Hawking Radiation of Rindler Modified Schwarzschild Black Hole

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ABSTRACT

One of the significant techniques to calculate the black hole (BH) temperature and understand BH thermodynamics is to employ the semi-classical BH tunneling method. In this thesis, by using the Parikh-Wilczek (PW), Hamilton-Jacobi (HJ) and Damour-Ruffini-Sannan (DRS) tunneling methods, we aim to explore the Hawking radiation (HR) of the Grumiller BH (GBH). This BH is also known as Rindler modified Schwarzschild BH. Generally, in the tunneling method the imaginary part of the action (classically forbidden process) is directly proportional to the Boltzmann factor or is inversely proportional to the temperature of the BH whose performing the Hawking radiation (HR). In the original work of PW, who considered the self-gravitational effect with the energy conservation, it was shown that the small deviations from pure thermal radiation in the HR may cause to a leakage of the information from the BH. This phenomenon is summarized and extended to the GBH in this dissertation.

Keywords: Quantum tunneling method, Hawking radiation, Relativistic Hamilton-Jacobi equation, Parikh-Wilczek method, Damour-Ruffini-Sannan method, Grumiller black hole, Rindler acceleration.

ÖZ

Kara delik (BH) sıcaklığını hesaplamak ve BH termodinamiğini anlamak için en önemli tekniklerden birisi yarı-klasik BH tünelleme yöntemini kullanmaktır. Bu tezde, Parikh-Wilczek (PW), Hamilton-Jacobi (HJ) ve Damour-Ruffini-Sannan (DRS) tünelleme yöntemleri kullanarak, biz Grumiller BH (GBH)'un ve Hawking radyasyonunu (HR) araştırmayı hedefliyoruz. Bu BH, Rindler modifiyeli Schwarzschild BH'u olarak da bilinmektedir. Genel olarak, tünelleme yönteminde eylemin (klasik olarak yasaklanmış işlem) sanal kısmı Boltzmann faktörü ile doğrudan orantılıdır ya da Hawking radyasyon (HR) gerçekleştiren BH'un sıcaklığına ters orantılıdır. Enerji korunumu ile kendini yerçekimi etkisinin dikkate alındığı PW'in orjinal çalışmasında, HR'nın saf termal radyasyonundaki küçük sapmaların BH'dan bir bilgi sızıntısına neden olduğu gösterilmiştir. Bu olgu, bu tezde özetlenmiş ve GBH için genişletilmiştir.

Anahtar kelimeler: Kuantum tünelleme metodu, Hawking radyasyonu, Relativisitik Hamilton-Jacobi denklemi, Parikh-Wilczek metodu, Damour-Ruffini-Sannan metodu, Grumiller kara deliği, Rindler ivmesi.

DEDICATION

To My Family

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Chapter 1

INTRODUCTION

A BH is a system in which gravity is so strong that in classical point of view no particle or radiation can escape from it. In other words, classically it corresponds to a system that everything would go into it but nothing will come out. But this phenomenon has a contradiction in defining the temperature for a BH. The reason is that it is impossible for any system to maintain the thermal equilibrium with a BH. This point was firstly considered by Hawking, who was able to show that BHs can radiate when quantum effects are taken in account. Actually, Hawking theoretically proved that the radiation from the BH possesses a particular thermal spectrum [1,2]. In short, he explained the existence of BH radiation as the tunneling of particles due to vacuum fluctuations taken place around the event horizon. These fluctuations cause to the generation of particle-antiparticle pair such that one member of the pair falls inside the horizon with a negative energy and thus decrease the mass of the BH, and the another one having positive energy (outgoing particle**)** which crosses the horizon and flies off from the BH is known as the HR. But in Hawking's seminal works it was not clear where the potential barrier for the tunneling is.

The discovery of the HR also opened up new mysteries which is called "information loss paradox". The information loss problem results from the argument of whether the HR is pure thermal or not. Because, in the case of HR is pure thermal, it should not contain or carry any information due to the quantum mechanics (QM). Namely, after the evaporation of the BH, the information of what made up the BH will be erased forever. However, this is against to the conservation of information rule.

In the same line of thought another method was proposed by Kraus and Wilczek (KW) [3,4] in 1995. They managed to describe the HR in the framework of tunneling process. In 2000, KW's idea was moved a step forward, and finishing touches were made by PW [5]. Their contribution to KW's work was about the potential barrier which is tunneled by the BH's particles. They threw an idea into the pot that the barrier depends on the tunneling particle itself. According to their method BH’s mass can fluctuate but to conserved energy, the energy of the spacetime should remain fixed. Thus, when a particle of energy ω is radiated, the BH mass reduces to $m - \omega$. In other words, particle’s self gravitation was also taken into account. So, the HR spectrum carries the loss of mass of the BH. The obtained radiation by this method is not a pure black body spectrum, which renders possible the leakage of the information. By introducing Painleve-Gullstrand (PG) coordinate transformation [6,7], which is well behaved across the horizon, the metric becomes stationary but not static. It is shown that such a choice of coordinate is particularly useful to describe the HR in the PW method. After [5]., several works have been done to explain the HR for a large class BHs using the PW’s tunneling process. For the topical review, one can refer to [8].

