Hawking Radiation of Non-asymptotically Flat Black Holes

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ABSTRACT

In this thesis, we study the Hawking radiation (HR) of non-asymptotically flat (NAF) four-dimensional ($4D$) static and spherically symmetric (SSS) black holes (BHs) via the Hamilton-Jacobi (HJ) and the Parikh-Wilczek tunneling (PWT) methods. Specifically for this purpose, linear dilaton BH (LDBH) and Grumiller BH (GBH) or alias Grumiller-Mazharimousavi-Halilsoy BH (GMHBH) are taken into consideration. We should state that the GMHBH has the same metric structure with the GBH. The most important difference between them is the theories in which they are derived. While the GBH belongs to the Einstein's theory, the GMHBH is the solution to the $f(\mathfrak{R})$ theory. For the GBH, we also study the quantization of its entropy/area via the quasinormal modes (QNMs).

We firstly apply the HJ method to the geometry of the LDBH. While doing this, in addition to its naive coordinates, we use four different regular (well behaved across the event horizon) coordinate systems which are isotropic, Painlevé-Gullstrand (PG), ingoing Eddington-Finkelstein (IEF) and Kruskal-Szekeres (KS) coordinates. Except the isotropic coordinates (ICs), direct computation of the HJ method leads us to obtain the standard Hawking temperature (T_H) in all other coordinate systems. With the aid of the Fermat metric, the ICs allow us to read the index of refraction of the medium around the LDBH. It is explicitly shown that the refractive index determines the value of the tunneling rate and its natural consequence horizon temperature. But, the ICs produce an imperfect result for the horizon temperature of the LDBH. We also explain how this discrepancy can be resolved by regularizing the integral which has a pole at the event horizon.

Secondly, we study the HR of scalar particles from the GMHBH via the HJ method. The GMHBH is also known as Rindler modified Schwarzschild BH, which is suitable to be tested in astrophysics. By considering the GMHBH, we aim not only to explore the effect of the Rindler parameter (a) on the T_H, but to examine if there is any disparateness between the computed horizon temperature and the standard T_H as well. For this purpose, we study on the three regular coordinate systems which are PG, IEF and KS coordinates. In all coordinate systems, we compute the tunneling probabilities of incoming and outgoing scalar particles from the event horizon by using the HJ equation. Thus, we show in detail that the HJ method is concluded with the conventional T_H in all these coordinate systems without giving rise to the famed factor-2 problem. Furthermore, in the PG coordinates we employ the PWT method in order to show how one can integrate the quantum gravity (QG) corrections to the semiclassical tunneling rate by taking into account of the effects of self-gravitation and back reaction. Then we reveal the effects of the QG corrections on the T_H.

Finally, we study the QNMs of the uncharged GBH. After reducing the radial equation of the massless Klein-Gordon (KG) equation to the Zerilli equation, we compute the complex frequencies of the QNMs of the GBH. To this end, an approximation method which considers small perturbations around the BH horizon is being used. Considering the highly damped QNMs in the process proposed by Maggiore, we obtain the quantum entropy/area spectra of the GBH. Although the QNM frequencies are governed by the a term, we prove that the spectroscopy does not depend on that term. Here, the dimensionless constant ε of the area spectrum appears as the double of the Bekenstein's result. The reason of that discrepancy is also discussed.

Keywords: Hawking radiation, Hamilton-Jacobi equation, quasinormal modes, linear dilaton black hole, Grumiller black hole, Rindler acceleration, quantization, spectroscopy.

ÖZ

Bu tezde, Hamilton-Jacobi (HJ) ve Parikh-Wilczek tünelleme (PWT) metotlarını kullanmak suretiyle asimtotik-düz-olmayan (NAF) dört boyutlu (4D) statik ve küresel simetrik kara deliklerin (BHs) Hawking ışınımını (HR) çalışıyoruz. Özellikle bu amaç için, lineer dilatonlu BH (LDBH) ile Grumiller BH (GBH) veya diğer adıyla Grumiller-Mazharimousavi-Halilsoy BH (GMHBH) dikkate alınmaktadır. Bu arada hemen belirtmeliyiz ki GMHBH ile GBH aynı metrik yapısına sahiptirler. Aralarındaki en önemli fark elde edildikleri teoridir. GBH Einstein'ın teorisine ait iken, GMHBH $f(\Re)$ teorisine ait bir çözümdür. Kuazinormal modlar (QNMs) yardımıyla GBH için ayrıca entropi/alan kuantizasyon çalışmasını yapmaktayız.

Biz ilk olarak LDBH geometrisine HJ yöntemini uyguluyoruz. Bunu yaparken, naif koordinatlara ek olarak, olay ufkunda tamamen düzenli olan dört farklı koordinat sistemini (izotropik, Painlevé-Gullstrand (PG), içeriye-giren Eddington-Finkelstein (IEF) ve Kruskal-Szekeres (KS)) kullanacağız. İzotropik koordinatlar (ICs) hariç, HJ yöntemi diğer tüm koordinat sistemlerinde bize standart Hawking sıcaklığını (T_H) vermektedir. Fermat metriğinin yardımıyla ICs, LDBH etrafındaki ortamın kırılma indeksini okumamıza olanak sağlar. Kırılma indeksinin, tünelleme oranı ve onun bir sonucu olan ufuk sıcaklığının değerini belirlediği açıkca gösterilmiştir. Ancak, ICs LDBH'un ufuk sıcaklığı için uygun olmayan bir sonuç vermiştir. Ortaya çıkan bu tutarsız sonucun, ufukta bir kutba sahip integralin düzenlenmesi ile nasıl düzeltilebileceğini de göstermekteyiz.

İkinci olarak, HJ yöntemi ile GMHBH'den saçılan skalar parçacıkların HR çalışacağız. GMHBH, astrofizikte test edilmeye uygun olan Rindler modifiyeli

Schwarzschild BH olarak da bilinir. GMHBH dikkate alarak, sadece Rindler parametresi (a)'nın T_H üzerindeki etkisini keşfetmeyi değil, aynı zamanda hesaplanan ufuk sıcaklığı ile standart T_H arasında farklılık var olup olmadığını da incelemeyi hedefliyoruz. Bu amaçla PG, IEF ve KS düzenli koordinat sistemlerinde çalışacağız. Bu koordinat sistemlerinde, HJ denklemi kullanarak olay ufkuna gelen ve giden skaler parçacıkların tünelleme olasılıklarını hesaplıyoruz. Böylelikle HJ yönteminin, ünlü faktör-2 sorununa neden olmaksızın, tüm bu koordinat sistemlerinde geleneksel T_H ile sonuçlandığı ayrıntılı olarak göstermekteyiz. Dahası PG koordinatlarında, PWT yöntemi sayesinde kuantum yerçekimi (QG) düzeltmeli yarıklasik tünelleme oranının, öz-yerçekimi ve geri reaksiyon etkilerini dahil ederek nasıl elde edileceğini göstermekteyiz. Sonra QG düzeltmelerinin T_H üzerindeki etkilerini ortaya koymaktayız.

Son olarak, yüksüz GBH'in QNMs'lerini çalışmaktayız. Kütlesiz Klein-Gordon (KG) denkleminden gelen radyal denklemi Zerilli denklemine indirgedikten sonra, GBH'a ait QNMs'ın kompleks frekanslarını hesaplamaktayız. Bu amaçla, BH ufku çevresinde, küçük perturbasyonları göz önünde bulunduran bir yaklaşım yöntemini kullanılmaktayız. Maggiore tarafından önerilen bir işlem sayesinde son derece sönümlü QNMs'ları dikkate alarak, GBHs'ların kuantum entropi/alan spektrumları elde etmekteyiz. QNM frekanslarının a terimi tarafından yönetilmesine karşın, biz spektroskopinin bu terime bağlı olmadığını kanıtladık. Burada, alan spektrumunun boyutsuz sabiti ε, Bekenstein'nın sonucunun iki katı olarak ortaya çıkmaktadır. Bu tutarsızlığı nedeni ayrıca tartışılmaktadır.

Anahtar kelimeler: Hawking radyasyonu, Hamilton-Jacobi denklemi, kuazinormal modlar, lineer dilatonlu kara delik, Grumiller kara deliği, Rindler ivmesi, kuantizasyon, spektroskopi.

To My Family

and

My Lovely Daughter **Parmida**

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Chapter 1

INTRODUCTION

In 1974, Stephan Hawking [1,2] proved that when a BH is considered as a thermodynamical system amalgamated with quantum effects, it can perform a thermal radiation. Thus, BHs should have a characteristic temperature. In fact, Hawking's discovery broke all taboos which were classically prohibited about the BHs until that day. Together with Bekenstein's work [3] it caused to born a new subject that is the so-called QG theory which has not been completed yet. After these pioneering studies, there has always been hippest to find new methods for the HR which can decode the underlying BH geometry. Today, we can see various methods about the HR in the literature (see [4] and references therein). Among them, the most popular one is the tunneling method which is derived by Kraus and Wilczek (KW) [5,6]. KW used the null geodesic method to develop the action for the tunneling particle which is considered as a self-gravitating thin spherical shell and then managed to quantize it. KW method's strong suit indeed is to provide a dynamical model of the HR in which BH shrinks as particles radiate. In this dynamical model, both energy conservation and self-gravitational effects are included which were not considered in the original derivation of HR. Six years later, their calculations were reinterpreted by Parikh and Wilczek (PW) [7]. They showed that the spectrum of the HR can deviate from pure thermally, which implies unitarity of the underlying quantum process and the resolution of the information loss paradox [8,9]. Nowadays, PW's pioneer work has been still preserving its popularity. A lot of works for various

BHs proves its validity (a reader may refer to [10]). As far as we know, the original PW's tunneling method only suffers from one of the NAF BHs which is the so-called LDBH. Unlike to the other well-known BHs employed in the PWT method, their evaporation does not admit non-thermal radiation Therefore, the original PWT method cannot be an answer for their information paradox problem. This event was firstly unraveled by Pasaoglu and Sakalli [11]. Then, it was shown that the weakness of the PW's method in retrieving the information from the LDBH can be overcome by adding the QG corrections to the entropy [12]. Furthermore, it has proven by another study of Sakalli et al. [13] in which the entropy of the LDBH can be adroitly tweaked by the QG effects that both its temperature and mass simultaneously decrease to zero at the end of the complete evaporation.

Based on the complex path analysis of Padmanabhan and his collaborators [14-16], Angheben et al. [17] developed an alternate method for calculating the imaginary part of the action belonging to the tunneling particles. To this end, they made use of the relativistic HJ equation. Their method neglects the effects of the particle self-gravitation and involves the WKB approximation. In general, the relativistic HJ equation can be solved by substituting a suitable ansatz. The chosen ansatz should consult the symmetries of the space-time in order to allow for the separability. Thus one can get a resulting equation which is solved by integrating along the classically forbidden trajectory that initiates inside the BH and ends up at the outside observer. However, the integral has always a pole located at the horizon. For this reason, one needs to apply the method of complex path analysis in order to circumvent the pole.

