# Basic Iterative Methods for Solving Elliptic Partial Differential Equation

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Approval of the Institute of Graduate Studies and Research

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We certify that we have read this thesis and that in our opinion; it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science in Mathematics.

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# ABSTRACT

In this thesis, we studied the numerical techniques for the solution of two dimensional Elliptic partial differential equations such as Laplace's and Poisson's equations. These types of differential equations have specific applications in physical and engineering models. The discrete approximation of both equations is based on finite difference method. In this research, five points finite difference approximation is used for Laplace's and Poisson's equations. To solve the resulting finite difference approximation basic iterative methods; Jacobi, Gauss-Seidel and Successive Over Relaxation (SOR) have been used.

Several model problems are solved by three different iterative methods and concluding remarks are presented.

**Keywords:** Elliptic partial differential equation, 5 point's finite difference scheme, basic iterative methods.

# ÖZ

Yapılan bu çalışma iki boyutlu Eliptik parçalı diferansiyel denklem problemlerinin sayısal analiz teknikleri kullanılarak yaklaşık olarak çözülmesi ile ilgilidir. Eliptik parçalı diferansiyel denklemler beş noktalı sonlu farklar yöntemi kullanılarak Poisson's ve Laplace denklemlerine uygulanmış ve bu denklemler temel iteratif çözüm prosedürü olan Jacobi, Gauss Seidel ve SOR yineleme yöntemleri kullanılarak iki farklı problem üzerinde nümerik olarak çözülmüştür. Ayrıca temel iteratif çözüm prosedürü teorik olarak incelenmiştir.

**Anahtar kelimeler:** Eliptik parçalı diferansiyel denklem, beş noktalı sonlu farklar şeması, temel iteratif yöntemler.

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# INTRODUCTION

## 1.1 Partial Differential Equation and their Classification

The solution of partial differential equations is the most important topic in all areas of science and engineering (e.g. in Heat transfer, fluid dynamics, quantum mechanics, boundary layer theory etc.). Equation involving one or more partial derivatives of a function of two or more independent variables is called partial differential equations [4].

The solution for PDEs can be classified into two parts: analytical and numerical solutions. The analytical solution finds a function that satisfies the PDE with the given boundary and initial conditions. The numerical solution is used to find an approximate solution since most of PDEs do not have analytical solutions. Usually, to obtain analytical solution is very difficult, instead of that through standard iterative methods the computers are used to get numerical approximation solution.

Many problems are controlled by lots of different partial differential equations. Consequently, the general form for linear PDEs of a second-order in more than two independent variables $x, y$ is

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0, \qquad (1.1)$$

where the coefficients $A, B, C, D, E, F$ and $G$ are constants or are functions of the independent variables $x$ and $y$. The first three parts are called principle part of PDEs contains the second derivatives [4].

The nature of the general solution to the equation (1.1) is determined by principle part. Indeed, the coefficients of them can be used to classify PDEs. The classification is based on the sign of the discriminate $B^2 - 4AC$ as follows [2]:

   a) If $B^2 - 4AC > 0$ the PDEs is called Hyperbolic. For instance, the wave equation has $A = 1, B = 0, C = -1$. Therefore, it is Hyperbolic.

   b) If $B^2 - 4AC = 0$ the PDEs is called Parabolic. For instance, the heat equation has $A = 1, B = 0, C = 0$. Therefore, it is Parabolic.

   c) If $B^2 - 4AC < 0$ the PDEs is called Elliptic. For instance, the Laplace and Poisson equations have $A = 1, B = 0, C = 1$. So, they are Elliptic.

In practical application, a particular solution for a differential equation requires boundary and / or initial conditions. An appropriate numerical method for the solution of differential problems depends upon the nature of these conditions. If there is only one solution to the differential problem, it is called "properly posed" or "well posed". Otherwise, problem is said to be "ill posed" [5]. In well posed problem, there exists a relatively small change in the solution and we can say that the differential problem is well conditioned. If the change in the solution is large, we say that the problem is "ill conditioned" [3][5].