As we mentioned above, the discovery of the PW method justifies the energy conservation and makes praiseworthy the treatment of the HR’s self-gravity. The intent of the HJ method is to give a particle description of the HR under the assumption that the emitted (scalar) particle’s action, ignoring its self-gravitation, does satisfy the Lorentzian HJ relativistic equation [9]. This method applies to any regular coordinate system across the event horizon.

In the DRS method [10,11], one uses an approach which only requires the existence of a future horizon. This method is completely independent of any dynamical details of the process leading to the formation of the horizon. It assumes analyticity properties of the wave function in the complexified manifold of the BH spacetime.

The aim of this thesis is to calculate explore the HR of the GBH by using three different methods, which are PW, HJ and DRs methods. The present thesis is organized as follows: In chapter 2, we firstly make a brief review of the PW's tunneling method. Then, we adopt the PW method to the GBH in order to evaluate the imaginary part of the action and read its tunneling rate. Chapter 3 is devoted to the application of the HJ method in the GBH geometry. DRS method is employed for the GBH in chapter 4. The conclusions are given in chapter 5.

Chapter 2

HR OF THE GBH VIA THE PW METHOD

2.1 The PW Method

In this section, for reviewing the PW method we shall consider the Schwarzschild BH as an example line-element.

The four dimensional (4D) static and spherically symmetric metric [12] is given by

$$ds^2 = -fdt^2 + f^{-1}dr^2 + Rd\Omega^2 \tag{2.1}$$

in which the line-element of the unit 2-sphere is

$$d\Omega^2 = d\theta^2 + sin^2\theta d\varphi^2 \tag{2.2}$$

The metric functions of the Schwarzschild read

$$f = f(r) = 1 - \frac{2M}{r} \quad and \quad R = R(r) = r^2, \tag{2.3}$$

When $f = 0$, then the root $\mathrm{r} = \mathrm{r}_h = 2M$ denotes the event horizon. The surface gravity of metric (2.1) is given by [12]

$$\kappa = \lim_{r \to r_h} \frac{1}{2} f'(r) \tag{2.4}$$

From now on, a prime over a quantity denotes the derivative with respect to its argument. After taking the derivative of the metric function (2.3) and substitute it into Eq. (2.4) we get surface gravity

$$\kappa = \frac{1}{4M} \tag{2.5}$$

Making use of the definition of the Hawking temperature (T_{BH}) which is $T_{BH} = \frac{\kappa}{2\pi}$ we obtain

$$T_{BH} = \frac{1}{2\pi} \lim_{r \to r_h} \frac{1}{2} f'(r) \Big|_{r=r_h} = \frac{1}{8\pi M} \tag{2.6}$$

Entropy of the BH is defined by

$$S_{BH} = \int \frac{dM}{T_{BH}} \tag{2.7}$$

where $dM = T_{BH} dS_{BH}$ the first law of thermodynamics. After evaluating the integral, the entropy of the BH becomes $S_{BH} = \frac{A_h}{4} = \pi r_h^2$.

2.2 PG Coordinates

PG coordinates [6,7] are known as the first coordinate system in the literature, which is regular at the event horizon. They are used for to describe the considered spacetime on either side of the event horizon of a static BH. In order to have the PG coordinate system, we can start to our operations with the following transformation.

$$dt = d\tau - Fdr \tag{2.8}$$

in which F is a function of r and τ is called as the PG time. Thus

$$dt^2 = d\tau^2 + F^2 dr^2 - 2F d\tau dr \tag{2.9}$$

Substituting Eq. (2.9) into the metric (2.1), we get

$$ds^2 = -f d\tau^2 - fF^2 dr^2 + 2fF d\tau dr + f^{-1} dr^2 + R d\Omega^2 \tag{2.10}$$

If we choose

$$F^2 = f^{-1}\left[f^{-1} - 1\right]$$

$$fF = \sqrt{1-f} \tag{2.11}$$

Eq. (2.10) reduces to

$$ds^2 = -fd\tau^2 + dr^2 + 2Ffd\tau dr + Rd\Omega^2 \tag{2.12}$$

Thus, we obtain the PG coordinates as follows.

$$ds^2 = -fd\tau^2 + dr^2 + 2\sqrt{1-f}\,d\tau dr + Rd\Omega^2 \tag{2.13}$$

Now, the coordinate singularity arising at the event horizon is completely removed. The radial null geodesics ($ds = 0$) for the metric Eq.(2.13) is found as

$$\dot{r} = (\pm 1 - \sqrt{1-f}\;) \tag{2.14}$$

where $\dot{\mathrm{r}} = \frac{\mathrm{dr}}{\mathrm{d\tau}}$. We can evaluate the imaginary part of the action for an outgoing positive energy particle which crosses the horizon outwards from r_{in} *to* r_{out} as

$$Im(I) = Im\int_{r_{in}}^{r_{out}} p_r dr = Im\int_{r_{in}}^{r_{out}}\int_0^{p_r} dp'_r\, dr \tag{2.15}$$

where r_{out} and r_{in} stand for the final and initial radius, respectively. Recalling the Hamilton's equations of motion

$$dp_r = \frac{dH}{\dot{r}} \tag{2.16}$$

where p_r shows the canonical momentum, and setting $dH = dM$, we have

$$Im(I) = Im\int_{r_{in}}^{r_{out}}\int_M^{M-\omega}\frac{dH}{\dot{r}}dr \tag{2.17}$$