A Friedmann-Robertson-Walker (FRW) universe assuming to be a good model for our universe has NAF geometry [18]. For this reason, we believe that most of the

BHs in the real universe should be necessarily described by NAF spacetimes. Hence, it is of our special interest to find out specific examples of NAF BHs as a test bed for HR problems via the HJ method. Starting from this idea, in this thesis we consider the LDBHs. First of all, the eponyms of these BHs are Clément and Galtsov [19]. Initially, they were found as a solution to Einstein-Maxwell-Dilaton (EMD) theory [20] in $4D$. Later on, it is shown that in addition to the EMD theory $D \geq 4$ dimensional LDBHs (even in the case of higher dimensions) are available in Einstein-Yang-Mills-Dilaton (EYMD) and Einstein-Yang-Mills-Born-Infeld-Dilaton (EYMBID) theories [21] (and references therein). The most intriguing feature of these BHs is that while radiating, they undergo an isothermal process. Namely, their temperature does not alter with shrinking of the BH horizon or with the mass loss. Our priority is to obtain the imaginary part of the action of the tunneling particle through the LDBH's horizon. This produces the tunneling rate that yields the T_H. In order to test the HJ method on the LDBH, in addition to the naive coordinates we will consider four more coordinate systems (all regular on the horizon) which are isotropic, PG, IEF and KS, respectively. Especially, we will mainly focus on the ICs. They require more straightforward calculations when comparing with the others. Furthermore, as it will be shown in the section (2.4), the usage of the standard HJ method with ICs reveals a discrepancy in the associated temperatures. For a more recent account in the same line of thought applied to Schwarzschild BH within the ICs, one may consult [22] in which a similar discrepancy problem in the HR has been studied. Gaining inspiration from [22] , we will discuss about how one can also remove the discrepancy to be appeared in the HR of the LDBH. Unlike from [22], we will also represent the calculation of the index of refraction of the LDBH medium, its effect on the tunneling rate and consequently on the T_H [23]. According to our knowledge, such a theoretical observation has not been reported before in the literature. Slightly different from the other coordinate systems, during the application of the HJ method in the KS coordinates; we will first reduce the LDBH spacetime to Minkowski space and then demonstrate in detail how one recovers the T_H.

Rindler acceleration, a, [24], which acts on an observer accelerated in a flat space-time has recently become rage anew. This is due to its similarity with the mysterious acceleration that revealed after the long period observations on the Pioneer spacecrafts –Pioneer 10 and Pioneer 11– after they run off a distance about 3×10^{9}km on their paths out of the Solar System [25]. Contrary to the expectations, that mysterious acceleration is attractive i.e., directed toward the Sun and this phenomenon is known as the Pioneer anomaly. Firstly, Grumiller [26] (and later together with his collaborators [27,28]) showed the correlation between the a and the Pioneer anomaly. On the other hand, Turyshev et al. [29] have recently made an alternative study to the Grumiller's ones in which the Pioneer anomaly is explained by thermal heat loss of the satellites.

Another intriguing feature of the a is that it may play the role of dark matter in galaxies [30,31]. Namely, the incorporation of the Newton's theory with the a might serve to explain rotation curves of spiral galaxies without the presence of a dark matter halo (a reader may refer to the study of Lin et al. [30]). For the galaxy-Sun pair, the a with the order of $\approx 10^{-11} m/s^2$ in physical units is a very close value to MOND's acceleration which successfully describes rotation curves without a dark matter halo (see [31] and references therein). However, very recently the studies [30,31] have been retested and criticized by Cervantes-Cota and Gómez-López [32].

As stated in [27,28], the main function of the a is to constitute a crude model which casts doubts on the description of rotation curves with a linear growing of the velocity with the radius. By virtue of this, in the novel study of [25] it was suggested that the effective potential of a point mass M should include r-dependent acceleration term. Therefore the problem effectively degrades to a $2D$ system in which the Newton's gravitational force modifies into $F_G = -m\left(\frac{M}{r^2} + a\right)$ where m is the mass of a test particle. For $a < 0$, the two forces represent repulsive property however while $a > 0$ gives an inward attractive force. Here, unless stated otherwise, throughout the thesis we shall use positive values of the a. Moreover, in the studies of [26,27] it is explicitly shown that dilatonic field source in general relativity (GR) is required for deriving a spacetime with the a. However, in performing this process one should be cautious about the physical energy conditions. It has been recently revived by [33] that the GBH spacetime does not satisfy the all essential energy conditions of the GR. Very recently, Mazharimousavi and Halilsoy (MH) [34] have shown that the GBH metric becomes physically acceptable in the $f(\Re)$ gravity. In other words, in the $f(\Re)$ gravity the problematic energy conditions are all fixed. Thus, in chapter 2, instead of saying GBH, we shall prefer to call the Rindler modified Schwarzschild BH as the GMHBH. The physical source that has been used in [34] possesses a perfect fluid-type energy momentum tensor, and the pressure of the fluid becomes negative with a particular choice. So, one can infer that the a plays role of the dark matter. Meanwhile, very recently detailed analysis of the geodesics of this BH has been made by Halilsoy et al. [35]. As a last remark about the GBH or the GMHBH, when the Rindler term in the GBH metric is terminated ($a = 0$), all results reduce to those of Schwarzschild BH as it must.

As mentioned above, the associated integral in the HJ method is evaluated by applying the method of complex path analysis in order to circumvent the pole. Result of the integral leads us to get the tunneling rate for the BH which renders possible to read the T_H. On the other hand, PWT method [7] uses the null geodesics to derive the T_H as a quantum tunneling process. In this method, self-gravitational interaction of the radiation and energy conservation are taken into account. As a result, the HR spectrum cannot be strictly thermal for many well-known BHs, like Schwarzschild, Reissner-Nordström etc. [7, 36]. Here we also investigate the HR of the GMHBH via the well-known PWT method.

In the next chapter, we shall review the GMHBH which has a fluid source in the context of f($\Re$) gravity [34]. Then we use the HJ method in order to calculate the imaginary part of the classical action for outgoing trajectories crossing the horizon. In addition to the naive coordinates, three more coordinate systems (all regular) which are PG, IEF and KS, respectively, are considered. By doing this, we aim not only to make an analysis about the influences of the a on the HR, but to examine whether the associated methods employing for the GMHBH with different coordinate systems yield the true T_H without admitting the factor-2 problem or not. For the review of the factor-2 problem arising in the HR, a reader may consult [10,37-40]. Slightly different from the other coordinate systems, during the application of the HJ method in the KS coordinates, we will first reduce the GMHBH spacetime to a Minkowski type space with a conformal factor, and then show in detail how one recovers the T_H. Furthermore, in the PG coordinate system we shall study the PWT method [7] for the GMHBH. Besides, with the PWT method it is possible to add QG corrections (see [12,13] and references therein) to the tunneling probability by considering the back reaction effect. To this end, the log-area correction to the Bekenstein-Hawking entropy will be taken into account. Finally, the modified T_H of the GMHBH due to the back reaction effect will be computed.

On the other hand, one of the trend subjects in the thermodynamics of BHs is the quantization of the BH horizon area and entropy. The pioneer works in this regard date back to 1970s, in which Bekenstein showed that BH entropy is proportional to the area of the BH horizon [3,41]. Furthermore, Bekenstein [42-44] conjectured that if the BH horizon area is an adiabatic invariant, according to Ehrenfest's principle [45] it has a discrete and equally spaced spectrum as $A_n = \varepsilon n l_p^2$ $(n = 0,1,2, \dots\dots)$ where ε is a dimensionless constant and l_p is the Planck length ($l_p^2 = \hbar$). A_n denotes the area spectrum of the BH horizon and n is the quantum number. One can easily see that when the BH absorbs a test particle, the minimum increase of the horizon area becomes $\Delta A_{min} = \varepsilon l_p^2$. Meanwhile, the undetermined dimensionless constant ε is considered as the order of unity. Bekenstein claimed that the BH horizon is formed by patches of equal area $\varepsilon\hbar$ with $\varepsilon = 8\pi$. After that, many studies have been done in order to obtain such equally spaced area spectrum, whereas the spacing could different than $\varepsilon = 8\pi$ (for the topical review, one may see [46] and references therein). Since a BH is characterized by mass, charge and angular momentum, it can be treated as an elementary particle. Since each object made by the elementary particles has its own characteristic vibrations known as the QNM frequencies, QNMs should reveal some information about the BH. Especially they are important for observational aspect of gravitational waves phenomena. In the same conceptual framework, Hod [47,48] suggested that ε can be obtained by using the QNM of a BH. Based on Bohr's correspondence principle [49], Hod theorized that the real part of the asymptotic QNM frequency (ω_R) of a highly damped BH is associated with the quantum transition energy between two quantum levels of the BH. This transition frequency allows a change in the BH mass as $\Delta M = \hbar\omega_R$. For the Schwarzschild BH, Hod calculated the value of the dimensionless constant as $\varepsilon = 4ln3$. Thereafter, Kunstatter [50] considered the natural adiabatic invariant I_{adb} for a system with energy E and vibrational frequency $\Delta\omega$ (for a BH, E is identified with the mass M) which is given by $I_{adb} = \int \frac{dE}{\Delta\omega}$. At large quantum numbers, the adiabatic invariant is quantized via the Bohr-Sommerfeld quantization; $I_{adb} \cong n\hbar$. Thus, Hod' result ($\varepsilon = 4ln3$) is also derived by Kunstatter. Then, Maggiore [51] developed another method in which the QNM of a perturbed BH is considered as a damped harmonic oscillator. This approach was more realistic since the QNM has an imaginary part. In other words, Maggiore considered the proper physical frequency of the harmonic oscillator with a damping term in the form of $\omega = \sqrt{\omega_R^2 + \omega_I^2}$ where ω_R and ω_I denote the real and imaginary parts of the frequency of the QNM, respectively.

In the $n \gg 1$ limit which is equal to the case of highly excited mode, $\omega_I \gg \omega_R$. Therefore, one infers that ω_I should be used rather than ω_R in the adiabatic quantity. As a result, it was found that $\varepsilon = 8\pi,$ which corresponds to the same area spectrum of Bekenstein's original result of the Schwarzschild BH [52,53]. Today, we can see numerous studies in the literature in which Maggiore's method (MM) was employed (see for instance [54-59]). In this thesis, our main purpose is to explore how the influence of the Rindler acceleration affects the GBH spectroscopy. To this end, we shall compute the QNMs of the GBH and subsequently use them in the MM.

The thesis is organized as follows: In chapter 2, we make a brief review of the LDBH with its naïve coordinates by giving some of their geometrical and thermodynamical features. Then in section 2.2, we show how the relativistic HJ equation can become separable on that geometry. The calculation of the tunneling rate and henceforth the T_H via the HJ method is also represented. The metric for a LDBH in ICs is derived in section 2.3. The effect of index of refraction on the tunneling rate is explicitly shown. The obtained horizon temperature is the half of the accepted value of the T_H. It is demonstrated that how the proper regularization of singular integrals resolves the discrepancy in the aforementioned temperatures. Sections 2.4 and 2.5 are devoted to the calculation of the T_H in PG and IEF coordinate systems, respectively. In section 2.6, we apply the HJ method to KS form of the LDBHs.

In chapter 3, we review some of the geometrical and thermodynamical features of the GMHBH given in $f(\Re)$ theory. We then show how the HJ equation is separated by a suitable ansatz within the naive coordinates of the GMHBH. In section 3.2, the calculations of the tunneling rate and henceforth the T_H via the HJ method are represented. Section 3.3 is devoted to the HR of the GMHBH in the PG coordinates via the HJ and the PWT methods. The back reaction effect on the T_H is also discussed. Sections 3.4 and 3.5 are devoted to the applications of the HJ method in the IEF and KS coordinate systems, respectively.

