Generalized Bernstein Polynomials

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ABSTRACT

This thesis consisting of three chapters is concerned with Bernstein polynomials. In the first chapter, an introduction to Bernstein polynomials is given. Then, basic properties of Bernstein polynomials are studied in the second chapter. Last chapter studies the generalized Bernstein polynomials and since it is known that generalized Bernstein polynomials are related to q-integers, we gave basic properties of q-integers. In this chapter, convergence properties of Bernstein polynomials are also given. In addition, we introduced some probabilistic considerations of generalized Bernstein polynomials.

Keywords: Bernstein polynomials; generalized Bernstein polynomials; q-integers; convergence

ÖZ

Bu çalışma üç bölümden oluşmaktadır. Bu tezde Bernstein polinomları çalışılmıştır. İlk olarak Bernstein polinomlarının tanımı yapılmış ve başlıca özellikleri incelenmiştir. İkinci bölümde genelleştirilmiş Bernstein polinomlari incelenmiş ve bu polinomlar q-tamsayılarıyla ilgili olduğundan q-tamsayılarının başlıca özellikleri de verilmiştir. Sonrasında Bernstein polinomlarının yakınsaklık özellikleri çalışılmıştır. Buna ek olarak Bernstein polinomlarının bazı olasılık metodlarıyla yakınsaklık özellikleri ele alınmıştır.

Anahtar kelimeler: Bernstein polinomları; genelleştirilmiş Bernstein polinomları; q-tamsayıları; yakınsaklık

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LIST OF SYMBOLS

$\forall$	for all
$\exists$	there exists
E	expectation
P	probability measure
Δ	difference operator
(Ω, F, P)	probability space
Var	variance
$B_{k,n}(t)$	k^{th} n^{th}- degree Bernstein polynomial
$B_n(f;x)$	a sequence of Bernstein polynomials
$C[a,b]$	space of a continuous functions on a domain [a,b]
$C^m[a,b]$	space of m-times continuously differentiable functions on a domain [a,b]
$[k]_q$	q-analog of k
$[k]_q!$	q-factorial
$\begin{bmatrix} n \\ k \end{bmatrix}_q$	q-binomial coefficient
$\rightrightarrows$	uniform convergence

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Chapter 1

INTRODUCTION

Bernstein polynomial basis is started to review the historical progress together with contemporary state of theory, algorithms and applying the method of polynomials for finite domains. Initially introduced by S. N. Bernstein to ease a useful proof of the Weierstrass Approximation Theorem, the slow convergence rate of Bernstein polynomial approximations to continuous functions result in them to fade in obscurity, till the arrival of digital computers.

The Bernstein form started to enjoy common use as a multifaceted means of intuitively creating and working on geometric shapes. At the same time, inciting further development of basic theory, identification of its excellent numerical stability properties and an increasingly variegation of its reportoire of applications, simple and efficient recursive algorithms, with the wish for utilizing power of computers for geometric design applications.

Karl Weierstrass gave the first proof of his (fundamental) theorem on approximation by algebraic and trigonometric polynomials, in 1885. This was important for development of Approximation Theory. It was a long and complicated proof and leaded a kind of mathematicians to find simpler and more useful proofs. In 1912, the Russian mathematician Sergei N. Bernstein formulated a sequence of polynomials namely the

Bernstein Polynomials:

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} \tag{1.0.1}$$

for any $f \in C[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}$.

Chapter 2

BERNSTEIN POLYNOMIALS

In this chapter we study basic properties of Bernstein Polynomials.

2.1 Properties of Bernstein Polynomials

Property 2.1.1 *[2] A Recursive Definiton of Bernstein Polynomials*

The Bernstein Polynomial of degree n can be introduced by combining two $(n-1)^{st}$ *degree Bernstein polynomials with each other. That is, the* k^{th} n^{th}*- degree Bernstein polynomial can be formulated by*

$$B_{k,n}(t) = (1-t)\,B_{k,n-1}(t) + tB_{k-1,n-1}(t).$$

Proof. *To prove this, we will use the basic definition of the Bernstein polynomials which is given by*

$$B_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n-k}$$

where

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

for $k = 0, 1, .., n$.

$$\begin{aligned}
(1-t)B_{k,n-1}(t) + tB_{k-1,n-1} &= (1-t)\binom{n-1}{k}t^k(1-t)^{n-1-k} + t\binom{n-1}{k-1}t^{k-1}(1-t)^{n-1-(k-1)} \\
&= \binom{n-1}{k}t^k(1-t)^{n-k} + \binom{n-1}{k-1}t^k(1-t)^{n-k} \\
&= \left[\binom{n-1}{k} + \binom{n-1}{k-1}\right]t^k(1-t)^{n-k} \\
&= \binom{n}{k}t^k(1-t)^{n-k} \\
&= B_{k,n}(t).
\end{aligned}$$

■

Property 2.1.2 *[2] The Bernstein Polynomials are All Non-Negative*

$f(t)$ *is a non-negative function over the closed interval* $[a,b]$ *if* $f(t) \geq 0$ *for* $t \in [a,b]$. *In this case the Bernstein polynomials with the degree n is non-negative over the interval* $[0,1]$.

Proof. *To prove this we use the mathematical induction with the recursive definition of Bernstein polynomials. It is shown that the functions* $B_{0,1}(t) = 1-t$ *and* $B_{1,1}(t) = t$ *are both non-negative over the interval* $[0,1]$. *If we suppose that all Bernstein polynomials of degree less than k are non-negative, the other case we can use the recursive definition of the Bernstein polynomial and it is written by*

$$B_{n,k}(t) = (1-t)B_{n,k-1}(t) + tB_{n-1,k-1}(t)$$

and prove that $B_{n,k}(t)$ *is also non-negative over the interval* $[0,1]$, *since all components on the right-hand side of the equation are non-negative components over the interval* $[0,1]$. *By induction, all Bernstein polynomials are non-negative over the interval* $[0,1]$.

At the same time, we have proved that each Bernstein polynomial is positive when $t \in (0,1)$. ■

Property 2.1.3 *[2] The Bernstein Polynomials form a Partition of Unity*

If the summation of all values of t is one, then $f_n(t)$ *is a called a partition unity. The* k^{th} *degree* $k+1$ *Bernstein polynomials form a partition of unity in that they all sum to one.*

Proof. *If we assume that this is true, it is easy to show an undistinguished different fact : for each k, the sum of the* $k+1$ *of degree k is equal to the sum of the k Bernstein polynomials of degree* $k-1$. *That is,*

$$\sum_{n=0}^{k} B_{n,k}(t) = \sum_{n=0}^{k-1} B_{n,k-1}(t).$$

This computation is crystal clear , using the recursive definition of Bernstein polynomial and rearranging the sums :

$$\begin{aligned}\sum_{n=0}^{k} B_{n,k}(t) &= \sum_{n=0}^{k} \left[(1-t)B_{n,k-1}(t) + tB_{n-1,k-1}(t)\right] \\ &= (1-t)\left[\sum_{n=0}^{k-1} B_{n,k-1}(t) + B_{k,k-1}(t)\right] + t\left[\sum_{n=1}^{k} B_{n-1,k-1}(t) + B_{-1,k-1}(t)\right]\end{aligned}$$

(where we have utilized $B_{k,k-1}(t) = B_{-1,k-1}(t) = 0$)

$$
\begin{aligned}
&= (1-t)\sum_{n=0}^{k-1} B_{n,k-1}(t) + t\sum_{n=1}^{k} B_{n-1,k-1}(t) \\
&= (1-t)\sum_{n=0}^{k-1} B_{n,k-1}(t) + t\sum_{n=0}^{k-1} B_{n,k-1}(t) \\
&= \sum_{n=0}^{k-1} B_{n,k-1}(t).
\end{aligned}
$$

Once we have established this equality, it is simple to write

$$
\sum_{n=0}^{k} B_{n,k}(t) = \sum_{n=0}^{k-1} B_{n,k-1}(t) = \sum_{n=0}^{k-2} B_{n,k-2}(t) = \cdots = \sum_{n=0}^{1} B_{n,1}(t) = (1-t)+t = 1.
$$

■

Property 2.1.4 *[2] Degree Raising*

Any of the lower-degree Bernstein Polynomials of degree less than n can be defined as a linear combination of n^{th} degree Bernstein polynomials. In this case, any $(n-1)^{th}$ degree Bernstein polynomial can be written as a linear combination of n^{th} degree Bernstein polynomials.