## 1.2 Elliptic Partial Differential Equation and Types of Boundary Conditions

In this thesis, we are mainly concerned with the linear Elliptic self-adjoint problems. Elliptic partial differential equations arise usually from equilibrium or steady- state problems and represent in many fields of engineering and science[6][5]. The partial differential equations are called self-adjoint Elliptic, if equation (1.1) can be replaced by

$$\frac{\partial}{\partial x}\left(A(x,y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(C(x,y)\frac{\partial u}{\partial x}\right) + Fu + G = 0 \qquad (1.2)$$

The self-adjoint Elliptic partial differential equation (1.2) is said to be Poisson's equation, when $A(x,y) = C(x,y) = 1$ and $F = 0$ in the domain $\Omega$ and it is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -G(x,y), \qquad (1.3)$$

and equation (1.2) is Laplace's equation, when $G(x,y) = 0$ in (1.3) , it is in the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 . \qquad (1.4)$$

The solution $u(x,y)$, satisfies the boundary conditions on $\partial\Omega$ , where $\partial\Omega$ is the boundary of domain $\Omega$ on the square region

$$\Omega = \{(x,y) \mid a < x < b , a < y < b\},$$

as shown in Figure (1.1).
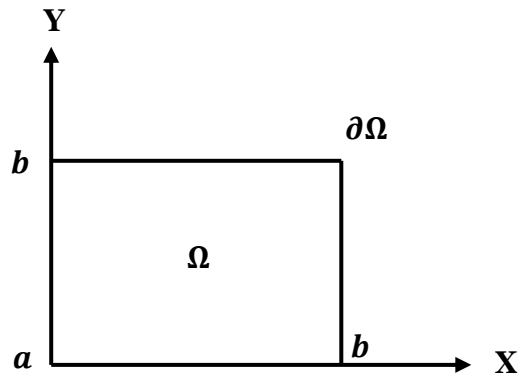
Figure 1.1: General characteristics of Elliptic PDFs

In generally, three distinct problems involving equations $(1.2), (1.3)$ and $(1.4)$ arise depending on the boundary conditions prescribed on $\partial\Omega$ [3][6].

### a) Dirichlet condition

The value of the function is prescribed, when $u(x, y) = g(x, y)$ on the boundary $\partial\Omega$

### b) Neumann condition

The value of the derivative normal to the boundary is prescribed, when $\frac{\partial u}{\partial n} = v(x, y)$ on the boundary $\partial\Omega$.

### c) Robin's condition

A combination of the function and its normal derivative is specified on the boundary, $au(x, y) + b\frac{\partial u}{\partial n}$ is prescribed on the boundary $\partial\Omega$.

There are different methods to find numerical solution such as finite element, finite difference and finite volume methods. Finite difference method is going to be evolved in the next Chapter in detail.

In Chapter 3, we presented a detailed analysis for the solution of sparse linear systems using three basic iterative methods: Jacobi, Gauss Seidel and Successive Over Relaxation (SOR).

In Chapter 4, we presented the numerical results from solving two model problems and concluding remarks are given for each model problem.

In Chapter 5, we presented overall chapters' conclusion.

# Chapter 2

# SOUTION OF ELLIPTIC PDEs BY FINITE DIFFERENCE METHOD

The finite difference method is one of the several techniques for the numerical solution of partial differential equations. The method is based on discrete approximation of the partial derivatives in partial differential equations obtained by Taylor's expansion near the point of interests [2].

In this Chapter, the finite difference method for the solution of the Elliptic partial differential equations is discussed.