In chapter 4, we represent that how the massless KG equation reduces to the one dimensional Schrödinger-type wave equation which is the so-called the Zerilli equation [60] in the GBH geometry. Section 4.2 is devoted to the calculation of the QNMs of the GBH by considering the small perturbations around the horizon. After that, we employ the MM for the GBH in order to compute its entropy/area spectra. Finally we draw our conclusions in chapter 5.

Throughout the thesis, the units $G = c = k_B = 1$ are used. Furthermore in chapters 2 and 3 we take $l_p = 1$, however in chapter 4 it is used as $l_p^2 = \hbar$.

Chapter 2

HAWKING RADIATION OF THE LINEAR DILATON BLACK HOLE VIA THE HAMILTON-JACOBI METHOD[1]

2.1 LDBH Spacetime

In general, the metric of a SSS BH in 4D is given by:

$$ds^2 = -f dt^2 + f^{-1} dr^2 + R^2 d\Omega^2, \tag{2.1}$$

where

$$d\Omega^2 = d\theta^2 + sin\theta^2 d\varphi^2, \tag{2.2}$$

is the line-element for the unit two-sphere S^2. Since we target to solve the relativistic HJ equation for a massive but uncharged scalar field in the LDBH background, let us first analyze the geometry of the LDBH. When the metric functions of the line-element (1) are given by:

$$f = \Sigma(r - r_h), \qquad \text{and} \qquad R^2 = A^2 r, \tag{2.3}$$

we designate the spacetime (2.1) as the LDBH [19,21]. In several theories (EMD, EYMD and EYMBID), metric functions (2.3) do not alter their form. Only non-zero

[1] This Chapter is mainly quoted from Ref. [23], which is *Sakalli, I., & Mirekhtiary, S.F. (2013). Journal of Experimental and Theoretical Physics. 117, 656-663.*

positive constants A and Σ take different values depending on which theory is taken into account [21].

It is obvious that a LDBH possesses a NAF geometry and r_h represents the horizon of it. For $r_h \neq 0$, the horizon hides the null singularity at $r = 0$. Even in the extreme case ($r_h = 0$) in which the central null singularity $r = 0$ is marginally trapped, such that outgoing waves are not allowed to reach the external observers, a LDBH still maintains its BH property.

One should consider the quasi-local mass definition [61] for our metric (1), since the present form of the metric represents NAF geometry. The relationship between the mass M and the horizon r_h is given as follows

$$r_h = \frac{4M}{\Sigma\, A^2}. \tag{2.4}$$

In general, the definition of the T_H is expressed in terms of the surface gravity κ as $T_H = \frac{\kappa}{2\pi}$ [62]. For the line-elements given in the form of Eq. (2.3), the surface gravity is given by

$$\kappa = \left[-\frac{1}{4} lim_{r\to r_h}\left(g^{tt} g^{ij} g_{tt,i} g_{tt,j}\right)\right]^{\frac{1}{2}}, \tag{2.5}$$

which yields $\kappa = \frac{\Sigma}{2}$. Thus, the T_H value of the LDBH becomes:

$$T_H = \frac{\Sigma}{4\pi}. \tag{2.6}$$

It is clear from the above equation that the obtained temperature is constant. We are familiar to such a phenomenon in standard thermodynamics. If we recall the isothermal process in the concept of heat engines, we remember that $\Delta T = 0$ in the process which corresponds to constant temperature. Therefore the LDBH's radiation is such a process that the energy transfer out of the BH typically processes at a proper slow rate that thermal equilibrium is always preserve.

2.2 HR of the LDBH via the HJ Method in the Naive Coordinates

In this section we shall consider the problem of a moving scalar particle in the LDBH geometry while the back-reaction and self gravitational effects are ignored. Staying in the semi-classical framework, the classical action I of the particle satisfies the relativistic HJ equation, which is given by

$$g^{\mu\nu}\partial_\mu I \partial_\nu I + m^2 = 0, \tag{2.7}$$

and for the metric (2.1) it takes the following form

$$-\frac{1}{f}(\partial_t I) + f(\partial_r I)^2 + \frac{1}{R^2}\left[(\partial_\theta I)^2 + \frac{1}{\sin^2\theta}\left(\partial_\varphi I\right)^2\right] + m^2 = 0, \tag{2.8}$$

where m represents the mass of scalar particle and $g^{\mu\nu}$ is the inverse of metric tensor. For the Eq. (2.8), it is common to use the separation of variables method for the action $I = I(t, r, \theta, \varphi)$ as follows:

$$I = -Et + W(r) + Z_i\left(x^i\right), \tag{2.9}$$

where

$$\partial_t I = -E \quad , \quad \partial_r I = \partial_r W(r) \quad , \quad \partial_i I = Z_i. \tag{2.10}$$

The Z_i are constants in which $i = 1, 2$ labels angular coordinates θ and φ, respectively. Since the norm of the time like Killing vector ∂_t is (negative) unity $(g_{\mu\nu}\xi^\mu\xi^\nu\big|_{r=\tilde{r}} = -1)$ at a particular location

$$r \equiv \tilde{r} = \frac{1}{\Sigma} + r_h. \tag{2.11}$$

E is referred as the energy of the particle detected by an observer located at $\tilde{r}$. Obviously, $\tilde{r}$ corresponds to a point that is outside the horizon. Solving for $W(r)$ yields

$$W(r) \equiv W_{(\pm)} = \pm \int \frac{\sqrt{E^2 - \frac{f}{A^2 r}[Z_\theta^2 + \frac{Z_\varphi^2}{\sin^2\theta} + (mA)^2 r]}}{f} dr. \tag{2.12}$$

Here $\pm$ naturally comes since the Eq.(2.8) was quadratic in terms of $W(r)$. Solution of the Eq. (2.12) with "$+$" sign corresponds to scalar particles moving away from the BH (outgoing) and the other solution i.e., the solution with "$-$" sign represents particles moving toward the BH (ingoing). After evaluating the above integral around the pole at the horizon (by using the Feynman's prescription [63]), one arrives at the following:

$$W_{(\pm)} \cong \pm \int \frac{E}{f} dr = \pm \frac{E}{\Sigma} \int \frac{1}{r - r_h} dr = \pm\left(\frac{i\pi E}{\Sigma} + c\right), \tag{2.13}$$

where c is a complex integration constant. The latter result is found by the aid of the Cauchy's integral formula

$$Res(y, \propto) = \frac{1}{\pi i} \oint_\delta \; y(z) dz, \tag{2.14}$$

where $Res(y, \propto)$ represents the residue of y-function and δ traces out a semi-circle around $\propto$ in a counterclockwise manner. Therefore, we infer that the imaginary parts of the I come both from the pole at the horizon and the complex constant c. From here, we can derive the probability of ingoing waves as P_{in} and the probability of outgoing waves with P_{out} , which are calculated as follows

$$P_{out} = \exp(-2ImI) = \exp\left(-\frac{2\pi E}{\Sigma} - 2Imc\right), \tag{2.15}$$

$$P_{in} = \exp(-2ImI) = \exp\left(\frac{2\pi E}{\Sigma} - 2Imc\right). \tag{2.16}$$

According to the classical definition of the BHs, all ingoing particles must be absorbed at the horizon, which means that there is no reflection probability for incoming particle. Namely,

$$P_{in} = 1. \tag{2.17}$$

This is possible if and only if

$$Imc = \frac{\pi E}{\Sigma}, \tag{2.18}$$

which yields

$$ImW_{(+)} = \frac{2\pi E}{\Sigma}, \tag{2.19}$$

and whence the tunneling rate of the LDBH becomes

$$\Gamma = P_{out} = e^{\left(\frac{-4\pi E}{\Sigma}\right)}. \tag{2.20}$$

According to the statistical physics, the tunneling rate is related with the temperature as follows

$$\Gamma = e^{-\beta E}, \tag{2.21}$$

β where is the Boltzmann factor, which is the inverse of the temperature. Hereby one can read the horizon temperature of the LDBH as

$$\widetilde{T_H} = \frac{1}{\beta} = \frac{\Sigma}{4\pi}. \tag{2.22}$$

which is exactly equal to the T_H obtained in Eq. (2.6).

2.3 HR of the LDBH via the HJ Method in the ICs and Effect of the Index of Refraction on the T_H

In general, when the metric (1) is transformed to the ICs, the resulting line-element admits a BH spacetime in which the metric functions are nonsingular at the horizon, the time direction is a Killing vector and the three dimensional subspace of the spatial part of the line-element (known as time slice) appears as Euclidean with a conformal factor. Furthermore, using of the ICs makes possible of the calculation of the index of refraction of the light rays (a subject of gravitational lensing) around a BH. So, the light propagation of a BH can be mimicked by the index of refraction. By this way, an observer may identify the type of the BH.

In this section, we firstly transform the LDBH to the ICs and then analyze the HJ equation. Next, we examine the horizon temperature whether it agrees with the T_H or not. At the final part, we discuss the discrepancy in the temperatures and its abolishment.

LDBHs can be expressed in the ICs by the following transformation

$$\frac{d\zeta}{\zeta} = \frac{dr}{A\sqrt{\Sigma(r^2 - rr_h)}}, \tag{2.23}$$

which yields that

$$\zeta = \left[2r - r_h + 2\sqrt{r\left(r - r_{h)}\right)}\right]^{\frac{1}{\gamma}}, \tag{2.24}$$

and inversely

$$r = \frac{1}{4\zeta^\gamma}(\zeta^\gamma + r_h)^2, \tag{2.25}$$

where

$$\gamma = A\sqrt{\Sigma}. \tag{2.26}$$

On the other hand, the horizon is now replaced with

$$\zeta_h = (r_h)^{-1/\gamma}. \tag{2.27}$$

This transformation modifies the metric (2.1) to the general form of the ICs as

$$ds^2 = -Fdt^2 + G(\mathrm{d}\zeta^2 + \zeta^2 d\Omega^2), \tag{2.28}$$

where

$$F = \frac{\Sigma}{4\zeta^\gamma}(\zeta^\gamma - r_h)^2, \tag{2.29}$$

$$G = \frac{A^2}{4\zeta^{\gamma+2}}(\zeta^\gamma + r_h)^2. \tag{2.30}$$

In this coordinate system, the region $\zeta > \zeta_h$ encloses the exterior region of the LDBH, which is static. In the naive coordinates (2.1) of the LDBH, all Killing vectors are spacelike in the interior region and we understand that the interior of the LDBH is not stationary. However, when we consider the interior region $\zeta < \zeta_h$ of the metric (2.28), it admits a hyper surface-orthogonal timelike Killing vector which implies the static region. Namely, the region $\zeta < \zeta_h$ does not cover the interior of the LDBH. Instead, it recloses the exterior region such that metric (2.28) is a double covering of the LDBH exterior.

