-3)Q^{2}}{r^{4}}\mathcal{L}_{F}\right]. \tag{3.14}$$

Here the trace of T_{μ}^{ν} becomes

$$T=T_{\mu}^{\mu}=d\mathcal{L}-4F\mathcal{L}_{F}, \tag{3.15}$$

and therefore with 3.3 we find

$$f=\frac{2}{d}\left[f_{R}R+(d-1)\Box f_{R}-\kappa\left(d\mathcal{L}-4F\mathcal{L}_{F}\right)\right]. \tag{3.16}$$

To proceed further we set the trace of energy momentum tensor to be zero i.e.,

$$d\mathcal{L} - 4F\mathcal{L}_F = 0 \tag{3.17}$$

which leads to a power Maxwell Lagrangian [17, 18, 19, 20, 21]

$$\mathcal{L} = -\frac{1}{4\pi}F^{\frac{d}{4}}. \tag{3.18}$$

Here for our convenience the integration constant is set to be $-\frac{1}{4\pi}$. On the other hand, the constant curvature $R = R_0$, and the zero trace condition together imply

$$f'(R_0)R_0 - \frac{d}{2}f(R_0) = 0. \tag{3.19}$$

This equation admits

$$f(R_0) = R_0^{\frac{d}{2}}, \tag{3.20}$$

where the integration constant is set to be one. One can easily write the Einstein equations as (analogy with equations 2.4 and 2.5)

$$G_\mu^\nu = \kappa\, \tilde{T}_\mu^\nu \tag{3.21}$$

where

$$\tilde{T}_\mu^\nu = \frac{2R_0}{f(R_0)\,d}T_\mu^\nu - \frac{\Lambda_{eff}}{\kappa}\delta_\mu^\nu, \tag{3.22}$$

$$\Lambda_{eff} = \frac{(d-2)R_0}{2d}, \tag{3.23}$$

and in which T_{μ}^{ν} is given by 3.4. The constancy of the Ricci scalar amounts to

$$-\frac{r^2A''+2(d-2)rA'+(d-2)(d-3)(A-1)}{r^2}=R_0 \tag{3.24}$$

which yields

$$A=1-\frac{R_0}{d(d-1)}r^2-\frac{m}{r^{d-3}}+\frac{\sigma}{r^{d-2}}, \tag{3.25}$$

where σ and m are two integration constants. From the Einstein equations one identifies the constant σ as

$$\sigma=\frac{8}{d(d-2)R_0^{\frac{d-2}{2}}}\left(\frac{(d-3)(d-2)Q^2}{4}\right)^{\frac{d}{4}}. \tag{3.26}$$

In the next section we study solutions in 4−dimensions.

3.2 4−dimensions

In 4−dimensions, we know that the nonabelian $SO(3)$ gauge field coincides with the abelian $U(1)$ Maxwell field [22]. Due to its importance we shall study the 4−dimensional case separately and give the results explicitly. First of all, in 4−dimensions the metric function becomes

$$A=1-\frac{R_0}{12}r^2-\frac{m}{r}+\frac{Q^2}{2R_0r^2}, \qquad 0<|R_0|<\infty \tag{3.27}$$

and the form of action reads as

$$S=\int d^4x\sqrt{-g}\left[\frac{f(R)}{2\kappa}+\mathcal{L}(F)\right] \tag{3.28}$$

in which

$$f(R) = R^2, \tag{3.29}$$

with $R = R_0$ and

$$\mathcal{L}(F) = -\frac{1}{4\pi}F. \tag{3.30}$$

By assumption, R_0 gets positive / negative values and the resulting spacetime becomes de-Sitter / anti de-Sitter, type in $f(R) = R^2$ theory respectively, with effective cosmological constant $\Lambda_{eff} = \frac{R_0}{4}$. Let us add that in order to preserve the sign of the charge term in 3.27 we must abide by the choice $R_0 > 0$. However, simultaneous limits $Q^2 \to 0$ and $R_0 \to 0$, so that $\frac{Q^2}{R_0} = \lambda_0$ =constant, leads also to an acceptable solution within $f(R)$ gravity [23]. It is not difficult to see here that m is the ADM mass of the resulting black hole. Viability of the pure $f(R) = R^2$ model which has recently been considered critically [4] is known to avoid the Dolgov-Kawasaki instability [12]. Further, in the late time behaviour of the expanding universe (i.e. for $r \to \infty$) it asymptotes to the de Sitter / anti de Sitter form. With reference to [4] we admit that sourceless $f(R) = R^2$ model doesn't possess a good record as far as the Solar System tests are concerned. Herein we have sources and wish to address the universe at large. In the next section we investigate energy conditions for these solutions in $4-$dimensions.

3.3 Energy Conditions

When a matter field couples to any system, energy conditions must be satisfied for physically acceptable solutions. The local energy density as measured by an observer with velocity u is nonnegative and the local energy flow vector q is non-spacelike. This condition must be satisfied if we replace u by a null vector k. We study energy conditions by calculating Weak, Strong and Dominant energy condition [24, 25, 26]. The Weak Energy Conditions (WEC) states that

$$\begin{aligned} \rho &\geq 0, \qquad (3.31) \\ \rho + p_i &\geq 0. \end{aligned}$$

in which ρ is the energy density of matter as measured by an observer. And p_i are the principal pressure components. The Strong Energy Conditions (SEC) imply

$$\begin{aligned} \rho + \sum_{i=1}^{d-1} p_i &\geq 0, \qquad (3.32) \\ \rho + p_i &\geq 0. \end{aligned}$$

In The Dominant Energy Condition (DEC), the effective pressure must not be negative, i. e.,

$$P_{eff} = \frac{1}{d-1} \sum_{i=1}^{d-1} T_i^i \geq 0, \qquad (3.33)$$

In addition to the energy conditions one can impose the Causality Condition (CC)

$$0 \leq \frac{P_{eff}}{\rho} < 1. \qquad (3.34)$$

Finally we introduce $\omega = \frac{P_{eff}}{\rho}$, which is bounded as

$$-1 \leq \omega < \frac{1}{d-1}. \qquad (3.35)$$

3.3.1 Energy Conditions for d -dimensions

We analyze the energy conditions thoroughly covering all dimensions for two cases, when $R_0 > 0$ and $R_0 < 0$.

3.3.1.1 $\underline{R_0 > 0.}$ The energy density and the pressure components given by

$$\begin{aligned}
\rho &= -\tilde{T}_0^0 = \frac{R_0}{2\pi d}\left(\frac{F^{\frac{d}{4}}}{R_0^{\frac{d}{2}}} + \frac{(d-2)}{8}\right), \\
p_i &= \tilde{T}_i^i = \frac{R_0}{2\pi d}\left(\frac{2}{(d-2)}\frac{F^{\frac{d}{4}}}{R_0^{\frac{d}{2}}} - \frac{(d-2)}{8}\right), \quad i = 2, \cdots, (d-1), \\
p_1 &= \tilde{T}_1^1 = -\frac{R_0}{2\pi d}\left(\frac{F^{\frac{d}{4}}}{R_0^{\frac{d}{2}}} + \frac{(d-2)}{8}\right).
\end{aligned} \tag{3.36}$$

As we can see WEC 3.31, is held. For SEC, 3.32, it is shown that the second condition is satisfied but first condition implies that

$$\rho + \sum_{i=1}^{d-1} p_i = \frac{R_0}{2\pi d}\left(2\frac{F^{\frac{d}{4}}}{R_0^{\frac{d}{2}}} - \frac{(d-2)^2}{8}\right) \geq 0 \tag{3.37}$$

or consequently

$$\left(2\left(\frac{F}{R_0^2}\right)^{\frac{d}{4}} - \frac{(d-2)^2}{8}\right) \geq 0. \tag{3.38}$$

By a substitution from 3.11 for F one finds that for $r < r_c$ the condition is satisfied in which

$$r_c = \sqrt[d]{\frac{16}{(d-2)^2}}\sqrt[4]{\frac{(d-2)(d-3)Q^2}{4R_0^2}}. \tag{3.39}$$

To have a positive effective pressure, DEC 3.33, amounts to

$$P_{eff} = \frac{1}{d-1}\sum_{i=1}^{d-1} T_i^i = \frac{1}{(d-1)}\frac{R_0}{2\pi d}\left(\frac{F^{\frac{d}{4}}}{R_0^{\frac{d}{2}}} - \frac{(d-2)(d-1)}{8}\right) \geq 0, \tag{3.40}$$

which for $r < \tilde{r}_c$ it is fulfilled in which

$$\tilde{r}_c = \sqrt[d]{\frac{8}{(d-2)(d-1)}}\sqrt[4]{\frac{(d-2)(d-3)Q^2}{4R_0^2}}. \tag{3.41}$$

One can find the causality condition, 3.34, such as

$$0 \leq \frac{P_{eff}}{\rho} = \frac{\left(F^{\frac{d}{4}}R_0^{\frac{-d}{2}} - \frac{(d-2)(d-1)}{8}\right)}{(d-1)\left(F^{\frac{d}{4}}R_0^{\frac{-d}{2}} + \frac{(d-2)}{8}\right)} < 1. \tag{3.42}$$

This is equivalent to

$$F^{\frac{d}{4}}R_0^{\frac{-d}{2}} - \frac{(d-2)(d-1)}{8} > 0 \tag{3.43}$$

which for $r < \tilde{r}_c$ is satisfied. The state function for this case becomes

$$\omega = \frac{\left(\left(\frac{F}{R_0^2}\right)^{\frac{d}{4}} - \frac{(d-2)(d-1)}{8}\right)}{(d-1)\left(\left(\frac{F}{R_0^2}\right)^{\frac{d}{4}} + \frac{(d-2)}{8}\right)}, \tag{3.44}$$

which is bounded as

$$-1 \leq \omega < \frac{1}{d-1}. \tag{3.45}$$

It is observed that

$$\begin{cases} 0 \leq \omega < \frac{1}{d-1} & \text{if} \quad r < \tilde{r}_c \\ -1 \leq \omega < 0 & \text{if} \quad \tilde{r}_c < r \end{cases}. \tag{3.46}$$

3.3.1.2 $\underline{R_0 < 0.}$ One may see, that presence of $R_0^{\frac{d}{2}}$ in the definition of ρ and p_i imposes that $d \neq 2n+1$ for $n = 2, 3, 4, ...$ For $d = 4n$, we get

$$\begin{aligned} \rho &= -\tilde{T}_0^0 = \frac{-|R_0|}{8\pi n}\left(\frac{F^n}{R_0^{2n}} + \frac{2n-1}{4}\right), \\ p_i &= \tilde{T}_i^i = \frac{-|R_0|}{8\pi n}\left(\frac{1}{2n-1}\frac{F^n}{R_0^{2n}} - \frac{2n-1}{4}\right), \\ p_1 &= \tilde{T}_1^1 = \frac{|R_0|}{8\pi n}\left(\frac{F^n}{R_0^{2n}} + \frac{2n-1}{4}\right), \end{aligned} \tag{3.47}$$