## 2.1 Basic Approximations

Assume that three points on $x - axis$ separated by a distance $h$, as shown in Figure$(2.1)(a)$, and consider the value of the function $u(x, y)$ at these three points $u(x - h, y), u(x, y), u(x + h, y)$ denoted by $u_{i-1}$, $u_i$ and $u_{i+1}$ respectively [8].
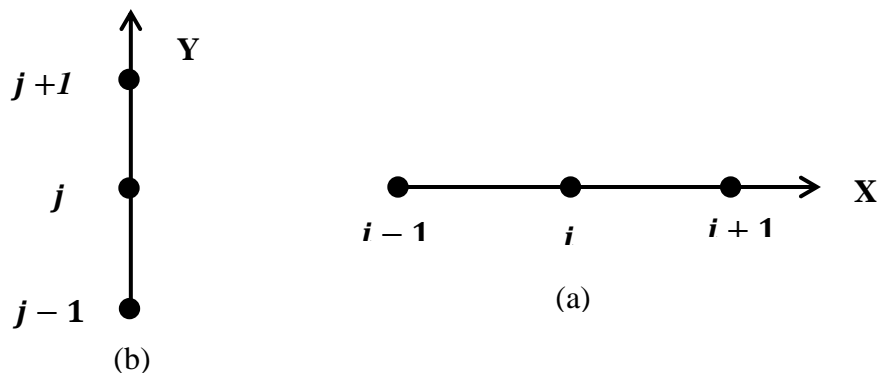


Figure 2.1: Finite Difference along x and y

We can write the two Taylor's expansion for $u_{i-1}$ and $u_{i+1}$ respectively as follows.

$$u_{i-1} = u_i - \frac{\partial u}{\partial x}\Big|_{x_i} h + \frac{\partial^2 u}{\partial x^2}\Big|_{x_i} \frac{h^2}{2!} - \frac{\partial^3 u}{\partial x^3}\Big|_{x_i} \frac{h^3}{3!} + \frac{\partial^4 u}{\partial x^4}\Big|_{x_i} \frac{h^4}{4!} - O(h^5) \tag{2.1}$$

$$u_{i+1} = u_i + \frac{\partial u}{\partial x}\Big|_{x_i} h + \frac{\partial^2 u}{\partial x^2}\Big|_{x_i} \frac{h^2}{2!} + \frac{\partial^3 u}{\partial x^3}\Big|_{x_i} \frac{h^3}{3!} + \frac{\partial^4 u}{\partial x^4}\Big|_{x_i} \frac{h^4}{4!} + O(h^5), \tag{2.2}$$

where $\Big|_{x_i}$ means that the derivative is computed at the point $x_i$, adding equations (2.1) and (2.2) side by side, we get;

$$u_{i-1} + u_{i+1} = 2u_i + \frac{\partial^2 u}{\partial x^2}\Big|_{x_i} h^2 + \frac{\partial^4 u}{\partial x^4}\Big|_{x_i} \frac{h^4}{12} + O(h^5). \tag{2.3}$$

After some rearrangement, equation (2.3) reduces to,

$$\frac{\partial^2 u}{\partial x^2}\Big|_{x_i} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2) \tag{2.4}$$

The right hand side of equation (2.4) is called the second order accurate central difference approximation of second derivative. Subtracting equation (2.1) from equation (2.2), we get

$$u_{i+1} - u_{i-1} = 2\frac{\partial u}{\partial x}\Big|_{x_i} h + \frac{\partial^3 u}{\partial x^3}\Big|_{x_i} \frac{h^3}{3} + O(h^5) \tag{2.5}$$

or

$$\frac{\partial u}{\partial x}\Big|_{x_i} = \frac{u_{i+1} - u_{i-1}}{2h} + O(h^2) \tag{2.6}$$

[2][8].

Equation (2.6) is called second order accurate central difference approximation of first derivative. In the similar way, we can take three points along the $y - axis$ by a distance $h$, as shown in Figure(2.1)(b). We get

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{y_j} = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + O(h^2) \qquad (2.7)$$

and

$$\left.\frac{\partial u}{\partial y}\right|_{y_j} = \frac{u_{j+1} - u_{j-1}}{2h} + O(h^2) \qquad (2.8)$$