Proof. *Firstly, we note that*

$$
\begin{aligned}
tB_{k,n}(t) &= \binom{n}{k} t^{k+1}(1-t)^{n-k} \\
&= \binom{n}{k} t^{k+1}(1-t)^{(n+1)-(k+1)} \\
&= \frac{\binom{n}{k}}{\binom{n+1}{k+1}} B_{k+1,n+1}(t) \\
&= \frac{k+1}{n+1} B_{k+1,n+1}(t)
\end{aligned}
$$

and

$$(1-t)\,B_{k,n}(t)=\binom{n}{k}t^{k}(1-t)^{n-k}$$

$$=\frac{\binom{n}{k}}{\binom{n+1}{k}}B_{k,n+1}(t)$$

$$=\frac{n-k+1}{n+1}B_{k,n+1}(t),$$

and finally

$$\begin{aligned}\frac{1}{\binom{n}{k}}B_{k,n}(t)+\frac{1}{\binom{n}{k+1}}B_{k+1,n}(t) &= t^{k}(1-t)^{n-k}+t^{k+1}(1-t)^{n-(k+1)}\\ &= t^{k}(1-t)^{n-k-1}((1-t)+t)\\ &= t^{k}(1-t)^{n-k-1}\\ &= \frac{1}{\binom{n-1}{k}}B_{k,n-1}(t).\end{aligned}$$

Using this final equation, it can be written as

$$\begin{aligned}B_{k,n-1}(t) &= \binom{n-1}{k}\left[\frac{1}{\binom{n}{k}}B_{k,n}(t)+\frac{1}{\binom{n}{k+1}}B_{k+1,n}(t)\right]\\ &= \left(\frac{n-k}{n}\right)B_{k,n}(t)+\left(\frac{k+1}{n}\right)B_{k+1,n}(t)\end{aligned}$$

which expresses a Bernstein polynomial of degree $n-1$ *in terms of a linear combination of Bernstein polynomials of degree n.* ■

Property 2.1.5 *[2] Converting from the Bernstein Basis to the Power Basis*

Any n^{th} degree Bernstein polynomial can be written in terms of the power basis which is expressed by $\left\{1, t, t^2, ..., t^n\right\}$.

Proof. *This can be directly computed by using the definition of the Bernstein polynomials and the binomial theorem, as follows :*

$$\begin{aligned} B_{i,n}(t) &= \binom{n}{i} t^i (1-t)^{n-i} \\ &= \binom{n}{i} t^i \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} t^k \\ &= \sum_{k=0}^{n-i} (-1)^k \binom{n}{i}\binom{n-i}{k} t^{k+i} \\ &= \sum_{k=i}^{n} (-1)^{k-i} \binom{n}{i}\binom{n-i}{k-i} t^k \\ &= \sum_{k=i}^{n} (-1)^{k-i} \binom{n}{k}\binom{k}{i} t^k. \end{aligned}$$

■

Property 2.1.6 *[2] Derivatives*

Polynomial of degree $n-1$ are derivatives of the n^{th} degree of the Bernstein polynomials. Also, these derivatives can be written as a linear combination of Bernstein polynomials using the definition of Bernstein polynomials. In this case,

$$\frac{d}{dt} B_{i,n}(t) = n\left(B_{i-1,n-1}(t) - B_{i,n-1}(t)\right)$$

for $0 \leq i \leq n$. This can be written by direct differentiation

$$
\begin{aligned}
\frac{d}{dt}B_{i,n}(t) &= \frac{d}{dt}\binom{n}{i}t^i(1-t)^{n-i} \\
&= \frac{in!}{i!(n-i)!}t^{i-1}(1-t)^{n-i} - \frac{(n-i)n!}{i!(n-i)!}t^i(1-t)^{n-i-1} \\
&= \frac{n(n-1)!}{(i-1)!(n-i)!}t^{i-1}(1-t)^{n-i} - \frac{n(n-1)!}{i!(n-i-1)!}t^i(1-t)^{n-i-1} \\
&= n\left(\frac{(n-1)!}{(i-1)!(n-i)!}t^{i-1}(1-t)^{n-i} - \frac{(n-1)!}{i!(n-i-1)!}t^i(1-t)^{n-i-1}\right) \\
&= n(B_{i-1,n-1}(t) - B_{i,n-1}(t)).
\end{aligned}
$$

The consequence is that, the derivative of Bernstein polynomials can be shown as the degree of the polynomial multiplied by the difference of two $(n-1)^{st}$ degree Bernstein polynomials.

Property 2.1.7 *[2] The Matrix Representation of Bernstein polynomials*

A matrix representation is useful for the Bernstein polynomials. The linear combination of Bernstein basis functions for a given polynomial is given by

$$B(t) = c_0B_{0,k}(t) + c_1B_{1,k}(t) + ... + c_kB_{k,k}(t).$$

It is easy to write this as a dot product of two vectors

$$B(t) = \begin{bmatrix} B_{0,k}(t) & B_{1,k}(t) & . & . & . & B_{k,k}(t) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ . \\ . \\ . \\ c_k \end{bmatrix}.$$

We can transform this to

$$B(t) = \begin{bmatrix} 1 & t & t^2 & . & . & . & t^k \end{bmatrix} \begin{bmatrix} b_{0,0} & 0 & 0 & ... & 0 \\ b_{1,0} & b_{1,1} & 0 & ... & 0 \\ b_{2,0} & b_{2,1} & b_{2,2} & ... & 0 \\ . & . & . & ... & . \\ . & . & . & ... & . \\ . & . & . & ... & . \\ b_{k,0} & b_{k,1} & b_{k,2} & ... & b_{k,k} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ . \\ . \\ . \\ c_k \end{bmatrix},$$

where the $b_{m,n}$ are the coefficients of the power basis that are used to determine the respective Bernstein polynomials. We note that the matrix in this case in lower triangular matrix. If we want to give an example, we can give the quadratic case $(n = 2)$ with the matrix expression

$$B(t) = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}.$$

Previously, we define the Bernstein polynomials in equation (1.0.1) for each positive integer n. It will be shown that if f is continuous over the interval $[0, 1]$, its sequence of Bernstein polynomials converges uniformly to f over the interval $[0, 1]$, thus giving a useful proof of Weierstrass's theorem. For the proof of Weierstrass Theorem, Bernstein composed incoming polynomials in place of the known polynomials . For instance, Taylor polynomials are not useful for all continuous functions, it can be applicable only infinitely differentiable functions.

It is clear from equation (1.0.1) that for all $n \geq 1$,

$$B_n(f;0) = f(0) \text{ and } B_n(f;1) = f(1), \tag{2.1.1}$$

so that a Bernstein polynomial of f interpolates f at both endpoints of the interval $[0,1]$.

Besides, from the binomial expansion it follows that

$$B_n(1;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = (x + (1-x))^n = 1. \tag{2.1.2}$$

Thus the Bernstein polynomial of the constant function 1 is also 1. In addition, the Bernstein polynomial of the function $f(t) = t$ is x. In fact since

$$\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$$

for $1 \leq k \leq n$, the Bernstein polynomial of the function t is

$$\begin{aligned} B_n(t,x) &= \sum_{k=0}^{n} \frac{k}{n}\binom{n}{k} x^k (1-x)^{n-k} = x \sum_{k=1}^{n} \binom{n-1}{k-1} x^{k-1} (1-x)^{n-k} \\ &= x \sum_{s=0}^{n-1} \binom{n-1}{s} x^s (1-x)^{n-1-s} = x. \end{aligned} \tag{2.1.3}$$

The Bernstein operator B_n maps a function f, defined over the interval $[0,1]$ to $B_n f$, which is the function $B_n f$ computed at x represented by $B_n(f;x)$. The Bernstein operator is clearly linear, since it comes from equation (1.0.1) that

$$B_n(\lambda f + \mu g) = \lambda B_n f + \mu B_n g \tag{2.1.4}$$

for all functions f and g defined over the interval $[0,1]$ and all real λ and μ.

B_n is a monotone operator from the equation (1.0.1), then it follows from the monotonicity of B_n and equation (2.1.2) that

$$p \le f(x) \le P,\ x \in [0,1] \Rightarrow p \le B_n(f;x) \le P, x \in [0,1]. \tag{2.1.5}$$

In this case, letting $p = 0$ in equation (2.1.5), we get

$$f(x) \ge 0, x \in [0,1] \Rightarrow B_n(f,x) \ge 0, x \in [0,1]. \tag{2.1.6}$$

Theorem 2.1.8 *[3] The Bernstein polynomial can be written in the following form*

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k f(0)\, x^k, \tag{2.1.7}$$

where Δ is the forward difference operator, shown as

$$\Delta f\left(x_j\right) = f\left(x_{j+1}\right) - f\left(x_j\right) = f\left(x_j + h\right) - f\left(x_j\right),$$

with step size $h = \frac{1}{n}$.