Here M and $M-\omega$ correspond to the mass of the BH before and after the HR, respectively. Using the radial null geodesics (2.13) in the above equation, one obtains

$$Im(I) = Im\int_{r_{in}}^{r_{out}}\int_{M}^{M-\omega}\frac{dMdr}{1-[1-f]^{\frac{1}{2}}} \tag{2.18}$$

which is equivalent to

$$Im(I) = Im\int_{r_{in}}^{r_{out}}\int_{M}^{M-\omega}\frac{1+[1-f]^{\frac{1}{2}}}{f}dM'dr \tag{2.19}$$

Taylor expansion of the metric function f around the event horizon r_h $(r \approx r_h)$ is given by

$$f = f(r_h) + f'(r_h)|_{r=r_h}(r-r_h)$$

$$+\frac{f''(r_h)}{2!}|_{r=r_h}\frac{(r-r_h)^2}{2!}+.... \tag{2.20}$$

Finally, it can be approximated to

$$f \approx f'(r_h)|_{r=r_h} = f'(r_h)(r-r_h) \tag{2.21}$$

Substituting Eq. (2.21) into Eq. (2.19) , we find out

$$Im(I) = Im\int_{M}^{M-\omega}\int_{r_{in}}^{r_{out}}\frac{1+[1-f]^{\frac{1}{2}}}{f(r_h)(r-r_h)}dr\,dM' \tag{2.22}$$

If we solely consider the integration which is with respect to r;

$$\gamma = \int\frac{1+[1-f]^{\frac{1}{2}}}{f'(r_h)(r-r_h)}dr \tag{2.23}$$

we should use the residue technique. Because in Eq. (2.23) we see that there is a simple pole ($r - r_h$) at horizon. This can be overcome by the contour around the pole. In short, we obtain

$$\gamma = -i\,\pi \frac{1+\left[1-f\,(r_h)\right]}{f\,'\left(r_h\right)} \tag{2.24}$$

Since $f\,(r_h) = 0$, it becomes

$$\gamma = -i\,\pi \frac{2}{f\,'\left(r_h\right)} \tag{2.25}$$

and imaginary part of the action reads

$$Im\left(I\right) = -2\pi \int_{M}^{M-\omega} \frac{dM^{'}}{f\,'(\,r_h\,)} \tag{2.26}$$

Then ,we have relationship between T_H and κ is

$$T_H = \frac{\kappa}{2\pi} \tag{2.27}$$

and

$$\kappa = \frac{f\,'(r_h\,)}{2} \tag{2.28}$$

After combing Eqs.(2.28) and (2.27), we get

$$f\,'(\,r_h\,) = 4\pi T_H \tag{2.29}$$

Now put it back to Eq. (2.26).

$$Im\left(I\right) = -\frac{1}{2}\int_{M}^{M-\omega} \frac{dM\,'}{T_H} \tag{2.30}$$

In general, the change in entropy is given by

$$\Delta S_{BH} = \int \frac{dQ}{T_H} \tag{2.31}$$

where it takes the following form for the BHs as

$$dS = \frac{dQ'}{T_H} = \frac{dM'}{T_H} \tag{2.32}$$

Thus it leads us to modify Eq. (2.26) as follows

$$Im(I) = -\frac{1}{2}\int_{M}^{M-\omega} dS \tag{2.33}$$

so that it gives

$$= -\frac{1}{2}\left[S(M-\omega) - S(\omega)\right]$$

$$= -\frac{1}{2}\Delta S \tag{2.34}$$

As one already knows from basic QM [13], the tunneling rate is defined as

$$\Gamma = exp\left[-2Im(I)\right] \tag{2.35}$$

Therefore it reads

$$\Gamma = exp(\Delta S) \tag{2.36}$$

where ΔS is the difference of the Bekenstein-Hawking entropy between before and after the HR. Since, the total entropy of a BH is given by

$$S_{BH} = \frac{1}{4}A_h \tag{2.37}$$

where $(A_h = 4\pi r^2)$ is called the area of the BH. Then

$$S_{BH} = \pi r_h^2 = 4\pi M^2 \qquad (2.38)$$

where M represents the total mass of the BH before the HR. After the emission of the particle, the mass of the BH become decreases to $M - \omega$ so that Eq. (2.38) becomes :

$$S(M - \omega) = 4\pi (M - \omega)^2 \qquad (2.39)$$

Then, the change of entropy results in

$$\Delta S_{BH} = 4\pi (M - \omega)^2 - 4\pi M^2$$

$$= -4\pi\omega(2M - \omega) \qquad (2.40)$$

Now, we have obtained the original result derived by PW. That produces the semi-classical tunneling rate as

$$\Gamma \approx e^{\Delta S_{BH}} = e^{-8\pi\omega(M - \frac{\omega}{2})} \qquad (2.41)$$

where we have expressed the result more naturally in terms of the change in S_{BH}. When the factor ω^2 is neglected in Eq.(2.40), the tunneling rate (2.41) reduces to the black body radiation, which is pure thermal that permits the leakage of the information from the system and expressed by a Boltzmann factor $e^{-\beta\omega}$ in which $\beta = \frac{1}{T}$. Namely, the existence of ω^2 is due to the the physics of energy conservation. Besides, it gives rise to a deflection from the pure thermal radiation of the BH and therefore leads to a information escaping from the BH. This phenomenon is significant on the resolution of the information loss problem.

