$\alpha\beta-$ Statistical Convergence

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ABSTRACT

In this thesis we studied $\alpha\beta$-statistical convergence. We started with the discussion of statistical convergence. Later, we gave a brief summary of λ-statistical, lacunary statistical and $A-$statistical convergences. The concept of $\alpha\beta$-statistical convergence which is the main interest of this thesis has been considered in the last chapter of the thesis. In this chapter we also show that $\alpha\beta$-statistical convergence is a non-trivial extension of statistical, λ-statistical and lacunary statistical convergences. Finally, we introduced boundedness of a sequence in the sense of $\alpha\beta$-statistical convergence.

Keywords: $\alpha\beta-$Statistical Convergence, $\lambda-$Statistical Convergence, Statistical Convergence, A-Statistical Convergence

ÖZ

Bu tezde, $\alpha\beta$-istatistiksel yakınsaklık kavramı incelenmiştir. Bu kapsamda, öncelikle istatistiksel yakınsaklık kavramı ve bu kavrama bağlı olarak, λ -istatistiksel, lacunary istatistiksel ve $A-$istatistiksel yakınsaklık konuları hatırlatılmıştır. Daha sonra bu tezin esas amacı olan $\alpha\beta$-istatistiksel yakınsaklık tanımı özellikleri ve λ-istatistiksel, lacunary istatistiksel ve $A-$istatistiksel yakınsaklık ile ilişkileri verilmiştir. Bu kapsamda $\alpha\beta$-istatistiksel anlamında sınırlılık tanımı ilk kez bu çalışmada verilmiştir.

Anahtar Kelimeler: $\alpha\beta-$İstatistiksel Yakinsaklik,$\lambda-$İstatistiksel Yakinsaklik, İstatistiksel Yakinsaklik, A-İstatistiksel Yakinsaklik

DEDICATION

TO MY FAMILY

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Chapter 1

INTRODUCTION

Statistical convergence, which is the generalization of the ordinary convergence was first introduced by Steinhaus [11] at a conference in Poland in 1951. Then it was developed by Fast [2] in 1951 and has become popular among researchers from the different fields of mathematics. After statistical convergence different researchers introduced some other methods. Fridy and Orhan [8], proposed and discussed lacunary statistical convergence. In this method an arbitrary lacunary sequence θ is used to define lacunary statistical convergence. Statistical convergence was extended to A-statistical convergence in [10] by using a nonnegative regular matrix instead of Cesáro matrix. Mursaleen[17] introduced the concept of $\lambda-$statistical convergence [17]. Çolak introduced concepts of statistical convergence of order α and $\lambda-$statistical convergence of order α in [14] and [15] respectively. All of these concepts are nontrivial extensions of ordinary convergence. In addition, implication relations are studied for each case. For example in the concept of lacunary statistical convergence sufficient conditions for statistical convergence and ordinary convergence are obtanied in [8]. Also they carried out differences and implication conditions between statistical convergence. In the case of $\lambda-$ statistical convergence, Mursaleen did the same as well by proving the condition for implication between statistical convergence and λ-statistical convergence. Later, using these new type of convergences many research papers have been published by different authors (see for example [4],[5],[6],[10],[12],[13], [16],[18] and [19].

Recently, Aktuglu introduced the concepts of $\alpha\beta-$statistical convergence and $\alpha\beta-$ statistical convergence of order γ in [1]. In this thesis we shall focus on $\alpha\beta-$statistical convergence and $\alpha\beta-$statistical convergence of order γ which are given in [1]. Some new definitions such as $\alpha\beta-$statistically bounded sequences, $\alpha\beta-$statistically Cauchy sequences, sequenses diverging to $-\infty$ or ∞ in the sense of $\alpha\beta-$statistical convergence are given and studied in this thesis.

In Chapter 1, we give a short summary of the theory of sequences including definition of convergent sequences, Cauchy sequences and liminf and limsup of a sequence. We also give basic properties of sequences and Cauchy sequences. Moreover, some needed properties of infinite matrices and matrix transformations are discussed in Chapter 1. Finally, density functions and their properties are studied at the last part of Chapter 1.

Chapter 2 is devoted to new type of convergences such as Statistical, $\lambda-$statistical, lacunary and $A-$statistical convergences. Definitions and discusssions about basic properties and implications between these new type of convergences.

The last Chapter is organized as follows. First definirions of are provided of $\alpha\beta-$statistical convergence and $\alpha\beta-$statistical convergence of order γ which are given in [1]. Next, the relation between $\alpha\beta-$statistical convergence and new type of convergences are analyzed (see also [1]). Finally, some new definitions such as boundedness in the sense of $\alpha\beta-$statistical convergence, divergence to ∞ and $-\infty$ in the sense of $\alpha\beta-$statistical convergence are discussed.

Chapter 2

PRELIMINARIES

Before starting to discuss $\alpha\beta-$statistical convergence , we consider a very short summary about the theory of sequences and some other related topics. Furthermare in the present Chapter, some needed properties of sequences, infinite matrices and matrix transformations and density functions are studies. These basic properies of sequences will help us to see the differences between the known theory of sequences and their $\alpha\beta-$statistical cases.

2.1 Sequences

This Section is devoted to the brief summary of the theory of sequences. New type of convergences like statistical convergence, $\lambda-$statistical convergence, Lacunary statistical convergence and more generally $\alpha\beta-$statistical convergence are all summability methods. Therefore to compare these type of convergences by the ordinary convergence first recall some well known properties of ordinary convergence. Starting with some basic definition which can be easly found in any real analysis text books, related with sequences.

Definition 1 *A sequence is a function defined on the set of natural numbers* $\mathbb{N}$*. Sequences get different names with respect to their range. If the range of the sequence is* $\mathbb{R}$ *then we call this sequence a real number sequence (or real sequences). If the terms are rational numbers* $\mathbb{Q}$*, then we call this sequence rational number sequence (or rational* *sequences). Generally we use the notation*

$$x = (x_n)$$

to represent sequences. For each value of n, *the term* x_n *is known as* n^{th} *term of* x.*The space of all sequences is denoted by* ω.

Example 2 *Consider the constant function* $f(n) = 1$ *then we have the following constant sequence* $x_n := (1, 1, 1, \ \ldots, 1, \ \ldots)$

Example 3 *Taking* $f(n) = (-1)^n$ *then*$x_n := (-1, 1, -1, \ldots, (-1)^n, \ldots)$

Definition 4 *Given* $x := (x_n)$ *and let*$k_1 < k_2 < \ldots < k_n \ldots$ *where* $k_n \in \mathbb{N}$. *Then the sequence* $\left(x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots\right)$ *is called a subsequence of* x.

Example 5 *Given* $x := \left(1, \frac{1}{2}, \frac{1}{3}, ...\right)$, *then obviously the sequence*

$$x = (x_{3n}) = \left(\frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \ldots, \frac{1}{3n}, \ldots\right)$$

is a subsequence of $x := (x_n)$.

Definition 6 *A sequence* $x := (x_n)$ *is called bounded above if* $\exists K_1 \in \mathbb{R}$, *which satisfies the inequality* $x_n \leq K_1$ *for all* $n \in \mathbb{N}$. *In this case we say* K_1 *is an upper bound for* x.

Definition 7 *We say* $x := (x_n)$ *is bounded below if* $\exists K_2$, *which satisfies the inequality*$K \leq x_n$ *for all* $n \in \mathbb{N}$. *In this case we say* K_2 *is a lower bound for* x.

Definition 8 *We say that a sequence* $x := (x_n)$ *is bounded if* $\exists \ K > 0$, *which satisfies the inequality*

$$|x_n| \leq K$$

for all $n \in \mathbb{N}$.

Lemma 9 *We say* $x := (x_n)$ *is bounded if and only if it is bounded below and bounded above.*

Recall that, $l_\infty = \{x \in \omega : x_n \text{ is bounded}\}$.