One can easily rewrite the metric (2.28) in the form of the Fermat metric [64] which is given by

$$ds^2 = F(-dt^2 + \tilde{g}), \tag{2.31}$$

where

$$\tilde{g} = n(\zeta)^2(d\zeta^2 + \zeta^2\, d\Omega^2), \tag{2.32}$$

in which $n(\zeta)$ is referred to as the index of refraction for the LDBH.

$$n(\zeta) = \sqrt{\frac{G}{F}} = \frac{A}{\sqrt{\Sigma}\zeta}\left(\frac{\zeta^\gamma + r_h}{\zeta^\gamma - r_h}\right). \tag{2.33}$$

The HJ equation for the metric (2.28) takes the following form

$$-\frac{1}{F}(\partial_t I)^2 + \frac{1}{G}(\partial_\zeta I)^2 + \frac{1}{\zeta^2 G}\left[(\partial_\theta I)^2 + \frac{1}{\sin^2\theta}\left(\partial_\varphi I\right)^2\right] + m^2 = 0. \tag{2.34}$$

Letting

$$I = -Et + W_{iso}(\zeta) + Z\left(x^i\right), \tag{2.35}$$

and then solving Eq. (2.34) for $W_{iso}(\zeta)$ we get

$$W(\zeta) \equiv W_{iso(\pm)} = \pm\int n(\zeta)\sqrt{E^2 - m^2 F - \frac{F}{G\zeta^2}\left[Z_\theta^2 + \frac{Z_\varphi^2}{\sin^2\theta}\right]}\, d\zeta. \tag{2.36}$$

Near the horizon ($\zeta \approx \zeta_h$) , the above expression behaves as

$$W_{iso(\pm)} \cong \pm E \int n(\zeta) d\zeta. \tag{2.37}$$

Now, one can clearly see that $W_{iso(\pm)}$ strictly depends on the integral of the refractive index of the LDBH. If we set

$$z = \zeta^{\gamma} \quad \rightarrow \quad \zeta = z^{\frac{1}{\gamma}} \quad \rightarrow \quad d\zeta = \frac{1}{\gamma} z^{(\frac{1-\gamma}{\gamma})} dz, \tag{2.38}$$

one can rewrite Eq. (2.37) as

$$W_{iso(\pm)} = \pm \frac{EA}{\gamma\sqrt{\Sigma}} \int \frac{z + r_h}{z^{\frac{1}{\gamma}}(z - r_h)} z^{(\frac{1-\gamma}{\gamma})} dz,$$

$$= \pm \frac{E}{\Sigma} \int \frac{z+r_h}{z(z-r_h)} dz. \tag{2.39}$$

Employing the Feynman's prescription, we then find

$$W_{iso(\pm)} = \pm \frac{i2\pi E}{\Sigma} + c_0, \tag{2.40}$$

where c_0 is another complex constant. By following the foregoing procedure, i.e.,

$$P_{in} = 1 \quad \rightarrow \quad Imc_0 = \frac{2\pi E}{\Sigma}. \tag{2.41}$$

Therefore

$$ImI = ImW_{iso(+)} = \frac{4\pi E}{\Sigma}. \tag{2.42}$$

We derive the tunneling rate of the LDBH within the ICs as

$$\Gamma = e^{-2ImI} = e^{\frac{-8\pi E}{\Sigma}}, \tag{2.43}$$

which adjusts the horizon temperature of the LDBH as

$$\widetilde{T_H} = \frac{\Sigma}{8\pi} \qquad \rightarrow \qquad \widetilde{T_H} = \frac{1}{2} T_H. \tag{2.44}$$

But the obtained temperature $\widetilde{T_H}$ is the half of the conventional Hawking temperature, T_H given in Eq. (2.6). So, the above result represents that transforming the naive coordinates to the ICs yields an apparent temperature of the BH which is less than the true temperature, T_H. This is analogous to the apparent depth $\hat{h}$ of a fish swimming at a depth d below the surface of a pool is less than the true depth d i.e., $\hat{h} < d$. This illusion is due to the difference of the index of refractions between the mediums. Particularly, such an event happens when $n_{object} > n_{observer}$ as in the present case. Because, it is obvious from Eq. (2.33) that the index of refraction of the medium of an observer who is located at the outer region is less than the index of refraction of the medium near to the horizon. Since the value of $W_{iso(\pm)}$ acts as a decision-maker on the value of the horizon temperature, one can deduce that the index of refraction and consequently the gravitational lensing effect, plays an important role on the observation of the true T_H.

On the other hand, it is doubtless that coordinate transformation of the naive coordinates to the ICs should not change the true temperature of the horizon of the BH. Since the appearances are deceptive, one should make a deeper analysis in order to find the real. Recently, a similar problem appeared in the Schwarzschild BH has been thoroughly discussed by Chatterjee and Mitra [22]. Since the isotropic coordinate ζ becomes complex inside the horizon (r< r_h) they have proven that while evaluating the integral (2.37) around the horizon, the path across the horizon involves a change of π/2 instead of π in the phase of the complex variable $(\zeta^{\gamma} - r_h)$. This could best be seen from Eq. (2.25), which is rewritten as

$$r = r_h + \frac{(\zeta^{\gamma} - r_h)^2}{\zeta^{\gamma}}. \tag{2.45}$$

It can be manipulated as

$$\frac{dr}{r - r_h} = -\gamma \frac{d\zeta}{\zeta} + 2 \frac{dz}{z - r_h}. \tag{2.46}$$

The first term on the right hand side of the above equation does not admit any imaginary part at the horizon. So, any imaginary contribution coming from $\frac{dr}{r-r_h}$ must be double of $2\frac{dz}{z-r_h}$. The latter remark produces a factor $\frac{i\pi}{2}$ for the integral in Eq. (2.39). This yields that

$$ImW_{iso(+)} = \frac{2\pi E}{\Sigma}, \tag{2.47}$$

which modifies the horizon temperature in Eq. (2.44) as

$$\widetilde{T_H} = \frac{\Sigma}{4\pi} \qquad \rightarrow \qquad \widetilde{T_H} = T_H. \tag{2.48}$$

So the proof of how one can recover the T_H in the ICs of the LDBH has been satisfactorily made.

2.4 HR of the LDBH via the HJ Method in the PG Coordinates

Generally, we use the PG coordinates [65,66] to describe the spacetime on either side of the event horizon of a static BH. In the PG coordinate system, an observer does not sense the surface of the horizon to be in any way special. In this section, we shall use the PG coordinates as another regular coordinate system in the HJ equation (2.7) and examine whether they result in the T_H or not.

We can pass to the PG coordinates by applying the following transformation [67] to the metric (2.1):

$$dT = dt + \frac{\sqrt{1-f}}{f}dr, \tag{2.49}$$

where T is called PG time. One of the main features of these coordinates is that the PG time corresponds to the proper time. Substituting Eq. (2.49) into metric (2.1), one gets

$$ds^2 = -f dT^2 + 2\sqrt{1-f}dTdr + dr^2 + R^2 d\Omega^2. \tag{2.50}$$

For this metric, the HJ equation (2.7) reads

$$-(\partial_T I)^2 + 2\sqrt{1-f}(\partial_T I)(\partial_r I) + f(\partial_r I)^2 +$$

$$\frac{1}{R^2}\left[(\partial_\theta I)^2 + \frac{1}{\sin^2\theta}\left(\partial_\varphi I\right)^2\right] + m^2 = 0. \tag{2.51}$$

Letting

$$I = -ET + W_{PG}(r) + Z_i\left(x^i\right), \tag{2.52}$$

one obtains

$$-E^2 + 2\sqrt{1-f}E(\partial_r W_{PG}) + f(\partial_r W_{PG})^2 + \frac{1}{R^2}\left[Z_\theta^2 + \frac{1}{\sin^2\theta}Z_\varphi^2\right] + m^2 = 0. \qquad (2.53)$$

Thus, we can derive an expression for $W_{PG}(r)$ as

$$W_{PG}(r) \equiv W_{PG(\pm)} = \int \frac{E}{\Sigma(r-r_h)}\left(\sqrt{1-\Sigma(r-r_h)} \pm \sqrt{1-\Sigma(r-r_h) - \frac{\forall f}{E^2}}\right)dr, \quad (2.54)$$

where

$$\forall = -E^2 + \frac{1}{R^2}\left({Z_\theta}^2 + \frac{Z_\varphi^2}{\sin^2\theta}\right) + m^2. \qquad (2.55)$$

Near the horizon, Eq. (2.54) reduces to:

$$W_{PG(\pm)} \cong \frac{E}{\Sigma}\int \frac{1}{(r-r_h)}(1 \pm 1)dr. \qquad (2.56)$$

According to our experience in the previous sections we know that $W_{PG(-)} = 0$, which guaranties that there is no reflection for the ingoing particle. Thus we have only

$$W_{PG(+)} = \frac{2\pi i E}{\Sigma}. \qquad (2.57)$$

From here, we derive the imaginary part of the action I as

$$ImI = ImW_{PG(+)} = \frac{2\pi E}{\Sigma}. \qquad (2.58)$$

With the aid of Eqs. (2.20) and (2.21), one can directly read the horizon temperature of the LDBH in the PG coordinates as

$$\widetilde{T_H} = \frac{\Sigma}{4\pi}. \qquad (2.59)$$

This result fully agrees with the standard value of the T_H (2.6).

2.5 HR of the LDBH via the HJ Method in the IEF Coordinates

IEF coordinate system is also regular at the event horizon. It was originally developed by Eddington [68] and Finkelstein [69]. These coordinates are fixed to radially moving photons. The line-element (2.1) takes the following form in the IEF coordinates (see for instance [69]).

$$ds^2 = -f dv^2 + 2\sqrt{1-f}\, dv dr + dr^2 + R^2 d\Omega^2, \tag{2.60}$$

in which v is a null coordinate, the so-called advanced time. It is given by:

$$v = t + r_*, \tag{2.61}$$

where r_* is known as the Regger-Wheeler coordinate or the tortoise coordinate. For the outer space of the LDBH, it is computed as

$$r_* = \int \frac{dr}{f} = \frac{1}{\Sigma} ln(r - r_h), \qquad f = \Sigma(r - r_h). \tag{2.62}$$

The timelike Killing vector for the metric (2.60) is given by $\xi^\mu = \partial_v$. So in this coordinate system an observer measures the scalar particle's energy by using the following expression

$$-\partial_v I = E. \tag{2.63}$$

Whence, the action I is assumed to be of the form:

$$I = -Ev + W_{IEF}(r) + Z(x^i). \tag{2.64}$$

Applying the HJ method for the metric (2.60), we obtain

$$W_{IEF}(r) \equiv W_{IEF(\pm)} = \int \frac{E}{\Sigma(r-r_h)} (1 \pm \sqrt{1 - \frac{\nexists\Sigma(r-r_h)}{E^2}}) dr. \tag{2.65}$$

where

$$\nexists = \frac{1}{R^2}\left(Z_\theta{}^2 + \frac{Z_\varphi^2}{\sin^2\theta}\right) + m^2. \tag{2.66}$$

In the vicinity of the horizon, Eq. (2.65) reads

$$W_{IEF(\pm)} \cong \frac{E}{\Sigma}\int \frac{1}{(r-r_h)}(1 \pm 1)dr. \tag{2.67}$$

Then,

$$W_{IEF(-)} = 0 \quad \rightarrow \quad W_{IEF(+)} = \frac{2\pi i E}{\Sigma} \;\; \rightarrow \;\; ImI = ImW_{IEF(+)} = \frac{2\pi E}{\Sigma}. \tag{2.68}$$

Therefore we infer from the above result, likewise to the PG coordinates, the use of the IEF coordinates in the HJ equation enables us to reproduce the T_H (2.6) from the horizon temperature of the LDBH.

2.6 HR of the LDBH via the HJ Method in the KS Coordinates

Another well-behaved coordinate system which encloses the entire spacetime manifold of the maximally extended BH solution is the so-called KS coordinates [71,72]. These coordinates have an effect of squeezing infinity into a finite distance, and thus the entire spacetime can be displayed on a stamp-like diagram. In this section, we will apply the HJ equation to KS metric of the LDBH in order to verify whether the horizon temperature $\widetilde{T_H}$ is going to be equal to the T_H or not.