2.3 GBH Geometry and its HR within the Concept of the PW Method

In this section, getting inspired from the previous section we shall apply the PW method to the GBH spacetime.

The line-element of GBH in spacetime [14-16] is given by

$$ds^2 = -fdt^2 + \frac{dr^2}{f} + r^2\left(d\theta^2 + sin^2\theta d\varphi^2\right) \tag{2.42}$$

where now the metric function f reads

$$f = 1 - \frac{2M}{r} + 2ar \tag{2.43}$$

by which a is called the Rindler acceleration. In the limit of $a = 0$, metric (2.42) reduces to the well-known Schwarzschild BH. GBH has only one horizon which makes $f(r) = 0$ is located at

$$r_h = \frac{-1 + \sqrt{1 + 16Ma}}{4a} \tag{2.44}$$

The metric (2.42) has coordinate singularity at the horizon which can be removed by transforming it into the PG coordinates. To this end, we change the time to

$$dt = d\tau + \frac{\sqrt{1-f}}{f}dr \tag{2.45}$$

Whence, the line-element (2.42) converts to

$$ds^2 = -fdt^2 + 2\sqrt{1-f}\,dtdr + dr^2 + R^2 d\Omega^2 \tag{2.46}$$

On the other hand, the radial outgoing null geodesics in this geometry can be expanded as:

$$\dot{r} = 1 - \sqrt{1-f} \approx 1 - \sqrt{1 - f\,(r_h) + \frac{1}{2} f\,'(r_h)(r - r_h) + \mathrm{O}(r - r_h)^2}$$
$$\dot{r} \approx \frac{1}{2} f\,'(r_h)(r - r_h) \tag{2.47}$$

Next , differentiate Eq.(2.43) with respect to r gives

$$f\,'(r_h) = \frac{2M}{r_h^{\;2}} + 2a \tag{2.48}$$

Substituting it into the null geodesics Eq. (2.47), one gets

$$\dot{r} = \frac{(\,r - r_h\,)}{2}\Big(\frac{2M}{r_h^2} + 2a\Big) \tag{2.49}$$

Combining the above with the event horizon (2.44), we obtain

$$\dot{r} = \frac{(r - r_h)}{2}\left(\frac{16Ma^2}{1 + 8Ma - \sqrt{1 + 16Ma}} + 2a\right) \tag{2.50}$$

Here M denotes the total mass of the GBH before HR. After the radiation, the null geodesics modifies to

$$\dot{r} = \left(\frac{16(M - \omega)a^2}{1 + 8(M - \omega)a - \sqrt{1 + 16(M - \omega)a}} + 2a\right) \tag{2.51}$$

Using the general form of the imaginary part of the action for the outgoing particles with positive energy, i.e.,

$$Im\,(I\,) = Im\int_{r_{in}}^{r_{out}} p_r dr = Im\int_{r_{in}}^{r_{out}}\int_0^{p_r} d\tilde{p}dr \tag{2.52}$$

and inserting the canonical momentum

$$d\tilde{p}_r = \frac{dH}{\dot{r}} \tag{2.53}$$

into Eq. (2.52), we find out

$$Im\left(I\right) = Im\int_{r_{in}}^{r_{out}}\int_0^{p_r}\frac{dr}{\dot{r}}dH \tag{2.54}$$

In this equation, we have used the fact that if the particles have tunneled out, then the BH will lose some of its energy. This means a change in the Hamiltonian as $H = M \to M - \omega$. Thus we can rewrite Eq. (2.54) as

$$Im\left(I\right) = Im\int_{r_{in}}^{r_{out}}\frac{dr}{\dot{r}}\int_0^{\omega}(-d\,\tilde{\omega}) \tag{2.55}$$

Near the horizon the radial outgoing null ($\dot{r}$) geodesics before radiation takes the following form

$$\dot{r} = \frac{r - r_h}{2}f\,'\left(r_h(M)\right) \tag{2.56}$$

However, after the radiation it changes to

$$\dot{r} = \frac{r - r_h}{2}f\,'\left(r_h\left(M - \omega\right)\right) = \left(r - r_h\right)\kappa_{QK} \tag{2.57}$$

Combining Eqs. (2.57) and (2.55), we obtain

$$Im\left(I\right) = -Im\int_0^{\omega}\left[\int_{r_{in}}^{r_{out}}\frac{dr}{\left(r - r_h\right)\kappa}\right]d\tilde{w} \tag{2.58}$$