Example 10 *Sequences defined by* $x_n := \frac{n-1}{n}$ *and* $y_n := \frac{1}{2n}$ *are both bounded.*

Definition 11 *A sequence* $x := (x_n)$ *converges to a number* $L \in \mathbb{R}$ *and denoted by* $x_n \to L$, *if for every* $\varepsilon > 0$ *there exists a* $N(\varepsilon) \in \mathbb{N}$, *such that for all* $n \geq N(\varepsilon)$,

$$|x_n - L| < \varepsilon.$$

In other words, x_n *converges to* L *if* $\forall \varepsilon > 0$, $|x_n - L| < \varepsilon$ *holds except finitely many terms of the sequence* x. *We use the notation* c, *to represent the space of convergent sequences,*

$$c = \{x \in \omega : x_n \textit{ is convergent}\}.$$

Definition 12 *A sequence which is not convergent is called divergent.*

Definition 13 *A sequence which is convergent to* 0 *is called a null sequence. The spaces of all null sequences is denoted by* c_0, *i.e.*

$$c_0 = \{x \in \omega : x_n \to 0\}.$$

Example 14 *The sequence* $y_k := \frac{1}{k}$ *is a null sequence.*

Remark 15 *Obviously* $c_0 \subset c \subset \omega$.

Now we recall some basic properties of sequences which are well known and can be found in any calculus text book.

Lemma 16 *If* $x := (x_n)$ *converges to* L*, then any subsequence* $x^{'} := (x_{n_k})$ *of* x *also converges to* L.

Lemma 17 *Any convergent sequence of real numbers is bounded.*

Remark 18 *In general, a bounded sequence need not be convergent. In fact, the sequence* $x_n := (-1)^n$ *is bounded but not convergent.*

Theorem 19 *(The Bolzano-Weierstrass Theorem) Any bounded sequence of real numbers has a convergent subsequence.*

Theorem 20 *Let* x *be a convergent sequence, then limit of* x *is unique.*

Proof. Assume that x converges to different limits L_1 and L_2, i.e.

$$\lim_{n\to\infty} x_n = L_1 \quad \text{and} \quad \lim_{n\to\infty} x_n = L_2 .$$

Given any $\varepsilon > 0$, there exists $N_1 > 0$, such that $\forall n \geq N_1$

$$|x_n - L_1| < \frac{\varepsilon}{2}$$

Similar there exists $N_2 > 0$, such that $\forall n \geq N_2$

$$|x_n - L_2| < \frac{\varepsilon}{2}$$

Take $N := max\{N_1, N_2\}$, then $\forall n \geq N$,

$$\begin{aligned} |L_1 - L_2| &= |L_1 - x_n + x_n - L_2| \\ &\leq |x_n - L_1| + x_n - L_2 \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

which implies

$$L_1 = L_2.$$

■

Definition 21 *For each $\varepsilon > 0$ and $a \in \mathbb{R}$. The set*

$$K_\varepsilon(a) = \{x \in \mathbb{R}, |x - a| < \varepsilon\}$$

is called the $\varepsilon-$neighbourhood of a.

Lemma 22 *Assume that $x_n \to L$. Then for every $\varepsilon > 0$, except finitely many terms of x_n, all other terms lie in $K_\varepsilon(L)$. In other words; the set*

$$\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$$

is finite.

Definition 23 *A sequence* $x := (x_n)$ *is called a Cauchy sequence if* $\forall\varepsilon > 0$, $\exists N(\varepsilon) \in \mathbb{N}$ *such that for all* $n, m \in \mathbb{N}$, *with* $n, m \geq N(\varepsilon)$, $|x_n - x_m| < \varepsilon$.

Lemma 24 *Every real valued Cauchy sequence is bounded.*

Theorem 25 *(Cauchy Convergent Criterion) A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Definition 26 *A sequence* $x := (x_n)$ *of real numbers is called increasing if it satisfies the inequality*

$$x_1 \leq x_2 \leq ... \leq x_n \leq x_{n+1} \leq ....$$

Definition 27 *A sequence* $x := (x_n)$ *of real numbers is called decreasing if it satisfies the inequality*

$$x_1 \geq x_2 \geq ... \geq x_n \geq x_{n+1} \geq ....$$

A sequence which is increasing or decreasing is called a monotone sequence.

Example 28 *The sequence* $x_n := (\frac{1}{2})^n$ *is decreasing.*

Example 29 *The sequence* $x_n := (2)^n$ *is incraesing.*

Theorem 30 *(Monotone Convergence Theorem) A monotone sequence of real numbers is convergent if and only if it is bounded.*

Theorem 31 *If (x_n) is monotone increasing (decreasing) and not bounded above (below), then $x_n \to \infty$ $(x_n \to -\infty)$ as $n \to \infty$.*

Definition 32 *Given a sequence $x := (x_n)$ of real numbers. The limit superior of (x_n) is denoted by $\lim\sup x_n$ (or $\overline{\lim}_{n\to\infty} x_n$) and defined as*

$$\begin{aligned}\lim\sup x_n &= \overline{\lim_{n\to\infty}}\, x_n \\ &= \inf_{k\in\mathbb{N}} \sup\{x_n : n \geq k\}.\end{aligned}$$

Definition 33 *Given a sequence $x := (x_n)$ of real numbers. The limit inferior of $x := (x_k)$ is denoted by $\lim\inf x_n$ (or $\underline{\lim}_{n\to\infty} x_n$) and defined as*

$$\begin{aligned}\lim\inf x_n &= \underline{\lim}_{n\to\infty} x_n \\ &= \sup_{k\in\mathbb{N}} \inf\{x_n : n \geq k\}.\end{aligned}$$

Example 34 *Consider the sequence $x_n := -1 + (-1)^n$. Then*

$$x_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -2 & \text{if } n \text{ is odd.} \end{cases}$$

Thus,

$$\overline{\lim_{n\to\infty}}\, x_n = 0 \quad \text{and} \quad \underline{\lim}_{n\to\infty} x_n = -2.$$

Example 35 *Let* $x_n := (-1)^n \frac{1}{2n+1}$ *then it is easy to see that*

$$\overline{\lim_{n\to\infty}} x_n = \underline{\lim}_{n\to\infty} x_n = 0.$$

Lemma 36 *Let x and y be two real sequence then,*

i) $\liminf x_n < \limsup x_n$

ii) $\limsup(x_n + y_n) \le \limsup x_n + \limsup y_n$

iii) $\liminf\,(a_n + b_n) \ge \liminf a_n + \liminf b_n$.

Let x be a real sequence then,

$$\lim_{n\to\infty} x_n = L \Longleftrightarrow \liminf x_n = \limsup x_n = L.$$

2.2 Matrix Transformation

Definition 37 *Let A and B be two infinite matrices and* λ *be a scalar then,*

i) $A + B = (a_{nk} + b_{nk})$ *(matrix addition)*

ii) $\lambda A = (\lambda a_{nk})$.

Definition 38 *An infinite matrix* $A = (a_{nk})$, *with non-negative entries (i.e.* $a_{nk} \ge 0$) *is called a non-negative infinite matrix.*

Assume that $A = (a_{nk})$ is an infinite matrix such that for all $x \in \omega$, the series

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k,$$

converges for each n. In this case the infinite matrix $A : \omega \to \omega$ defines a transformation on ω.

Definition 39 *An infinite matrix which maps a convergent sequence to a convergent sequence is called conservative. In other words, A is conservative if and only if for each $x \in c$, $Ax \in c$.*

Theorem 40 *(Kojima-Shurer) Let $A = (a_{nk})$ be an infinite matrix. $A = (a_{nk})$ is conservative if and only if*

(i) $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$,

(ii) $a_k := \lim_n a_{nk} = \delta_k$ for all k,

(iii) $\lim_n \sum_{k=1}^{\infty} a_{nk} = \delta$.

Definition 41 *An infinite matrix A is called regular if and only if for each $x \in c$, with $x \to L$, $\lim_n (Ax)_n = L$. Necessary and sufficient conditions for regularity of an infinite matrix $A = (a_{ij})$ is given by the following Silverman-Toeplitz Theorem.*

Theorem 42 *(Silverman-Toeplitz conditions) Let $A = (a_{nk})$ be an infinite matrix. $A = (a_{nk})$ is reguler if and only if*

(i) $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$,

(ii) For all k we have $a_k := \lim_n a_{nk} = 0$,

(iii) $\lim_n \sum_{k=1}^{\infty} a_{nk} = 1$.