Proof. Beginning with equation (1.0.1) and extending the term $(1-x)^{n-k}$, we have

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k \sum_{s=0}^{n-k} (-1)^s \binom{n-k}{s} x^s.$$

Let us put $t = k + s$. We might write

$$\sum_{k=0}^{n}\sum_{s=0}^{n-k} \cdots = \sum_{t=0}^{n}\sum_{k=0}^{t} \cdots. \tag{2.1.8}$$

Also we have

$$\binom{n}{k}\binom{n-k}{s} = \binom{n}{t}\binom{t}{k},$$

and so we might write the double summation as

$$\sum_{t=0}^{n}\binom{n}{t}x^{t}\sum_{k=0}^{t}(-1)^{t-k}\binom{t}{k}f\left(\frac{k}{n}\right) = \sum_{t=0}^{n}\binom{n}{t}\Delta^{t}f(0)\,x^{t},$$

on using the expansion for a higher-order forward difference. ■

Theorem 2.1.9 *[3] The derivative of the Bernstein polynomial $B_{n+1}(f;x)$ can be written in the following form*

$$B'_{n+1}(f;x) = (n+1)\sum_{k=0}^{n}\Delta f\left(\frac{k}{n+1}\right)\binom{n}{k}x^{k}(1-x)^{n-k} \tag{2.1.9}$$

for $n \geq 0$, where Δ is applied with step size $h = \frac{1}{(n+1)}$. Otherwise, if f is monotonically increasing or monotonically decreasing over the interval $[0,1]$, so are all its Bernstein polynomials.

Theorem 2.1.10 *[3] For any integer $m \geq 0$, the m^{th} derivative of $B_{n+m}(f;x)$ can be expressed in terms of m^{th} differences of f as*

$$B^{(m)}_{n+m}(f;x) = \frac{(n+m)!}{n!}\sum_{k=0}^{n}\Delta^{m}f\left(\frac{k}{n+m}\right)\binom{n}{k}x^{k}(1-x)^{n-k} \tag{2.1.10}$$

for all $n \geq 0$. Here Δ is applied with step size $h = \frac{1}{n+k}$.

Proof. We write

$$B_{n+m}(f;x) = \sum_{k=0}^{n+m} f\left(\frac{k}{n+m}\right)\binom{n+m}{k} x^k (1-x)^{n+m-k}$$

and differentiate m times to get

$$B_{n+m}^{(m)}(f;x) = \sum_{k=0}^{n+m} f\left(\frac{k}{n+m}\right)\binom{n+m}{k} p(x), \tag{2.1.11}$$

where

$$p(x) = \frac{d^m}{dx^m} x^k (1-x)^{n+m-k}.$$

Now, we use the Leibniz rule which is

$$\frac{d^m}{dx^m}(f(x)g(x)) = \sum_{k=0}^{m}\binom{m}{k}\frac{d^k}{dx^k}f(x)\frac{d^{m-k}}{dx^{m-k}}g(x),$$

to differentiate the product of x^k and $(1-x)^{n+m-k}$. First we find that

$$\frac{d^s}{dx^s}x^k = \begin{cases} \frac{k!}{(k-s)!}x^{k-s}, k-s \geq 0, \\ 0, k-s < 0 \end{cases}$$

and

$$\frac{d^{m-s}}{dx^{m-s}}(1-x)^{n+m-k} = \begin{cases} (-1)^{m-s}\frac{(n+m-k)!}{(n+s-k)!}(1-x)^{n+s-k}, k-s \leq n \\ 0, k-s > n. \end{cases}$$

Accordingly the m^{th} derivative of $x^k(1-x)^{n+m-k}$ is

$$p(x) = \sum_s (-1)^{m-s}\binom{m}{s}\frac{k!}{(k-s)!}\frac{(n+m-k)!}{(n+s-k)!}x^{k-s}(1-x)^{n+s-k}, \tag{2.1.12}$$

where the last summation is over all s from 0 to m, with the limitations $0 \le k - s \le n$. Now, we replace l with $k - s$, such that

$$\sum_{k=0}^{n+m} \sum_{s} \cdots = \sum_{l=0}^{n} \sum_{s=0}^{m} \cdots . \tag{2.1.13}$$

We also note that

$$\binom{n+m}{k} \frac{k!}{(k-s)!} \frac{(n+m-k)!}{(n+s-k)!} = \frac{(n+m)!}{n!} \binom{n}{k-s}. \tag{2.1.14}$$

It then follows from equations (2.1.11), (2.1.12), (2.1.13) and (2.1.14) that the m^{th} derivative of $B_{n+m}(f;x)$ is

$$\frac{(n+m)!}{n!} \sum_{l=0}^{n} \sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} f\left(\frac{l+s}{n+m}\right) \binom{n}{l} x^l (1-x)^{n-l}.$$

Finally, we note that

$$\sum_{s=0}^{m} (-1)^{m-s} \binom{m}{s} f\left(\frac{l+s}{n+m}\right) = \Delta^m f\left(\frac{l}{n+m}\right),$$

where the operator Δ is applied with step size $h = \frac{1}{n+m}$. Whence the result. ■

Theorem 2.1.11 *[3] If $f \in C^m[0,1]$, for some $m \ge 0$, then*

$$p \le f^{(m)}(x) \le P, x \in [0,1] \Rightarrow c_m p \le B_n^{(m)}(f;x) \le c_m P, x \in [0,1],$$

for all $n \ge m$, where $c_0 = c_1 = 1$ and

$$c_m = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)...\left(1 - \frac{m-1}{n}\right), 2 \le m \le n.$$

Proof. The former relation in the theorem can also be seen in equation (2.1.5) if we let $m = 0$. For $m \geq 1$ we begin with equation (2.1.10) and replace n by $n-m$. Then, we use

$$\frac{\Delta^p f(x_0)}{h^p} = f^{(p)}(\xi), \text{ where } \xi \in \left(x_0, x_p\right), x_p = x_0 + ph$$

with $h = \frac{1}{n}$, we write

$$\Delta^m f(\frac{k}{n}) = \frac{f^{(m)}(\xi_k)}{n^m}, \tag{2.1.15}$$

where $\frac{k}{m} < \xi_k < \frac{k+m}{n}$. Thus

$$B_n^{(m)}(f;x) = \sum_{k=0}^{n-m} c_m f^{(m)}(\xi_k) x^k (1-x)^{n-m-k},$$

and the theorem follows easily from the latter equation. ■

Theorem 2.1.12 *[3] If f is a function of $C[0,1]$ and $\varepsilon > 0$ is arbitrary, then there exists an integer N such that*

$$|f(x) - B_n(f;x)| < \varepsilon, 0 \leq x \leq 1,$$

for all $n \geq N$.

The above statement says that Bernstein polynomials of a function f is continuous over the interval $[0,1]$ converging uniformly to f over the interval $[0,1]$.

Proof. We start with the identity

$$\left(\frac{k}{n} - x\right)^2 = \left(\frac{k}{n}\right)^2 - 2\left(\frac{k}{n}\right)x + x^2,$$

multiply each term by $\binom{n}{k}x^k(1-x)^{n-k}$ and sum from $k=0$ to n, to give

$$\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k} = B_n(t^2;x)-2xB_n(t;x)+x^2B_n(1;x).$$

It then follows from equations (2.1.2), (2.1.3) and

$$B_n(t^2;x) = x^2+\frac{1}{n}x(1-x),$$

that

$$\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k} = \frac{1}{n}x(1-x). \tag{2.1.16}$$

For any fixed $x \in [0,1]$, let us approximate the sum of the polynomials $p_{n,k}(x)$ over all values of k for which $\frac{k}{n}$ is not close to x. To make this notation exact, we take a number $\delta > 0$ and let S_δ indicate the set of all values of k satisfying $\left|\frac{k}{n}-x\right| \geq \delta$ implies that

$$\frac{1}{\delta^2}\left(\frac{k}{n}-x\right)^2 \geq 1. \tag{2.1.17}$$

Then, using equation (2.1.17), we have

$$\sum_{k\in S_\delta}\binom{n}{k}x^k(1-x)^{n-k} \leq \frac{1}{\delta^2}\sum_{k\in S_\delta}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k}.$$

The last mentioned sum is not greater than the sum of the same expression over all k. Using equation (2.1.16), we have

$$\frac{1}{\delta^2}\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k} = \frac{x(1-x)}{n\delta^2}.$$

Since $0 \leq x(1-x) \leq \frac{1}{4}$ on $[0,1]$, we have

$$\sum_{k \in S_\delta} \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{1}{4n\delta^2}. \tag{2.1.18}$$

Let us write

$$\sum_{k=0}^{n} \cdots = \sum_{k \in S_\delta} \cdots + \sum_{k \notin S_\delta} \cdots,$$

where the last mentioned sum is therefore over all k such that $\left|\frac{k}{n} - x\right| < \delta$. Having seperate the summation into two parts, which depend on a choice of δ that we still have to make. Now we are ready to approximate the difference between $f(x)$ and its Bernstein polynomial. Using equation (2.1.2), we have

$$f(x) - B_n(f;x) = \sum_{k=0}^{n} \left(f(x) - f\left(\frac{k}{n}\right) \right) \binom{n}{k} x^k (1-x)^{n-k}$$

and hence

$$f(x) - B_n(f;x) = \sum_{k \in S_\delta} \left(f(x) - f\left(\frac{k}{n}\right) \right) \binom{n}{k} x^k (1-x)^{n-k} + \sum_{k \notin S_\delta} \left(f(x) - f\left(\frac{k}{n}\right) \right) \binom{n}{k} x^k (1-x)^{n-k}.$$