The minus sign appears in Eq. (2.58) is because of $(dH = -d\,\tilde{\omega})$ which implies the shrinking of the horizon. Now the integral can be evaluated by deforming the contour where its semicircle centered at pole r_h. Therefore Eq. (2.58) turns out to be

$$Im\left(I\right)=-\pi\int_0^{\omega}\frac{d\,\tilde{\omega}}{\kappa} \tag{2.59}$$

Recalling the Hawking temperature:

$$T_H=\frac{\kappa}{2\pi} \tag{2.60}$$

we can rewrite expression (2.59) as

$$Im\left(I\right)=-\frac{1}{2}\int_0^{\omega}\frac{d\tilde{\omega}}{T_H}=-\frac{1}{2}\int_M^{M-\omega}\frac{d\tilde{M}}{T_H}=-\frac{1}{2}\int_M^{M-\omega}dS \tag{2.61}$$

$$=-\frac{1}{2}\left[S\left(M-\omega\right)-S\left(M\right)\right]=-\frac{1}{2}\Delta S_{\mathrm{BH}} \tag{2.62}$$

where ΔS_{BH} is the change in entropy of the GBH before and after the HR. According to Eq. (2.62) which originally comes from [5], it is also equal to

$$\Delta S_{BH}=-2Im\left(I\right) \tag{2.63}$$

For the GBH, its explicit form can be computed as

$$\Delta S_{GBH}=\frac{\pi}{8a^2}\left[\sqrt{1+16Ma}-8a\omega-\sqrt{1+16\left(M-\omega\right)a}\right] \tag{2.64}$$

One can see that if we expand the above to the series by using the Taylor series for the Rindler parameter a, the leading term gives us the Schwarzschild entropy, i.e.,

$$\Delta S_{GBH}\approx\frac{\pi}{8a^2}\left[-32M^2a^2+32(M-\omega)^2a^2\right]+\mathcal{O}\left(a^3\right) \tag{2.65}$$

$$\Delta S_{GBH}\rightarrow\Delta S_{Schw.}=-8\pi\omega\left[M-\frac{\omega}{2}\right] \tag{2.66}$$

Now, with the aid of Eq. (2.63) , one can easily read the tunneling rate of the GBH as:

$$\Delta\Gamma \sim e^{-2\,Im(I)} = e^{+\Delta S_{GBH}} \tag{2.67}$$

Chapter 3

HR OF THE GBH VIA THE HJ METHOD

In this chapter , we will consider an alternate method for calculation the imaginary part of the action which is the so-called HJ method based on the relativistic HJ equation (see for instance [17] and references therein). The relativistic HJ equation is given by

$$g^{\mu\upsilon}\partial_{\mu}I\,\partial_{\upsilon}I+m^{2}=0 \tag{3.1}$$

where $g^{\mu\upsilon}$ stands for the metric tensor and m for mass of the test particle. For the GBH line-element (2.42), we read its covariant and contravariant forms as

$$g^{\mu\upsilon}=\begin{pmatrix}-1/f & 0 & 0 & 0\\ 0 & f & 0 & 0\\ 0 & 0 & 1/r^{2}d\theta^{2} & 0\\ 0 & 0 & 0 & 1/r^{2}\sin^{2}\theta d\varphi^{2}\end{pmatrix} \tag{3.2}$$

$$g_{\mu\upsilon}=\begin{pmatrix}-f & 0 & 0 & 0\\ 0 & 1/f & 0 & 0\\ 0 & 0 & r^{2}d\theta^{2} & 0\\ 0 & 0 & 0 & r^{2}\sin^{2}\theta d\varphi^{2}\end{pmatrix} \tag{3.3}$$

After substituting them into Eq. (3.1), the resulting equation is

$$g^{00}\partial_{0}I\,\partial_{0}I+g^{11}\partial_{1}I\,\partial_{1}I+g^{22}\partial_{2}I\,\partial_{2}+g^{33}I\,\partial_{3}I\,\partial_{3}+m^{2}=0 \tag{3.4}$$

which is equivalent to

$$g^{tt}(\partial_t I)^2 + g^{rr}(\partial_r I)^2 + g^{\theta\theta}(\partial_\theta I)^2 + g^{\varphi\varphi}(\partial_\varphi I)^2 + m^2 = 0 \tag{3.5}$$

Therefore, we get

$$-f^{-1}(\partial_t I)^2 + f(\partial_r I)^2 + r^{-2}(\partial_\theta I)^2 + r^{-2}\sin^{-2}\theta(\partial_\varphi I)^2 + m^2 = 0 \tag{3.6}$$

If we set the following the Ansatz for the action

$$I = -Et + W(r) + J(x^i) \tag{3.7}$$

which yields

$$\partial_t I = -E, \quad \partial_r I = W', \quad \partial_\theta I = J_\theta = \partial_\theta J, \quad \partial_\varphi I = J_\varphi = \partial_\varphi J \tag{3.8}$$

Next ,we substitute the above action (3.6) in to the H.J Eq.(3.4) and obtain

$$-f^{-1}E^2 + fW'^2 + r^{-2}J_\theta^2 + r^{-2}\sin^{-2}\theta J_\varphi^2 + m^2 = 0 \tag{3.9}$$