Example 43 *The Cesaro matrix $C = (c_{nk})$, of order one is an infinite matrix where*

$$c_{nk} = \begin{cases} \frac{1}{n} & \textit{if} \quad 1 \leq k \leq n, \\ 0 & \textit{otherwise} \end{cases}$$

Example 44 *The matrix $A = (a_{nk})$ where*

$$a_{nk} = \begin{cases} 1 - \frac{1}{n^2} & \textit{if } k = n-1, \\ \frac{1}{n^2} & \textit{if } k = n, \\ 0 & \textit{otherwise} \end{cases}$$

or equivalently,

$$A = (a_{nk}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & \cdots \\ 0 & \frac{8}{9} & \frac{1}{9} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & \cdots & 1 - \frac{1}{n^2} & \frac{1}{n^2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

is a regular matrix.

Definition 45 *Consider a sequence x, and an infinite matrix A. Then we say that x is A-summable to L if:*

$$\lim_{n \to \infty} (Ax)_n = L$$

In the rest of the thesis, we will consider the matrix A, as an infinite, non-negative and regular matrix unless it is mentoined otherwise.

2.3 Densities

As it is well known, the theory of statistical convergence and other type of convergences are all based on a density function. This is why we need to explain the idea and basic properties of densities. Therefore through this section, we give definitions and some basic properties of density functions, which will be used in next Chapters.

Definition 46 *For any subset $D \subseteq \mathbb{N}$, the function x_D of D is defined by:*

$$x_D(k) := \begin{cases} 1, & k \in D \\ 0, & k \notin D \end{cases}, \; k = 1, 2, 3, ...$$

Example 47 *For a set $D = \{3n \mid n \in \mathbb{N}\}$ the characteristic function is* $\chi_D(k) := \begin{cases} 1, & k \in D \\ 0, & k \notin D \end{cases}$

or as a sequence it is,

$$\chi_K = (0, 0, 1, 0, 0, 1, ..)$$

Definition 48 *Symmetric difference of two subsets A and B of natural numbers, $\mathbb{N}$ is defined as below:*

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

If the symmetric difference above is finite, then we can say that they have "$\sim$" relation and is denoted by:

$$A \sim B \Leftrightarrow A \triangle B.$$

Definition 49 *(See [3]) A set funtion*

$$\delta : 2^{\mathbb{N}} \to [0,1]$$

satisfying the following conditions;

$$\begin{aligned} &(1) \quad if \;\; A \sim B \;\; then \; \delta(A) = \delta(B); \\ &(2) \quad if \; A \cap B = \emptyset, \; then \; \delta(A) + \delta(B) \leq \delta(A \cup B); \\ &(3) \quad for \; all \;\; A, B; \; \delta(A) + \delta(B) \leq 1 + \delta(A \cap B); \\ &(4) \quad \delta(\mathbb{N}) = 1. \end{aligned} \tag{2.1}$$

is called an asymptotic density function.

For a given density function δ, the set function defined by;

$$\overline{\delta}(A) = 1 - \delta(\mathbb{N} \backslash A)$$

also satisfies conditions given in (2.1). Therefore $\overline{\delta}$ is also a density function which is called upper asymptotic density of δ.

Proposition 50 *Let δ be a density and $\overline{\delta}$ be its upper density then any subsets A and B of natural numbers we have;*

$i)$ $A \subseteq B \Rightarrow \delta(A) \leqslant \delta(B)$,

$ii)$ $A \subseteq B \Rightarrow \overline{\delta}\,(A) \leqslant \overline{\delta}(B)$,

$iii)$ $\overline{\delta}(A) + \overline{\delta}(B) \geq \overline{\delta}(A \cup B)$,

$iv)$ $\delta(\varnothing) = \overline{\delta}(\varnothing) = 0$,

$v)$ $\overline{\delta}(\mathbb{N}) = 1$,

$vi)$ $A \sim B \Rightarrow \overline{\delta}\,(A) = \overline{\delta}(B)$,

$vii)$ $\delta(A) \leq \overline{\delta}(A)$.

Proof. (i) Using $A \cap (B \backslash A) = \emptyset$ and (2) of (2.1) one can write that,

$$\delta(A) + \delta(B \backslash A) \leq \delta(A \cup (B \backslash A)) = \delta(B),$$

since $\delta(B \backslash A) \geq 0$, we have,

$$\delta(A) \leq \delta(B).$$

(ii) If $A \subseteq B$ then $B^c = \mathbb{N} \backslash B \subseteq \mathbb{N} \backslash A = A^c$, using $i)$, we get

$$\delta(\mathbb{N} \backslash B) \;\leq\; \delta(\mathbb{N} \backslash A)$$

or equivalently,

$$\overline{\delta}(A) = 1 - \delta(\mathbb{N} \backslash A) \;\leq 1 - \delta(\mathbb{N} \backslash B) = \overline{\delta}(B).$$

(iii) As a consequence of the definition we can write that,

$$\begin{aligned}\overline{\delta}(A)+\overline{\delta}(B) &= 2-\delta(\mathbb{N}\backslash A)-\delta(\mathbb{N}\backslash B) \\ &= 2-(\delta(\mathbb{N}\backslash A)+\ \delta(\mathbb{N}\backslash B)) \\ &\geq 2-(1+\delta((\mathbb{N}\backslash A)\cap(\mathbb{N}\backslash B))).\end{aligned}$$

But $\delta((\mathbb{N}\backslash A)\cap(\mathbb{N}\backslash B)) = \delta(\mathbb{N}\backslash(A\cup B))$ gives

$$\overline{\delta}(A)+\overline{\delta}(B) \geq 1-\delta(\mathbb{N}\backslash(A\cup B)) = \overline{\delta}(A\cup B).$$

(iv) Take $A = \phi$ and $B = \mathbb{N}$ in (2) of (2.1), we have

$$\delta(\emptyset)+\delta(\mathbb{N}) \leq \delta(\mathbb{N}\cup\emptyset) = \delta(\mathbb{N}),$$

which implies that

$$\delta(\emptyset) = 0.$$

Moreover if we take $A = \emptyset$, we have

$$\overline{\delta}(\emptyset) = 1-\delta(\mathbb{N}\backslash\emptyset) = 1-\delta(\mathbb{N}) = 0.$$

(v) This is a direct consequence of the definition of density (vi) Assume that $A \sim B$ then we have

$$\begin{aligned}(\mathbb{N}\backslash A) \quad \triangle \quad (\mathbb{N}\backslash B) &= ((\mathbb{N}\backslash A)\setminus(\mathbb{N}\backslash B))\cup((\mathbb{N}\backslash B)\setminus(\mathbb{N}\backslash A)) \\ &= (B\backslash A)\cup(A\backslash B) = A \sim B,\end{aligned}$$

which means that

$$\delta(\mathbb{N}\backslash A) = \delta(\mathbb{N}\backslash B).$$

Hence

$$\overline{\delta}(A) = \overline{\delta}(B).$$

(vii) Take $B = \mathbb{N}\backslash A$ in (2) of (2.1), gives

$$\delta(\mathbb{N}\backslash A) + \delta(A) \leq \ \delta((\mathbb{N}\backslash A)\cup A) = \delta(\mathbb{N}) = 1$$

thus

$$\delta(A) \leq 1 - \delta(\mathbb{N}\backslash A) = \overline{\delta}(A).$$

■

Definition 51 *([3]) We say that, a subset A of natural numbers has natural density if* *and only if,*

$$\delta(A) = \overline{\delta}(A).$$

Example 52 *Let d be a function defined from power set of natural numbers to the interval* $[0,1]$ *as follows,*

$$d(A) = \lim_{n\to\infty} \frac{|A_n|}{n}$$

where $|A_n|$ denotes the number of elements in $A \cap \{1,2,3,\ldots,n\}$, then d defines a natural density.