Thus we get the inequality

$$|f(x) - B_n(f;x)| \leq \sum_{k \in S_\delta} \left| f(x) - f\left(\frac{k}{n}\right) \right| \binom{n}{k} x^k (1-x)^{n-k} + \sum_{k \notin S_\delta} \left| f(x) - f\left(\frac{k}{n}\right) \right| \binom{n}{k} x^k (1-x)^{n-k}.$$

Since $f \in C[0,1]$, it is bounded over the interval $[0,1]$ and we have $|f(x)| \leq M$, for some $M > 0$. Hence we can denote

$$\left| f(x) - f\left(\frac{k}{n}\right) \right| \leq 2M$$

for all k and all $x \in [0,1]$, and thus

$$\sum_{k \in S_\delta} \left| f(x) - f\left(\frac{k}{n}\right) \right| \binom{n}{k} x^k (1-x)^{n-k} \leq 2M \sum_{k \in S_\delta} \binom{n}{k} x^k (1-x)^{n-k}.$$

Using equation (2.1.18), we obtain

$$\sum_{k \in S_\delta} \left| f(x) - f\left(\frac{k}{n}\right) \right| \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{M}{2n\delta^2}. \tag{2.1.19}$$

Since f is continuous, it is also uniformly continuous over the interval $[0,1]$. Hence, related to any selection of $\varepsilon > 0$ there is a number $\delta > 0$, depending on ε and f such that

$$|x - x'| < \delta \Longrightarrow |f(x) - f(x')| < \frac{\varepsilon}{2}$$

for all $x, x' \in [0,1]$. Hence, for some $k \notin S_\delta$, we have

$$\begin{aligned} \sum_{k \notin S_\delta} \left| f(x) - f\left(\frac{k}{n}\right) \right| \binom{n}{k} x^k (1-x)^{n-k} &< \frac{\varepsilon}{2} \sum_{k \notin S_\delta} \binom{n}{k} x^k (1-x)^{n-k} \\ &< \frac{\varepsilon}{2} \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}, \end{aligned}$$

and using equation (2.1.2) one more time we find that

$$\sum_{k \notin S_\delta} \left| f(x) - f\left(\frac{k}{n}\right) \right| \binom{n}{k} x^k (1-x)^{n-k} < \frac{\varepsilon}{2}. \tag{2.1.20}$$

On combining the equations (2.1.19) and (2.1.20), we get

$$|f(x) - B_n(f;x)| < \frac{M}{2n\delta^2} + \frac{\varepsilon}{2}.$$

It comes from the line above that if we select $N > \frac{M}{(\varepsilon\delta^2)}$, then

$$|f(x) - B_n(f;x)| < \varepsilon$$

for all $n \geq N$ and hence the result. ■

Theorem 2.1.13 *[3] If $f \in C^m[0,1]$, for some integer $m \geq 0$, then $B_n^{(m)}(f;x)$ converges uniformly to $f^{(m)}(x)$ on $[0,1]$.*

Proof. The case when $m = 0$ holds by Theorem (2.1.12). For $m \geq 1$ we begin with the expression for $B_{n+m}^{(m)}(f;x)$ given in equation (2.1.10) and write

$$\Delta^m f\left(\frac{k}{n+m}\right) = \frac{f^{(m)}(\xi_k)}{(n+m)^t},$$

where $\frac{k}{n+m} < \xi_k < \frac{k+m}{n+m}$, as we do computations in equation (2.1.15). We then approximate $f^{(m)}(\xi_k)$, writing

$$f^{(m)}(\xi_k) = f^{(m)}\left(\frac{k}{n}\right) + \left(f^{(m)}(\xi_k) - f^{(m)}\left(\frac{k}{n}\right)\right).$$

We thus obtain

$$\frac{n!(n+m)^m}{(n+m)!} B_{n+m}^{(m)}(f;x) = S_1(x) + S_2(x), \tag{2.1.21}$$

where

$$S_1(x) = \sum_{k=0}^{n} f^{(m)}\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

and

$$S_2(x) = \sum_{k=0}^{n} \left(f^{(m)}(\xi_k) - f^{(m)}\left(\frac{k}{n}\right) \right) \binom{n}{k} x^k (1-x)^{n-k}.$$

In $S_2(x)$, we can make $\left|\xi_k - \frac{k}{n}\right| < \delta$ for all k, for any selection of $\delta > 0$, by taking n sufficiently large. So, given any $\varepsilon > 0$, we can select a positive value of δ such that

$$\left| f^{(m)}(\xi_k) - f^{(m)}\left(\frac{k}{n}\right) \right| < \varepsilon,$$

for all k, by the uniform continuity of $f^{(m)}$. Hence $S_2(x) \to 0$ uniformly over the interval $[0,1]$ as $n \to \infty$. We can simply justify that

$$\frac{n!(n+m)^m}{(n+m)!} \to 1 \text{ as } n \to \infty,$$

and we can see from Theorem (2.1.12) with $f^{(m)}$ in place of f that $S_1(x)$ converges uniformly to $f^{(m)}(x)$ over the interval $[0,1]$. Whence the result. ■

Chapter 3

GENERALIZED BERNSTEIN POLYNOMIALS

In this chapter we mention about the generalized Bernstein polynomials based on the $q-$integers, which were introduced by Phillips as given below ;

$$B_n(f,q;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right)\begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k}(1-q^s x), \qquad n = 1,2,... \tag{3.0.1}$$

When we put $q = 1$ in this equation, we get the classical Bernstein polynomial expressed by

$$B_n(f;x) = \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k} x^k (1-x)^{n-k}.$$

3.1 Basic Information on q-Bernstein Polynomials

First of all, basic information on q-Bernstein polynomials will be given. The function f is evaluated at ratios of the $q-$integers $[k]_q$ and $[n]_q$, where q is a positive real number and

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q}, q \neq 1 \\ k, q = 1. \end{cases}$$

Let us define

$$\mathbb{N}_q = \left\{[k]_q, \text{with } k \in \mathbb{N}\right\}, \text{ for any given } q > 0$$

and use the definition

$$\mathbb{N}_q = \left\{0, 1, 1+q, 1+q+q^2, 1+q+q^2+q^3, \cdots\right\}.$$

It is obvious that the set of $q-$integers $\mathbb{N}_q$ generalizes the set of non-negative integers $\mathbb{N}$, which we recover by putting $q = 1$.

Let $q > 0$ be given. We define a $q-$factorial, $[k]_q!$, of $k \in \mathbb{N}$, as

$$[k]_q! = \begin{cases} [k]_q\,[k-1]_q \ldots [1]_q\,, k \geq 1, \\ 1, k = 0. \end{cases}$$

The $q-$binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q\,[n-1]_q \ldots [n-k+1]_q}{[k]_q!} = \frac{[n]_q!}{[k]_q!\,[n-k]_q!},$$

for integers $n \geq k \geq 0$.

The $q-$binomial coefficients are also called Gaussian polynomials, named after C. F. Gauss.

Lemma 3.1.1 *The Gaussian polynomials satisfy the Pascal-type relations*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k\begin{bmatrix} n-1 \\ k \end{bmatrix}_q. \tag{3.1.1}$$

Proof. Considering the equation (3.1.1), we get

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix}_q &= \frac{[n-1]_q!}{[k-1]_q!\,[n-k]_q!} + q^k \frac{[n-1]_q!}{[k]_q!\,[n-k-1]_q!} \\
&= \frac{[k]_q\,[n-1]_q \ldots [n-k+1]_q}{[k]_q!} + q^k \left[\frac{[n-1]_q \ldots [n-k]_q}{[k]_q!} \right] \\
&= \frac{[n-1]_q \ldots [n-k+1]_q}{[k]_q!} \left[[k]_q + q^k\,[n-k]_q \right] \\
&= \frac{[n-1]_q \ldots [n-k+1]_q}{[k]_q!} \left[\frac{1-q^k}{1-q} + q^k \left[\frac{1-q^{n-k}}{1-q} \right] \right] \\
&= \frac{[n-1]_q \ldots [n-k+1]_q}{[k]_q!} \left[\frac{1-q^k+q^k-q^n}{1-q} \right] \\
&= \frac{[n-1]_q \ldots [n-k+1]_q}{[k]_q!} \left[\frac{1-q^n}{1-q} \right] \\
&= \frac{[n]_q!}{[k]_q!\,[n-k]_q!}.
\end{aligned}$$

■

In this chapter, we will give problems of convergence properties of the sequence $\{B_n(f,q;x)\}_{n=1}^{\infty}$. Here, it is shown that in general, these properties are essentially different from those in the classical case $q = 1$.

Property 3.1.2 *Let $B_n(f,q;x)$ be defined by the equation (3.0.1). Then,*

$$B_n(at+b,q;x) = ax+b \tag{3.1.2}$$

for all $q > 0$ and all $n = 1,2,\cdots$

$$B_n(f,q;0) = f(0) \ \ ; \ B_n(f,q;1) = f(1) \tag{3.1.3}$$

for all $q > 0$ and all $n = 1,2,\cdots$.