Then , if we solve it for W , it yields

$$W'^2 = f^{-2}\left[E^2 - r^{-2}f\,(J_\theta^2 + J_\varphi^2\sin^{-2}\theta - m^2r^2\right] \tag{3.10}$$

which corresponds to

$$W' = \pm\sqrt{f^{-2}\left[E^2 - r^{-2}f\,(J_\theta^2 + J_\varphi^2\sin^{-2}\theta - m^2r^2\right]} \tag{3.11}$$

and

$$W = \pm\int f^{-1}\sqrt{\left[E^2 - r^{-2}f\,(J_\theta^2 + J_\varphi^2\sin^{-2}\theta - m^2r^2\right]} \tag{3.12}$$

In the GBH metric, f can be redefined as

$$f = 1 - 2Mr^{-1} + 2ar = r^{-1}(2ar^2 + r - 2M) \tag{3.13}$$

So for finding the horizon the equation $2ar^2+r-2M=0$ should be solved. Therefore, we obtain the event horizon as

$$r_h=\frac{-1+\sqrt{1+16Ma}}{4a} \tag{3.14}$$

The other root is

$$r_0=\frac{-1-\sqrt{1+16aM}}{4a} \tag{3.15}$$

which cannot be considered as a kind of horizon due to its negative value. With the aid of above roots, one can set the metric function as

$$f=2ar^{-1}(r-r_0)(r-r_h) \tag{3.16}$$

Around the horizon Eq. (3.12) behaves as

$$W\approx\pm\int\frac{E}{f}dr \tag{3.17}$$

If we expand the metric function f around the horizon r_h,

$$f(r)\approx f(r_h)+f'(r_h)(r-r_h)+....... \tag{3.18}$$

we can get

$$f(r)\approx f'(r_h)(r-r_h) \tag{3.19}$$

so that

$$W\approx\pm\int\frac{E}{f'(r-r_h)}dr$$

$$=\pm i\pi\frac{E}{f'(r_h)}+c \tag{3.20}$$

where c is an integration constant. Since $\kappa = \frac{f'(r_h)}{2}$, we rewrite the above equation as

$$W = \pm \frac{i\pi E}{2\kappa} + c \tag{3.21}$$

It means that we have two solutions. One of them belongs to an outgoing scalar particle (moving away from the BH) and the other one possesses to an ingoing particle (moving toward the BH). Namely, the outgoing one has

$$W_{+} = \frac{i\pi E}{2\kappa} + c \tag{3.22}$$

and ingoing one belongs to

$$W_{-} = \frac{-i\pi E}{2\kappa} + c \tag{3.23}$$

According to the definition of the tunneling probability:

$$P = exp[-2\,Im(I)] \tag{3.24}$$

Hence, we read the ingoing and outgoing probabilities at the horizon as follows

$$P_{in} = exp\left[-2\,Im(W_{-})\right] \tag{3.25}$$

and

$$P_{out} = exp\left[-2\,Im(W_{+})\right] \tag{3.26}$$

According to the definition of a classical BH, a BH is a system which must absorb all the particles crossing its horizon. This sentence implies that $P_{in} = 1$. In the other words

$$-2\,Im(W_-) = \frac{\pi E}{\kappa} - 2\,Im(c) = 0 \tag{3.27}$$

That means that the probability is normalized, so that any incoming particle crossing the horizon has a 100% chance of entering the BH. Thus, we understand that

$$Im(c) = \frac{\pi E}{2\kappa} \tag{3.28}$$

If we substitute the above result into Eq. (3.26):

$$P_{out} = exp(-\frac{2\pi E}{\kappa}) \tag{3.29}$$

so that we can derive the tunneling rate from the following definition

$$\Gamma \sim e^{-2Im(I)} = P_{out} = e^{-\frac{E}{T}} \tag{3.30}$$

and

$$e^{-\frac{2\pi E}{\kappa}} = e^{-\frac{E}{T}} \tag{3.31}$$

Therefore

$$T = \frac{\kappa}{2\pi} = \frac{a(r_h - r_0)}{2\pi r_h} \tag{3.32}$$

which is equivalent to the Hawking temperature of the GBH.

Chapter 4

HR OF THE GBH VIA THE DRS METHOD

In this section, we employ the DRS method [10, 11] for calculating the entropy and the horizon temperature of the GBH. The Klein-Gordon (KG) equation in a curved spacetime [12] is given by

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}(\sqrt{-g}g^{\mu\nu}\frac{\partial\Phi}{\partial x^{\nu}})-\eta^{2}\Phi=0 \tag{4.1}$$

where η is the mass of the scalar particle (bosons). If we use the above equation for the metric described in Eqs. (3.2) and (3.3) with

$$\sqrt{-g}=r^{2}\sin\theta \tag{4.2}$$

we have

$$\frac{1}{r^{2}\sin\theta}\partial_{t}\left[r^{2}\sin\theta g^{tt}\partial_{t}\Phi\right]+$$

$$\frac{1}{r^{2}\sin\theta}\partial_{r}\left[r^{2}\sin\theta g^{rr}\partial_{r}\Phi\right]+$$

$$\frac{1}{r^{2}\sin\theta}\partial_{\theta}\left[r^{2}\sin\theta g^{\theta\theta}\partial_{\theta}\Phi\right]+$$

$$\frac{1}{r^{2}\sin\theta}\partial_{\phi}\left[r^{2}\sin\theta g^{\phi\phi}\partial_{\phi}\Phi\right]-\eta^{2}\Phi=0 \tag{4.3}$$