These expressions reveal that the condition $\rho \geq 0$ and $\rho + p_i \geq 0$ (WEC)are not satisfied. Similarly the SEC is also violated and since the source is exotic we shall not consider it any further here. A case of interest for $R_0 < 0$ is the choice $d = 4n+2$ for $n = 1, 2, 3, ...$ in which

$$\begin{aligned} \rho &= -\tilde{T}_0^0 = \frac{|R_0|}{4\pi(2n+1)}\left(\frac{F^{\frac{2n+1}{2}}}{|R_0|^{2n+1}} - \frac{n}{2}\right), \\ p_i &= \tilde{T}_i^i = \frac{|R_0|}{4\pi(2n+1)}\left(\frac{1}{2n}\frac{F^{\frac{2n+1}{2}}}{|R_0|^{2n+1}} + \frac{n}{2}\right), \quad i = 2, \cdots, (d-1), \\ p_1 &= \tilde{T}_1^1 = -\frac{|R_0|}{4\pi(2n+1)}\left(\frac{F^{\frac{2n+1}{2}}}{|R_0|^{2n+1}} - \frac{n}{2}\right). \end{aligned} \tag{3.48}$$

WEC: $\rho \geq 0$ yields

$$\frac{F^{\frac{2n+1}{2}}}{|R_0|^{2n+1}} - \frac{n}{2} \geq 0 \tag{3.49}$$

or

$$r < \bar{r}_c \tag{3.50}$$

where

$$\bar{r}_c = \sqrt[4n+2]{\frac{2}{n}}\sqrt[4]{\frac{n\,(4n-1)\,Q^2}{|R_0|^2}}. \tag{3.51}$$

SEC: The conditions are simply satisfied.

DEC: This amounts to

$$P_{eff} = \frac{1}{4n+1}\frac{|R_0|}{4\pi\,(2n+1)}\left(\frac{F^{\frac{2n+1}{2}}}{|R_0|^{2n+1}} + \frac{n}{2} + 2n^2\right) \geq 0, \tag{3.52}$$

which is also satisfied.

CC: The causality condition implies

$$0 \leq \frac{P_{eff}}{\rho} = \frac{\left(\frac{F^{\frac{2n+1}{2}}}{|R_0|^{2n+1}} + \frac{n}{2} + 2n^2\right)}{(4n+1)\left(\frac{F^{\frac{2n+1}{2}}}{|R_0|^{2n+1}} - \frac{n}{2}\right)} < 1, \tag{3.53}$$

or equivalently

$$|R_0|^{2n+1}\,\frac{1+4n}{4} < F^{\frac{2n+1}{2}} \tag{3.54}$$

which is satisfied for

$$r < \breve{r}_c \tag{3.55}$$

where

$$\breve{r}_c = \sqrt[4n+2]{\frac{4}{1+4n}}\sqrt[4]{\frac{n(4n-1)Q^2}{|R_0|^2}}. \tag{3.56}$$

Here the state function $\omega = \frac{P_{eff}}{\rho}$ becomes

$$\omega = \frac{\left(\frac{F^{\frac{2n+1}{2}}}{|R_0|^{2n+1}} + \frac{n}{2} + 2n^2\right)}{(4n+1)\left(\frac{F^{\frac{2n+1}{2}}}{|R_0|^{2n+1}} - \frac{n}{2}\right)}, \tag{3.57}$$

which is bounded as

$$-1 \leq \omega < \frac{1}{4n+1}. \tag{3.58}$$

One can show that

$$\begin{cases} 0 \leq \omega < \frac{1}{4n+1} & \text{if} \quad r < \bar{r}_c \\ -1 \leq \omega < 0 & \text{if} \quad \bar{r}_c < r \end{cases}. \tag{3.59}$$

3.3.2 Energy Condition for 4-dimensions

The energy density and the principal pressure are given as

$$\begin{aligned} \rho &= -\tilde{T}_0^0 = \frac{1}{8\pi R_0}\left(F + \frac{1}{4}R_0^2\right), \\ p_1 &= \tilde{T}_1^1 = -\frac{1}{8\pi R_0}\left(F + \frac{1}{4}R_0^2\right), \\ p_i &= \tilde{T}_i^i = \frac{1}{8\pi R_0}\left(F - \frac{1}{4}R_0^2\right), \quad i = 2, 3. \end{aligned} \tag{3.60}$$

These conditions imply that for $R_0 \geq 0$, both the WEC and SEC are satisfied. DEC implies, on the other hand, from 3.40 that

$$P_{eff} = \frac{1}{3}\sum_{i=1}^{3}\tilde{T}_i^i = \frac{1}{24\pi R_0}\left(F - \frac{3}{4}R_0^2\right) \geq 0, \tag{3.61}$$

which yields

$$R_0 \geq 0 \text{ and } F \geq \frac{3}{4}R_0^2 \rightarrow r \leq \sqrt[4]{\frac{2Q^2}{3R_0^2}}. \tag{3.62}$$

In addition to the energy conditions one can impose the causality condition (CC) from 3.42

$$0 \leq \frac{P_{eff}}{\rho} = \frac{\left(F - \frac{3}{4}R_0^2\right)}{3\left(F + \frac{1}{4}R_0^2\right)} < 1, \tag{3.63}$$

which is satisfied if $F \geq \frac{3}{4}R_0^2$ or $r \leq \sqrt[4]{\frac{2Q^2}{3R_0^2}}$.

By $\omega = \frac{P_{eff}}{\rho}$, one observes that in the range for $0 < r < \infty$ we have

$$-1 \leq \omega < \frac{1}{3}. \tag{3.64}$$

In terms of the physical parameters, if

$$\sqrt[4]{\frac{2Q^2}{3R_0^2}} \leq r \tag{3.65}$$

then $-1 \leq \omega \leq 0$, and if

$$\sqrt[4]{\frac{2Q^2}{3R_0^2}} > r \tag{3.66}$$

we have $0 < \omega < \frac{1}{3}$. It is clearly seen that the foregoing bounds serve to define possible critical distances where the sign of the effective pressure changes sign. This may be interpreted as changing phase for example, from contraction to expansion or vice versa in a universe centered by a black hole [27].

Chapter 4

SOLUTIONS FOR $f(R)$ GRAVITY COUPLED WITH THE ELECTROMAGNETIC FIELD

Starting from a known function of $f(R)$ a priori is an alternative approach which hosts its own shortcoming from the outset. Keeping a set of free parameters to be fixed by observational data can be employed in favour of $f(R)$ gravity to explain a number of cosmological phenomena. First of all, to be on the safe side along with the successes of general relativity most researchers prefer an ansatz of the form $f(R) = R + \alpha g(R)$, so that with $\alpha \to 0$ one recovers the Einstein limit. The struggle now is for the new function $g(R)$ whose equations are not easier than those satisfied by $f(R)$ itself.

We assume $f(R) = \xi(R + R_1) + 2\alpha\sqrt{R + R_0}$, in which ξ, α, R_0 and R_1 are constants, a priori to secure the Einstein limit by setting the constants $R_0 = R_1 = \alpha = 0$ and $\xi = 1$. This extends a previous study without sources [28, 29, 30] to the case with sources. Why the square root term in the Lagrangian?. It will be shown that for $R_0 = R_1 = 0$ and without external sources such a choice of square root Lagrangian gives the curvature energy-momentum tensor components as $T_t^t = T_r^r$, $T_\theta^\theta = T_\varphi^\varphi = 0$, which signify a global monopole [25, 31, 32, 33, 34]. A global monopole which arises from spontaneous breaking of gauge symmetry is the minimal structure that yields non-zero curvature even with zero mass. We test the analogous concept in $f(R)$ gravity to obtain similar structures. Unlike the case of [35] our concern here will be restricted to the 4−dimensional spacetime. As source, we take electromagnetic fields, both from the linear (Maxwell) and the nonlinear theories. For the linear Maxwell source we obtain a black hole solution with electric charge (Q) and magnetic charge (P) reminiscent of the Reissner-Nordstrom (RN) solution with different asymptotic behaviors. That is, our spacetime is non-asymptotically flat with a deficit angle. For the nonlinear, pure electric source we choose the standard Maxwell invariant superposed with the square root invariant, i.e. the Lagrangian is given by $\mathcal{L}(F) \sim F + 2\beta\sqrt{-F}$, where $F = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant and β is a coupling constant. This particular choice has the feature that it breaks the scale invariance [36, 37] , gives a linear electric potential which plays role in quark confinement [38]. We find out that the scale breaking parameter β modifies the mass of the black hole. For this reason Lagrangians supplemented by a square-root Maxwell Lagrangian may find rooms of applications in black hole physics.

4.1 $f(R)$ Gravity Coupled with Maxwell Field

The action for $f(R)$ gravity coupled with Maxwell field in 4-dimensions is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{f(R)}{2\kappa} - \frac{1}{4\pi}F\right] \tag{4.1}$$

in which $f(R)$ is a real function of Ricci scalar R and $F = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant. (We choose $\kappa = 8\pi$ and $G = 1$). The Maxwell two-form is chosen to be

$$\mathbf{F} = \frac{Q}{r^2}dt \wedge dr + P\sin\theta d\theta \wedge d\phi \tag{4.2}$$

where Q and P are the electric and magnetic charges, respectively. Our static spherically symmetric metric ansatz is

$$ds^2 = -A(r)\,dt^2 + \frac{dr^2}{A(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{4.3}$$

where $A(r)$ stands for the only metric function to be found. The Maxwell equations $($i.e. $dF = 0 = d^*F)$ are satisfied and the field equations are given by

$$f_R R_\mu^\nu + \left(\Box f_R - \frac{1}{2}f\right)\delta_\mu^\nu - \nabla^\nu\nabla_\mu f_R = \kappa T_\mu^\nu$$

our notation here is the same as in chapter 3, 3.3-3.5. The energy momentum tensor is

$$4\pi T_\mu^\nu = -F\delta_\mu^\nu + F_{\mu\lambda}F^{\nu\lambda}. \tag{4.4}$$

The non-zero energy momentum tensor components are

$$T_\mu^\nu = \frac{P^2 + Q^2}{8\pi r^4}diag\,[-1,-1,1,1] \tag{4.5}$$

with zero trace and consequently from 3.6 when d=4

$$f = \frac{1}{2}f_R R + 3\Box f_R. \tag{4.6}$$

One finds

$$R = -\frac{r^2A'' + 4rA' + 2(A-1)}{r^2}, \tag{4.7}$$

$$R_t^t = R_r^r = -\frac{1}{2}\frac{rA'' + 2A'}{r}, \tag{4.8}$$

$$R_\theta^\theta = R_\phi^\phi = -\frac{rA' + A - 1}{r^2}. \tag{4.9}$$

in which a prime denotes derivative with respect to r. Overall, the field equations read now

$$
\begin{aligned}
f_R\left(-\frac{1}{2}\frac{rA''+2A'}{r}\right)+\left(\Box f_R-\frac{1}{2}f\right)-\nabla^t\nabla_t f_R &= \kappa T_0^0, && (4.10)\\
f_R\left(-\frac{1}{2}\frac{rA''+2A'}{r}\right)+\left(\Box f_R-\frac{1}{2}f\right)-\nabla^r\nabla_r f_R &= \kappa T_1^1, && (4.11)\\
f_R\left(-\frac{rA'+(A-1)}{r^2}\right)+\left(\Box f_R-\frac{1}{2}f\right)-\nabla^\theta\nabla_\theta f_R &= \kappa T_2^2. && (4.12)
\end{aligned}
$$