It is not difficult to see that the function $d(A)$ defined above satisfies the conditions for density functions. We can also define $d(A)$ in another way using the non-negative regular matrix C_1, the Cesaro matrix of order one, since $\frac{A(n)}{n}$ is n th term of the sequence $(C_1 x_A)$ we have,

$$d(A) = \liminf (C.x_A)_n.$$

Proposition 53 *Let M be a non-negative, regular, infinite matrix and let* δ_M *be defined as follows;*$\delta_M = \lim_{n\to\infty} \inf(M.\varkappa_A)_n$*,then* δ_M *is a natural density function (i.e. satisfies (2.1) and furthermore,*

$$\bar{\delta}_M = \limsup_{n\to\infty} (M.\varkappa_A)_n.$$

Example 54 *It is easy to see that,*

i) $\delta(\mathbb{N}) = 1$

ii) $\delta(\mathbb{N}^2) = \delta(\left\{n^2 : n\epsilon\mathbb{N}\right\}) = 0$

iii) $\delta(\{2n : n\epsilon\mathbb{N}\}) = \delta(\{2n+1 : n\epsilon\mathbb{N}\}) = \frac{1}{2}$.

Example 55 *The natural density of all finite sets are zero.*

Example 56 *In genaral, for a set* $K = \{ak + b : k\epsilon\mathbb{N}\}$*, we have*

$$\delta(K) = \frac{1}{a}.$$

Example 57 *Given,*

$$x_k = (1,0,0,1,1,1,1,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0,...)$$

and the subset $K = \{k \in \mathbb{N} \ : x_k = 1\}$ *of natural numbers, then for large m we have*

$$\left|K(2^m)\right| = \begin{cases} \geq 2^{m-1} + 2^{m-3} & \text{if } m \text{ is odd,} \\ \\ \leq 2^{m-2} + 2^{m-3} & \text{if } m \text{ is even.} \end{cases}$$

moreover,

$$\lim_m \frac{|K(2^m)|}{2^m} = \begin{cases} \geq \frac{5}{8} & \text{if } m \text{ is odd,} \\ \\ \leq \frac{3}{8} & \text{if } m \text{ is even.} \end{cases}.$$

therefore $\delta(K)$ *does not exist.*

Chapter 3

NEW TYPE CONVERGENCES

The concept of new type of convergences has been initiated by Fast in [2]. After that, there is an increasing interest to new type of convergences, among researchers. Different type of converges has been introduced by many researchers. This section is devoted to the definitions and properties of these new type of convergences such as statistical convergence, lacunary statistical convergence and $\lambda-$statistical convergence. The aim here is not to give these type of convergences with all details but is to briefly explain the idea for each case.

3.1 Statistical Convergence

Definition 58 ([2]) *Any sequence $x := (x_k)$ satisfying the condition,*

$$\delta(\{k : |x_k - L| \geq \varepsilon\}) = 0.$$

for every $\varepsilon > 0$, is called statistically convergent to L and is denoted by $st - \lim_n x_n = L$.

Recall that the natural density of all finite sets are zero. If we combine this by the fact that ordinary convergence of a sequence to a number L, implies that

$$\{k : |x_k - L| \geq \varepsilon\}$$

is a finite set we get the following Theorem.

Theorem 59 *Ordinary convergence implies statistical convergence.*

The following remark gives the most important difference between ordinary and statistical convergence.

Remark 60 *If x is statistically convergent to L, then every $\varepsilon-$neigborhood of L, contains all terms of the sequence except terms with their indices having density* 0.

Now we are ready to consider examples of statistical convergent sequences, and also try to show the differences between ordinary and statistical convergenses on examples.

Example 61 *Let $x := (x_n)$ be the sequence*

$$x_n = \begin{cases} 3 & if \quad n = m^2, \\ \frac{1}{m} & if \quad n \neq m^2. \end{cases}$$

Since $\delta\left(\left\{n^2 : n \in \mathbb{N}\right\}\right) = 0$ we have $st - \lim_n x_n = 0$, but x is not convergent in the ordinary sense.

The boundedness property is not hold by the statistical convergence. Recall that in the sense of ordinary convergence, convergent sequences are all bounded. So, this also shows that statistical and ordinary convergence are different from each other.

Example 62 *Let $x := (x_n)$ be the sequence*

$$x_k = \begin{cases} n^2 & if \quad n = m^2, \\ 1 & if \quad n \neq m^2. \end{cases}$$

It is easy to see that x is not bounded but $st-\lim_n x_n = 1$.

Lemma 63 [2] *Assume that $st-\lim x_n = L_1$ and $st-\lim y_n = L_2$ then*

(i) $st-\lim(x_n + y_n) = L_1 + L_2$.

(ii) $st-\lim(x_n y_n) = L_1 L_2$.

(iii) $st-\lim(kx_n) = kL_1$ for any $k \in \mathbb{R}$.

Proof. (i) Given, $\varepsilon > 0$. Then, since

$$\{n : |(x_n - y_n) - (L_1 + L_2)| \geq \varepsilon\} \subset \left\{n : |x_n - L_1| \geq \frac{\varepsilon}{2}\right\} \cup \left\{n : |y_n - L_2| \geq \frac{\varepsilon}{2}\right\}.$$

we have $st-\lim(x_n + y_n) = L_1 + L_2$. (ii) Assume that $st-\lim x_n = L_1$. By the definition of statistical convergence

$$\delta(A) = \delta(\{n : |x_n - L_1| < 1\}) = 1.$$

On the other hand

$$|x_n y_n - L_1 L_2| \leq |x_n|\,|y_n - L_2| + |L_2|\,|x_n - L_1|\,.$$

For each $n \in A$, $|x_n| < |L| + 1$. This implies that,

$$|x_n y_n - L_1 L_2| \leq (|L_1| + 1)\,|y_n - L_2| + |L_2|\,|x_n - L_1|\,. \tag{3.1}$$

Now given $\varepsilon > 0$ and choose $\delta > 0$ such that

$$0 < 2\delta < \frac{\varepsilon}{|L_1| + |L_2| + 1}. \tag{3.2}$$

Let $F_1 = \{n : |x_n - L_1| < \delta\}$ and $F_2 = \{n : |y_n - L_2| < \delta\}$ then $\delta(F_1) = \delta(F_2) = 1$ and $\delta(A \cap F_1 \cap F_2) = 1$. For each $n \in A \cap F_1 \cap F_2$ we have from 3.1 and 3.2

$$|x_n y_n - L_1 L_2| < \varepsilon.$$

Now $\delta\{n : |x_n y_n - L_1 L_2| \geq \varepsilon\} = 0$ and $st - \lim(x_k y_k) = L_1 L_2$.

(iii) Take $y_k = \lambda$ for all $n \in \mathbb{N}$, then it follows from (ii).

■

Remark 64 *Asume that* $st\text{-}\lim\ x = L \Leftrightarrow \exists(n_k)$ *such that* $\delta\{n_k : k \in \mathbb{N}\} = 1$ *and* $\lim_k x_{n_k} = L$.

Example 65 *Consider the sequences,*

$$x = (x_n) := \begin{cases} 1 & n = k^2, \textit{ for some } k \\ 0 & n = k^2 + 1, \textit{ for some } k \\ 2 & \textit{otherwise} \end{cases}$$

and

$$y = (y_n) := \begin{cases} \frac{1}{n} + 1 & \textit{otherwise} \\ 0 & n = k^2, \textit{ for some } k \end{cases}$$

then x and y are not convergent in the ordinary sense but

$$st - \lim x_n = 2 \textit{ and } st - \lim y_n = 1.$$

Using Lemma 2.1.6 we have,

$$st - \lim(x_n + y_n) = 3$$

$$st - \lim(x_n y_n) = 2$$

and

$$st - \lim(3x_n) = 6.$$

Definition 66 *A sequence x is statistically divergent to ∞ if for any real number M,*

$$\delta(\{n \in \mathbb{N} : x_n > M\}) = 1.$$

Example 67 *Consider the sequence*

$$x = (x_n) := \begin{cases} \sqrt{n} & \textit{otherwise} \\ 1 & n = k^2 + 2, \textit{ for some } k \end{cases}$$

then x is statistically diverges to ∞.