Metric (2.1) can be rewritten as follows [70]

$$ds^2 = -f du dv + R^2 d\Omega^2, \tag{2.69}$$

where

$$du = dt - dr_* \quad \text{and} \quad dv = dt + dr_*. \tag{2.70}$$

We can now define new coordinates (U, V) in terms of the surface gravity as:

$$U = -e^{-\kappa u} \qquad \text{and} \qquad V = e^{\kappa v}, \tag{2.71}$$

so that metric (2.69) transforms to the KS metric as follows

$$ds^2 = \frac{f}{\kappa^2 UV} dUdV + R^2 d\Omega^2. \tag{2.72}$$

From the definitions given in the Eqs. (2.3),(2.5) and (2.71), one can derive the KS metric of the LDBH:

$$ds^2 = \frac{-16M}{\Sigma^2 A^2} dUdV + R^2 d\Omega^2. \tag{2.73}$$

This metric is well-behaved everywhere outside the physical singularity $r = 0$. Alternatively, metric (2.73) can be recast as

$$ds^2 = -d\bar{T}^2 + dX^2 + R^2 d\Omega^2. \tag{2.74}$$

This is done by the following transformation:

$$\bar{T} = \frac{4\sqrt{M}}{\Sigma A}(V + U) = \frac{4\sqrt{M}}{\Sigma A}\sqrt{\frac{r}{r_h} - 1} sinh\left(\frac{\Sigma t}{2}\right), \tag{2.75}$$

$$X = \frac{4\sqrt{M}}{\Sigma A}(V - U) = \frac{4\sqrt{M}}{\Sigma A}\sqrt{\frac{r}{r_h} - 1} cosh\left(\frac{\Sigma t}{2}\right). \tag{2.76}$$

These new coordinates satisfy

$$X^2 - \bar{T}^2 = \frac{16M}{\Sigma^2 A^2}\left(\frac{r}{r_h} - 1\right). \tag{2.77}$$

This means that $X = \pm\bar{T}$ corresponds to the future $(+)$ and past horizons $(-)$. On the other hand, $\partial_{\bar{T}}$ is not a timelike Killing vector anymore for the metric (2.74); instead we should consider the timelike Killing vector as:

$$\partial_{\tilde{T}} = N(X\partial_{\bar{T}} + \bar{T}\partial_X). \tag{2.78}$$

where N denotes the normalization constant. This constant can admit such a specific value that the norm of the Killing vector becomes negative unity $(g_{\mu\nu}\xi^{\mu}\xi^{\nu} = -1)$ at the specific location (2.11). Thus, we compute its value as $N = \frac{\Sigma}{2}$, which is nothing but the surface gravity (2.5). Since the energy is in general defined by:

$$-\partial_{\tilde{T}} I = E, \tag{2.79}$$

one finds

$$-E = \frac{\Sigma}{2}(X\partial_{\bar{T}} I + \bar{T}\partial_X I). \tag{2.80}$$

Without loss of generality, we may only consider the two dimensional form of the KS metric (2.74)

$$ds^2 = -d\bar{T}^2 + dX^2, \tag{2.81}$$

which appears as Minkowskian. Thus, the calculation of the HJ method becomes more straightforward. The HJ equation (2.7) for the above metric reads

$$-(\partial_{\bar{T}} I)^2 + (\partial_X I)^2 + m^2 = 0. \tag{2.82}$$

This equation implies that the action I to be used in the HJ equation (2.7) for the metric (2.74) can be

$$I = g(u) + Z(x^i), \tag{2.83}$$

where $u = X - \bar{T}$. For simplicity, we may further set $Z(x^i) = m = 0$. Using Eq. (2.82) with ansatz (2.83), we derive an expression for $g(u)$ as .

$$g(u) = \int \frac{2E}{\Sigma u} du. \tag{2.84}$$

This expression develops a divergence at the future horizon $u = 0$, namely $X = \bar{T}$. Thus, it leads to a pole at the horizon (doing a semi-circular contour of integration in the complex plane) and the result is found to be

$$g(u) = \frac{2i\pi E}{\Sigma}, \tag{2.85}$$

which implies the correct imaginary part of the action:

$$ImI = \frac{2\pi E}{\Sigma}. \tag{2.86}$$

As a consequence, the above result implies that the true horizon temperature (i.e., T_H) can also be recovered in the background of the KS metric of the LDBH.

Chapter 3

HAWKING RADIATION OF THE GRUMILLER-MAZHARIMOUSAVI-HALILSOY BLACK HOLE IN THE $f(\mathfrak{R})$ THEORY VIA THE HAMILTON-JACOBI METHOD[2]

3.1 GMHBH Spacetime in the $f(\mathfrak{R})$ Theory and HJ Method

In this section we firstly introduce the geometry and some thermodynamically properties of the GMHBH. Secondly, we demonstrate how one can derive the radial equation of the relativistic HJ equation in the background of the GMHBH. Finally and most importantly, we represent how the HJ method eventuates in the T_H.

Let us start from the $4D$ action of $f(\mathfrak{R})$ gravity

$$S = \frac{1}{2\lambda}\int \sqrt{(-g)} f(\mathfrak{R}) d^4x + S_M\,, \tag{3.1}$$

where $\lambda = 8\pi G = 1$, $\mathfrak{R}$ is the curvature scalar and $f(\mathfrak{R}) = \mathfrak{R} - 12a\xi ln|\mathfrak{R}|$ in which a and ξ are positive constants. Here, S_M denotes the physical source for a perfect fluid-type energy momentum tensor which is given by

$$T_\mu^\nu = diag.[-\rho, p, q, q], \tag{3.2}$$

[2] This Chapter is mainly quoted from Ref. [73], which is *Mirekhtiary, S.F., & Sakalli, I. (2014). Communication in Theoretical Physics. 61, 558-564.*

with the thermodynamic pressure p being a function of the rest mass density of the matter ρ only, so that $p = -\rho$. Besides, q is also a state function which is to be determined. Recently, MH has obtained the GMHBH solution to the above action in their landmark manuscript [34]. Their solution is described by the following $4D$ SSS line-element

$$ds^2 = -Hdt^2 + H^{-1}dr^2 + r^2 d\Omega^2. \tag{3.3}$$

MH ingeniously have arranged their solution in such a manner that the metric function H exactly matches with the GBH solution without the cosmological constant [26], i.e.,

$$H = 1 - \frac{2M}{r} + 2ar = \frac{2a}{r}(r - r_h)(r - r_o), \tag{3.4}$$

where M is the constant mass and as mentioned in the Abstract a shows the Rindler parameter which is assumed to be positive throughout this thesis. Besides, the other parameters seen in Eq. (3.4) are

$$r_h = \frac{\sqrt{1+16Ma}-1}{4a} \qquad \text{and} \qquad r_0 = -\frac{\sqrt{1+16Ma}+1}{4a}. \tag{3.5}$$

The GMHBH has only one horizon, since r_0 cannot be horizon due to its negative signature. After computing the scalars of the metric, we obtain

$$\begin{aligned} K = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} &= 32\frac{a^2}{r^2} + 48\frac{M^2}{r^6}, \\ R &= -12\frac{a}{r}, \\ R_{\alpha\beta}R^{\alpha\beta} &= 40\frac{a^2}{r^2}. \end{aligned} \tag{3.6}$$

On the other hand, energy-momentum components become

$$p = -\rho = \frac{([6a\xi - f(\Re)]r^2 + 4(\xi - a)r - 6M\xi)}{2r^2}, \tag{3.7}$$

and

$$q = -\frac{f(\Re)r - 2\xi + 8a}{2r}, \tag{3.8}$$

where

$$f(\Re) = -\left[12a\xi ln\left(\frac{12a}{r}\right) + \frac{12a}{r}\right]. \tag{3.9}$$

One can easily observe from the last three equations that the parameter a is decisive for the fluid source. This can be best seen by simply taking the limit of $a \to 0$ which corresponds to the vanishing fluid and Ricci scalar, and so forth $\xi \to 0$. Thus, $f(\Re)$ gravity reduces to the usual $\Re$-gravity of the theory of Einstein. In other words, while $a \to 0$, the GMBH solution reduces to the Schwarzschild geometry.

By using the definitions made in Eqs. (2.5) and (2.), T_H of the GMHBH is expressed as follows:

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi}\partial_r H|_{r=r_h} = \frac{a(r_h - r_0)}{2\pi r_h} = \left(\frac{a\sqrt{1+16aM}}{\pi(\sqrt{(1+16aM)}-1}\right). \tag{3.10}$$

From the above expression, it is obvious that while the GMHBH losing its mass M by virtue of the HR, T_H increases (i.e., $T_H \to \infty$) with $M \to 0$ in such a way that its divergence speed is tuned by the term a. Also, one can immediately check that $\log_{a\to 0} T_H = \frac{1}{8\pi M}$, which is the most known Hawking temperature: T_H of the Schwarzschild BH. In general, The Bekenstein-Hawking entropy of the metrics like Eq. (3.3) is given by

$$S_{BH} = \frac{A_h}{4} = \pi r_h^2. \tag{3.11}$$

Accordingly, its differential form becomes

$$dS_{BH} = \frac{4\pi r_h}{\sqrt{1+16aM}} dM. \tag{3.12}$$

Therefore one can prove the validity of the first law of thermodynamics for the GMHBH via the following thermodynamical law,

$$T_H dS_{BH} = dM. \tag{3.13}$$

Now, we want to proceed to our calculations by considering the problem of a scalar particle which moves in this spacetime. By doing this, we initially ignore all the back-reaction or the self-gravitational effects. Within the semi-classical framework, the classical action I of the particle satisfies the relativistic HJ equation was given in Eq. (2.7). So, for the metric (3.3), the HJ equation becomes

$$\frac{-1}{H}(\partial_t I) + H(\partial_r I)^2 + \frac{(\partial_\theta I)^2}{r^2} + \frac{1}{r^2 \sin^2\theta}\left(\partial_\varphi I\right)^2 + m^2 = 0. \tag{3.14}$$

Substituting the ansatz (2.9) for the I into the above equation, we get

$$\partial_t I = -E, \qquad \partial_r I = \partial_r W(r), \qquad \partial_k I = Z_k, \tag{3.15}$$

As stated before, Z_k's are constants in which $k = 1,2$ labels angular coordinates θ and φ, respectively. In this geometry, the norm of the time-like Killing vector $(\partial_{t)}$ is negative unit at only the following particular location

$$r \equiv R_d = \frac{r_h + r_0}{2} + \frac{1+\sqrt{4(r_h+r_0)^2+4(r_h+r_0)a+1}}{4a}, \tag{3.16}$$

which satisfies $(g_{\mu\nu}\xi^{\mu}\xi^{\nu}\big|_{r=R_d} = -1)$. Hence, E in Eq. (3.15) is designated with the "energy" of the particle detected by an observer (or by a probe of a radiation detector) at R_d, where is outside the horizon. Solving Eq. (3.14) for $W(r)$ yields

$$W(r) \equiv W_{(\pm)} = \pm \int \frac{\sqrt{E^2 - \frac{H}{r^2}[Z_\theta^2 + \frac{Z_\varphi^2}{sin^2\theta} + m^2 r^2]}}{H} dr. \tag{3.17}$$

Therefore, the above solution with "$+$" sign corresponds to scalar particles moving away from the BH (outgoing ones) and the other solution with "$-$" sign belongs to the ingoing particles. After evaluating the above integral around the pole existing at the horizon (cf. the Feynman's prescription [63]), we have

$$W_{(\pm)} \cong \pm \frac{i\pi E r_h}{2\mathrm{a}(r_h - r_0)} + \delta, \tag{3.18}$$

in which δ is another complex integration constant. Thence, we can determine the probabilities of ingoing and outgoing particles while crossing the GMHBH horizon as

$$P_{out} = exp(-2ImI) = exp\left(-\frac{\pi E r_h}{a(r_h - r_0)} - 2Im\delta\right), \tag{3.19}$$

$$P_{in} = e^{-2ImI} = \exp\left(\frac{\pi E r_h}{\mathrm{a}(r_h - r_0)} - 2Im\delta\right). \tag{3.20}$$

Because of the condition of being BH, there should not be any reflection for the ingoing waves, which means that P_{in}=1. This is possible with $Im\delta = \frac{\pi E r_h}{2a(r_h - r_0)}$. This choice also implies that the imaginary part of the action I for a tunneling particle can only come out of $W_{(+)}$. Briefly, we get

$$ImI = ImW_+ = \frac{\pi E r_h}{a(r_h - r_0)}. \tag{3.21}$$

Consequently, the tunneling rate for the GMHBH turns out to be as

$$\Gamma = e^{-2ImI} = e^{\frac{-2\pi E r_h}{a(r_h - r_0)}} = e^{-\frac{E}{T}}. \tag{3.22}$$

Thus, one can easily read the horizon temperature of the GMHBH as

$$\check{T}_H = \frac{a(r_h - r_0)}{2\pi r_h}. \tag{3.23}$$

which is exactly equal to the T_H given in Eq. (3.10).