Herein

$$
\Box f_R = A' f_R' + A f_R'' + \frac{2}{r} A f_R',\ \nabla^t\nabla_t f_R = \frac{1}{2}A' f_R',\ \nabla^r\nabla_r f_R = A f_R'' + \frac{1}{2}A' f_R',
$$

$$
\nabla^\phi\nabla_\phi f_R = \nabla^\theta\nabla_\theta f_R = \frac{A}{r} f_R' \tag{4.13}
$$

and for the details we refer to [2]. The tt and rr components of the field equations imply

$$
\nabla^r\nabla_r f_R = \nabla^t\nabla_t f_R \tag{4.14}
$$

or equivalently

$$
f_R'' = 0. \tag{4.15}
$$

This leads to the solution

$$
f_R = \xi + \eta r \tag{4.16}
$$

where ξ and η are two positive constants [8]. The other field equations become

$$f_R\left(-\frac{1}{2}\frac{rA''+2A'}{r}\right)+\frac{1}{2}\eta A'+\frac{2}{r}A\eta-\frac{1}{2}f = \kappa T_0^0, \tag{4.17}$$
$$f_R\left(-\frac{rA'+(A-1)}{r^2}\right)+A'\eta+\frac{1}{r}A\eta-\frac{1}{2}f = \kappa T_2^2. \tag{4.18}$$

Now, we make the choice

$$f(R)=\xi\left(R+\frac{1}{2}R_0\right)+2\alpha\sqrt{R+R_0} \tag{4.19}$$

which leads to

$$R=\frac{\alpha^2}{\eta^2 r^2}-R_0 \tag{4.20}$$

where α, R_0 and ξ from 4.16 are constants. As a result one obtains for $f(r)$

$$f=\frac{\xi\alpha^2}{\eta^2 r^2}+\frac{2\alpha^2}{\eta r}-\frac{1}{2}\xi R_0 \tag{4.21}$$

and from 4.7 we have

$$-\frac{r^2A''+4rA'+2(A-1)}{r^2}=\frac{\alpha^2}{\eta^2 r^2}-R_0. \tag{4.22}$$

This equation admits a solution for the metric function given by

$$A(r)=1-\frac{\alpha^2}{2\eta^2}+\frac{C_1}{r}+\frac{C_2}{r^2}+\frac{1}{12}R_0 r^2. \tag{4.23}$$

Herein the two integration constants C_1 and C_2 are identified through the other field equations 4.17 and 4.18 as

$$C_1 = \frac{\xi}{3\eta} \text{ and } C_2 = \frac{\left(Q^2 + P^2\right)}{\xi}, \tag{4.24}$$

while for the free parameters we have $\alpha = \eta > 0$. Finally the metric function becomes

$$A(r) = \frac{1}{2} - \frac{m}{r} + \frac{q^2}{r^2} - \frac{\Lambda_{eff}}{3} r^2 \tag{4.25}$$

where $m = -\frac{\xi}{3\eta} < 0$, $\Lambda_{eff} = \frac{-R_0}{4}$ and $q^2 = \frac{\left(Q^2+P^2\right)}{\xi}$. The choice of the free parameters in terms of each other prevents us from obtaining the general relativity limit, namely the Reissner-Nordström (RN)-de Sitter (dS) solution. It is observed that the parameter ξ acts as a scale factor for mass and charge and for the case $\xi = 1$ and $Q = P = 0$ the solution reduces to the known solution given by [28, 29, 30, 39, 40]. The properties of this solution can be summarized as follow: The mass term has the opposite sign to that of Schwarzschild and the solution is not asymptotically flat, giving rise to a deficit angle. The latter property is reminiscent of a global monopole term with a fixed charge. To see the case of a global monopole we set $R_0 = 0 = q^2$ (i.e. zero external charges and zero cosmological constant) and find the energy-momentum components. This reveals that the non-zero components are $T_t^t = T_r^r = -\frac{1}{2r^2}$, which identifies a global monopole [32, 33, 34]. The solution 4.25 can therefore be interpreted as an Einstein-Maxwell plus a global monopole solution in $f(R)$ gravity. The area of a sphere of radius r (for $q^2 = R_0 = 0$) is not $4\pi r^2$ but $2\pi r^2$. Further, it can be shown easily that the surface $\theta = \frac{\pi}{2}$ has the geometry of a cone with a deficit angle $\Delta = \frac{\pi}{2}$. It can also be anticipated that a global monopole modifies perihelion of circular orbits, light bending and other physical properties. Although in the linear Maxwell theory the sign of mass is opposite, in the next section we shall show that this can be overcome by going to the nonlinear electrodynamics with a square root Lagrangian. Another aspect of the solution is that since $f_R > 0$ we have no ghost states.

4.2 $f(R)$ Gravity Coupled with Nonlinear Electromagnetism

4.2.1 Solution within Nonlinear Electrodynamics

In this section we use an extended model for the Maxwell Lagrangian given in the action

$$S = \int d^4x\sqrt{-g}\left[\frac{f(R)}{2\kappa} + \mathcal{L}(F)\right] \tag{4.26}$$

where $f(R) = \xi(R+R_1) + 2\alpha\sqrt{R+R_0}$, in which R_1 and R_0 are constants to be found while

$$\mathcal{L}(F) = -\frac{1}{4\pi}\left(F + 2\beta\sqrt{-F}\right). \tag{4.27}$$

Here β is a free parameter such that $\lim_{\beta\to 0}\mathcal{L}(F) = -\frac{1}{4\pi}F$, which is the linear Maxwell Lagrangian. The main reason for adding this term is to break the scale invariance and create a mass term. The normal Maxwell action is known to be invariant under the scale transformation, $x \to \lambda x$, $A_\mu \to \frac{1}{\lambda}A_\mu$, ($\lambda =$const.), while $\sqrt{-F}$ violates this rule. We shall show how a similar term modifies the mass term in $f(R)$ gravity. Our choice of the Maxwell 2-form is written as

$$\mathbf{F} = E(r)\, dt \wedge dr \tag{4.28}$$

and the spherical line element as 4.3. The nonlinear Maxwell equation reads

$$d\left({}^{\star}\mathbf{F}\frac{\partial \mathcal{L}}{\partial F}\right) = 0 \tag{4.29}$$

which yields the solution

$$E(r) = \sqrt{2}\beta + \frac{Q}{r^2} \tag{4.30}$$

with a confining electric potential as $V(r) = -\sqrt{2}\beta r + \frac{Q}{r}$. This is known as the "Cornell potential" for quark confinement in quantum chromodynamics (QCD). The Einstein equations implies the same equations as 3.3-3.6 and the energy momentum tensor

$$\begin{aligned} T_{\mu}^{\nu} &= \mathcal{L}(F)\,\delta_{\mu}^{\nu} - F_{\mu\lambda}F^{\nu\lambda}\frac{\partial \mathcal{L}}{\partial F} = \\ &\frac{F}{4\pi}\mathrm{diag}\left[1, 1, \frac{2\beta}{\sqrt{-F}} - 1, \frac{2\beta}{\sqrt{-F}} - 1\right], \end{aligned} \tag{4.31}$$

with the additional condition that the trace $T_{\mu}^{\mu} = T \neq 0$, here. Upon substitution into the field equations one gets

$$R_1 = \frac{4\beta^2}{\xi} + \frac{1}{2}R_0. \tag{4.32}$$

$$\alpha = \eta \tag{4.33}$$

and a black hole solution results with the metric function

$$A(r) = \frac{1}{2} - \frac{4\sqrt{2}\beta Q - \xi}{3\eta r} + \frac{Q^2}{\xi r^2} + \frac{R_0}{12} r^2. \tag{4.34}$$

This is equivalent to the solution given in 4.25 with the same Λ_{eff} but with the new $m = \frac{4\sqrt{2}\beta Q - \xi}{3\eta}$ and $q = \frac{Q^2}{\xi}$. This is how the scale breaking term in the Lagrangian modifies the mass.

For the sake of completeness we comment here that, choosing a magnetic ansatz for the field two-form as

$$\mathbf{F} = P \sin\theta d\theta \wedge d\varphi \tag{4.35}$$

together with a nonlinear Maxwell Lagrangian

$$\mathcal{L}(F) = -\frac{1}{4\pi}\left(F + 2\beta\sqrt{F}\right) \tag{4.36}$$

and

$$R_1 = \frac{1}{2} R_0 \tag{4.37}$$

admits the magnetic version of the solution as

$$A(r) = \frac{1}{2} - \frac{4\sqrt{2}\beta P - \xi}{3\eta r} + \frac{P^2}{\xi r^2} + \frac{R_0}{12} r^2. \tag{4.38}$$

The magnetic solution, however, is not as interesting as the electric one.