Definition 68 *A sequence x is statistically divergent to −∞ if for any real number K,*

$$\delta(\{n \in \mathbb{N} : x_n < K\}) = 1.$$

Example 69 *Consider the sequence*

$$x = (x_n) := \begin{cases} \sqrt{n} & n = k^2 + 2, \textit{ for some } k \\ -n + 1 & \textit{otherwise} \end{cases}$$

then x is statistically diverges to -∞.

Definition 70 *A sequence $x := (x_k)$ is statistically Cauchy sequence if for each $\varepsilon > 0$, $\exists N(\varepsilon)$ such that,*

$$\lim_n \frac{1}{n} |\{k \le n : |x_k - x_N| \ge \varepsilon\}| = 0.$$

Lemma 71 *A sequence $x := (x_k)$ is statistically Cauchy sequence if and only if $\exists D \subset \mathbb{N}$ with $\delta(D) = 1$ and x is Cauch on D.*

Example 72 *Consider the sequence*

$$x = (x_n) := \begin{cases} 1 & n = k^2 + 2, \text{ for some } k \\ \frac{1}{n} & \text{otherwise} \end{cases}$$

then x is statistically Cauchy sequence.

Parallel to the ordinary case, one state the following theorem,

Theorem 73 *A sequence x is statistically convergent if and only if it is statistically Cauchy sequence.*

Proof. Assume that x is statistically convergent to L. By the definition there exists a subset D of natural numbers with $\delta(D) = 1$ and $x \to L$ on D in the ordinary sense. This means that x is a cauch sequence on D or equivalently x is a statistical Cauchy sequence.

Conversely, assume that x is statistical Cauch sequence then by Lemma, there exists a subset D of natural numbers with $\delta(D) = 1$ and x is Cauchy on D. Therefore it is convergent on D, which means that x is statistical cauchy sequence. ■

Theorem 74 ([4]) *If x is a sequence such that $st - \lim x_k = L$ and $\triangle x_k = o(\frac{1}{k})$, then $\lim x_k = L$.*

3.2 Lacunary Statistical Convergence

Definition 75 ([8]) *A sequence $\theta = \{k_r\}$ satisfiying,*

i) $k_0 = 0$

ii) $h_r = k_r - k_{r-1} \to \infty, \; r \to \infty.$

is called a lacunary sequence.

For each lacunary sequence θ one define the interval $I_r := (k_{r-1,} \, k_r]$ and the fraction $q_r := \frac{k_r}{k_{r-1}}$. Lacunary statistical convergence has been introduced by Fridy and Orhan in the following way.

Example 76 *The sequence $\theta = \{k_r\} = \{2^r\}$ is a lacunary sequence with $I_r := \left(2^{r-1}, 2^r\right]$ and $q_r := 2$.*

Definition 77 ([8]) *A sequence x is called Lacunary statistical convergent and denoted by $\varkappa_k \to L(\theta - st)$ if for every $\varepsilon > 0$,*

$$\lim_r \frac{1}{h_r} |\{k \in I_r : \; |x_k - L| \geq \epsilon\}| = 0$$

Lemma 78 ([8]) *For a lacunary sequence $\theta = \{k_r\}$, $\varkappa_k \to L$ implies $\varkappa_k \to L(\theta - st)$ if and only if* $\liminf_r q_r > 1$.

Lemma 79 ([8]) *For a lacunary sequence $\theta = \{k_r\}$, $\varkappa_k \to L(\theta - st)$ implies $\varkappa_k \to L$ if and only if* $\limsup_r q_r < \infty$.

As a consequence of following Lemmas we have;

Theorem 80 ([8]) *Let $\theta = \{k_r\}$ be a lacunary sequence. Then statistical convergence and $\theta-$statistical convergence of $\varkappa_k \to L$ are equal if and only if*

$$1 < \liminf_r q_r < \limsup_r q_r < \infty.$$

Example 81 *The lacunary sequence $\theta = \{k_r\} = \{2^r\}$, satisfies the conditions of the above theorem for $r > 0$.*

3.3 $\lambda-$Statistical Convergence

The concept of λ-statistical convergence for sequences of numbers was first introduced by Mursaleen in [17] where he showed that statistical convergence is an extension of ordinary convergence. Fridy introduced sufficent conditions for ordinary and and statistical convergence. Also Orhan and Fridy provided $\theta-$convergence and introduced implication conditions and also the differences between θ-convergent and statistical convergence. Mursaleen applied the same method for λ-statistical convergence and obtained implication conditions. Later this theory was extended by Çolak, introducing statistical convergence of order α.

To define λ-statistical convergence first we need a sequence (λ_r) of positive, non-decreasing numbers such that $\lambda_r \to \infty$, as $r \to \infty$, $\lambda_1 = 1$ and $\lambda_{r+1} \leq \lambda_r + 1$. Assume that w is the space of all sequences satisfying these conditions. Then for each $(\lambda_r) \in w$ and for each r, we can define intervals,

$$M_r = [r - \lambda_r + 1, r].$$

Definition 82 ([*17*]) *A sequence x is said to be $\lambda-$statistical convergent to L if, for all $\varepsilon > 0$,*

$$\lim_r \frac{1}{\lambda_r} \{|k \epsilon M_r : \ |x_k - L| \geq \epsilon|\} = 0.$$

The λ-statistical convergence of $\varkappa$ to L is represented by the notation $\varkappa_k \to L(\lambda - st)$.

Remark 83 *For $\lambda_r = r$, $\lambda-$statistical convergence coincides with statistical convergence.*

3.4 A-Statistical Convergence

As we discussed in the previous sections, density was defined on Cesáro matrix A of order one. Freedman and Sember [3] used a non-negative regular matrix instead of A, and defined the concept of A-density. In [10] Kolk, used $A-$density to define $A-$statistical convergence. Later, many mathematicians have used $A-$statistical convergence in their research studies.

Definition 84 ([*3*]) *Let $A = (a_{nk})$ be a nonnegative regular matrix an $K \subseteq \mathbb{N}$. Then the* *set function or density*

$$\delta_A(K) := \lim_n (A\varkappa)_n = \lim_n \sum\nolimits_{k\in K} (a_{nk})$$

if the limit above exists, is called the A−density of the set K and denoted by $\delta_A(K)$.

Lemma 85 *For an existing* $\delta_A(K)$ *or* $\delta_A(\mathbb{N}\diagdown K)$ *we have the following relation.*

$$\delta_A(K) = 1 - \delta_A(\mathbb{N}\diagdown K)$$

Remark 86 $\delta_A(K) = 0$ *when K is finite.*

Definition 87 ([*10*]) *Suppose* $A = (a_{nk})$ *is nonnegative regular matrix, if* $\exists L$ *such that for all* $\varepsilon > 0$

$$\lim_{n\to\infty} \sum_{k:|x_k - L|\geq\varepsilon} (a_{nk}) = 0$$

then we say that the sequence $x = (x_K)$ *is A-statistical convergent to L. In this case we will denote it as bellow ;*

$$\varkappa_K \to L(A - st)$$

Lemma 88 ([*10*]) *Let*

$$K(\varepsilon) = \{k \in \mathbb{N} : |\varkappa_K - L| \geq \varepsilon\}$$

and let $x_{K(\varepsilon)}$ *be the characteristic function of* $K(\varepsilon)$ *then,* x_K *is A-statistical convergent* *to L if and only if* $\forall \epsilon > 0$,

$$\lim_{n\to\infty} (A\varkappa_{K(\varepsilon)})_n = 0.$$

Example 89 *Consider the matrix* $C = (c_{nk})$ *where*

$$c_{nk} = \begin{cases} \frac{1}{n} & k \leq n \\ 0 & otherwise \end{cases}$$

or

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \end{bmatrix}$$

which is known as Cesáro matrix. Then if we use C_1 *instead of A given at the begining of this section, we reach the definiton of natural density see below;*

$$\delta(K) = \lim_{n\to\infty} (C_1.\varkappa_K)_n = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|.$$