Proof. Let

$$
\begin{aligned}
B_n(at+b,q;x) &= \sum_{k=0}^{n}\left(a\frac{[k]_q}{[n]_q}+b\right)\begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k}(1-q^s x) \\
&= a\sum_{k=0}^{n}\frac{[k]_q}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k}(1-q^s x)+b\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k}(1-q^s x).
\end{aligned}
$$

Then we set

$$
P_{nk}(q;x)=\begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k}(1-q^s x). \tag{3.1.4}
$$

Since $P_{nk}(q;x)$ is a probability function, $P_{nk}(q;x)\geq 0$ for $q\in(0,1)$ and $x\in[0,1]$, then

$$
\sum_{k=0}^{n} P_{nk}(q;x)=1 \tag{3.1.5}
$$

for all $n=1,2,\ldots$.

According to this, we have

$$
\begin{aligned}
B_n(at+b,q;x) &= a\sum_{k=0}^{n}\frac{[k]_q}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k}(1-q^s x)+b\sum_{k=0}^{n}P_{nk}(q;x) \\
&= a\sum_{k=0}^{n}\frac{[k]_q}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k}(1-q^s x)+b \\
&= a\sum_{k=0}^{n}\frac{[k]_q}{[n]_q}\frac{[n]_q!}{[k]_q!\,[n-k]_q!}x^k \prod_{s=0}^{n-1-k}(1-q^s x)+b \\
&= a\sum_{k=0}^{n}\frac{[n-1]_q!}{[k-1]_q!\,[n-k]_q!}x^k \prod_{s=0}^{n-1-k}(1-q^s x)+b \\
&= ax\sum_{k=1}^{n-1}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q x^{k-1} \prod_{s=0}^{n-1-(k-1)-1}(1-q^s x)+b \qquad \begin{pmatrix} n\to n-1 \\ k\to k-1 \end{pmatrix} \\
&= ax+b.
\end{aligned}
$$

■

Theorem 3.1.3 *[1] Let a sequence (q_n) satisfy $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. Then for any function $f \in C[0,1]$,*

$$B_n(f, q_n; x) \rightrightarrows f(x) \qquad [x \in [0,1]\,; n \to \infty]\,.$$

We always assume that $q \in (0,1)$ and f is a real continuous function over the interval $[0,1]$.

Let (Ω, F, P) be a probability space and $Z : \Omega \to \mathbb{R}$ be a random variable. We use the standard notation EZ for the mathematical expectation and $VarZ$ for the variance of the random variable Z and define :

$$EZ := \int_{\Omega} Z(\varpi)\, P(d\omega) \quad ; \quad VarZ := E\left(Z^2\right) - (EZ)^2\,.$$

Consider a random variable $Y_n(q;x)$ having the probability distribution

$$P\left\{Y_n(q;x) = \frac{[k]_q}{[n]_q}\right\} = P_{nk}(q;x),\ k = 0, 1, , n\ ; n = 1, 2, \ldots. \tag{3.1.6}$$

Obviously, by definition of B_n and expectation, $B_n(f, q; x) = Ef(Y_n(q;x))$. Let us now show this relation.

$$
\begin{aligned}
B_n(f,q;x) &= \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right)\begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k}(1-q^s x) \\
&= \sum_{k=0}^{n} f(Y_n(q;x))\, P_{nk}(q;x) \\
&= \sum_{k=0}^{n} f(Y_n(q;x))\, P(Y_n(q;x)) \\
&= Ef(Y_n).
\end{aligned}
$$

It is not difficult to see that the limits as $n \to \infty$ of both the values of $Y_n(q;x)$ and the probabilities of these values exist, which will be shown below.

Theorem 3.1.4 *[1] For all $k = 0, 1, 2, \ldots$*

$$\lim_{n\to\infty} \frac{[k]_q}{[n]_q} = 1 - q^k, \tag{3.1.7}$$

and

$$\lim_{n\to\infty} P_{nk}(q;x) = \frac{x^k}{(1-q)^k\, [k]_q!} \prod_{s=0}^{\infty}(1-q^s x) =: P_{\infty k}(q;x). \tag{3.1.8}$$

Proof. Let us consider the equation (3.1.7). We know that

$$[k]_q = \frac{1-q^k}{1-q}$$

and

$$[n]_q = \frac{1-q^n}{1-q}.$$

However,

$$\frac{[k]_q}{[n]_q} = \frac{1-q^k}{1-q^n},$$

then take a limit when $n \to \infty$ to get

$$\lim_{n\to\infty} \frac{[k]_q}{[n]_q} = 1 - q^k.$$

Also, consider the equation (3.1.8), we know that

$$P_{nk}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1-q^s x).$$

If we expand $\begin{bmatrix} n \\ k \end{bmatrix}_q$,

$$P_{nk}(q;x) = \frac{[n]_q!}{[k]_q!\,[n-k]_q!} x^k \prod_{s=0}^{n-k-1} (1-q^s x)$$

where

$$[n]_q! = \frac{1-q^n}{1-q} \cdot \frac{1-q^{n-1}}{1-q} \cdot \ldots \cdot 1 = \frac{(1-q^n)\left(1-q^{n-1}\right)\ldots\left(1-q^{n-k+1}\right)\left(1-q^{n-k}\right)\ldots 1}{(1-q)^n}$$

and

$$[n-k]_q! = \frac{1-q^{n-k}}{1-q} \cdot \frac{1-q^{n-k-1}}{1-q} \cdot \ldots \cdot 1 = \frac{\left(1-q^{n-k}\right)\left(1-q^{n-k-1}\right)\ldots 1}{(1-q)^{n-k}},$$

then

$$\frac{[n]_q!}{[k]_q!\,[n-k]_q!} = \frac{(1-q^n)\left(1-q^{n-1}\right)\ldots\left(1-q^{n-k+1}\right)}{[k]_q!\,(1-q)^k}.$$

Now, we put these equations into $P_{nk}(q;x)$ to get

$$P_{nk}(q;x)=\frac{(1-q^n)\left(1-q^{n-1}\right)...\left(1-q^{n-k+1}\right)}{[k]_q!(1-q)^k}x^k\prod_{s=0}^{n-k-1}(1-q^sx),$$

then take limit when $n\to\infty$

$$\begin{aligned}\lim_{n\to\infty}P_{nk}(q;x) &= \lim_{n\to\infty}\frac{(1-q^n)\left(1-q^{n-1}\right)...\left(1-q^{n-k+1}\right)}{[k]_q!(1-q)^k}x^k\prod_{s=0}^{n-k-1}(1-q^sx)\\ &= \lim_{n\to\infty}\frac{x^k}{(1-q)^k[k]_q!}\prod_{s=0}^{\infty}(1-q^sx)=P_{\infty k}(q;x).\end{aligned}$$

■

Note that unlike $P_{nk}(q;x)$, the functions $P_{\infty k}(q;x)$ are transcendental entire functions rather than polynomials.

Theorem 3.1.5 *[1] $P_{\infty k}(q;x)\geq 0$ for $x\in[0,1]$ and by the Euler's identity, we have*

$$\sum_{k:=0}^{\infty}P_{\infty k}(q;x)=1 \tag{3.1.9}$$

for all $x\in[0,1]$.

Proof. Since $P_{nk}(q;x)$ is a probability function so does $P_{\infty k}(q;x)$. We also know that

$$P_{\infty k}(q;x)=\frac{x^k}{(1-q)^k[k]_q!}\prod_{s=0}^{\infty}(1-q^sx)$$

then

$$
\begin{aligned}
\sum_{k:=0}^{\infty} P_{\infty k}(q;x) &= \sum_{k:=0}^{\infty} \frac{x^k}{(1-q)^k [k]_q!} \prod_{s=0}^{\infty} (1-q^s x) \\
&= \prod_{s=0}^{\infty} (1-q^s x) \left[1 + \sum_{k=1}^{\infty} \frac{x^k}{(1-q)^k [k]_q!} \right] \\
&= \prod_{s=0}^{\infty} (1-q^s x) \left[1 + \sum_{k=1}^{\infty} \frac{x^k}{(1-q)^k} \cdot \frac{(1-q)^k}{(1-q)(1-q^2)\ldots(1-q^k)} \right] \\
&= \prod_{s=0}^{\infty} (1-q^s x) \left[1 + \sum_{k=1}^{\infty} \frac{x^k}{(1-q)(1-q^2)\ldots(1-q^k)} \right].
\end{aligned}
$$

Here we will use the Euler Identity where the Euler Identity is

$$
1 + \sum_{n=1}^{\infty} \frac{t^n}{(1-q)(1-q^2)\ldots(1-q^n)} = \prod_{n=0}^{\infty} (1-tq^n)^{-1}.
$$