3.2 HR of the GMHBH via the HJ and PWT Methods in the PG Coordinates

By following the works that we made in section (2.4), we can transform the naive coordinates of the GMHBH to the PG coordinates by using the following transformation.

$$dt_{PG} = dt + \frac{\sqrt{1-H}}{H} dr. \tag{3.24}$$

Then the GMHBH metric (3.3) transforms to

$$ds^2 = -H dt_{PG}{}^2 + 2\sqrt{1-H} dt_{PG} dr + dr^2 + r^2 d\Omega^2, \tag{3.25}$$

and consequently its HJ equation takes the form

$$-\left(\partial_{t_{PG}} I\right)^2 + 2\sqrt{1-H}(\partial_{t_{PG}} I)(\partial_r I) + H(\partial_r I)^2 +$$

$$\frac{1}{r^2}(\partial_\theta I)^2 + \frac{1}{r^2 sin^2\theta}\left(\partial_\varphi I\right)^2 + m^2 = 0. \tag{3.26}$$

Similar to the ansatz (2.52), one may set

$$I = -Et_{PG} + W_{PG}(r) + J\left(x^i\right), \tag{3.27}$$

such that Eq. (3.26) becomes

$$2E\sqrt{1-\mathrm{H}}\acute{W} + \mathrm{H}(\acute{W})^2 + g = 0, \tag{3.28}$$

where $\acute{W} = \partial_r W_{PG}$ and

$$g = -E^2 + \frac{1}{r^2}\left(J_\theta{}^2 + \frac{J_\varphi^2}{sin\theta^2}\right) + m^2. \tag{3.29}$$

Then we obtain

$$W_{PG}(r) \equiv W_{PG(\pm)} = E\int \frac{\sqrt{1-H}}{H}\left(1 \pm \sqrt{1 - \frac{Hg}{(1-H)E^2}}\right)dr. \tag{3.30}$$

Near the horizon, it reduces to

$$W_{PG(\pm)} \cong E\int \frac{1\pm 1}{H}dr. \tag{3.31}$$

According to the our former experiences, we set $W_{PG(-)} = 0$, and this leads us to find out

$$W_{PG(+)} = \frac{\pi i E r_h}{a(r_h - r_0)}, \tag{3.32}$$

where we now have

$$ImI = ImW_{PG(+)} = \frac{\pi E r_h}{\mathrm{a}(r_h - r_0)}. \tag{3.33}$$

Recalling the Eqs. (2.20) and (2.21), we read the horizon temperature of the GMHBH within the PG coordinates as

$$\breve{T}_H = \frac{a(r_h - r_0)}{2\pi r_h}. \tag{3.34}$$

This result is in agreement with the standard value of the T_H (3.10).

Now, we want to continue to our calculations by employing the tunneling method prescribed by PW [7]. To this end, we recalculate the imaginary part of the I for an outgoing positive energy particle which crosses the horizon outwards in the PG coordinates. In the metric (3.25), the radial null geodesics of a test particle has a rather simple form

$$\dot{r} = \frac{dr}{dt_{PG}} = \pm 1 - \sqrt{1 - H}, \tag{3.35}$$

where upper (lower) sign corresponds to outgoing (ingoing) geodesics. After expanding the metric function H around the horizon r_h, we get

$$H = H'(r_h)(r - r_h) + O(r - r_h)^2. \tag{3.36}$$

And hence by using Eq. (3.10), the radial outgoing null geodesics (3.35) can be approximated to

$$\dot{r} \cong \kappa(r - r_h). \tag{3.37}$$

In general, the imaginary part of the I for an outgoing positive energy particle which crosses the horizon from inside (r_{in}) to outside (r_{out}) is defined as

$$ImI = Im \int_{r_{in}}^{r_{out}} p_r dr = Im \int_{r_{in}}^{r_{out}} \int_0^{p_r} d\tilde{p}_r \, dr \, . \tag{3.38}$$

Hamilton's equation for the classical trajectory is given by

$$dp_r = \frac{d\mathcal{H}}{\dot{r}}, \tag{3.39}$$

where p_r and $\mathcal{H}$ denote radial canonical momentum and Hamiltonian, respectively. Thus, we have

$$ImI = Im\int_{r_{in}}^{r_{out}}\int_0^{\mathcal{H}}\frac{d\widetilde{\mathcal{H}}}{\dot{r}}dr. \tag{3.40}$$

Now, if we consider the whole system as a spherically symmetric system of total mass (energy) M, which is kept fixed. Then we suppose that this system consists of a GMHBH with varying mass $M-\omega$, emitting (in each time) a spherical shell of mass ω such that $\omega \ll M$. This phenomenon is known as self-gravitational effect [5]. After taking this effect into account, the above integration is expressed as

$$ImI = Im\int_{r_{in}}^{r_{out}}\int_M^{M-\omega}\frac{d\widetilde{\mathcal{H}}}{\dot{r}}dr = -Im\int_{r_{in}}^{r_{out}}\int_M^{\omega}\frac{d\widetilde{\omega}}{\dot{r}}dr, \tag{3.41}$$

in which the Hamiltonian $\mathcal{H} = M-\omega \therefore d\mathcal{H} = -d\omega$ is used. Furthermore, Eq. (3.37) can now be redefined as follows

$$\dot{r} \cong \kappa_{QG}\,(r-r_h), \tag{3.42}$$

where $\kappa_{QG} = \kappa(M-\omega)$ is the modified horizon gravity, which is the so-called QG corrected surface gravity, cf. [74,75]. Thus, after evaluating the integral (3.41) with respect to r which is done by deforming the contour, the imaginary part of the action reads

$$ImI = -\pi\int_0^{\omega}\frac{d\widetilde{\omega}}{T_{QG}}dr, \tag{3.43}$$

where the "modified Hawking temperature" is expressed in the form of

$$T_{QG} = \frac{\kappa_{QG}}{2\pi}. \tag{3.44}$$

From here on, we make use of the above expression to determine how the action I is related with the QG corrected entropy, S_{QG} . Namely

$$ImI = -\frac{1}{2}\int_0^{\omega} \frac{d\tilde{\omega}}{T_{QG}} dr = -\frac{1}{2}\int_{S_{QG}(M)}^{S_{QG}(M-\omega)} dS = -\frac{1}{2}\Delta S_{QG} \,. \tag{3.45}$$

Then the modified tunneling rate is computed via

$$\Gamma_{QG} \sim e^{-2ImI} = e^{\Delta S_{QG}}. \tag{3.46}$$

In string and loop QG theories, S_{QG} is introduced with a logarithmic correction (see for instance [12,76])

$$S_{QG} = \frac{A_h}{4} + \alpha ln A_h + \mathrm{O}\left(\frac{1}{A_h}\right), \tag{3.47}$$

where α is a dimensionless constant, and it symbolizes the back reaction effects. It possesses positive values in the string theory, however for the loop QG theory it appears as negative. In other words, it takes different values according to which theory is considered [75]. Thus, with the aid of Eqs. (3.11) and (3.47), one can compute ΔS_{QG} as follows

$$\Delta S_{QG} = -\frac{\pi\left(8a\omega + \sqrt{1+16a(M-\omega)} - \sqrt{1+16aM}\right)}{8a^2} + \alpha ln\left(\frac{1+8a(M-\omega)-\sqrt{1+16a(M-\omega)}}{1+8aM-\sqrt{1+16aM}}\right). \tag{3.48}$$

According to the fundamental law in thermodynamics

$$T_{QG} dS_{QG} = dM, \tag{3.49}$$

one can derive the T_{QG} with the back reaction effects. After a straightforward calculation, we can derive T_{QG} from Eq. (3.47) in terms of the T_H as follows

$$T_{QG} = (1 + \frac{\alpha}{\pi r_h^2})^{-1} T_H. \tag{3.50}$$

Thus, one can easily see that once we terminates the back reaction effects (i.e., α=0), the standard T_H is precisely reproduced. On the other hand, it is also possible to retrieve the T_{QG} from Eq. (3.48). For this purpose, we expand ΔS_{QG} and recast terms up to leading order in ω. So, one finds

$$\Delta S_{QG} \cong -\left[\frac{\pi}{a}\left(\frac{\sqrt{1+16aM}-1}{\sqrt{1+16aM}}\right) + \frac{16a\alpha}{(1+16aM-\sqrt{1+16aM}}\right]\omega + O(\omega^2),$$

$$= -\left(\frac{1}{T_H} + \alpha\frac{16\pi T_H}{1+16aM}\right)\omega + O(\omega^2). \tag{3.51}$$

Considering Eqs. (2.21) and (3.46), we obtain

$$\Gamma_{QG} \sim e^{\Delta S_{QG}} = e^{-\frac{\omega}{T}}, \tag{3.52}$$

Thus, the inverse temperature that is identified with the coefficient of ω reads

$$T = (\frac{1}{T_H} + \alpha\frac{16\pi T_H}{1+16aM})^{-1}\ , \tag{3.53}$$

After manipulating the above equation, one can see that

$$T = (1 + \frac{\alpha}{\pi r_h^2})^{-1} T_H. \tag{3.54}$$

obviously it is nothing but the T_{QG} (3.50).