Example 90 *Consider the following nonnegative regular matrix,*

$$A = (a_{nk}) = \begin{cases} 1 & k = n^2 \\ 0 & k \neq n^2 \end{cases} \quad n = 1, 2, 3, \ldots$$

and the sequence $\varkappa$ given below,

$$\varkappa = \begin{cases} \frac{1}{2} & k = n^2 \\ 1 & k \neq n^2 \end{cases} .$$

Then for any $\varepsilon > 0$ and

$$K(\varepsilon) = \left\{ k \in \mathbb{N} : \left| \varkappa_K - \frac{1}{2} \right| \geq \varepsilon \right\}$$

we get

$$\lim(A\varkappa_{K(\varepsilon)}) = 0$$

Therefore

$$A - st \lim_{n \to \infty} = \frac{1}{2}.$$

Remark 91 *Consider λ_n with the following properties*

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_{n+1} &\leq \lambda_n + 1 \\ \lambda_n &\geq 0 \end{aligned}$$

and define the matrix $A = (a_{rk})$ as below;

$$A_\lambda = \begin{cases} \frac{1}{\lambda_K} & k \in I_n \\ 0 & otherwise \end{cases}$$

then $A_\lambda-$statistical convergence coincides with $\lambda-$statistical convergence.

Example 92 *Let $\theta = \{k_r\}$ be a lacunary sequence then consider the matrix $A_\theta = (a_{rk})$ where*

$$A_\theta = \begin{cases} \frac{1}{h_r} & k \in I_r \\ 0, & k \notin I_r \end{cases}$$

then $A_\theta-$statistical convergence is lacunary statistical convergence.

Example 93 *:Given the matrix below*

$$A = a_{nk} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

and the sequences

$$\varkappa_n = \begin{cases} 0 & n = 2n \\ 1 & n \neq 2n \end{cases}$$

we will see that $\varkappa$ is A-statistical convergent to zero while it is not statistical convergent. To see that it is enough to see that, if $\varepsilon > 1$ then the set below is empty

$$K(\varepsilon) = \{k : |\varkappa_k - 0| \geq \varepsilon\}.$$

But if $0 < \varepsilon \leq 1$ *then we have*

$$K(\varepsilon) = \{1, 3, 5, 7, ..\}$$

In this case, since

$$\varkappa_k = (1, 0, 1, 0, 1, ...)$$

we get

$$(A.\varkappa_k) = (0, 0, 0, ..).$$

Therefore, calculating the A-density of K we have,

$$\delta(K) = \lim_{n \to \infty} \sum_{k \in K(\varepsilon)} a_{nk} = \lim_{n \to \infty} \sum_{k \in \{1,3,5,...\}} a_{nk} = 0.$$

Chapter 4

$\alpha\beta$-STATISTICAL CONVERGENCE

This chapter is devoted to the concept of $\alpha\beta-$statistical convergence which is introduced in ([1]). Recall that for each type of convergence, there exists a density function and it plays a basic role in the definition of different type of convergences. The idea which is used to define new type of convergences was the following, a sequence may have infinitely many terms which are not including in $\varepsilon-$neigborhoods of the limit point for ε small enough but the set of indicies of such terms have density zero. As it is well known this is not possible in ordinary sense. Therefore, new type of convergences defined in this way give us a new type convergence which is different from ordinary convergence. In many years researchers focus on convergences which are obtained from different density functions. But a careful observation shows that all density functions are based on different class of intervals. For example statistical convergence and lacunary statistical convergences are based on intervals $[1,n]$ and $(k_{n-1},k_n]$ respectively. In [1], it is shown that a generalization of set of intervals gives us a generalization of these new type convergences. This is the basic idea of $\alpha\beta-$statistical convergence. After this brief idea of $\alpha\beta-$statistical convergence, we shall discuss details of this generalization of new type of convergences.

Now let α and β be two sequences such that,

$$\begin{aligned} P_1 &: \alpha(n), \beta(n) \geq 0 \;\; \forall n \epsilon N, \\ P_2 &: \beta(n) \geq \alpha(n), \forall n \epsilon N \\ P_3 &: \beta(n) - \alpha(n) \to \infty \text{ as } n \to \infty. \end{aligned} \tag{4.1}$$

For the simplicity we shall use the notation Λ to represent the set of pairs of sequences α and β satisfying (4.1) i.e. ,

$$\Lambda := \{(\alpha, \beta) \mid \alpha \text{ and } \beta \text{ satisfies } P_1, P_2, P_3\} \subset s \times s$$

Definition 94 ([*1*])*For any $K \subset N$ and for each pair $(\alpha, \beta) \in \Lambda$*

we define :

$$\delta^{\alpha,\beta}(K) = \lim_{n\to\infty} \frac{\left|K \cap [\alpha(n), \beta(n)]\right|}{(\beta(n) - \alpha(n) + 1)} \tag{4.2}$$

where $|S|$ is the cardinality of the set S and $P_n^{\alpha,\beta} = [\alpha(n), \beta(n)]$

Lemma 95 ([*1*]) *Let M and K be any subset of $\mathbb{N}$ and $(\alpha, \beta) \in \Lambda$*

i) $\delta^{\alpha,\beta}(\phi) = 0$

ii) $\delta^{\alpha,\beta}(\mathbb{N}) = 1$

iii) If K is finite then $\delta^{\alpha,\beta}(K) = 0$

iv) $K \subset M \Longrightarrow \delta^{\alpha,\beta}(K) \leq \delta^{\alpha,\beta}(M)$

Proof. *i*) Taking $K = \phi$ in (4.2) gives,

$$\begin{aligned}
\delta^{\alpha,\beta}(\phi) &= \lim_{n\to\infty} \frac{\left|\phi \cap [\alpha(n),\beta(n)]\right|}{(\beta(n)-\alpha(n)+1)} \\
&= \lim_{n\to\infty} \frac{0}{(\beta(n)-\alpha(n)+1)} \\
&= 0.
\end{aligned}$$

ii) Using (4.2), we have

$$\begin{aligned}
\delta^{\alpha,\beta}(\mathbb{N}) &= \lim_{n\to\infty} \frac{\left|K \cap [\alpha(n),\beta(n)]\right|}{(\beta(n)-\alpha(n)+1)} \\
&= \lim_{n\to\infty} \frac{(\beta(n)-\alpha(n)+1)}{(\beta(n)-\alpha(n)+1)} \\
&= 1.
\end{aligned}$$

iii) Assume that $K \subset \mathbb{N}$ is finite with $|K| = c$,

$$\begin{aligned}
\delta^{\alpha,\beta}(K) &= \lim_{n\to\infty} \frac{\left|K \cap [\alpha(n),\beta(n)]\right|}{(\beta(n)-\alpha(n)+1)} \\
&\leq \lim_{n\to\infty} \frac{c}{(\beta(n)-\alpha(n)+1)} \\
&= 0.
\end{aligned}$$

iv) If $K \subset M$ then

$$K \cap [\alpha(n),\beta(n)] \subset M \cap [\alpha(n),\beta(n)]$$

which means that

$$\left|K \cap [\alpha(n),\beta(n)]\right| \leq \left|M \cap [\alpha(n),\beta(n)]\right|$$

and

$$\frac{\left|K\cap[\alpha(n),\beta(n)]\right|}{(\beta(n)-\alpha(n)+1)}\leq\frac{\left|M\cap[\alpha(n),\beta(n)]\right|}{(\beta(n)-\alpha(n)+1)}.$$