Then we get

$$
\begin{aligned}
\sum_{k:=0}^{\infty} P_{\infty k}(q;x) &= \prod_{s=0}^{\infty} (1-q^s x) \left[1 + \sum_{k=1}^{\infty} \frac{x^k}{(1-q)(1-q^2)\ldots(1-q^k)} \right] \\
&= \prod_{s=0}^{\infty} (1-q^s x) \prod_{k=0}^{\infty} \left(1-q^k x\right)^{-1} \\
&= \prod_{s=0}^{\infty} (1-q^s x) \cdot (1-q^s x)^{-1} = 1.
\end{aligned}
$$

Hence the result. ■

We can now consider the random variables $Y_\infty(q;x)$ given by the following probability distributions:

$$
P\left\{Y_\infty(q;x) = 1-q^k\right\} = P_{\infty k}(x), \quad k = 0,1,\ldots \text{ for } x \in [0,1] \tag{3.1.10}
$$

$$
P\{Y_\infty(q;1) = 1\} = 1. \tag{3.1.11}
$$

For $f \in C[0,1]$, we set

$$B_\infty(f,q;x) := Ef(Y_\infty(q;x)).$$

It follows from equations (3.1.10) and (3.1.11) that

$$B_\infty(f,q;x) = \begin{cases} \sum_{k=0}^{\infty} f\left(1-q^k\right) P_{\infty k}(q;x), \text{ if } x \in [0,1) \\ f(1), \text{ if } x = 1. \end{cases} \tag{3.1.12}$$

3.2 Main Results on Convergence

Our main results on convergence of generalized Bernstein polynomials are Theorems below.

Theorem 3.2.1 *[1] For any $f \in C[0,1]$,*

$$B_\infty(f,q;x) \rightrightarrows f(x) \qquad [x \in [0,1]; q \uparrow 1].$$

Proof. By the equations (3.1.3) and (3.1.12)

$$B_n(f,q;1) = B_\infty(f,q;1) = f(1),$$

for all $q > 0$. It sufficies to prove that

$$B_\infty(f,q;x) \rightrightarrows f(x) \qquad [x \in [0,1]; q \uparrow 1].$$

We know that

$$P_{\infty k}(q;x) = \frac{x^k}{(1-q)^k\,[k]_q!}\prod_{s=0}^{\infty}(1-q^s x),$$

then we set

$$\psi := \psi(q;x) := \prod_{s=0}^{\infty}(1-q^s x) \tag{3.2.1}$$

and write for $k = 1,2,\ldots$

$$P_{\infty k}(q;x) = \frac{x^k \psi}{(1-q)(1-q^2)\ldots(1-q^k)}.$$

By direct computations we get,

$$\begin{aligned}
E(Y_\infty(q;x)) &= \sum_{k=1}^{\infty}\left(1-q^k\right)P_{\infty k}(q;x) \\
&= \sum_{k=1}^{\infty}\left(1-q^k\right)\frac{x^k\psi}{(1-q)(1-q^2)...(1-q^k)} \\
&= x\psi + x\sum_{k=2}^{\infty}\frac{x^{k-1}\psi}{(1-q)...(1-q^{k-1})} \\
&= x\left(\psi + \sum_{k=2}^{\infty}\frac{x^{k-1}\psi}{(1-q)...(1-q^{k-1})}\right).
\end{aligned}$$

Replacing $(k-1)$ with k and then k with j, we get

$$E(Y_\infty(q;x)) = x\sum_{j=0}^{\infty}P_{\infty j}(q;x) = x,$$

and

$$
\begin{aligned}
E\left((Y_\infty(q;x))^2\right) &= \sum_{k=1}^{\infty}\left(1-q^k\right)^2 \frac{x^k\psi}{(1-q)(1-q^2)\dots(1-q^k)} \\
&= \sum_{k=1}^{\infty}\frac{\left(1-q+q-q^k\right)x^k\psi}{(1-q)(1-q^2)\dots(1-q^{k-1})} \\
&= \sum_{k=1}^{\infty}\frac{(1-q)\left[q\left(1-q^{k-1}\right)\right]x^k\psi}{(1-q)(1-q^2)\dots(1-q^{k-1})} \\
&= qx^2\sum_{k=2}^{\infty}\frac{x^{k-2}\psi}{(1-q)(1-q^2)\dots(1-q^{k-2})} \\
&\quad +(1-q)x\sum_{k=1}^{\infty}\frac{x^{k-1}\psi}{(1-q)(1-q^2)\dots(1-q^{k-1})} \\
&= qx^2+(1-q)x.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
Var(Y_\infty(q;x)) &= E\left(x^2\right)-(E(x))^2 \\
&= qx^2+(1-q)x+x^2 \\
&= -x^2(1-q)+(1-q)x \\
&= (1-q)x(1-x)\le\frac{(1-q)}{4}\ \ (\text{max. value of } x(1-x)\le\frac{1}{4})
\end{aligned}
$$

and it tends to 0 uniformly with respect to $x\in[0,1]$ as $q\uparrow 1$. Now, we show that

$$
B_\infty(f,q;x)=E(f(Y_\infty(q;x)))\rightrightarrows f(x)\quad [x\in[0,1),q\uparrow 1].
$$

Let $\varepsilon>0$ be given. We choose $\delta>0$ in such a way that $|f(t')-f(t'')|<\frac{\varepsilon}{2}$ for $|t'-t''|<\delta$, $t',t''\in[0,1]$ denote $C=\max\{|f(t)|:t\in[0,1]\}$ and $A=\{\omega\in\Omega:|Y_\infty(q;x)-x|\ge\delta\}$. We use Chebyshev's inequality

$$P(|x_n - x| \geq \delta) \leq \frac{var(x)}{\delta^2},$$

to get

$$\begin{aligned} P(|Y_\infty(q;x) - x| \geq \delta) &\leq \frac{VarY_\infty(q;x)}{\delta^2} \\ &= \frac{1-q}{4\delta^2} \to 0; q \uparrow 1. \end{aligned}$$

Thus,we obtain

$$\begin{aligned} |B_\infty(f,q;x) - f(x)| &\leq \left(\int_A + \int_{\Omega\backslash A}\right) |f(Y_\infty(q;x) - f(x))| P(d\omega) \\ &\leq 2CP(|Y_\infty(q;x) - x| \geq \delta) + \frac{\varepsilon}{2} \\ &\leq 2C\frac{(1-q)}{4\delta^2} + \frac{\varepsilon}{2} \\ &\leq \frac{C(1-q)}{2\delta^2} + \frac{\varepsilon}{2} \langle \varepsilon, \quad \text{if } q \uparrow 1. \end{aligned} \tag{3.2.2}$$

■

Theorem 3.2.2 *[1] Let* $0 < \alpha < 1$. *Then for any* $f \in C[0,1]$

$$B_n(f,q;x) \rightrightarrows B_\infty(f,q;x), \quad [x \in [0,1], q \in [\alpha,1]; n \to \infty].$$

Before passing to the proof of the above Theorem, use need to give the following lemmas.

Lemma 3.2.3 *[1] For any* $\varepsilon > 0$, *there exists a small* $\eta_\varepsilon > 0$ *and a positive integer* N_ε

such that

$$|B_n(f,q;x) - f(x)| \; < \; \varepsilon$$

for all $x \in [0,1]$, $q \in [1-\eta_\varepsilon, 1)$ *and* $\eta > N_\varepsilon$.

Proof. We use Korovkin's Theorem such that

$$B_n(1,q;x) = 1\ ,\ \ B_n(t,q;x) = x\ ,\ B_n\left(t^2,q;x\right) = x^2 + \frac{x(1-x)}{[n]_q}$$

and have

$$EY_n(q;x) = x\ ,\ EY_n(q;x)^2 = x^2 + \frac{x(1-x)}{[n]_q}.$$

Thus

$$\begin{aligned} VarY_n(q;x) &= x^2 + \frac{x(1-x)}{[n]_q} - x^2 \\ &= x(1-x)\left(1+q+q^2+...+q^{n-1}\right)^{-1}. \end{aligned}$$

Let $|f(x)| \leq C$ for all $x \in [0,1]$. Let $\delta > 0$ be chosen to such a degree that $|f(t) - f(x)| \leq \frac{\varepsilon}{2}$, whenever $|t-x| \leq \delta$.