By combining the arguments of Figure(2.1)(a) and (2.1)(b), we obtain the five point stencil in Figure(2.2) and $u$ gets two subscripts, one of them for the $i$ on the $x$ direction and the other for $j$ on the $y$ direction. By combining the equations (2.4) and (2.7), we write

$$\left.\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right|_{i,j} \cong \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} \qquad (2.9)$$
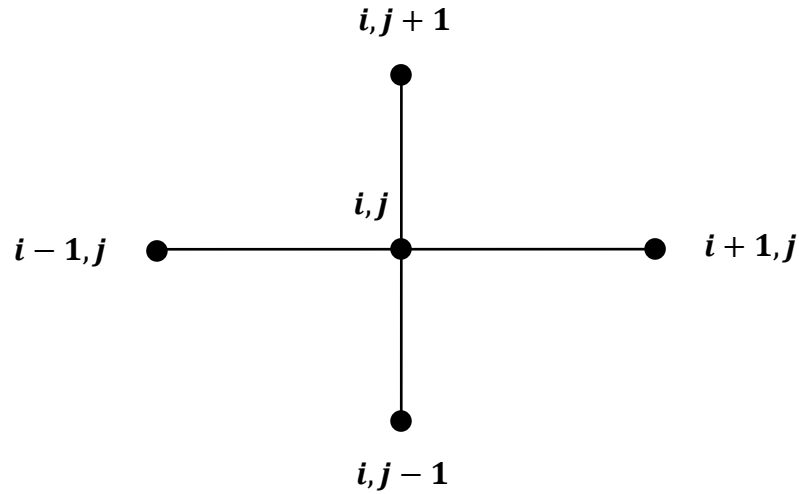


Figure 2.2: Five point stencil

## 2.2 Difference Schemes for the Elliptic PDE's

Consider Laplace's equation and Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{2.10}$$

and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \tag{2.11}$$

defined over $\Omega = \{(x, y)/0 < x, y < 1\}$ respectively with Dirichlet boundary conditions

$$u(x, y) = g(x, y), \quad \text{for all } (x, y) \in \partial\Omega. \tag{2.12}$$

Using square grid and by implementing the five point approximation scheme Figure (2.3), the finite difference approximation for Laplace's and Poisson's equation at the point $i, j$ has the following form;

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0 \tag{2.13}$$

and

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f_{ij} \tag{2.14}$$

Equations (2.13) and (2.14) are called the five point finite difference approximation for Laplace's and Poisson's equation respectively. The five point stencil of these approximations is shown in Figure (2.3) [8][9].
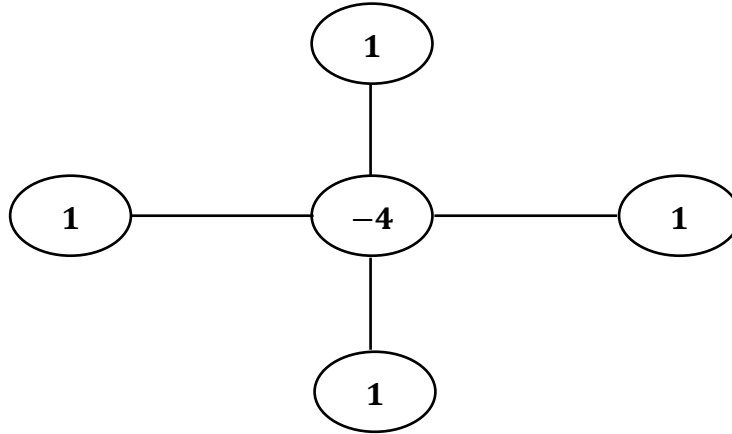
Figure 2.3: Five point stencils for the Laplace and Poisson equation

Equations (2.13) and (2.14) are then assembled into a linear system

$$Au = b_1 \quad and \quad Au = b_2 \,. \tag{2.15}$$

If the difference equations are taken in the natural order of the points, the coefficient matrix $A$ is