Taking limits from both sides as $n\to\infty$ we have

$$\delta^{\alpha,\beta}(K)\leq\delta^{\alpha,\beta}(M).$$

■

Definition 96 ([*1*]) *We say the sequence x is $\alpha\beta-$statistically convergent to L and denote by $x_n\to L$ $(\alpha\beta-st)$ if* $\forall\varepsilon>0$

$$\begin{aligned}\delta^{\alpha,\beta}\left(\left\{k\in P_n^{\alpha,\beta}:|x_k-L|\geq\varepsilon\right\}\right) &= \lim_{n\to\infty}\frac{\left|\left\{k\epsilon P_n^{\alpha,\beta}:|x_k-L|\geq\varepsilon\right\}\right|}{(\beta(n)-\alpha(n)+1)}\\ &= \lim_{n\to\infty}\frac{\left|\left\{k\epsilon[\alpha(n),\beta(n)]:|x_k-L|\geq\varepsilon\right\}\right|}{(\beta(n)-\alpha(n)+1)}\\ &= 0.\end{aligned}$$

Example 97 *Consider the sequence*

$$x=(x_n):=\begin{cases}0 & n=k^2,\text{ for some }k\\ 1 & \text{otherwise}\end{cases}$$

and take $\alpha(n) = n, \beta(n) = n^2$, *then*

$$\begin{aligned}
\delta^{\alpha,\beta}\left(\left\{k \in P_n^{\alpha,\beta} : |x_k - 1| \geq \varepsilon\right\}\right) &= \lim_{n\to\infty} \frac{\left|\left\{k\epsilon P_n^{\alpha,\beta} : |x_k - L| \geq \varepsilon\right\}\right|}{(\beta(n) - \alpha(n) + 1)} \\
&= \lim_{n\to\infty} \frac{\left|\left\{k\epsilon \left[n, n^2\right] : |x_k - L| \geq \varepsilon\right\}\right|}{n^2 - n + 1} \\
&= \lim_{n\to\infty} \frac{\left|\left\{k\epsilon \left[n, n^2\right] : |x_k - L| \geq \varepsilon\right\}\right|}{n^2 - n + 1} \\
&\leq \lim_{n\to\infty} \frac{n}{n^2 - n + 1} = 0
\end{aligned}$$

therefore x is $\alpha\beta-$*statistical convergent to 1.*

Example 98 *Consider the sequence*

$$x = (x_n) := \begin{cases} 1 & n = k^3, \textit{ for some } k \\ 0 & \textit{otherwise} \end{cases}$$

and take $\alpha(n) = 1, \beta(n) = n^3$, *then*

$$\begin{aligned}
\delta^{\alpha,\beta}\left(\left\{k \in P_n^{\alpha,\beta} : |x_k - 1| \geq \varepsilon\right\}\right) &= \lim_{n\to\infty} \frac{\left|\left\{k\epsilon P_n^{\alpha,\beta} : |x_k - L| \geq \varepsilon\right\}\right|}{(\beta(n) - \alpha(n) + 1)} \\
&= \lim_{n\to\infty} \frac{\left|\left\{k\epsilon\left[1, n^3\right] : |x_k - L| \geq \varepsilon\right\}\right|}{n^3} \\
&= \lim_{n\to\infty} \frac{\left|\left\{k\epsilon\left[1, n^3\right] : |x_k - L| \geq \varepsilon\right\}\right|}{n^3} \\
&\leq \lim_{n\to\infty} \frac{n}{n^3} = 0
\end{aligned}$$

therefore x is $\alpha\beta-$*statistical convergent to 0.*

Lemma 99 ([*1*]) *Assume that* $x_n \to L_1$ $(\alpha\beta - st)$ *and* $y_n \to L_2$ $(\alpha\beta - st)$ *then*

(i) $(x_n + y_n) \to L_1 + L_2$ $(\alpha\beta - st)$

(ii) $(x_n y_n) \to L_1 L_2$ $(\alpha\beta - st)$

(iii) $(kx_n) \to kL_1$ $(\alpha\beta - st)$ *for any* $k \in \mathbb{R}$.

Proof. (i) Given, $\varepsilon > 0$. Since

$$\{n : |(x_n - y_n) - (L_1 + L_2)| \geq \varepsilon\} \subset \left\{n : |x_n - L_1| \geq \frac{\varepsilon}{2}\right\} \cup \left\{n : |y_n - L_2| \geq \frac{\varepsilon}{2}\right\}.$$

we have $(x_n + y_n) \to L_1 + L_2$ $(\alpha\beta - \mathrm{st})$.

(ii) Assume that $x_n \to L_1 \ (\alpha\beta - \mathrm{st})$. By the definition of statistical convergence

$$\delta^{\alpha,\beta}(A) = \delta^{\alpha,\beta}(\{n : |x_n - L_1| < 1\}) = 1.$$

On the other hand

$$|x_n y_n - L_1 L_2| \leq |x_n| |y_n - L_2| + |L_2| |x_n - L_1|,$$

for each $n \in A$, $|x_n| < |L| + 1$. This implies that,

$$|x_n y_n - L_1 L_2| \leq (|L_1| + 1) |y_n - L_2| + |L_2| |x_n - L_1|. \tag{4.3}$$

Now given $\varepsilon > 0$ and choose $\delta > 0$ such that

$$0 < 2\delta < \frac{\varepsilon}{|L_1| + |L_2| + 1}. \tag{4.4}$$

Let

$$F_1 = \{n : |x_n - L_1| < \delta\}$$

and

$$F_2 = \{n : |y_n - L_2| < \delta\}$$

then

$$\delta^{\alpha,\beta}(F_1) = \delta^{\alpha,\beta}(F_2) = 1$$

and

$$\delta^{\alpha,\beta}(A \cap F_1 \cap F_2) = 1.$$

For each $n \in A \cap F_1 \cap F_2$ we have from (4.3) and (4.4)

$$|x_n y_n - L_1 L_2| < \varepsilon.$$

Now

$$\delta^{\alpha,\beta}\{n : |x_n y_n - L_1 L_2| \geq \varepsilon\} = 0$$

and

$$(x_k y_k) \to L_1 L_2 \, (\alpha\beta - \mathrm{st}) \, .$$

(*iii*) Take $y_k = \lambda$ for all $n \in \mathbb{N}$, then it follows from (*ii*). ■

The following Lemma shows that $\alpha\beta-$statistical convergence is an extension of ordinary convergence.

Lemma 100 ([*1*]) *Let x be a convergent sequence (in the ordinary sense) then x is $\alpha\beta-$statistically convergent.*

Proof. Assume that $x \to L$ in the ordinary sense. Then for each $\varepsilon > 0$, and for all $(\alpha,\beta) \in \Lambda$ the set

$$\left\{k \epsilon P_n^{\alpha,\beta} : |x_k - L| \geq \varepsilon\right\}$$

is finite. Therefore,

$$\delta^{\alpha,\beta}(\{K : |x_k - L| \geq \varepsilon\}) = 0$$

which implies that

$$x_n \to L(\alpha\beta - \mathrm{st}).$$

■

Remark 101 ([*1*]) *Choose* $\alpha(n) = 1$ *and* $\beta(n) = n$ *then* $P_n^{\alpha,\beta} = [1,n]$ *and*

$$\begin{aligned}
\delta^{\alpha,\beta}(\left\{k \in P_n^{\alpha,\beta} : |x_k - L| \geq \varepsilon\right\}) &= \lim_{n\to\infty} \frac{|\{k \epsilon [1,n] : |x_k - L| \geq \varepsilon\}|}{n} \\
&= \lim_{n\to\infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{n}
\end{aligned}$$

which is the density function used in the definition of statistical convergence. In other words, for $\alpha(n) = 1$ *and* $\beta(n) = n$, $\alpha\beta-$*statistically convergence reduces to statistical convergence.*

Remark 102 ([*1*]) *Assume that* λ_n *is an arbitrary sequence in* ω, *then take* $\alpha(n) = n -$

$\lambda_n + 1$ *and* $\beta(n) = n$, *it is easy to see that* $(\alpha,\beta) \in \Lambda$ *and*

$$\begin{aligned} \beta(n) - \alpha(n) &= n - (n - \lambda_n + 1) \\ &= \lambda_n - 1. \end{aligned}$$