Applying Chebyshev's Inequality, we obtain

$$\begin{aligned}
|B_n(f,q;x)-f(x)| &\leq \int_{[0,1]} |f(t)-f(x)|\, P_{Y_n(q;x)}(dt) \\
&\leq \int_{|t-x|\leq\Delta} + \int_{|t-x| \,\rangle\, \Delta} \\
&\leq \frac{\varepsilon}{2} + 2C\delta^{-2} Var Y_n(q;x) \\
&\leq \frac{\varepsilon}{2} + \frac{2C}{\delta^2}\cdot\frac{1}{4}\left(1+q+q^2+...+q^{n-1}\right)^{-1} \\
&\leq \frac{\varepsilon}{2} + \frac{C}{2\delta^2}\left(1+q+q^2+...+q^{n-1}\right)^{-1} \quad (q\to 1-\eta_\varepsilon) \\
&\leq \frac{\varepsilon}{2} + \frac{C}{2\delta^2}\cdot\frac{1-(1-\eta_\varepsilon)^n}{\eta_\varepsilon} \quad \left(1-(1-\eta_\varepsilon)^n \geq \frac{1}{2},\ (1-\eta_\varepsilon)^n \leq \frac{1}{2}\right).
\end{aligned}$$

Now we set $\eta_\varepsilon = \frac{\varepsilon\delta^2}{2C}$ and take N_ε in a such way that for all $n \geq N_\varepsilon$. The following inequality holds

$$1+(1-\eta_\varepsilon)+(1-\eta_\varepsilon)^2+...+(1-\eta_\varepsilon)^{n-1} \geq \frac{1}{2}(1-(1-\eta_\varepsilon)) = \frac{1}{2\eta_\varepsilon}.$$

Then for $q > 1-\eta_\varepsilon$, $n \geq N_\varepsilon$ and all $x \in [0,1]$, we have

$$\begin{aligned}
|B_n(f,q;x)-f(x)| &\leq \frac{\varepsilon}{2} + \frac{C}{2\delta^2}\left(1+q+q^2+...+q^{n-1}\right)^{-1} \\
&\leq \frac{\varepsilon}{2} + \frac{C}{2\delta^2}\cdot 2\cdot\frac{\varepsilon\delta^2}{2C} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

With the above result, the theorem is proved. ■

Lemma 3.2.4 *[1] Let* $0 < \alpha < \beta < 1$ *and let* $P_{nk}(q;x)(k = 0,...,n\ ,\ n = 1,2,...)$ *and* $P_{\infty k}(q;x)(k = 0,1,...)$ *be functions given by equations (3.1.4) and (3.1.8), respectively.*

Then for any $k = 0,1,2,\ldots$

$$P_{nk}(q;x) \rightrightarrows P_{\infty k}(q;x) \quad [x \in [0,1],\ q \in [\alpha,\beta];\ n \to \infty].$$

Proof. We note that $\begin{bmatrix} n \\ k \end{bmatrix}_q \to \left((1-q)^k [k]_q!\right)^{-1}$ as $n \to \infty$ uniformly with respect to $q \in [\alpha,\beta]$. Therefore, it suffices to prove that

$$\prod_{s=0}^{n-1-k} (1-q^s x) \to \prod_{s=0}^{\infty} (1-q^s x) \quad (n \to \infty)$$

uniformly with respect to $q \in [\alpha,\beta]$. This follows from the estimate :

$$\begin{aligned} 0 &\le \prod_{s=0}^{n-1-k} (1-q^s x) - \prod_{s=0}^{\infty} (1-q^s x) \text{ (Take common parentheses)} \\ &\le \prod_{s=0}^{n-1-k} (1-q^s x) \left[1 - \prod_{s=n-k}^{\infty} (1-q^s x)\right] \\ &\le 1 - \prod_{s=n-k}^{\infty} (1-q^s x) \le 1 - \prod_{s=n-k}^{\infty} (1-\beta^s) \to 0,\ n \to \infty \ . \end{aligned}$$

Now, let $\varepsilon > 0$ be given. By Theorem 3.2.1, there exists a small number $\zeta_\varepsilon > 0$ such that for all $x \in [0,1]$ and all $q \in [1-\zeta_\varepsilon, 1)$, we have

$$|B_\infty(f,q;x) - f(x)| \le \varepsilon.$$

Let $\eta_\varepsilon > 0$ and N_ε be numbers pointed out in Lemma 3.2.3. We set $\zeta = \min\{\eta_\varepsilon, \zeta_\varepsilon\}$. Then for all $x \in [0,1]$, $n > N_\varepsilon$ and $q \in [1-\zeta_\varepsilon, 1)$ we get

$$|B_n(f,q;x) - B_\infty(f,q;x)| \le 2\varepsilon.$$

To complete the proof of the Theorem, it sufficies to show that $B_n(f,q;x) \to B_\infty(f,q;x)$ uniformly with respect to $x \in [0,1]$ and $q \in [\alpha, 1-\zeta_\varepsilon]$. By equations (3.1.3) and (3.1.12),

$$B_n(f,q;1) = f(1) = B_\infty(f,q;1)$$

for all q.

We choose $a \in (0,1)$ in such a way that $|f(t) - f(1)| < \frac{\varepsilon}{3}$ for $a \le t \le 1$. Let R be a positive integer satisfying the condition $1 - q^{R+1} \ge a$ for all $q \in [\alpha, 1-\zeta_\varepsilon]$. We estimate the difference

$$\Delta := |B_n(f,q;x) - B_\infty(f,q;x)|$$

for $n > R$ and $x \in [0,1)$. Using equations (3.1.5) and (3.1.9) we obtain

$$\begin{aligned}
\Delta &= |B_n(f,q;x) - B_\infty(f,q;x)| \\
&= |B_n(f,q;x) - f(1) + f(1) - B_\infty(f,q;x)| \\
&= |B_n(f,q;x) - B_n(f,q;1) + B_\infty(f,q;1) - B_\infty(f,q;x)| \\
&= \left|\sum_{k=0}^{n}\left(f\left(\frac{[k_q]}{[n]_q}\right) - f(1)\right)P_{nk}(q;x) - \sum_{k=0}^{\infty}\left(f\left(1-q^k\right) - f(1)\right)P_{\infty k}(q;x)\right| \\
&\le \left|\sum_{k=0}^{R}\left(f\left(\frac{[k_q]}{[n]_q}\right) - f(1)\right)P_{nk}(q;x) - \sum_{k=0}^{R}\left(f\left(1-q^k\right) - f(1)\right)P_{\infty k}(q;x)\right| \\
&\quad + \sum_{k=R+1}^{n}\left|f\left(\frac{[k_q]}{[n]_q}\right) - f(1)\right|P_{nk}(q;x) + \sum_{k=R+1}^{\infty}\left|f\left(1-q^k\right) - f(1)\right|P_{\infty k}(q;x) \\
&= S_1 + S_2 + S_3.
\end{aligned}$$

Using Lemma 3.2.4 and the fact that $f\left(\frac{[k]_q}{[n]_q}\right) \to f\left(1-q^k\right)$ as $n \to \infty$ for all $k = 1,2,...,R$ uniformly with respect to $q \in [\alpha, 1-\zeta_\varepsilon]$,we compute S_1 as follows:

$$
\begin{aligned}
S_1 &= \left|\sum_{k=0}^{R}\left(f\left(\frac{[k_q]}{[n]_q}\right)-f(1)\right)P_{nk}(q;x)-\sum_{k=0}^{R}\left(f\left(1-q^k\right)-f(1)\right)P_{\infty k}(q;x)\right| \\
&= -f(1)\sum_{k=0}^{R}[P_{nk}(q;x)-P_{\infty k}(q;x)]+\sum_{k=0}^{R}\left|f\left(\frac{[k]_q}{[n]_q}\right)P_{nk}(q;x)-f\left(1-q^k\right)P_{\infty k}(q;x)\right| \\
&\quad -f\left(\frac{[k]_q}{[n]_q}\right)P_{\infty k}(q;x)+f\left(\frac{[k]_q}{[n]_q}\right)P_{\infty k}(q;x) \\
&= -f(1)\sum_{k=0}^{R}[P_{nk}(q;x)-P_{\infty k}(q;x)]+\sum_{k=0}^{R}f\left(\frac{[k]_q}{[n]_q}\right)[P_{nk}(q;x)-P_{\infty k}(q;x)] \\
&\quad -\left[f\left(1-q^k\right)-f\left(\frac{[k]_q}{[n]_q}\right)\right]P_{\infty k}(q;x) \\
&= \sum_{k=0}^{R}\left[f\left(\frac{[k]_q}{[n]_q}\right)-f(1)\right][P_{nk}(q;x)-P_{\infty k}(q;x)].
\end{aligned}
$$

If $n \to \infty$, $P_{nk}(q;x) \to P_{\infty k}(q;x)$, thus we conclude that $S_1 < \frac{\varepsilon}{3}$. Using equation (3.1.5) and positivity of $P_{nk}(q;x)$, we estimate S_2

$$
S_2 < \frac{\varepsilon}{3}\sum_{k=R+1}^{n} P_{nk}(q;x) \le \frac{\varepsilon}{3}.
$$

Similarly,

$$
S_3 < \frac{\varepsilon}{3}\sum_{k=R+1}^{\infty} P_{\infty k}(q;x) \le \frac{\varepsilon}{3}.
$$

Hence, $\Delta < \varepsilon$. ■

Corollary 3.2.5 *[1] If f is a polynomial of degree $\le m$, then $B_\infty(f,q;x)$ is also a polynomial of degree $\le m$.*

Theorem 3.2.6 *[1] If f is a polynomial, then* $\deg B_\infty(f,q;x) = \deg f$.