4.2.2 Thermodynamical Aspects

The solution we found in the previous section is feasible as far as a physical solution is concerned. Here we set our parameters, including the condition ξ and η positive, to get $4\sqrt{2}\beta Q-\xi>0$ such that the solution admits a black hole solution with positive mass as

$$A(r)=\frac{1}{2}-\frac{m}{r}+\frac{q^2}{r^2}+\frac{R_0}{12}r^2. \tag{4.39}$$

Now we wish to discuss some of the thermodynamical properties by using the Misner-Sharp [41, 42, 43, 44, 45, 46] energy to show that the first law of thermodynamics is satisfied. To do so first we set $R_0=0$ and introduce the possible event horizon as $r=r_h$ such that $A(r_h)=0$. This yields

$$r_{\pm} = m\pm\sqrt{m^2-2q^2} \tag{4.40}$$

$$(r_h=r_+)$$

in which

$$A(r)=\frac{(r-r_-)(r-r_+)}{2r^2} \tag{4.41}$$

and the constraint $m\geq m_{cri}$ is imposed with $m_{crit}=\sqrt{2}q$. If one sets $Q>0$, this condition is satisfied if $Q>\frac{\xi}{\sqrt{2}\left(4\beta+\frac{3}{\sqrt{\xi\eta}}\right)}$ (providing $4\beta+\frac{3}{\sqrt{\xi\eta}}\neq 0$). The choice $m=$

m_{crit} leads to the extremal black hole. The Hawking temperature is defined as

$$T_H = \frac{A'(r_+)}{4\pi} = \frac{r_+^2 - 2q^2}{8\pi r_+^3} \tag{4.42}$$

and the entropy [47]

$$S = \frac{\mathcal{A}_+}{4G} f_R|_{r=r_+} \tag{4.43}$$

with $\mathcal{A}_+ = 4\pi r_+^2$, the surface area of the black hole at the horizon. The heat capacity of the black hole also is given by

$$C_q = T\left(\frac{dS}{dT}\right)_q = -\frac{2}{3}\frac{r_+^2\pi\left(2q^2 - r_+^2\right)\left(12q^4 + 4q^2r_+^2 + r_+^4\right)}{\left(2q^2 + r_+^2\right)^2\left(6q^2 - r_+^2\right)}. \tag{4.44}$$

which takes both $(+)$ and $(-)$ values. Both the vanishing / diverging C_q values indicate special points at which the system attains thermodynamical phase changes. The first law of thermodynamics can be written as

$$TdS - dE = PdV \tag{4.45}$$

in which

$$dE = \frac{1}{2\kappa}\left[\frac{2}{r_h^2}f_R + (f - Rf_R)\right]\mathcal{A}_+ dr_+ \tag{4.46}$$

with E the Misner-Sharp energy and $T = \frac{A'}{4\pi}$ the Hawking temperature. Further, $S = \frac{\mathcal{A}_+}{4} f_R$ stands for the black hole entropy, $p = T_r^r = T_0^0$ is the radial pressure of matter fields at the horizon and finally the change of volume of the black hole at the horizon is given by $dV = \mathcal{A}_+ dr_+$. One can easily show that the first law of thermodynamics in the form introduced above is satisfied [48].

Chapter 5

QUANTUM SINGULARITIES IN A MODEL OF $f(R)$ GRAVITY

In the previous chapter, it was shown that the solution with linear electromagnetic field does not admit a black hole while the solution with nonlinear electromagnetic source admits a black hole solution. The solution sourced by linear electromagnetic field resulted with a naked curvature singularity at $r = 0$ which is a typical central singularity peculiar to spherically symmetric systems. As was mentioned the solution given in chapter 4, is a kind of extension of a global monopole solution which represents spherically symmetric solution of the Einstein's equations with matter that extends to infinity. It can also be interpreted as a cloud of cosmic string with spherical symmetry [49]. And, hence, the spacetime is conical. However, with the inclusion of linear or nonlinear electromagnetic field, the spacetime is no more conical in the context of $f(R)$ gravity.

Within the framework of ETG gravity, black hole solutions have been widely studied in the literature (see [1, 50] and references therein for a complete review). However, the solutions that result with naked singularities have not been studied in detail. In physics, naked singularities are considered to be a threat to a cosmic censorship hypothesis. Furthermore, as in the classical general relativity, compared to the black hole solutions, naked singularities are not well-understood in the context of $f(R)$ gravity.

This still remains a fundamental problem in general relativity as well as in ETG to be solved. Another important diffuculty in resolving this problem is the scale where the curvature singularity occurs. In these small scales, it is believed that the classical methods should be replaced with quantum techniques in resolving the singularity problems that necessitate the use of quantum gravity. Since the quantum theory of gravity is still "under construction", an alternative method is proposed by Wald [51] which was further developed by Horowitz and Marolf (HM) [52] in determining the character of classically singular spacetime and to see if quantum effects have any chance to heal or regularise the dynamics and restore the predictability if the singularity is probed with quantum particles/fields.

In this chapter, we investigate the occurence of naked singularities in the context of $f(R)$ gravity in the view of quantum mechanics. We believe that this will be the unique example that the formation of classically naked curvature singularities in $f(R)$ gravity will be probed with quantum fields/particles that obey the Klein-Gordon, Dirac and Maxwell equations. The criterion proposed by HM will be used in this study to investigate the occurence of naked singularities.

This criterion has been used succesfully for other spacetimes to check whether the classically singular spacetimes are quantum mechanically regular or not. As an example; negative mass Schwarzshild spacetime, charged dilatonic black hole spacetime and fundamental string spacetimes are considered in [52]. An alternative function space namely the Sobolev space instead of the Hilbert space has been introduced in [53], for analysing the singularities within the framework of quantum mechanics. Helliwell and

Konkowski have studied quasiregular [54], Gal'tsov - Letelier - Tod spacetime [55], Levi-Civita spacetimes [56, 57], and recently, they also consider conformally static spacetimes [58]. Pitelli and Letelier have studied spherical and cylindrical topological defects [59], Banados-Teitelboim-Zanelli (BTZ) spacetimes [60], the global monopole spacetime [61] and cosmological spacetimes [62]. Quantum singularities in matter coupled $2+1$ dimensional black hole spacetimes are considered in [63]. Quantum singularities are also considered in Lovelock theory [26]. Recently, the occurence of naked singularities in $2+1$ dimensional magnetically charged solution in Einstein-Power-Maxwell theory have also been considered [64].

The main theme in these studies is to understand whether these clasically singular spacetimes turn out to be quantum mechanically regular if it is probed with quantum fields rather than classical particles.

The solution to be investigated in this chapter is a kind of $f(R)$ gravity extension of the analysis presented in [50] for the global monopole spacetime. The inclusion of the linear Maxwell field within the context of $f(R)$ gravity affects the topology significantly and removes the conical nature at infinity. Furthermore, the true timelike naked curvature singularity is created at $r=0$ which is peculiar to spherically symmetric systems. We investigate this singularity within the context of quantum mechanics by employing three different quantum fields/particles obeying the Klein-Gordon, Dirac and Maxwell fields with different spin structures.

5.1 Quantum Singularities

The metric function 4.25 is the solution in the presence of linear Maxwell field that results with naked singularity. In this chapter, this solution will be denoted as;

$$B(r) = \frac{1}{2} - \frac{m}{r} + \frac{q^2}{r^2} - \frac{\Lambda_{eff}}{3} r^2. \tag{5.1}$$

This solution can also be considered as a spherically symmetric cloud of cosmic string which gives rise to a deficit angle [49]. Therefore, the solution given in equation 4.25, is a kind of Einstein-Maxwell extension of the global monopole solution in the $f(R)$ gravity. One of the striking effects of the additional fields is the removal of the conical geometry of the global monopole spacetime. The Kretschmann scalar which indicates the formation of curvature singularity is given by

$$\mathcal{K} = \frac{1}{3} \frac{8\Lambda_{eff}^2 r^8 + 4\Lambda_{eff} r^6 + 3r^4 + 12mr^3 + 12r^2\left(3m^2 - q^2\right) - 144mq^2 r + +168q^4}{r^8}.$$

It is obvious that $r = 0$ is a typical central curvature singularity. This is a timelike naked singularity because the behavior of the new radial coordinate defined by $r_* = \int \frac{dr}{B(r)}$ is finite when $r \to 0$. Hence, the new solution obtained in chapter 4 and given in 5.1 is classically a singular spacetime.

Our aim in this chapter is to investigate this classically singular spacetime with regards to quantum mechanical point of view.

One of the important predictions of the Einstein's theory of general relativity is the formation of spacetime singularities. In classical general relativity, singularities are defined as the point in which the evolution of timelike or null geodesics is not defined after a proper time. According to the classification of the classical singularities devised by Ellis and Schmidt, scalar curvature singularities are the strongest ones in the sense that the spacetime can not be extended and all physical quantities such as the gravitational field, energy density and tidal forces diverge at the singular point. In black hole spacetimes, the location of the curvature singularity is at $r = 0$ and is covered by horizon(s). The singularities hidden by horizon(s) do not constitute a threat to the Penrose's cosmic censorship hypothesis. However, there are some cases that the singularity is not hidden and hence, it is *naked*. In the case of naked singularities, further care is required because they violate the cosmic censorship hypothesis. The resolution of the naked singularities stand as one of the most drastic problems in general relativity to be solved.

Naked singularities that occur at $r = 0$, are very small scales where classical general relativity is expected to be replaced by quantum theory of gravity. Herein, the occurence of naked singularities in $f(R)$ gravity will be analysed through a quantum mechanical point of view. In probing the singularity, quantum test particles/fields obeying the Klein-Gordon, Dirac and Maxwell equations are used. In other words, the singularity will be probed with spin 0, spin $1/2$ and spin 1 fields. The reason for using three different types of fields is to clarify whether or not the classical singularity is sensitive to the spin of the fields.

Our analysis will be based on the work of Wald, which was further developed by HM to probe the classical singularities with quantum test particles obeying the Klein-

Gordon equation in static spacetimes having timelike singularities. According to HM, the singular character of the spacetime is defined as the ambiguity in the evolution of the wave functions. That is to say, the singular character is determined in terms of the ambiguity when attempting to find self-adjoint extension of the operator to the entire Hilbert space. If the extension is unique, it is said that the space is quantum mechanically regular. The brief review is as follows:

Consider a static spacetime $\left(M, g_{\mu\nu}\right)$ with a timelike Killing vector field ξ^{μ}. Let t denote the Killing parameter and Σ denote a static slice. The Klein-Gordon equation in this space is

$$\left(\nabla^{\mu}\nabla_{\mu} - M^2\right)\psi = 0. \tag{5.2}$$

This equation can be written in the form

$$\frac{\partial^2\psi}{\partial t^2} = \sqrt{f}D^i\left(\sqrt{f}D_i\psi\right) - fM^2\psi = -A\psi, \tag{5.3}$$

in which $f = -\xi^{\mu}\xi_{\mu}$ and D_i is the spatial covariant derivative on Σ. The Hilbert space $\mathcal{H}$, $\left(L^2\left(\Sigma\right)\right)$ is the space of square integrable functions on Σ. The domain of an operator A, $D(A)$ is taken in such a way that it does not enclose the spacetime singularities. An appropriate set is $C_0^{\infty}\left(\Sigma\right)$, the set of smooth functions with compact support on Σ. Operator A is real, positive and symmetric therefore its self-adjoint extensions always exist. If it has a unique extension A_E, then A is called essentially self-adjoint [65, 66]. Accordingly, the Klein-Gordon equation for a free particle satisfies

$$i\frac{d\psi}{dt} = \sqrt{A_E}\psi, \tag{5.4}$$

with the solution

$$\psi(t) = \exp\left[-it\sqrt{A_E}\right]\psi(0)\,. \tag{5.5}$$

If A is not essentially self-adjoint, the future time evolution of the wave function 5.5 is ambiguous. Then, HM criterion defines the spacetime quantum mechanically singular. However, if there is only a single self-adjoint extension, the operator A is said to be essentially self-adjoint and the quantum evolution described by 5.5 is uniquely determined by the initial conditions. According to the HM criterion, this spacetime is said to be quantum mechanically non-singular. In order to determine the number of self-adjoint extensions, the concept of deficiency indices is used. The deficiency subspaces $N_{\pm}$ are defined by (see Ref. [53] for a detailed mathematical background),

$$N_{+} = \{\psi \in D(A^{*}), \quad A^{*}\psi = Z_{+}\psi, \quad ImZ_{+} > 0\} \quad \text{with dimension } n_{+} \tag{5.6}$$

$$N_{-} = \{\psi \in D(A^{*}), \quad A^{*}\psi = Z_{-}\psi, \quad ImZ_{-} < 0\} \quad \text{with dimension } n_{-}$$

The dimensions (n_{+}, n_{-}) are the deficiency indices of the operator A. The indices $n_{+}(n_{-})$ are completely independent of the choice of $Z_{+}(Z_{-})$ depending only on whether or not Z lies in the upper (lower) half complex plane. Generally one takes $Z_{+} = i\lambda$ and $Z_{-} = -i\lambda$, where λ is an arbitrary positive constant necessary for dimensional reasons. The determination of deficiency indices are then reduced to counting the number of solutions of $A^{*}\psi = Z\psi$; (for $\lambda = 1$),

$$A^{*}\psi \pm i\psi = 0 \tag{5.7}$$

that belong to the Hilbert space $\mathcal{H}$. The Theorem given below was presented by Von Newmann in 1929 which is very important for present application. Theorem:

For an operator A with deficiency indices (n_+, n_-) there are three possible cases.

(a) If $n_+ = n_- = 0$, then A is essentially self-adjoint.

(b) If $n_+ = n_- = n \geq 1$, then A is many self-adjoint extensions, parametrized by a unitary $n \times n$ matrix.

(c) If $n_+ \neq n_-$, then A has no self-adjoint extension.

If there is no square integrable solutions (i.e. $n_+ = n_- = 0$), the operator A possesses a unique self-adjoint extension and it is essentially self-adjoint. As a result, a neccessary condition for the operator A to be essentially self-adjoint is to examine the solutions satisfying 5.7 that do not belong to the Hilbert space.

5.2 Klein - Gordon Fields

A scalar field describing by the Klein-Gordon equation for a scalar particle with mass M is given by,

$$\Box\psi = g^{-1/2}\partial_\mu \left[g^{1/2} g^{\mu\nu}\partial_\nu\right]\psi = M^2\psi. \tag{5.8}$$

For the metric 5.1, the Klein-Gordon equation becomes,

$$\begin{aligned}\frac{\partial^2\psi}{\partial t^2} &= -B(r)\left\{B(r)\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta}\right. \\ &\quad \left. + \left(\frac{2B(r)}{r} + B^{'}(r)\right)\frac{\partial\psi}{\partial r}\right\} + B(r)M^2\psi. \end{aligned} \tag{5.9}$$

In analogy with the equation 5.3, the spatial operator A for the massless case is

$$\begin{aligned} A &= B(r)\left\{B(r)\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\right. \\ &\quad \left.+\left(\frac{2B(r)}{r}+B^{'}(r)\right)\frac{\partial}{\partial r}\right\}, \end{aligned} \tag{5.10}$$

and the equation to be solved is $(A^{*}\pm i)\psi=0$.Using separation of variables, $\psi=R(r)Y_l^m(\theta,\varphi)$, we get the radial portion of equation 5.7 as,

$$\frac{d^2R(r)}{dr^2}+\frac{\left(r^2B(r)\right)^{'}}{r^2B(r)}\frac{dR(r)}{dr}+\left(\frac{-l(l+1)}{r^2B(r)}\pm\frac{i}{B^2(r)}\right)R(r)=0. \tag{5.11}$$

where a prime denotes the derivative with respect to r.

5.2.1 The case of r$\to\infty$

The case $r\to\infty$ is topologically different compared to the analysis reported in [61]. In the present problem the geometry is not conical. The approximate metric when $r\to\infty$ is

$$ds^2\simeq-(\frac{R_0r^2}{12})dt^2+\left(\frac{12}{R_0r^2}\right)dr^2+r^2\left(d\theta^2+\sin^2\theta d\varphi^2\right). \tag{5.12}$$

For the above metric, the radial equation 5.11 becomes,

$$\frac{d^2R(r)}{dr^2}+\frac{4}{r}\frac{dR(r)}{dr}=0, \tag{5.13}$$

whose solution is

$$R(r)=C_1+\frac{C_2}{r^3},$$

where C_1 and C_2 are arbitrary integration constants. It is clear to observe that the above solution is square integrable as $r \to \infty$ if and only if $C_1 = 0$. Hence, the asymptotic behaviour of $R(r)$ is given by $R(r) \simeq \frac{C_2}{r^3}$.

5.2.2 The case of r→ 0

Near the origin there is a true timelike curvature singularity resulting from the existence of charge. Therefore, the approximate metric near the origin is given by

$$ds^2 \simeq -(\frac{q^2}{r^2})dt^2 + \left(\frac{r^2}{q^2}\right)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{5.14}$$

The radial equation 5.11 for the above metric reduces to

$$\frac{d^2R(r)}{dr^2} - \frac{l(l+1)}{q^2}R(r) = 0, \tag{5.15}$$

whose solution is

$$\begin{aligned} R(r) &= C_3 e^{\alpha r} + C_4 e^{-\alpha r} \\ \alpha &= \frac{\sqrt{l(l+1)}}{q} \end{aligned} \tag{5.16}$$

where C_3 and C_4 are arbitrary integration constants. The square integrability of the above solution is checked by calculating the squared norm of the above solution in which the function space on each $t =$ constant hypersurface Σ is defined as $\mathcal{H} = \{R \mid \| R \| < \infty\}$. The squared norm for the metric 5.14 is given by,

$$\| R \|^2 = \int_0^{\text{constant}} \frac{|R(r)|^2 r^4}{q^2} dr. \tag{5.17}$$

Our calculation has revealed that the solution above is always square integrable near $r = 0$ even if $l = 0$ which corresponds to the S-wave solutions.

Consequently, the spatial operator A has deficiency indices $n_+ = n_- = 1$, and is not essentially self-adjoint. Hence, the classical singularity at $r = 0$ remains quantum mechanically singular when probed with fields obeying the Klein-Gordon equation.

5.3 Maxwell fields

The Newman-Penrose formalism will be used to find the source free Maxwell fields propagating in the space of $f(R)-$gravity. Let us note that the signature of the metric 5.1 is changed to -2 in order to use the source free Maxwell equations in Newman-Penrose formalism. Thus, the metric is rewritten as,

$$ds^2 = B(r)\,dt^2 - \frac{dr^2}{B(r)} - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{5.18}$$

The four coupled source-free Maxwell's equation for electromagnetic field in Newman-Penrose formalism is given by

$$\begin{aligned} D\phi_1 - \bar{\delta}\phi_0 &= (\pi - 2\alpha)\,\phi_0 + 2\rho\phi_1 - \kappa\phi_2, \\ \delta\phi_2 - \Delta\phi_1 &= -\nu\phi_0 + 2\mu\phi_1 + (\tau - 2\beta)\,\phi_2, \\ \delta\phi_1 - \Delta\phi_0 &= (\mu - 2\gamma)\,\phi_0 + 2\tau\phi_1 - \sigma\phi_2, \\ D\phi_2 - \bar{\delta}\phi_1 &= -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\varepsilon)\,\phi_2, \end{aligned} \tag{5.19}$$

where $B(r)$ is the metric function given in 5.1, ϕ_0, ϕ_1 and ϕ_2 are the Maxwell spinors, $\varepsilon, \rho, \pi, \alpha, \mu, \gamma, \beta$ and τ are the spin coefficients to be found and the "bar" denotes the complex conjugation. The null tetrad vectors for the metric 5.18 are defined by

$$\begin{aligned} l^a &= \left(\frac{1}{B(r)}, 1, 0, 0\right), \\ n^a &= \left(\frac{1}{2}, -\frac{B(r)}{2}, 0, 0\right), \\ m^a &= \frac{1}{\sqrt{2}}\left(0, 0, \frac{1}{r}, \frac{i}{r\sin\theta}\right). \end{aligned} \tag{5.20}$$

The directional derivatives in the Maxwell's equations are defined by $D = l^a\partial_a, \Delta = n^a\partial_a$ and $\delta = m^a\partial_a$. We define operators in the following way

$$\begin{aligned} \mathbf{D}_0 &= D, \\ \mathbf{D}_0^\dagger &= -\frac{2}{B(r)}\Delta, \\ \mathbf{L}_0^\dagger &= \sqrt{2}r\,\delta \text{ and } \mathbf{L}_1^\dagger = \mathbf{L}_0^\dagger + \frac{\cot\theta}{2}, \\ \mathbf{L}_0 &= \sqrt{2}r\,\bar{\delta} \text{ and } \mathbf{L}_1 = \mathbf{L}_0 + \frac{\cot\theta}{2}. \end{aligned} \tag{5.21}$$

The nonzero spin coefficients are,

$$\mu = -\frac{1}{r}\frac{B(r)}{2}, \quad \rho = -\frac{1}{r}, \quad \gamma = \frac{1}{4}B^{'}(r), \quad \beta = -\alpha = \frac{1}{2\sqrt{2}}\frac{\cot\theta}{r}. \tag{5.22}$$