$$A = \begin{bmatrix} B & C & 0 \\ C & B & C \\ 0 & C & B \end{bmatrix}$$

where

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$$
B = \begin{bmatrix}
-4 & 1 & . & . & . & . & . & 0 \\
1 & -4 & 1 & & & & & . \\
. & 1 & -4 & 1 & & & & . \\
. & & 1 & . & & & & . \\
. & & & & . & & & . \\
. & & & & . & 1 & & . \\
. & & & & 1 & -4 & 1 & \\
0 & . & . & . & . & 1 & -4
\end{bmatrix}
\quad and \quad
C = \begin{bmatrix}
1 & 0 & . & . & . & . & . & 0 \\
0 & 1 & 0 & & & & & . \\
. & 0 & 1 & 0 & & & & . \\
. & & 0 & . & & & & . \\
. & & & & . & & & . \\
. & & & & . & 0 & & . \\
. & & & & 0 & 1 & 0 & \\
0 & . & . & . & . & . & 0 & 1
\end{bmatrix}
$$

The structure of the matrix $A$ is sparse matrices with 5 nonzero diagonals [8].

## 2.3 Boundary conditions

For Elliptic PDEs, there are given boundary conditions where in some property of $u$ is specified $\partial\Omega$ with either Laplace's or Poisson's equations. We can define three types of boundary conditions [3][6].

### a) Dirichlet condition

When the node of grid $u_{i,j}$ is close to the boundary condition. Then this node is either close to one boundary node or two boundary nodes. See Figure(2.4).
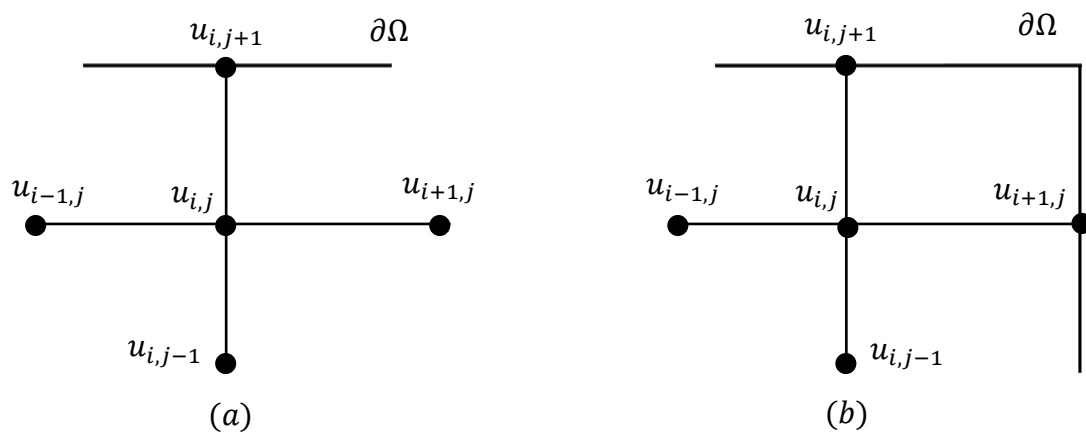


Figure 2.4: 5 points stencil close to boundary

In the Figure $(2.4)(a)$ the boundary node $\left(u_{i,j+1}\right)$ is known value (given). So, the

five points of finite differences for Laplace's equation can be written as:

$$u_{i+1,j} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = -u_{i,j+1}, \qquad (2.16)$$

and for the Poisson's equation can be written as:

$$u_{i+1,j} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = h^2 f_{ij} - u_{i,j+1}. \qquad (2.17)$$

Also, in the Figure $(2.4)(b)$ the boundary nodes $\left(u_{i,j+1}, u_{i+1,j}\right)$ are known the values

(given) [3]. So, the five points of finite differences for the Laplace's equation can be

written as:

$$u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = -u_{i,j+1} - u_{i+1,j}, \qquad (2.18)$$

and for the Poisson's equation can be written as:

$$u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = h^2 f_{ij} - u_{i,j+1} - u_{i+1,j} \qquad (2.19)$$