Moreover,

$$\begin{aligned} \delta^{\alpha,\beta}(\left\{k \in P_n^{\alpha,\beta} : |x_k - L| \geq \varepsilon\right\}) &= \lim_{n\to\infty} \frac{|\{k \epsilon [n - \lambda_n + 1, n] : |x_{k-L}| \geq \varepsilon\}|}{n - (n - \lambda_n + 1) + 1} \\ &= \lim_{n\to\infty} \frac{|\{k \epsilon [n - \lambda_n + 1, n] : |x_{k-L}| \geq \varepsilon\}|}{\lambda_n} \end{aligned}$$

which is the density function used in the definition of $\lambda-$*statistical convergence. In other words for* $\alpha(n) = n - \lambda_n + 1$ *and* $\beta(n) = n$, $\alpha\beta-$*statistical convergence reduces to* $\lambda-$*statistical convergence.*

Remark 103 ([*1*]) *Assume that* $\theta = \{k_n\}$ *is an arbitrary lacunary sequence, then take* $\alpha(n) = k_{n-1} + 1$ *and* $\beta(n) = k_n$, *it is easy to see that* $(\alpha,\beta) \in \Lambda$ *and*

$$\beta(n) - \alpha(n) = k_n - k_{n-1} + 1.$$

Moreover,

$$\begin{aligned}
\delta^{\alpha,\beta}(\left\{k \in P_n^{\alpha,\beta} : |x_k - L| \geq \varepsilon\right\}) &= \lim_{n\to\infty} \frac{|\{k\epsilon\,[k_{n-1}+1, k_n] : |x_{k-L}| \geq \varepsilon\}|}{k_n - k_{n-1}} \\
&= \lim_{n\to\infty} \frac{|\{k\epsilon\,[k_{n-1}+1, k_n] : |x_{k-L}| \geq \varepsilon\}|}{h_n} \\
&= \lim_{n\to\infty} \frac{|\{k\epsilon\,(k_{n-1}, k_n] : |x_{k-L}| \geq \varepsilon\}|}{h_n}
\end{aligned}$$

which is the density function used in the definition of lacunary statistical convergence. In other words for $\alpha(n) = k_{n-1} + 1$ and $\beta(n) = k_n$, $\alpha\beta-$statistical convergence reduces to lacunary statistical convergence.

Definition 104 *A sequence x is $\alpha\beta-$statistically divergent to ∞ if for any real number M,*

$$\delta^{\alpha,\beta}(\{n \in \mathbb{N} : x_n > M\}) = 1.$$

Example 105 *Consider the sequence*

$$x = (x_n) := \begin{cases} n & \text{otherwise} \\ 0 & n = k^2, \text{ for some } k \end{cases}$$

and choose $\alpha(n) = 1, \beta(n) = n^2$, then for any real number M

$$\begin{aligned}
\delta^{\alpha,\beta}(\{n \in \mathbb{N} : x_n < K\}) &= \lim_{n\to\infty} \frac{\left|\left\{k \in P_n^{\alpha,\beta} : x_k > M\right\}\right|}{(\beta(n) - \alpha(n) + 1)} \\
&= \lim_{n\to\infty} \frac{\left|\left\{k \in \left[1, n^2\right] : x_k > M\right\}\right|}{n^2} \\
&= \lim_{n\to\infty} \frac{\left|\left\{k \in \left[1, n^2\right] : x_k > M\right\}\right|}{n^3} \\
&\leq \lim_{n\to\to\infty} \frac{n^2 - (n + M)}{n^2} = 1.
\end{aligned}$$

then x is $\alpha\beta-$statistically diverges to ∞.

Remark 106 *Since $\alpha\beta-$statistical convergence includes statistical, $\lambda-$statistical and lacunary statistical convergences, any sequence x which is statistically divergent to ∞, is $\alpha\beta-$statistically diverges to ∞, for the appropriate choice of α and β.*

Definition 107 *A sequence x is $\alpha\beta-$statistically divergent to $-\infty$ if for any real number K,*

$$\delta^{\alpha,\beta}(\{n \in \mathbb{N} : x_n < K\}) = 1.$$

Example 108 *Consider the sequence*

$$x = (x_n) := \begin{cases} 0 & n = k^3, \text{ for some } k \\ -n & \text{otherwise} \end{cases}$$

and choose sequences $\alpha(n) = 1, \beta(n) = n^3$ *then for any reel number* K,

$$\begin{aligned}
\delta^{\alpha,\beta}(\{n \in \mathbb{N} : x_n < K\}) &= \lim_{n\to\infty} \frac{\left|\left\{k \in P_n^{\alpha,\beta} : x_k < K\right\}\right|}{(\beta(n) - \alpha(n) + 1)} \\
&= \lim_{n\to\infty} \frac{\left|\left\{k \in \left[1, n^3\right] : x_k < K\right\}\right|}{n^3} \\
&= \lim_{n\to\infty} \frac{\left|\left\{k \in \left[1, n^3\right] : x_k < K\right\}\right|}{n^3} \\
&\leq \lim_{n\to\infty} \frac{n^3 - (n+K)}{n^3} = 1.
\end{aligned}$$

x is $\alpha\beta-$*statistically diverges to* $-\infty$ *for any real number* K.

Remark 109 *Since* $\alpha\beta-$*statistical convergence includes statistical,* $\lambda-$*statistical and lacunary statistical convergences, any sequence* x *which is statistically divergent to* $-\infty$, *is* $\alpha\beta-$*statistically diverges to* $-\infty$, *for the appropriate choice of* α *and* β.

Definition 110 *A sequence* $x := (x_k)$ *is called* $\alpha\beta-$*statistically bounded if there exists a positive constant* M, *such that*

$$\delta^{\alpha\beta}(\{n : |x_n| > M\}) = 0.$$

Example 111 *Choose* $\alpha(n) = 1, \beta(n) = n^3$ *and consider the sequence*

$$x = (x_n) := \begin{cases} 0 & \textit{otherwise} \\ n & n = k^3, \textit{for some } k \end{cases}$$

then for any $M > 0$,

$$\delta^{\alpha,\beta}(\{n : |x_n| \geq M\}) = \lim_{n\to\infty} \frac{\left|\left\{k \in P_n^{\alpha,\beta} : |x_k| > M\right\}\right|}{(\beta(n) - \alpha(n) + 1)}$$

$$= \lim_{n\to\infty} \frac{\left|\left\{k \in \left[1, n^3\right] : |x_k| > M\right\}\right|}{n^3}$$

$$= \lim_{n\to\infty} \frac{\left|\left\{k \in \left[1, n^3\right] : |x_k| > M\right\}\right|}{n^3}$$

$$\leq \lim_{n\to\infty} \frac{n}{n^3} = 0,$$

which means that x is $\alpha\beta-$statistically bounded.

Definition 112 *A sequence x is said to be $\alpha\beta-$statistically convergent of order γ to L and denoted by $x_n \to L \quad (\alpha\beta^{\gamma} - st)$, if $\forall \varepsilon > 0$*

$$\lim_{n\to\infty} \frac{\left|\left\{k \in P_n^{\alpha,\beta} : |x_k - L| \geq \varepsilon\right\}\right|}{(\beta(n) - \alpha(n) + 1)^{\gamma}} = 0.$$

Lemma 113 ([*1*]) *If $0 < \gamma \leq \delta \leq 1$ and $x_n \to L \quad (\alpha\beta - st)$ then $x_n \to L \quad (\alpha\beta - st)$.*

Chapter 5

CONCLUTION

$\alpha\beta$-statistical convergence is studied in this thesis. First, the definitions of density, matrix transitions and sequence are studied in order to discuss the concept of statistical convergence. Then, a brief summary of λ-statistical, lacunary statistical and A-statistical convergences is given.

In the last chapter the concept of $\alpha\beta$-statistical convergence, which is the main interest of this thesis has been considered. It is shown that $\alpha\beta$-statistical convergence is a non-trivial extension of statistical,λ-statistical and lacunary statistical convergences. Finally, we introduced boundedness of a sequences in the sense of $\alpha\beta$-statistical convergence, which is firstly discussed in this thesis.

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