Maxwell spinors are defined by [67],

$$\begin{aligned} \phi_0 &= F_{13} = F_{\mu\nu}l^\mu m^\nu \\ \phi_1 &= \frac{1}{2}\left(F_{12} + F_{43}\right) = \frac{1}{2}F_{\mu\nu}\left(l^\mu n^\nu + \overline{m}^\mu m^\nu\right), \\ \phi_2 &= F_{42} = F_{\mu\nu}\overline{m}^\mu n^\nu, \end{aligned} \tag{5.23}$$

where $F_{ij}\,(i,j=1,2,3,4)$ and $F_{\mu\nu}\,(\mu,\nu=0,1,2,3)$ are the components of the Maxwell tensor in the tetrad and tensor bases respectively. Substituting 5.21 into the Maxwell's equations together with nonzero spin coefficients, the Maxwell's equations become

$$\left(\mathbf{D}_0+\frac{2}{r}\right)\phi_1-\frac{1}{r\sqrt{2}}\mathbf{L}_1\phi_0=0, \tag{5.24}$$

$$\left(\mathbf{D}_0+\frac{1}{r}\right)\phi_2-\frac{1}{r\sqrt{2}}\mathbf{L}_0\phi_1=0, \tag{5.25}$$

$$\frac{B(r)}{2}\left(\mathbf{D}_0^{\dagger}+\frac{B^{'}(r)}{B(r)}+\frac{1}{r}\right)\phi_0+\frac{1}{r\sqrt{2}}\mathbf{L}_0^{\dagger}\phi_1=0, \tag{5.26}$$

$$\frac{B(r)}{2}\left(\mathbf{D}_0^{\dagger}+\frac{2}{r}\right)\phi_1+\frac{1}{r\sqrt{2}}\mathbf{L}_1^{\dagger}\phi_2=0. \tag{5.27}$$

The equations above will become more tractable if the variables are changed to

$$\Phi_0=\phi_0 e^{i\omega t+im\varphi},\quad \Phi_1=\sqrt{2}r\phi_1 e^{i\omega t+im\varphi},\quad \Phi_2=2r^2\phi_2 e^{i\omega t+im\varphi}.$$

So we have,

$$\left(\mathbf{D}_0+\frac{1}{r}\right)\Phi_1-\mathbf{L}_1\Phi_0=0, \tag{5.28}$$

$$\left(\mathbf{D}_0-\frac{1}{r}\right)\Phi_2-\mathbf{L}_0\Phi_1=0, \tag{5.29}$$

$$r^2B(r)\left(\mathbf{D}_0^{\dagger}+\frac{B^{'}(r)}{B(r)}+\frac{1}{r}\right)\Phi_0+\mathbf{L}_0^{\dagger}\Phi_1=0, \tag{5.30}$$

$$r^2B(r)\left(\mathbf{D}_0^{\dagger}+\frac{1}{r}\right)\Phi_1+\mathbf{L}_1^{\dagger}\Phi_2=0. \tag{5.31}$$

The commutativity of the operators $\mathbf{L}$ and $\mathbf{D}$ enables us to eliminate each Φ_i from above equations and hence, we have

$$\left[\mathbf{L}_0^{\dagger}\mathbf{L}_1+r^2B(r)\left(\mathbf{D}_0+\frac{B^{'}(r)}{B(r)}+\frac{3}{r}\right)\left(\mathbf{D}_0^{\dagger}+\frac{B^{'}(r)}{B(r)}+\frac{1}{r}\right)\right]\Phi_0(r,\theta)=0, \tag{5.32}$$

$$\left[\mathbf{L}_0\mathbf{L}_1^{\dagger}+r^2B(r)\left(\mathbf{D}_0^{\dagger}+\frac{1}{r}\right)\left(\mathbf{D}_0-\frac{1}{r}\right)\right]\Phi_2(r,\theta)=0, \tag{5.33}$$

$$\left[\mathbf{L}_1\mathbf{L}_0^{\dagger}+r^2B(r)\left(\mathbf{D}_0^{\dagger}+\frac{B^{'}(r)}{B(r)}+\frac{1}{r}\right)\left(\mathbf{D}_0+\frac{1}{r}\right)\right]\Phi_1(r,\theta)=0. \tag{5.34}$$

The variables r and θ can be separated by assuming a separable solution in the form of,

$$\Phi_0(r,\theta)=R_0(r)\Theta_0(\theta),\quad \Phi_1(r,\theta)=R_1(r)\Theta_1(\theta),\quad \Phi_2(r,\theta)=R_2(r)\Theta_2(\theta).$$

The separation constants for 5.32 and 5.33 are the same, because $\mathbf{L}_n=-\mathbf{L}_n^{\dagger}(\pi-\theta)$ or in other words the operator $\mathbf{L}_0^{\dagger}\mathbf{L}_1$ acting on $\Theta_0(\theta)$ is the same as the operator $\mathbf{L}_0\mathbf{L}_1^{\dagger}$ acting on $\Theta_2(\theta)$ if we replace θ by $\pi-\theta$. However, for Eq. (47) we will assume another separation constant. Furthermore, by defining $R_0(r)=\frac{f_0(r)}{rB(r)}$, $R_1(r)=\frac{f_1(r)}{r}$ and $R_2(r)=\frac{f_2(r)}{r}$, the radial equations can be written as,

$$f_0^{''}(r)+\frac{2}{r}f_0^{'}(r)+ \tag{5.35}$$
$$\left[-i\omega\left(\frac{2}{rB(r)}-\frac{B^{'}(r)}{B^2(r)}\right)+\frac{\omega^2}{B^2(r)}-\frac{\varepsilon^2}{r^2B(r)}\right]f_0(r)=0,$$

$$f_2^{''}(r)-\frac{2}{r}f_2^{'}(r)+ \tag{5.36}$$
$$\left[i\omega\left(\frac{2}{rB(r)}-\frac{B^{'}(r)}{B^2(r)}\right)+\frac{\omega^2}{B^2(r)}-\frac{\varepsilon^2}{r^2B(r)}\right]f_2(r)=0,$$

$$f_1''(r)+\frac{B'(r)}{B(r)}f_1'(r)+\left[\frac{\omega^2}{B^2(r)}-\frac{\eta^2}{r^2B(r)}\right]f_1(r)=0,$$

where ε and η are the separability constants.

5.3.1 The case of r$\rightarrow\infty$

For the case $r\rightarrow\infty$, the corresponding metric is given in 5.12. Hence, the radial part of the Maxwell's equations 5.35, 5.36 and 5.37 becomes

$$f_j''(r)+\frac{2}{r}f_j'(r) = 0, \quad j=0,1 \tag{5.37}$$

$$f_2''(r)-\frac{2}{r}f_2'(r) = 0 \tag{5.38}$$

Thus, the solutions in the asymptotic case are

$$f_j(r) = C_1+\frac{C_2}{r}, \quad j=0,1 \tag{5.39}$$

$$f_2(r) = C_3+C_4r^3, \tag{5.40}$$

in which C_i are integration constants. The solution above is square integrable, if $C_1 = C_4 = 0$.

5.3.2 The case r$\rightarrow 0$

The metric near $r\rightarrow 0$ is given in 5.14. Hence, the radial part of the Maxwell equations 5.35, 5.36 and 5.37 for this case are given by

$$f_j''(r)-\frac{2}{r}f_j'(r)-\frac{\alpha^2}{q^2}f_j(r) = 0\,, \ j=1,2, \tag{5.41}$$

$$f_0''(r)+\frac{2}{r}f_0'(r)-\frac{\eta^2}{q^2}f_0(r) = 0 \tag{5.42}$$

whose solutions are obtained as,

$$f_j(r) = C_3 e^{\frac{\alpha}{q}r}(\alpha r - 1) + C_4 e^{-\frac{\alpha}{q}r}(\alpha r + 1), \qquad j = 1, 2, \tag{5.43}$$

$$f_0(r) = \frac{C_5}{r}\sinh\left(\frac{\eta}{q}r\right) + \frac{C_6}{r}\cosh\left(\frac{\eta}{q}r\right) \tag{5.44}$$

where C_i are constants. The above solution is checked for square integrability. Calculations have revealed that,

$$\parallel f_i \parallel^2 = \int_0^{\text{constant}} \frac{|f_i(r)|^2 r^4}{q^2} dr < \infty,$$

which indicates that the obtained solutions are square integrable. The definition of quantum singularity for Maxwell fields will be the same as for the Klein-Gordon fields. Here since we have three equations governing the dynamics of the photon waves, the unique self-adjoint extension condition on the spatial part of the Maxwell operator should be examined for each of the three equations. As a result, the occurrence of the naked singularity in $f(R)$ gravity is quantum mechanically singular if it is probed with photon waves.

5.4 Dirac Fields

The Newman-Penrose formalism will also be used here to find the massless Dirac fields (fermions) propagating in the space of $f(R)-$gravity. The Chandrasekhar-Dirac (CD) equations in Newman-Penrose formalism are given by

$$\begin{aligned}
(D+\varepsilon-\rho)\,F_1+\left(\bar{\delta}+\pi-\alpha\right)F_2 &= 0, \\
(\Delta+\mu-\gamma)\,F_2+(\delta+\beta-\tau)\,F_1 &= 0, \\
(D+\bar{\varepsilon}-\bar{\rho})\,G_2-(\delta+\bar{\pi}-\bar{\alpha})\,G_1 &= 0, \\
(\Delta+\bar{\mu}-\bar{\gamma})\,G_1-\left(\bar{\delta}+\bar{\beta}-\bar{\tau}\right)G_2 &= 0,
\end{aligned} \tag{5.45}$$

where F_1, F_2, G_1 and G_2 are the components of the wave function, $\varepsilon, \rho, \pi, \alpha, \mu, \gamma, \beta$ and τ are the spin coefficients to be found. The nonzero spin coefficients are given in 5.22. The directional derivatives in the CD equations are the same as in the Maxwell's equations. Substituting nonzero spin coefficients and the definitions of the operators given in 5.21 into the CD equations leads to

$$\begin{gathered}
\left(\mathbf{D}_0+\frac{1}{r}\right)F_1+\frac{1}{r\sqrt{2}}\mathbf{L}_1F_2=0, \\
-\frac{B(r)}{2}\left(\mathbf{D}_0^{\dagger}+\frac{B^{'}(r)}{2B(r)}+\frac{1}{r}\right)F_2+\frac{1}{r\sqrt{2}}\mathbf{L}_1^{\dagger}F_1=0, \\
\left(\mathbf{D}_0+\frac{1}{r}\right)G_2-\frac{1}{r\sqrt{2}}\mathbf{L}_1^{\dagger}G_1=0, \\
\frac{B(r)}{2}\left(\mathbf{D}_0^{\dagger}+\frac{B^{'}(r)}{2B(r)}+\frac{1}{r}\right)G_1+\frac{1}{r\sqrt{2}}\mathbf{L}_1G_2=0.
\end{gathered} \tag{5.46}$$