**b) Neumann condition:**

When $\partial u/\partial n = v(x, y)$ is on the boundary $\partial\Omega$, then when using the 5 point

difference scheme, three of the node lie on the boundary line, one inside the

boundary at $u_{i-1,j}$ and the fifth at $u_{i+1,j}$, we consider a fictitious grid point outside

the domain as shown in Figure (2.5) [3].
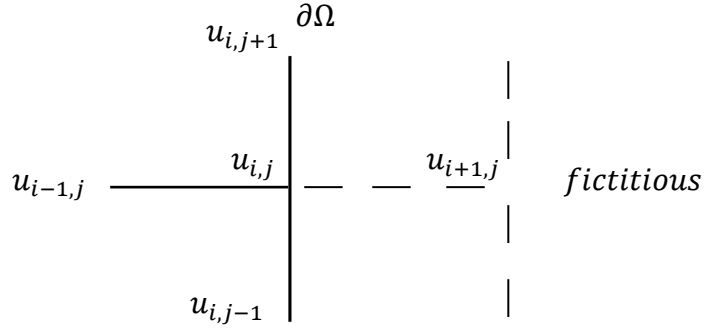
Figure 2.5: Fictitious grid point outside the domain $\partial\Omega$

To find the node $u_{i,j}$ we use the central differences approximation:

$$v(x,y) = \frac{u_{i+1,j} - u_{i-1,j}}{2h} \tag{2.20}$$

we rewrite equation (2.20) as

$$u_{i+1,j} = u_{i-1,j} + 2hv_{ij} \tag{2.21}$$

So, the five points of finite differences for Laplace's equation can be written as:

$$-2u_{i-1,j} - u_{i,j+1} - u_{i,j-1} + 4u_{i,j} = 2hv_{ij} \tag{2.22}$$

and for Poisson's equation can be written as:

$$-2u_{i-1,j} - u_{i,j+1} - u_{i,j-1} + 4u_{i,j} = 2hv_{ij} - h^2 f_{ij} \tag{2.23}$$

For Laplace's equation, the totality of equations at the $(M+1)^2$ grid points of the square leads to the matrix equation [3]

$$Au = 2hv \tag{2.24}$$

where $A$ is a matrix of order $(M+1)^2$ given by

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$$
A = \begin{bmatrix}
K & -2I & & & & & 0 \\
-I & K & . & & & & \\
& . & . & . & & & \\
& & . & . & . & & \\
& & & . & . & . & \\
& & & & . & . & . \\
& & & & . & K & -I \\
0 & & & & & -2I & K
\end{bmatrix}
$$

With $I$ the identity matrix of order $(M + 1)$, and $K$ matrix of order $(M + 1)$ given by

$$
K = \begin{bmatrix}
4 & -2 & & & & & 0 \\
-1 & 4 & . & & & & \\
& . & . & . & & & \\
& & . & . & . & & \\
& & & . & . & . & \\
& & & & . & . & . \\
& & & & . & 4 & -1 \\
0 & & & & & -2 & 4
\end{bmatrix}
$$

The vectors $u$ and $v$ of equation $(2.24)$ are respectively given by:

$$
u = \left[u_{0,0}, \dots, u_{N,0}, u_{0,1}, \dots, u_{N,1}, \dots, u_{0,N}, \dots, u_{N,N}\right]^T
$$

$$
v = \left[2v_{0,0}, v_{1,0}, \dots, 2v_{N,0}, v_{0,1}, 0, \dots, 0, v_{N,1}, \dots, 2v_{0,N}, v_{1,0}, \dots, v_{N-1,N}, 2v_{N,N}\right]^T [3].
$$

### c) Robin condition:

The boundary conditions of the form $(2.13)$ and $(2.14)$ can be incorporated into the difference equations for the boundary nodal points by an extension of the methods outlined in the Dirichlet and Neumann problems $[6]$.

# Chapter 3

# SOLUTION TECHNIQUES FOR ELLIPTIC PDE's

## 3.1 Introduction

In Chapter 2, we saw that the discretization of an Elliptic partial differential equation led to the solution of a large sparse linear algebraic system. In this Chapter, our target is to solve linear algebraic system

$$Au = b, \tag{3.1}$$

generated from the discretization of an Elliptic partial differential, where $A$ is large, sparse and typically positive definite $n \times n$ matrix. This means that relatively few entries of positive definite matrix $A$ are nonzero. Methods of solution equation (3.1) belong essentially to the class of iterative methods and such that if a direct solver such as Gauss elimination is applied to the sparse system, we introduce a large number of additional entries into the coefficient matrix. These fill in values will destroy the sparse structure of the problem and those increase the storage requirements significantly [8].