3.3 HR of the GMHBH via the HJ Method in the IEF Coordinates

In order to transform the naive coordinates of the GMHBH (3.3) to the IEF coordinates:

$$ds^2 = -Hdv^2 + 2\sqrt{1-H}dvdr + dr^2 + r^2 d\Omega^2, \tag{3.55}$$

its tortoise coordinate (for the outer region of the GMHBH)

$$r_* = \int \frac{dr}{H} = \frac{1}{2a(r_h - r_0)} \ln \left[\frac{(\frac{r}{r_h} - 1)^{r_h}}{(r - r_0)^{r_0}} \right], \tag{3.56}$$

should be used in the advanced time coordinate, $v = t + r_*$. By following the calculations that we already made in section (2.5), one can find

$$W_{IEF}(r) \equiv W_{IEF(\pm)} = E \int \frac{1}{\mathrm{H}} (1 \pm \sqrt{1 - \frac{\Im H}{E^2}}) dr, \tag{3.57}$$

where

$$\Im = \frac{1}{r^2} \left(Z_\theta{}^2 + \frac{Z_\varphi^2}{\sin^2 \theta} \right) + m^2. \tag{3.58}$$

Approaching to the r_h, we have

$$W_{IEF(\pm)} \cong E \int \frac{1 \pm 1}{H} dr, \tag{3.59}$$

which yields $W_{IEF(-)} = 0$, which automatically satisfies the necessity condition for having a BH. So the only non-zero expression that we have is

$$W_{IEF(+)} = \frac{\pi i E r_h}{a(r_h - r_0)}. \tag{3.60}$$

From here, we get

$$ImI = ImW_{IEF(+)} = \frac{\pi E r_h}{\mathrm{a}(r_h - r_0)}. \tag{3.61}$$

This expression is in accordance with the former results like Eq. (3.33), hence it guaranties that the horizon temperature of the GMHBH in the IEF coordinates is the T_H obtained in Eq. (3.10). Namely,

$$\breve{T}_H = \frac{a(r_h - r_0)}{2\pi r_h} = T_H. \tag{3.62}$$

3.4 HR of the GMHBH via HJ Method in the KS Coordinates

The aim of this section is to compute the T_H of the GMHBH when it is described in the KS coordinates. By doing this, we mainly follow the calculations made in the section (2.6).

Recalling the KS transformations given in the Eqs.(2.70) and (2.71), we put the metric of the GMHBH (3.3) into the following form

$$ds^2 = \frac{H}{\kappa^2 UV} dUdV + r^2 d\Omega^2, \tag{3.63}$$

which can be reorganized as

$$ds^2 = -\varrho dUdV + r^2 d\Omega^2 , \tag{3.64}$$

where

$$\varrho = \frac{2r_h^3}{ar(r_h - r_0)^2}(r - r_0)^{1+\frac{r_0}{r_h}}. \tag{3.65}$$

As an aside, the metric (3.64) is regular everywhere except $r = 0$, which represents the location of the physical singularity. It is also possible to recast the metric (3.65) to the following form

$$ds^2 = -\varrho(d\mathcal{M}^2 - d\mathbb{N}^2) + r^2 d\Omega^2, \tag{3.66}$$

where

$$\mathcal{M} = \frac{1}{2}(V+U) = \frac{\sqrt{\frac{r}{r_h}-1}}{(r-r_0)^{\frac{r_0}{2r_h}}} sinh(\kappa t), \tag{3.67}$$

$$\mathbb{N} = \frac{1}{2}(V-U) = \frac{\sqrt{\frac{r}{r_h}-1}}{(r-r_0)^{\frac{r_0}{2r_h}}} cosh(\kappa t). \tag{3.68}$$

Therefore we have

$$\mathbb{N}^2 - \mathcal{M}^2 = \frac{\sqrt{\frac{r}{r_h}-1}}{(r-r_0)^{\frac{r_0}{2r_h}}}, \tag{3.69}$$

So we deduce that while $\mathbb{N} = +\mathcal{M}$ represents the future horizon, $\mathbb{N} = -\mathcal{M}$ stands for the past horizon. Furthermore, the timelike Killing vector for the metric (3.66) becomes

$$\partial_{\bar{\bar{T}}} = \Pi(\mathbb{N}\partial_{\mathcal{M}} + \mathcal{M}\partial_{\mathbb{N}}), \tag{3.70}$$

where Π denotes the normalization constant. The particular value of the Π, which makes the norm of the Killing vector as negative unity can be found at the R_d location (3.16) as .

$$\Pi = \frac{r_h - r_0}{r_h}\sqrt{\frac{ar}{2(r-r_h)(r-r_0)}}\Bigg|_{r=R_d} = \frac{a(r_h - r_0)}{r_h}. \tag{3.71}$$

Since the energy of the scalar particle emitted by the BH is given by

$$\partial_{\bar{\bar{T}}} I = -E, \tag{3.72}$$

then we find

$$E = -\frac{a(r_h - r_0)}{r_h}(\mathbb{N}\partial_{\mathcal{M}} I + \mathcal{M}\partial_{\mathbb{N}} I). \tag{3.73}$$

Without loss of generality, we may ignore the angular part of the KS metric (3.66) and consider it as

$$ds^2 = -\varrho(d\mathcal{M}^2 - d\mathbb{N}^2). \tag{3.74}$$

In this case, the calculation of the HJ method becomes more straightforward. Thus, by substituting the metric (3.74) into the HJ equation (2.7), we take the following differential equation

$$-\varrho^{-1}[-(\partial_{\mathcal{M}} I)^2 + (\partial_{\mathbb{N}} I)^2] + m^2 = 0. \tag{3.75}$$

For simplicity, we may also set $m = 0$. Then, we can postulate the following ansatz motivated by the argument in section (2.6)

$$I = \varkappa(\hat{u}), \tag{3.76}$$

where $\hat{u} = \mathcal{M} - \mathbb{N}$. From the Eqs. (3.75) and (3.76), we derive the function $\varkappa(\hat{u})$:

$$\varkappa(\hat{u}) = \int \frac{E r_h}{a(r_h - r_0)\hat{u}} d\hat{u}. \tag{3.77}$$

This expression develops a divergence at the future horizon $\hat{u} = 0$ (i.e., $\mathbb{N} = +\mathcal{M}$). This leads to a pole at the horizon which can be overcome by doing a semi-circular contour of integration in the complex plane, and therefore the result is found to be

$$\varkappa(\hat{u}) = \frac{i\pi E r_h}{a(r_h - r_0)} \qquad \rightarrow \qquad Im I = \frac{\pi E r_h}{a(r_h - r_0)}, \tag{3.78}$$

which admits the horizon temperature as

$$\breve{T}_H = \frac{a(r_h - r_0)}{2\pi r_h} = T_H. \tag{3.79}$$

In short, we prove that the correct Hawking temperature is precisely recovered in the background of the KS metric of the GMHBH.

Chapter 4

SPECTROSCOPY OF THE GRUMILLER BLACK HOLE[3]

4.1 Scalar Perturbation of the GBH and its Zerilli Equation

In this section, we shall explicitly show how one gets the radial equation for a massless scalar field in the background of the GBH described by the metric (3.3) and its function (3.4), cf. [26-28]. Then, we will derive the form of the Zerilli equation (one-dimension wave equation) [60] for the GBH. Finally, following a particular approximation method, we will also show how one computes the QNMs and entropy/area spectra of the GBH.

In order to find the entropy spectrum by using the MM, here we firstly consider the massless scalar wave or the so-called KG equation on the geometry of the GBH. The KG equation of a massless scalar field in a curved spacetime is given by

$$\Box\Psi = 0, \tag{4.1}$$

where $\Box$ denotes the Laplace-Beltrami operator. Therefore, the explicit form of the above equation is given by

$$\frac{1}{\sqrt{-g}}\partial_j\left(\sqrt{-g}\partial^j\Psi\right) = 0, \qquad j = 0,1,2,3, \tag{4.2}$$

where $\sqrt{-g} = r^2 sin\theta$. Because of the spherical symmetry, we can write the field as

[3] This Chapter is mainly quoted from Ref. [77], which is *Sakalli, I., & Mirekhtiary, S.F. (2014). Astrophysics and Space Science. 350, 727-731.*

$$\Psi = \mathrm{F}(r)e^{i\omega t}Y_L^m(\theta,\varphi), \qquad Re(\omega) > 0, \tag{4.3}$$

in which $Y_L^m(\theta,\varphi)$is the well-known spheroidal harmonics which has the eigenvalue $-L(L+1)$ [78] and ω denotes the energy or the frequency of the scalar wave. Use of the above ansatz is the standard method of separation of variables, which enables us to reduce the Eq. (4.2) into a radial equation of $\mathrm{F}(r)$. In our case, the resulting equation is

$$\frac{1}{r^2}\left[\partial_r\left(r^2 H\frac{dF}{dr}\right)\right] + \left[\frac{\omega^2}{H} - \frac{L(L+1)}{r^2}\right]F(r) = 0. \tag{4.4}$$

If we change the radial function as

$$F(r) = \frac{\mathbb{R}(r)}{r}, \tag{4.5}$$

one gets

$$[H^2\partial_r^2 + H\partial_r(H)\partial_r]\mathbb{R}(r) - \left\{H\left[\frac{L(L+1)}{r^2} + \frac{\partial_r(H)}{r}\right] - \omega^2\right\}\mathbb{R}(r) = 0. \tag{4.6}$$

In order to simplify even more this equation, we use the tortoise coordinate (3.56), so that

$$\partial_{r_*} = H\partial_r \qquad \text{and} \qquad \partial_{r_*}^2 = \ H^2\partial_r^2 + H\partial_r(H)\partial_r. \tag{4.7}$$

Finally, the radial equation reduces to the famous Zerilli equation [60], which is considered as a one-dimensional wave equation in a scattering potential barrier $V(r)$

$$\left[-\frac{d^2}{dr^{*2}} + V(r)\right]\ \mathbb{R}(r) = \omega^2\mathbb{R}(r). \tag{4.8}$$

The effective potential $V(r)$ is called the Zerilli or the Regge-Wheeler potential, and now we have

$$V(r) = H\left[\frac{L(L+1)}{r^2} + \frac{\partial_r(H)}{r}\right]. \tag{4.9}$$

If we substitute the GBH function (3.4) into the above equation, we can express the Zerilli potential of the GBH in more explicit form as

$$V(r) = H\left[\frac{L(L+1)}{r^2} + \frac{2M}{r^3} + \frac{2a}{r}\right]. \tag{4.10}$$

Now, let us check the asymptotic limits of the tortoise coordinate (3.56)

$$lim_{r\to r_h} r^* = -\infty \quad \text{and} \quad lim_{r\to\infty} r^* = \infty, \tag{4.11}$$

which imply that the potential $V(r)$ is positive towards the horizon and spatial infinity, and satisfies

$$V(r^*) \to 0 \quad \text{as} \quad r^* \to -\infty,$$

$$V(r^*) \to 4a^2 \quad \text{as} \quad r^* \to \infty\,. \tag{4.12}$$

However

$$\int_{-\infty}^{+\infty} V(r^*)d\,r^*, \tag{4.13}$$

is infinite. Therefore, we remark that such potentials admit of bound states. In physics, a bound state describes a system where a wave (or its associated particle) is subject to a potential such that the wave has a tendency to remain localized in one or more regions of space. This means that, in order to preclude a possible QNM analysis, we must consider the solution of the Zerilli equation (4.8) either at the horizon or at the spatial infinity. As demonstrated in the next section, the simplest way of reading the QNMs is to consider the near horizon form of the Zerilli equation.

4.2 QNMs and Entropy/Area Spectra of the GBH

In this section, we shall attempt to compute the entropy and area spectra of the GBH by using the MM. Gaining inspiration from the pioneer studies [79-83] in which a particular approximation method was employed; here we will first calculate the QNMs by using the associated method. Meanwhile, the QNMs are defined to be those for which we have only ingoing plane wave at the horizon. Since we know the behaviors of the Zerilli potential from Eq. (4.12), we can express our condition as follows.

$$\mathbb{R}(r)|_{QNM} \approx e^{i\omega r^*} \quad \text{as} \quad (r^* \to -\infty \;\; or \;\; r \to r_h), \tag{4.14}$$

Now our strategy is as follows: First we solve the Zerilli equation (4.8) around the event horizon r_h, and subsequently impose the above boundary condition to compute the frequency of QNMs (i.e., ω).