For the solution of the CD equations, we assume separable solution in the form of

$$
\begin{aligned}
F_1 &= f_1(r)Y_1(\theta)e^{i(kt+m\varphi)}, \\
F_2 &= f_2(r)Y_2(\theta)e^{i(kt+m\varphi)}, \\
G_1 &= g_1(r)Y_3(\theta)e^{i(kt+m\varphi)}, \\
G_2 &= g_2(r)Y_4(\theta)e^{i(kt+m\varphi)},
\end{aligned} \tag{5.47}
$$

where m is the azimuthal quantum number and k is the frequency of the Dirac fields which is assumed to be positive and real .Since $\{f_1, f_2, g_1, g_2\}$ and $\{Y_1, Y_2, Y_3, Y_4\}$ are functions of r and θ respectively, by substituting 5.48 into 5.47 and applying the assumptions given below,

$$
f_1(r) = g_2(r) \quad \text{and} \quad f_2(r) = g_1(r) \ , \tag{5.48}
$$

$$
Y_1(\theta) = Y_3(\theta) \quad \text{and} \quad Y_2(\theta) = Y_4(\theta). \tag{5.49}
$$

Dirac equations transform into 5.51. In order to solve the radial equations , the separation constant λ should be defined. This is achieved from the angular equations. In fact, it is already known from the literature that the separation constant can be expressed in terms of the spin-weighted spheroidal harmonics. The radial parts of the Dirac equations become

$$
\begin{aligned}
\left(\mathbf{D}_0 + \frac{1}{r}\right) f_1(r) &= \frac{\lambda}{r\sqrt{2}} f_2(r), \\
\frac{B(r)}{2}\left(\mathbf{D}_0^{\dagger} + \frac{B^{'}(r)}{2B(r)} + \frac{1}{r}\right) f_2(r) &= \frac{\lambda}{r\sqrt{2}} f_1(r).
\end{aligned} \tag{5.50}
$$

We further assume that

$$\begin{aligned} f_1(r) &= \frac{\Psi_1(r)}{r}, \\ f_2(r) &= \frac{\Psi_2(r)}{r}, \end{aligned}$$

then 5.51 transforms into,

$$\begin{gathered} \mathbf{D}_0\Psi_1 = \frac{\lambda}{r\sqrt{2}}\Psi_2, \\ \frac{B(r)}{2}\left(\mathbf{D}_0^{\dagger} + \frac{B^{'}(r)}{2B(r)}\right)\Psi_2 = \frac{\lambda}{r\sqrt{2}}\Psi_1. \end{gathered} \tag{5.51}$$

Note that $\sqrt{\frac{B(r)}{2}}\mathbf{D}_0^{\dagger}\sqrt{\frac{B(r)}{2}} = \mathbf{D}_0^{\dagger} + \frac{B^{'}(r)}{2B(r)} + \frac{1}{r}$, using this together with the new functions as below

$$\begin{aligned} R_1(r) &= \Psi_1(r), \\ R_2(r) &= \sqrt{\frac{B(r)}{2}}\Psi_2(r), \end{aligned}$$

and defining the tortoise coordinate r_* as,

$$\frac{d}{dr_*} = B\frac{d}{dr}, \tag{5.52}$$

the 5.52 become,

$$\begin{aligned} \left(\frac{d}{dr_*} + ik\right)R_1 &= \frac{\sqrt{B}\lambda}{r}R_2, \\ \left(\frac{d}{dr_*} - ik\right)R_2 &= \frac{\sqrt{B}\lambda}{r}R_1, \end{aligned} \tag{5.53}$$

In order to write the 5.54 in more compact form, we combine the solutions in the following way,

$$
\begin{aligned}
Z_{+} &= R_1 + R_2, \\
Z_{-} &= R_2 - R_1.
\end{aligned}
$$

After doing some calculations we end up with a pair of one - dimensional Schrödinger-like wave equations with effective potentials,

$$\left(\frac{d^2}{dr_*^2} + k^2\right) Z_{\pm} = V_{\pm} Z_{\pm}, \tag{5.54}$$

$$V_{\pm} = \left[\frac{B\lambda^2}{r^2} \pm \lambda \frac{d}{dr_*}\left(\frac{\sqrt{B}}{r}\right)\right]. \tag{5.55}$$

In analogy with the equation 5.3, the radial operator A for the Dirac equations can be written as,

$$A = -\frac{d^2}{dr_*^2} + V_{\pm},$$

If we write above operator in terms of usual coordinates r by using 5.53, we have

$$A = -\frac{d^2}{dr^2} - \frac{B'}{B}\frac{d}{dr} + \frac{1}{B^2}\left[\frac{B\lambda^2}{r^2} \pm \lambda B \frac{d}{dr}\left(\frac{\sqrt{B}}{r}\right)\right], \tag{5.56}$$

Our aim now is to show whether this radial part of the Dirac operator is essentially self-adjoint or not. This will be achieved by considering 5.7 and counting the number of solutions that do not belong to Hilbert space. Hence, 5.7 becomes,

$$\left(\frac{d^2}{dr^2}+\frac{B^{'}}{B}\frac{d}{dr}-\frac{1}{B^2}\left[\frac{B\lambda^2}{r^2}\pm\lambda B\frac{d}{dr}\left(\frac{\sqrt{B}}{r}\right)\right]\mp i\right)\psi(r)=0. \tag{5.57}$$

5.4.1 The case of r$\rightarrow\infty$

For the asymptotic case $r\rightarrow\infty$, the above equation transforms to

$$\frac{d^2\psi}{dr^2}+\frac{2}{r}\frac{d\psi}{dr}=0, \tag{5.58}$$

whose solution is

$$\psi(r)=C_1+\frac{C_2}{r}. \tag{5.59}$$

Clearly the solution is square integrable if $C_1=0$. Hence, the solution is asmptotically well behaved.

5.4.2 The case r$\rightarrow 0$

Near $r\rightarrow 0$, 5.58 becomes

$$\frac{d^2\psi}{dr^2}-\frac{2}{r}\frac{d\psi}{dr}+\frac{\sigma}{r^3}\psi=0, \tag{5.60}$$

$$\sigma=\mp 2\lambda q$$

whose solution is given by

$$\psi(r)=\left(\frac{4\sigma}{x^2}\right)^{\frac{3}{2}}\left\{C_3J_3(x)+C_4N_3(x)\right\}. \tag{5.61}$$

where $J_3(x)$ and $N_3(x)$ are the first and second kind Bessel functions and $x = 2\sqrt{\frac{\sigma}{r}}$. As $r \to 0$, $x \to \infty$. The behavior of the Bessel functions for real $\nu \geq 0$ as $x \to \infty$ is given by

$$\begin{aligned} J_\nu(x) &\simeq \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \\ N_\nu(x) &\simeq \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \end{aligned} \tag{5.62}$$

thus the Bessel functions asymptotically behave as $J_3(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{7\pi}{4}\right)$ and $N_3(x) \sim \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{7\pi}{4}\right)$. Checking for the square integrability has revealed that both solutions are square integrable. Hence, the radial operator of the Dirac field fails to satisfy a unique self-adjoint extension condition. As a result, the occurence of the timelike naked singularity in the context of $f(R)$ gravity remains singular from the quantum mechanical point of view if it is probed with fermions[68].

Chapter 6

CONCLUSION

In this thesis, we aimed to obtain some new exact analytic solutions together with their physical properties in a model of $f(R)$ gravity which constitutes one of the important branches of ETG. The first solution presented in the thesis is obtained by imposing a constant scalar curvature R_0 (both $R_0 > 0$ and $R_0 < 0$). Furthemore, vanishing trace of energy - momentum tensor is another condition that is imposed for the sake of analytic exact solution. The general spherically symmetric spacetime minimally coupled with nonlinear Yang-Mills (YM) field is presented in all dimensions $(d \geq 4)$. The YM field can even be considered in the power-law form in which the YM Lagrangian is expressed by $L(F) \sim (F^a.F^a)^{\frac{d}{4}}$. Since exact solutions in $f(R)$ gravity with external matter sources, are rare, such solutions must be interesting. The equation of state for effective matter is considered in the form $P_{eff} = \omega\rho$, which is analyzed in Energy conditions. The general forms of $\omega(r)$ given in 3.54 determine ω within the ranges of $-1 < \omega < \frac{1}{d-1}$ and $0 < \omega < \frac{1}{d-1}$ respectively. The fact that $\omega < -1$ doesn't occur eliminates the possibility of ghost matter, leaving us with the YM source and the scalar curvature R_0. In case that the YM field vanishes $(Q \to 0)$ the only source to remain is the effective cosmological constant $\Lambda_{eff} = \frac{(d-2)R_0}{2d}$, which arises naturally in $f(R_0)$ gravity. Another interesting result to be drawn from this study is that the effective pressure P_{eff} changes sign before / after a critical distance. Thus, it is not possible to introduce a simple ω =constant, so that the pressure preserves its sign in the presence of a physical field (here YM) in the entire spacetime. From cosmological considerations the interesting case is when the critical distance lies outside the event horizon. Finally it should be added that although $f(R) = R^{d/2}$ gravities face viability problems in experimental tests the occurrence of sources may render them acceptable in this regard [27].

In the second solution, we considered external electromagnetic fields (both linear and nonlinear) in $f(R)$ gravity with the ansatz $f(R) = \xi(R+R_1) + 2\alpha\sqrt{R+R_0}$ in chapter 4. In this choice R_0 is a constant related to the cosmological constant, the constant R_1 is related to R_0 while α is the coupling constant for the correction term. This covers both the cases of linear Maxwell and a special case of power-law nonlinear electromagnetism. The non-asymptotically flat black hole solution obtained for the Maxwell source is naturally different and has no limit of the RN black hole solution. In the limit of $Q = P = \Lambda_{eff} = 0$ we obtain the metric for a global monopole in $f(R)$ gravity. Our solution can appropriately be interpreted as a global monopole solution in the presence of the electromagnetic fields. The thermodynamical properties of our black hole solution is analyzed by making use of the Misner-Sharp formalism and shown to obey the first law. As the nonlinear electromagnetic Lagrangian we choose the normal Maxwell, supplemented with the square root Maxwell invariant which amounts to a linear electric field. This latter form is known to break the scale invariance yielding a linear potential which is believed to play role in quark confinement problem. Within

$f(R)$ gravity the presence of scale breaking term modifies the mass of the resulting black hole. The advantage of employing square-root Maxwell Lagrangian as a non-linear correction can be stated as follows: Beside confinement in the linear Maxwell case we have in $f(R)$ gravity an opposite mass term while with the coupling of the square-root Maxwell Lagrangian we can rectify the sign of this term.

Finally, the formation of the naked singularity in the context of a model of the $f(R)$ gravity is investigated within the framework of quantum mechanics, by probing the singularity with the quantum fields obeying the Klein-Gordon, Maxwell and Dirac equations. We have investigated the essential self-adjointness of the spatial part of the wave operator A in the natural Hilbert space of quantum mechanics which is the linear function space with square integrability. Our analysis has shown that the timelike naked curvature singularity remains quantum mechanically singular against the propagation of the aforementioned quantum fields. Another notable outcome of our analysis is that the spin of the fields is not effective in healing of the naked singularity for the considered model of the $f(R)$ gravity spacetime.