As an alternative, to the direct methods, we may apply an iterative solver to the system of equations (3.1). Such methods start with initial guess $u^{(0)}$ for the solution $u = A^{-1}b$ and compute a sequence of approximation $\{u^{(k)}\}$ which hopefully converges to $u$. When using an iterative method, the coefficient matrix $A$ is involved only in terms of matrix by vector products. Thus, there is no need for storing the zero

entries of $A$, and is possible to implement algorithms that are extremely cost effective with respect to computing time as well as storage [2][8]. Another advantage of an iterative procedure is that does not accumulate rounding in the same way as a direct method: traditionally, iterative methods for problems like equation (3.1) have been closely connected to splitting of the coefficient matrix $A$ as follows;

Consider the splitting

$$A = M - N \tag{3.2}$$

Where $M$ and $N$ are $n \times n$ matrices and $M$ is nonsingular and easily invertible. Using the splitting equation (3.2) we may write the original system equation (3.1) is.

$$(M - N)u = b$$

$$Mu = Nu + b$$

$$u = (M^{-1}N)u + M^{-1}b$$

$$u = Gu + \tilde{b} \tag{3.3}$$

The iterative form is

$$Mu^{(k+1)} = Nu^{(k)} + b \tag{3.4}$$

which is equivalently as

$$u^{(k+1)} = (M^{-1}N)u^{(k)} + M^{-1}b \tag{3.5}$$

or

$$u^{(k+1)} = Gu^{(k)} + \tilde{b} \tag{3.6}$$

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Where $G = (M^{-1}N) \ and \ \tilde{b} = M^{-1}b$.

Then the recursive error equation generated from subtracting equations (3.3) and (3.6).

$$u - u^{(k+1)} = Gu + \tilde{b} - Gu^{(k)} - \tilde{b}$$

$$u - u^{(k+1)} = G(u - u^{(k)})$$

$$e_{k+1} = Ge_k \tag{3.7}$$

Where

$e_k = u - u^{(k)}$, and $u$ is the exact solution of equation (3.1) [8][9].

Definition and theorem are given according to following;

1) Directed graph, connectedness and irreducibility.

2) M-matrices.

3) Regular splitting.

**Definition 3.1: (graph of a matrix)** [10]

Let $A \in \mathbb{R}^{n \times n}$ be matrix corresponding to index set $n$. The following subset of all pairs from $n \times n$ is denoted as the graph $G(A)$ of the matrix $A$.

$$G(A) = \{(i, j) \in n \times n : a_{ij} \neq 0\} \qquad \qquad \square$$

**Definition 3.2:** [1]

A directed graph is said to be strongly connected if to each ordered pair of disjoint points $p_i$, $p_j$ there exists a directed path in the graph, $\overrightarrow{p_{i_0} p_{i_1}}, \overrightarrow{p_{i_1} p_{i_2}}, \dots \dots, \overrightarrow{p_{i_{r-1}} p_{i_r}}$ with $i_0 = i$, $i_r = j$ $\square$

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**Theorem 3.1:** [1]

A matrix is irreducible if and only if its directed graph is connected. □

According from the definition $(3.1 - 3.2)$ and theorem $(3.1)$ we may conclude that the directed graph $G(A)$ is strongly connected and also irreducible. □

In the following we will outline some of the special type of matrices and their characteristics.

**Definition 3.3: (M-matrix)** [1] [11]

A matrix $A = (a_{ij})$ is an M-matrix if $a_{ij} \leq 0$ for all $i \neq j, A$ is nonsingular and $A^{-1} \geq 0$. If $A$ is irreducible and $|a_{ii}| \geq \sum |a_{ij}|$ with strict inequality for at least on $i$, then $A$ is an M-matrix □

**Definition 3.4: (diagonally dominant and irreducible diagonally dominant)** [11]

The matrix $A$ is said to be strictly diagonally dominant if

$$\sigma_i = \sum_{i \neq j}^{n} |a_{ij}| < |a_{ii}|$$

Diagonally dominant if

$$\sigma_i = \sum_{i \neq j}^{n} |a_{ij}| \leq |a_{ii}|$$

And irreducible diagonally dominant if $A$ is irreducible and

1) $\sigma_i = |a_{ii}|$.