Then, we get the resulting equation as

$$\left[-r^2 f^{-1}(r)\partial_{tt} + \partial_r(r^2\partial_r) + \partial_\theta(\sin\theta\partial_\theta) + \frac{1}{\sin^2\theta}\partial_{\phi\phi}) - \eta^2 r^2\right]\Phi = 0 \qquad (4.4)$$

If we use the method of separation of variables, we may set the Ansatz for the scalar field as

$$\Phi = \frac{1}{(4\pi\omega)^{\frac{1}{2}}}\frac{1}{r}R_\omega(\mathrm{r,t})\,\mathrm{Y}_{\ell m}(\theta,\phi) \qquad (4.5)$$

Thus

$$\Phi_{,tt} = \frac{-f^{-1}Y(r)r^2}{\sqrt{4\pi\omega}}\frac{R_{tt}}{r}$$

$$\Phi_{,rr} = \frac{Y}{\sqrt{4\pi\omega}}\partial_r\left(r^2 f(r)\partial_r\left(\frac{R}{r}\right)\right)$$

$$\Phi_{,\theta\theta} = \frac{1}{\sqrt{4\pi\omega}}\frac{R}{r}\left(cos\,\theta Y_\theta + sin\,\theta Y_{\theta\theta}\right)$$

$$\Phi_{,\phi\phi} = \frac{1}{\sqrt{4\pi\omega}}\frac{R}{r}\mathrm{Y}_{\phi\phi} \qquad (4.6)$$

Now, we can transform the angular part of the equation to the spherical harmonics which has an eigenvalue $L(L+1)$ where L denotes the orbital quantum number. Furthermore, ω becomes the energy of the particle. After a straightforward calculation, we get the radial equation as

$$\left[-r^2 f^{-1}\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t}\left(r^2 f\partial_r\right) - \eta^2 r^2\right]\frac{R_\omega}{r} = -L(L+1)\frac{R_\omega}{r} \qquad (4.7)$$

Introducing $\Re = \frac{R_\omega}{r}$, $s = r^2$, $y = sf$ and $\lambda = L\left(L+1\right)$, Eq. (4.7) reduces to

$$\left[-sf^{-1}\frac{\partial^2}{\partial t}+\frac{\partial}{\partial r}(y)\frac{\partial}{\partial r}-\eta^2 s\right]\Re=-\lambda\Re \tag{4.8}$$

In order to change this equation to a Schrödinger like wave equation which is the so-called Zerilli equation [18], we introduce the tortoise coordinate:

$$dr_*=\frac{dr}{f} \tag{4.9}$$

Whence

$$\partial_r=f^{-1}\partial_{r_*}=f^{-2}\left(\partial^2_{r_*}-f\,'\partial r_*\right) \tag{4.10}$$

by which

$$\partial_r\left(y\,\partial_r\right)=y\,'\partial_r+y\,\partial r^2$$

$$=\frac{y\,'}{f}\partial r_*+\frac{y}{f^{\,2}}\left(\partial^2_{r_*}-f\,'\partial r_*\right) \tag{4.11}$$

Thus, we can rewrite the Eq. (4.8) as

$$\left[-\frac{\partial^2}{\partial t}+\frac{y'}{s}\partial_{r_*}-\frac{yf\,'}{sf}\partial_{r_*}+\frac{y}{sf}\frac{\partial^2}{\partial_{r_*}}-f\eta^2\right]\Re=-\frac{\lambda f}{s}\Re \tag{4.12}$$

Let us simplify this equation. To this end, one may set

$$\left(\frac{y\,'}{s}-\frac{yf}{sf}\right)\partial r_*=\left(\frac{y\,'}{s}-f\,'\right)\partial r_*$$

$$=f\,\frac{s'}{s}\partial r_* \tag{4.13}$$

Thus we have

$$\left[-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial^2_{r_*}}+\frac{s'}{s}f\partial r_*-\eta^2 f\right]\Re=-\frac{\lambda}{s}f\Re \tag{4.14}$$

Near the horizon $r \to r_h$ or $f \to 0$, the above wave equation reduces to

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial^2_{r_*}}\right)\Re=0 \tag{4.15}$$