Another alternative function space for analysing the singularity in this context is to use the Sobolev space instead of the natural Hilbert space [53]. The Analysis in the Sobolev space involves both the wave function and its derivative to be square integrable. Although the details are not given in this study, the analysis by using the Sobolev space has revealed that irrespective of the spin structure of the fields used to probe the singularity, the considered model of the $f(R)$ gravity spacetime remains quantum mechanically singular.

Hence, the generic conclusion that has emerged from our analysis is that in the considered model of the $f(R)$ gravity, the formation of timelike naked singularity is quantum mechanically singular.

REFERENCES

[1] S. Capozziello and M. De Laurentis. *Phys. Rep.*, 509:167, (2011).

[2] Y. Bisbar. *Phys. Rev. D*, 82:124041, (2010).

[3] L. G. Jaime, L. Patino and M. Salgado. arXiv: 1212.2604v1, (2012).

[4] V. Faraoni. *Phys. Rev. D*, 83:124044, (2011).

[5] L. Pogosian and A. Silvestri. *Phys. Rev. D*, 77:023503, (2008).

[6] T. P. Sotiriou and V. Faraoni. *Phys. Rev. Lett.*, 108:081103, (2012).

[7] Y. L. Bolotin, O.A. Lemets and D. A. Yerokhin. *Phys. Usp.*, 55:9, (2012).

[8] T. P. Sotiriou and V. Faraoni. *Rev. Mod. Phys.*, 82:451, (2010).

[9] T. P. Sotiriou and S. Liberati. *Ann. Phys.*, 322:935–966, (2007).

[10] V. Miranda, S. E. Joras and I. Waga. *Phys. Rev. Lett.*, 102:221101, (2009).

[11] V. Faraoni. *Phys. Rev. D*, 74:104017, (2006).

[12] A.D. Dolgov and M. Kawasaki. *Phys. Lett. B*, 573:1, (2003).

[13] T. P. Sotiriou. *Phys. Lett. B*, 645:389, (2007).

[14] S. Habib Mazharimousavi, M. Halilsoy and Z. Amirabi. *Gen. Relativ. Gravit.*, 42:261, (2010).

[15] S. Habib Mazharimousavi and M. Halilsoy. *Phys. Lett. B*, 659:471, (2008).

[16] S. Habib Mazharimousavi and M. Halilsoy. *Phys. Lett. B*, 694:54, (2010).

[17] M. Hassaine and C. Martinez. *Class. Quant. Grav.*, 25:195023, (2008).

[18] H. Maeda, M. Hassaine and C. Martinez. *Phys. Rev. D*, 79:044012, (2009).

[19] S. H. Hendi and H.R. Rastegar-Sedehi. *Gen. Relativ. Gravit.*, 41:1355, (2009).

[20] S. H. Hendi. *Phys. Lett. B*, 677:123, (2009).

[21] M. Hassaine and C. Martnez. *Phys. Rev. D*, 75:027502, (2007).

[22] P. B. Yasskin. *Phys. Rev. D*, 12:2212, (1975).

[23] S. H. Hendi. *Phys. Lett. B*, 690:220, (2010).

[24] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge University Press, (1973).

[25] M. Salgado. *Class. Quant. Grav.*, 20:4551, (2003).

[26] S. Habib Mazharimousavi, O. Gurtug and M. Halilsoy. *Int. J. Mod. Phys. D*, 18:2061, (2009).

[27] S. Habib Mazharimousavi, M. Halilsoy and T. Tahamtan. *Eur. Phys. J. C*, 72:1958, (2012).

[28] L. Sebastiani and S. Zerbini. *Eur. Phys. J. C*, 71:1591, (2011).

[29] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini. *Phys. Rev. D*, 73:084007, (2006).

[30] S. Nojiri and S.D. Odintsov. *Int. J. Geom. Meth. Mod. Phys.*, 4:115, (2007).

[31] M. Barriola and A. Vilenkin. *Phys. Rev. Lett.*, 63:341, (1989).

[32] S. Habib Mazharimousavi, M. Halilsoy and O. Gurtug. *Class. Quant. Grav.*, 27:205022, (2010).

[33] N. Dadhich. On the Schwarzschild Field. arXiv:gr-qc/9704068.

[34] N. Dadhich, K. Narayan, U. A. Yajnik and Paramana. *Phys. Rev. D*, 50:307, (1998).

[35] S. Habib Mazharimousavi and M. Halilsoy. *Phys. Rev. D*, 84:064032, (2011).

[36] P. Gaete and E. Guendelman. *Phys. Lett. B*, 640:201, (2006).

[37] P. Gaete, E. Guendelman and E. Spallucci. *Phys. Lett. B*, 649:218, (2007).

[38] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane and T.-M. Yan. *Phys. Rev. D*, 17:3090, (1978).

[39] T. Multamaki and I. Vilja. *Phys. Rev. D*, 74:064022, (2006).

[40] T. R. P. Carames and E.R. B. de Mello. *Eur. Phys. J. C*, 64:113, (2009).

[41] M. Akbar and R. G. Cai. *Phys. Lett. B*, 648:243, (2007).

[42] R. G. Cai, L. M. Cao, Y. P. Hu and N. Ohta. *Phys. Rev. D*, 80:104016, (2009).

[43] H. Maeda and M. Nozawa. *Phys. Rev. D*, 77:064031, (2008).

[44] M. Akbar and R. G. Cai. *Phys. Rev. D*, 75:084003, (2007).

[45] M. Akbar and R. G. Cai. *Phys. Lett. B*, 635:7, (2006).

[46] R. G. Cai, L. M. Cao and N. Ohta. *Phys. Rev. D*, 81:084012, (2010).

[47] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, S. Zerbini and J. Cosmol. *Astropart. Phys.*, 02:010, (2005).

[48] S. Habib Mazharimousavi, M. Halilsoy and T. Tahamtan. *Eur. Phys. J. C*, 72:1851, (2012).

[49] P. S. Letelier. *Phys. Rev. D*, 20:1294, (1979).

[50] S. Nojiri and S. D. Odintsov. *Phys. Rep.*, 505:59, (2011).

[51] R. M. Wald. *J. Math. Phys. (N. Y.)*, 21:2082, (1980).

[52] G. T. Horowitz and D. Marolf. *Phys. Rev. D*, 52:5670, (1995).

[53] A. Ishibashi and A. Hosoya. *Phys. Rev. D*, 60:104028, (1999).

[54] D. A. Konkowski and T. M. Helliwell. *Gen. Rel. Grav.*, 33:1131, (2001).

[55] T. M. Helliwell, D. A. Konkowski and V. Arndt. *Gen. Rel. Grav.*, 35:79, (2003).

[56] D. A. Konkowski, C. Reese, T. M. Helliwell and C. Wieland. *Classical and Quantum Singularities of Levi-Civita Spacetimes with and without a Cosmological Constant*. (2004).

[57] D. A. Konkowski, T. M. Helliwell and C. Wieland. *Class. Quant. Grav.*, 21:265, (2004).

[58] D. A. Konkowski and T. M. Helliwell. *Int. J. Mod. Phys. A*, 26(22):3878–3888, (2011).

[59] J. P. M. Pitelli and P. S. Letelier. *J. Math. Phys.*, 48:092501, (2007).

[60] J. P. M. Pitelli and P. S. Letelier. *Phys. Rev. D*, 77:124030, (2008).

[61] J. P. M. Pitelli and P. S. Letelier. *Phys. Rev. D*, 80:104035, (2009).

[62] P. S. Letelier and J. P. M. Pitelli. *Phys. Rev. D*, 82:104046, (2010).

[63] O. Unver and O. Gurtug. *Phys. Rev. D*, 82:084016, (2010).

[64] S. Habib Mazharimousavi, O. Gurtug, M. Halilsoy and O. Unver. *Phys. Rev. D*, 84:124021, (2011).

[65] M. Reed and B. Simon. *Functional Analysis*. Academic Press, New York, (1980).

[66] M. Reed and B. Simon. *Fourier Analysis and Self-Adjointness*. Academic Press, New York, (1975).

[67] S. Chandrasekhar. *The Mathematical Theory of Black Holes*. Oxford University Press, (1992).

[68] O. Gurtug and T. Tahamtan. *Eur. Phys. J. C.*, 72:2091, (2012).

[69] A. Marie Nzioki, S. Carloni, R. Goswami and P. K. S. Dunsby. *Phys. Rev. D*, 81:084028, (2010).

[70] S. Habib Mazharimousavi, M. Halilsoy and T. Tahamtan. *Eur. Phys. J. C*, 73:2264, (2013).

[71] M. Jafarpour and T. Tahamtan. *Int. Theo. Phys.*, 48:487, (2009).

[72] S. Habib Mazharimousavi, M. Halilsoy and T. Tahamtan. Revisiting the Charged BTZ Metric in Nonlinear Electrodynamics. arXiv:1201.5831.

[73] T. Tahamtan and M. Halilsoy. *Astrophysics and Space Science*, 343:435, (2013).

[74] S. Habib Mazharimousavi, M. Halilsoy and T. Tahamtan. *Phys. Lett. A*, 376:893, (2012).

[75] S. Nojiri and S. D. Odintsov. *Phys. Lett. B*, 657:238, (2007).

[76] A. D. Dolgov and M. Kawasaki. *Phys. Lett. B*, 573:1, (2003).

[77] L. Hollenstein and F. S. N. Lobo. *Phys. Rev. D*, 78:124007, (2008).

[78] C. W. Misner and D. H. Sharp. *Phys. Rev.*, 136:B571, (1964).

[79] T. Multamaki and I. Vilja. *Phys. Rev. D*, 76:064021, (2007).

[80] K. Bamba, S. Nojiri and S. D. Odintsov. *Phys. Rev. D*, 77:123532, (2008).

[81] G. J. Olmo and D. R.-Garcia. *Phys. Rev. D*, 84:124059, (2011).

[82] L. Amendola, R. Gannouji, D. Polarski and S. Tsujikawa. *Phys. Rev. D*, 75:083504, (2007).

[83] M. D. Seifert. *Phys. Rev. D*, 76:064002, (2007).

[84] A. Silvestri. *Phys. Rev. D*, 77:023503, (2008).

[85] I. Seggev. *Class. Quant. Grav.*, 21:2651, (2004).

[86] T. Moon, Y. S. Myung and E. J. Son. *Gen. Relativ. Gravit.*, 43:3079, (2011).

[87] S. Habib Mazharimousavi, M. Halilsoy and T. Tahamtan. Colliding Plane Wave Solution in F(R)=R$^N Gravity. arXiv$: 1110.0094.