The function $B_\infty(f,q;x)$ is the limit of the sequence of generalized Bernstein polynomials $B_n(f,q;x)$ when $q \in (0,1)$ is fixed. We say that $f \in C[0,1]$ satisfies the Lipschitz condition at the point 1 *if there exist* $\alpha > 0$, $M > 0$ *such that*

$$|f(t) - f(1)| \leq M|t-1|^\alpha$$

for $t \in [0,1]$.

Proof. We use mathematical induction on $m = \deg f$.

$$B_n(f^m,q;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right)\begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-1-k}(1-q^s x).$$

For $m=1$ the statement true. Let us suppose that the statement is true for degree $1 \leq m$ and consider $B_\infty\left(t^{m+1},q;x\right)$. By equation (3.2.1) we have

$$\begin{aligned}
B_\infty\left(t^{m+1},q;x\right) &= \sum_{k=1}^{\infty}\left(1-q^k\right)^{m+1}\frac{x^k\psi}{(1-q)\dots(1-q^k)} \\
&= \sum_{k=1}^{\infty}\left((1-q)+q\left(1-q^{k-1}\right)\right)^m\frac{x^k\psi}{(1-q)\dots(1-q^{k-1})} \\
&= \sum_{k=0}^{m}\binom{m}{k}q^k(1-q)^{m-k}\,x\sum_{r=1}^{\infty}\left(1-q^{r-1}\right)^k\frac{x^{r-1}\psi}{(1-q)\dots(1-q^{r-1})} \\
&= \sum_{k=0}^{m}\binom{m}{k}q^k(1-q)^{m-k}\,xB_\infty\left(t^k,q;x\right).
\end{aligned}$$

By the induction assumption this is a polynomial of degree $m+1$. ■

Theorem 3.2.7 *[1] For any $f \in C[0,1]$, the function $B_\infty(f,q;x)$ is continuous on* $[0,1]$ *and analytic in the unit disk $x : |x| < 1$. If* f *satisfies the Lipschitz condition at 1, then $B_\infty(f,q;x)$ is differentiable from the left at 1.*

Proof. Continuity of $B_\infty(f,q;x)$ with respect to x on $[0,1]$ follows immediately from the fact that $B_\infty(f,q;x)$ is a limit of uniformly convergent sequence of polynomials. To prove analyticity we write for $|x| < 1$,

$$B_\infty(f,q;x) = \psi(q;x)\sum_{k=0}^{\infty}\frac{f\left(1-q^k\right)}{(1-q)\ldots(1-q^k)}x^k \tag{3.2.3}$$

where $\psi(q;x)$ defined by equation (3.2.1) is an entire function. If $k = 0$ in equation (3.2.3) the denominator is taken to be 1, since

$$\lim_{k\to\infty}(1-q)\ldots\left(1-q^k\right) = \prod_{s=1}^{\infty}(1-q^s) \neq 0$$

It follows that the sequence $\left\{\frac{f\left(1-q^k\right)}{\prod_{s=1}^{k}(1-q^s)}\right\}_{k=1}^{\infty}$ is bounded. Thus the sum in equation (3.2.3) is an analytic function for $|x| < 1$ and so is $B_\infty(f,q;x)$. Now assume that f satisfies Lipschitz condition at the point 1. Using equations (3.1.9), (3.1.12) and (3.2.1), we get

$$\begin{aligned}
B_\infty(f,q;x) - B_\infty(f,q;1) &= B_\infty(f,q;x) - f(1) \\
&= \sum_{k=0}^{\infty}\left(f\left(1-q^k\right) - f(1)\right)P_{\infty k}(q;x) \\
&= \psi(q;x)\sum_{k=0}^{\infty}\frac{\left(f\left(1-q^k\right) - f(1)\right)}{(1-q)^k[k]_q!} \\
&= (1-x)\psi_1(q;x)u(q;x),
\end{aligned}$$

where

$$\psi_1(q;x) = -\prod_{s=1}^{\infty}(1-q^s x)$$

and

$$u(q;x) = \sum_{k=0}^{\infty} \frac{\left(f\left(1-q^k\right) - f(1)\right)}{(1-q)^k [k]_q!}. \tag{3.2.4}$$

Since the sequence $\left\{\left(\left(1-q^k\right)[k]_q!\right)^{-1}\right\}_{k=0}^{\infty}$ is bounded and $\left|f\left(1-q^k\right) - f(1)\right| \leq M(q^{\alpha})^k$, it follows that the series in equation (3.2.4) is uniformly convergent on $[0,1]$. Here, the function $u(q;x)$ is continuous on $[0,1]$. Thus,

$$\lim_{x \uparrow 1} \frac{B_{\infty}(f,q;x) - B_{\infty}(f,q;1)}{x-1} = -\psi_1(q;1)\, u(q;1),$$

and so $B_{\infty}(f,q;x)$ is differentiable at 1 from the left. ■

Theorem 3.2.8 *[1] If $f\left(1-q^k\right) = 0$ for all $k = 0,1,2,...$ then $B_{\infty}(f,q;x) = 0$ on $[0,1]$. If $B_{\infty}(f,q;x) = 0$ for an infinite number of points having an accumulation point on $[0,1]$, then $f\left(1-q^k\right) = 0$ for all $k = 0,1,2,..$*

Theorem 3.2.9 *[1] Let $f \in C[0,1]$. Then $B_{\infty}(f,q;x) = f(x)$ for all $x \in [0,1]$ if and only if $f(x) = ax + b$, where a and b are constants.*

Proof. It can readily be seen from equation (3.1.12) that for a fixed $q \in (0,1)$ there exist different continuous functions $f \neq g$ such that $B_{\infty}(f,q;x) = B_{\infty}(g,q;x)$. This is because $B_{\infty}(f,q;x)$ is defined only by the values of f at the points $\left\{1-q^k\right\}_{k=0}^{\infty}$. In particular, there exist non-linear function f such that $B_{\infty}(f,q;x)$ are linear function. If $f(x) = ax+b$, then by equation (3.1.2) $B_n(f,q;x) = ax+b = f(x)$ for all $n = 1,2,...$ and therefore

$$B_{\infty}(f,q;x) = \lim B_n(f,q;x) = f(x).$$

Now we suppose that $B_\infty(f,q;x) = f(x)$ for every $x \in [0,1]$. Let us consider the function $g(x) = f(x) - (f(1) - f(0))x$. It is evident that $g(0) = g(1)$ and $B_\infty(f,q;x) = g(x)$. Now, we will prove that $g(x) = g(0) = g(1)$ for all $x \in [0,1]$. Let $M = \max_{x\in[0,1]} g(x)$. Now assume that $M > g(1)$, then $M = g(z)$ for some $z \in (0,1)$ and $g\left(1-q^k\right) < M$ for sufficiently large k. Using equation (3.1.9) and positively of $P_{\infty k}(q;x)$ we have

$$M = g(z) = \sum_{k=0}^{\infty} g\left(1-q^k\right) P_{\infty k}(q;x) \quad < \ M.$$

The contradiction show that $g(x) \le g(1)$ for all $x \in [0,1]$. In a similar way, it can be proven that $g(x) \ge g(1)$ for all $x \in [0,1]$. Hence, $g(x) \equiv b$ for some $b \in \mathbb{R}$ and as a result $f(x) = ax + b$. ■

Theorem 3.2.10 *[1] Let $f \in C[0,1]$ and*

$$B_\infty\left(f,q_j;x\right) = a_j x + b_j \ , \ (x \in [0,1])$$

for a sequence q_j such that $q_j \uparrow 1$. Then f is a linear fuction.

Proof. Let $B_\infty\left(f,q_j;x\right) = a_j x + b_j$, $(x \in [0,1])$. From equation (3.1.2) and Theorem 3.2.8 it follows that

$$f(x) = a_j x + b_j \ \text{ for } x \in \left\{1-q_j^k\right\}_{k=0}^{\infty}.$$

Since $q_j \uparrow 1$ and $f \in C[0,1]$, we obtain $f(x) = ax + b$ for some $a,b \in \mathbb{R}$. ■

Chapter 4

CONCLUSION

The thesis contains basic properties of Bernstein polynomials and generalized Bernstein polynomials and convergence rate of Bernstein polynomial also we introduced some probabilistic considerations.

We proved that the most important properties of Bernstein polynmials and as it is a recursive definition of Bernstein polynmials, degree raising, the Bernstein polynomials form a partition of unity, converting from the Bernstein basis to the power basis, the Bernstein polynomials are all nonnegative, derivatives and a matrix representation for Bernstein polynomials.

The final part of the study is concerned with the generalized Bernstein polynomials and related with the q- integers. We gave the basic definition and properties of q- integers. Then we studied the convergence rate of Bernstein polynomials and we proved some related theorems. After all we preserved some probabilistic considerations of Bernstein polynomials.

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