Its solution gives us both the ingoing wave

$$\Re^{in}=e^{-i\omega(t+r_*)} \tag{4.16}$$

and the outgoing one

$$\Re^{out}=e^{-i\omega(t-r_*)} \tag{4.17}$$

On the other hand, GBH spacetime has singularity at the horizon. So we may transform it to a new coordinate system which is non-singular at r_h. For this goal, we introduce the advanced Eddington-Finkelstein coordinate system [19,20] which is performed by

$$v=t+r_* \tag{4.18}$$

By using the tortoise coordinate given in Eq. (4.9), we get a relationship as follows

$$dt^2=dv^2+dr_*^2-2dv dr_* \tag{4.19}$$

After substituting Eqs. (4.18) and (4.19) in the line-element (2.42), the GBH metric becomes

$$ds^2=-fdv^2-2fdvdr_*+r^2d\Omega^2 \tag{4.20}$$

recalling that $f dr_* = dr$

$$ds^2 = -f d\nu^2 - 2 d\nu dr + r^2 d\Omega^2 \tag{4.21}$$

As it can be seen above, the coordinate singularity appearing at the horizon is removed. Thus, this coordinate transformation leads to modify the solutions of ingoing and outgoing waves at the event horizon i.e., Eqs. (4.17) and (4.18) as follows

$$\Re^{out} = e^{2i\omega r_*} e^{-i\omega\nu} \tag{4.22}$$

$$\Re^{in} = e^{-i\omega\nu} \tag{4.23}$$

We are only interested in the outgoing waves. But it is vanishes at the horizon. This can be best seen from the following calculations.

According to the definition of tortoise coordinate $r_* = \int \frac{dr}{f}$, we get

$$r_* = \frac{1}{2a} \int \frac{r dr}{\left(r - r_h\right)\left(r - r_\circ\right)} \tag{4.24}$$

By solving this integral, one obtains the result as the following

$$r_* = \frac{1}{2a\left(r_h - r_0\right)} \left[r_h \, ln\left(\frac{r - r_h}{r_h}\right) - r_0 \, ln(r - r_0) \right] \tag{4.25}$$

and

$$exp(2i\omega r_*) = exp\left[ln\left(\frac{(\frac{r}{r_h} - 1)^{r_h}}{(r - r_0)^{r_0}} \right) \right]^{\frac{i\omega}{\kappa r_h}} \tag{4.26}$$

$$exp\left(2i\omega r_*\right)=\left(\frac{r}{r_h}-1\right)^{\frac{i\omega}{\kappa}}\left(r-r_0\right)^{-\frac{i\omega r_0}{\kappa r_h}} \tag{4.27}$$

Finally, by using the above equation we can rewrite outgoing wave solution (4.22) as

$$\Re^{out}=\left(\frac{r-r_h}{r_h}\right)^{\frac{i\omega}{\kappa}}\left(r-r_0\right)^{-\frac{i\omega r_0}{\kappa r_h}}e^{-i\omega v} \tag{4.28}$$

As mentioned above it is terminates at the horizon. Therefore, its present form describes only the outgoing particles outside the horizon and cannot represent the ones belonging to the inside of the horizon. In order to include the particle behavior at the inside of the horizon, we analytically extend Eq. (4.28) into the horizon as follows

$$\left(r-r_h\right)\rightarrow\left|r-r_h\right|e^{-i\pi}=\left(r_h-r\right)e^{-i\pi} \tag{4.29}$$

Thus, we express the outgoing wave inside the horizon as

$$\Re^{out}\left(r<r_h\right)=\left[\frac{\left(r_h-r\right)e^{-i\pi}}{r_h}\right]^{\frac{i\omega}{\kappa}}\left(r-r_\circ\right)^{-\frac{i\omega r_0}{\kappa r_h}}e^{-i\omega v}$$

$$=e^{\pi\frac{\omega}{\kappa}}e^{2i\omega r_*}e^{-i\omega v} \tag{4.30}$$

With the aid of the Eqs. (3.54) and (3.64), the emission rate of the outgoing particles reads

$$\Gamma=\left|\frac{\Re^{out}\left(r>r_h\right)}{\Re^{out}\left(r<r_h\right)}\right|^2=\left|\frac{e^{2i\omega r_*}e^{-i\omega v}}{e^{\pi\frac{\omega}{\kappa}}e^{2i\omega r_*}e^{-i\omega v}}\right|^2=e^{-\frac{2\pi\omega}{\kappa}} \tag{4.31}$$

So it reveals that

$$\Gamma = e^{-\frac{2\pi\omega}{\kappa}} = e^{-\frac{\omega}{T}}$$

$$T = \frac{\kappa}{2\pi} \tag{4.32}$$

the Hawking temperature of the GBH (3.32) is impeccably recovered via the DRS method.

Chapter 5

CONCLUSION

In this thesis, we have derived the emission rate from the GBH by using three different methods: the PW method, the HJ method and the DRS method. Firstly, we have employed the PW method. For this purpose, the energy conservation and self-gravitational effects during the HR of the GBH have been taken into account. As a result, we have found that the GBH's HR is no longer a purely thermal spectrum. This is meant the information conservation. For the sake of QM, the obtained result is consistent with the unitarity. Secondly, the HJ method has been considered. According to this approach, the action of the tunneling particle of the GBH is found by the relativistic HJ equation. Consequently, it has been featly shown that the action yields the true horizon temperature of the GBH, i.e., Eq. (3.32). Finally, we have applied the DRS method to study the HR of scalar particles of the GBH. In accordance with this purpose, we have used the KG equation in the GBH geometry. It has been revealed that the outgoing wave solution (3.54) is logarithmically singular at the horizon. After extending it to the inside of the BH by performing an analytical continuation, we have managed to find out the tunneling rate of the radiating scalar particles. From the thermal spectrum (3.63), we can obtain the Hawking temperature of the GBH, which takes the same form as Eq. (3.32).

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