Expansion of the GBH's metric function (3.4) around the r_h reads

$$H = \partial_r(H)(r - r_h) + O(r - r_h)^2 \cong 2\kappa(r - r_h), \tag{4.15}$$

which means that

$$r^* = \int \frac{dr}{H} \cong \int \frac{dr}{2\kappa(r - r_h)} = \frac{1}{2\kappa}\ln(r - r_h). \tag{4.16}$$

Defining a new variable $y = r - r_h$, we get

$$r = y + r_h, \qquad dr = dy, \qquad r^* = \frac{1}{2\kappa} lny, \qquad \partial r^* = 2\kappa y \partial y. \tag{4.17}$$

If one substitutes them with Eq. (4.15) into the Zerilli potential (4.10) and then performs the Taylor expansion around the horizon:

$$\frac{1}{r} = \frac{1}{y+r_h} \cong \frac{1}{r_h}\left(1 - \frac{y}{r_h}\right) \quad \text{and} \quad \frac{1}{r^2} = \frac{1}{(y+r_h)^2} \cong \frac{1}{r_h^2}\left(1 - \frac{y}{r_h}\right), \tag{4.18}$$

we obtain the near horizon limit of the Zerilli potential (4.10) as follows

$$V(y) \cong 2\kappa y \left[\frac{L(L+1)}{{r_h}^2}(1 - \frac{2y}{r_h}) + \frac{2\kappa}{r_h}(1 - \frac{y}{r_h})\right]. \tag{4.19}$$

By making those changes, the wave equation (4.8) becomes

$$-4\kappa^2 y^2 \frac{d^2\mathbb{R}(y)}{dy^2} - 4\kappa^2 y \frac{d\mathbb{R}(y)}{dy} + V(y)\mathbb{R}(y) = \omega^2\mathbb{R}(y). \tag{4.20}$$

The solution of the above complicated differential equation has been obtained by the help of the famous mathematical computer program, Maple [84]. After a rigorous and exhausting simplification process, we finally obtain the solution in terms of the confluent hypergeometric function (see for instance [85]) as follows

$$\mathbb{R}(y) \approx y^{i\varpi} U(\hat{a}, \hat{b}; \hat{c}y). \tag{4.20}$$

where $\varpi = \frac{\omega}{2\kappa}$. The parameters of the confluent hypergeometric function $U(\hat{a}, \hat{b}; \hat{c}y)$ are found to be

$$\hat{a} = \frac{1}{2} + i\left(\varpi - \frac{\hat{\alpha}}{\hat{\beta}\sqrt{\kappa}}\right),$$

$$\hat{b} = 1 + 2i\varpi,$$

$$\hat{c} = i\frac{\hat{\beta}}{2r_h\sqrt{\kappa}}, \tag{4.21}$$

in which

$$\hat{\beta} = 4\sqrt{r_h}\sqrt{L(L+1) + \kappa r_h}, \quad \text{and} \quad \hat{\alpha} = L(L+1) + 2\kappa r_h. \tag{4.22}$$

In the limit of $y \ll 1$, the solution (4.20) can be rewritten as (cf. [85])

$$R(y) \approx D_1 y^{-i\varpi} \frac{\Gamma(2i\varpi)}{\Gamma(\hat{a})} + D_2 y^{i\varpi} \frac{\Gamma(-2i\varpi)}{\Gamma(1+\hat{a}-\hat{b})}, \tag{4.23}$$

where the symbol of Γ stands for the Gamma function, and the constants D_1 and D_2 represent the amplitudes of the near-horizon outgoing and ingoing waves, respectively. According to the definition of the QNMs which imposes the termination of the outgoing waves at the horizon, the first term of Eq. (4.23) should be cancelled. This is possible with the poles of the Γ shown in the denominator. Namely if $1/\Gamma(\hat{a})$ is an entire function with zeros at $\hat{a} = -s, (s = 0,1,2,3, \dots .)$, there would not be any outgoing wave. Thus, the frequencies of the QNM of the GBH are obtained from

$$\frac{1}{2} + i\left(\varpi - \frac{\hat{\alpha}}{\hat{\beta}\sqrt{\kappa}}\right) = -s, \tag{4.24}$$

which yields

$$\omega_s = 2\sqrt{\kappa}\frac{\hat{\alpha}}{\hat{\beta}} + i\kappa(2s+1), \tag{4.25}$$

where s is now called the overtone quantum number of the QNMs. Thus, the imaginary part of the frequency of the QNM reads

$$\omega_I = a(2s+1)\frac{r_h - r_0}{r_h}. \tag{4.26}$$

As it can be seen from above, the Rindler acceleration a plays a crucial role on ω_{I} . While $a \to 0$, $\omega_I = \frac{(2s+1)}{4M}$ which agrees with the QNMs of the Schwarzschild BH

[83,86]. Hence the transition frequency between two highly damped neighboring states becomes

$$\Delta\omega = \omega_{s+1} - \omega_s = 4\pi \frac{T_H}{\hbar}. \tag{4.27}$$

We point out that in this section $\hbar \neq 1$ so that $T_H = \frac{\hbar\kappa}{2\pi}$. Recalling the adiabatic invariant quantity that we mentioned in the Introduction, we now have

$$I_{adb} = \frac{\hbar}{4\pi} \int \frac{dM}{T_H}. \tag{4.28}$$

According to the first law of BH thermodynamics (3.12), the above expression yields

$$I_{adb} = \frac{\hbar}{4\pi} S_{BH}. \tag{4.29}$$

where S_{BH} was given in Eq. (3.11). Finally, with the aid of the Bohr-Sommerfeld quantization rule $I_{adb} = n\hbar$ $(n = 0,1,2 ...)$, we can now quantize the entropy as

$$S_{BH_n} = 4\pi n. \tag{4.30}$$

Since $S_{BH} = \frac{A_h}{4\hbar}$; the corresponding area spectrum becomes

$$A_{h_n} = 16\pi n\hbar. \tag{4.31}$$

From the above expression, one can also measure the spacing of the area as

$$\Delta A_h = 16\pi\hbar. \tag{4.32}$$

It is easily seen that unlike to ω_I the spectroscopy of the GBH is completely independent of the Rindler term a. Additionally here $\varepsilon = 16\pi$, which means that the obtained spacings between subsequent levels are the double of the Bekenstein's original result [42,46]. We shall discuss the possible reason of this discrepancy in the next chapter.

Chapter 5

CONCLUSION

In this thesis, we have considered the HR of the LDBH and the GBH (or the GMHBH) spacetimes, both have $4D$ SSS geometry. The original motivation to study this class of metrics was the fact that they are NAF BHs. We believe that most of the BHs in the real universe should be necessarily described by NAF geometries like the FRW spacetime, which is assumed to be one of the best models for describing our universe. Furthermore, the spectroscopy of the GBH has been also studied.

Perhaps the most interesting feature of the LDBHs is that their radiation acts as an isothermal process which corresponds to no change in the temperature ($\Delta T = 0$). This can be easily seen from its T_H (2.6), which possesses a constant value. In other words, T_H of the LDBH is independent from the mass M (or from the event horizon r_h) of the BH. In addition to the naive coordinates, four different regular coordinate systems are examined for the LDBHs during their HR calculations. It was shown that the computed horizon temperatures in the naive, PG, IEF and KS coordinates via the HJ method exactly matched with the conventional T_H (2.6). Here, we should notice that the way that followed up for the HJ method in the section (2.6), which considers the KS coordinates, was slightly different than the sections (2.2-2.5). In the section (2.6), without loss of generality, we discarded the mass of the scalar field and neglected the angular dependence of the HJ equation. This turned out to be the application of the HJ method for the Minkowski metric. As a result, matching of the temperatures was successfully shown. On the other hand, the most interesting section of chapter (2) is definitely the section (2.3) in which the LDBH metric was expressed in terms of the ICs. By using the Fermat metric, we have identified the index of refraction of the medium around the LDBH. In particular, it is proven that the index of the refraction plays a decisive role on the tunneling rate of the scalar particles emitting from the LDBH. Unlike to the other coordinate systems, in the ICs the our standard integration technique that circumvents the pole at the event horizon produced unacceptable value for the horizon temperature. The obtained temperature was the half of what is expected. In order to overcome this discrepancy, we have inspired from a recent study [22] which considers the Schwarzschild BH. In that paper, it has shown how the proper regularization of singular integrals leads to the standard T_H for the ICs. By this way, we have also managed to regularize the horizon temperature of the LDBH in the ICs. It has been shown in detail that the path across the horizon entails the value ($i\pi/2$) on the integration instead of $i\pi$. The underlying reason of this may arise from the fact that the isotropic coordinate ζ is real outside the BH, however it becomes complex inside the BH. We believe that the phenomenon is also closely related with the gravitational lensing effect on the HR. For the most recent work in the same line of thought, a reader may refer to [87].

In chapter (3), the HR of the GMHBH is also examined by the HJ method. This time, in addition to the naive coordinates, we have considered only three different regular coordinate systems which are PG, IEF and KS coordinates. The main reason of the exclusion of the ICs from the GMHBH is the transcendental form of the transformation from ζ to r. Without giving rise to any factor-2 problem like the one happened in the section (2.3), it was shown that the whole computed horizon temperatures of the GMHBH have exactly matched with its T_H. In the PG coordinates, we have also taken account of the back reaction and self-gravitational effects. To do so, we have employed the PWT method for the HR of the GMHBH. The modified tunneling rate (3.46) has been computed via the log-area correction to the Bekenstein-Hawking entropy (3.47). From this, the QG corrected Hawking temperature T_{QG} (3.50) have been also derived.

Chapter (4) has been devoted to the computation of the GBH spectroscopy via the MM. To this end, we have applied an approximation method [79-83] to the Zerilli equation (4.8) in order to compute the QNMs of the GBH. After a straightforward calculation, the QNM frequencies of the GBH are analytically found. Although the imaginary part of the frequency ω_I is governed by the Rindler term a, however the entropy/area spectra of the GBH are found to be independent of the a. Moreover, the obtained spectroscopy is equally spaced, which agrees with the Bekenstein's conjecture [42-44]. Furthermore, we have evaluated the dimensionless constant as $\varepsilon = 16\pi$, which means that the equispacing is the double of its Schwarzschild value: $\varepsilon = 8\pi$ [51-53]. This differentness may be caused by the Schwinger mechanism [88]. In the Bekenstein's original work [42], one derives the entropy spectrum by combining both the Schwinger mechanism and the Heisenberg quantum uncertainty principle via the Bohr-Sommerfeld quantization. However, the present method does not consider the Schwinger mechanism; only the Bohr-Sommerfeld quantization rule is taken into account. Therefore, as stated in [48], the spacings between two neighboring levels could be different according to the applied method. Thus, having $\varepsilon = 16\pi$ rather than its well-known value $\varepsilon = 8\pi$ is not an unexpected result. Besides, equidistant structure of the area spectrum of the GBH is also in agreement with the Wei et al.'s [89], which postulates that static BHs of the Einstein's theory have evenly spaced quantum spectroscopy.

Finally, I plan to extend the same analysis to the rotating and higher dimensional BHs. This is going to be my new problems in the near future.

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