2) $\sigma_k < |a_{kk}|$ at least one index $k$. □

**Definition 3.5: (L-matrix)** [10]

A matrix $A = (a_{ij})$ is an L-matrix if $a_{ii} > 0$ $and$ $a_{ij} \leq 0$ for $i, j = 1, 2, \ldots \ldots, n.$ □

**Lemma 3.1:** [1]

A is monotone matrix if and only if $A$ is nonsingular and $A^{-1} \geq 0$ □

**Theorem 3.2:** [1][10]

If $A = [a_{ii}]$ is a real $n \times n$ matrix with $a_{ij} \leq 0$ for all $i \neq j$, then the following are equivalent.

1. $A$ is nonsingular and $A^{-1} > 0$

2. The diagonal entries of $A$ are positive real numbers □

If we consider the matrix $A$ in equation (3.1), we can say that matrix $A$ is $M$ and $L$ matrix, irreducible diagonally dominant and also monotone according to definitions $(3.3 - 3.4)$ lemma (3.1) and theorem (3.2).

We turn now to the question of convergence of the basic iterative method equation (3.6)

**Definition 3.6:** [1]

Let $M, N \in \mathbb{R}^{n \times n}$. Then $A = M - N$ is called

1) A regular splitting if $M$ is monotone and $N \geq 0$.

2) A convergent splitting if $M$ is nonsingular and $\rho(M^{-1}N) < 1.$ □

A necessary and sufficient condition for convergence of the iterative method equatin (3.6) or corresponding formula equation (3.7) is $\rho(M^{-1}N) < 1$ [1][10][11]. Illustrate the necessary and sufficient conditions for convergence of the method given by equatin (3.6).

**Theorem 3.3:** [10]

If $A = M - N$ is a regular splitting of the matrix $A$ and $A^{-1} \geq 0$ then

$$\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1 + \rho(A^{-1}N)} < 1 \qquad (3.8)$$

Thus the matrix $M^{-1}N$ is convergent and iterative method of equation (3.6) converges for any initial vector $u^{(0)}$.

**Proof:**

Assume that $A$ is nonsingular with $A^{-1} \geq 0$, since

$$A = M - N \Longrightarrow M = A + N = N(I + N^{-1}A)$$

$$M = N(I + N^{-1}A) \Longrightarrow N^{-1}M = (I + N^{-1}A)$$

$$(N^{-1}M)^{-1} = (I + N^{-1}A)^{-1} \Longrightarrow M^{-1}N = (I + N^{-1}A)^{-1}$$

$$M^{-1}N = [N^{-1}(N + A)]^{-1} = (N + A)^{-1}N$$

$$M^{-1}N = [A(A^{-1}N + I]^{-1}N = [A^{-1}N + I]^{-1}A^{-1}N$$

$$M^{-1}N = [A^{-1}N + I]^{-1}A^{-1}N$$

Let     $G = A^{-1}N$

$$M^{-1}N = (G + I)^{-1}G$$

Because this relation if $x$ is any eigenvector of $G$ corresponding to the eigenvalue $\lambda$,

Then $1 + \lambda \neq 0$

$$Gx = \lambda x$$

and

$$(G + I)^{-1}Gx = \frac{\lambda}{1 + \lambda}x$$

If follows that $x$ is also eigenvector of $M^{-1}N$ corresponding to the eigenvalue $\mu$ given by

$$(M^{-1}N)x = \mu x = (G + I)^{-1}Gx = \frac{\lambda}{1 + \lambda}x$$

$$\Rightarrow \quad \mu = \frac{\lambda}{1 + \lambda}$$

We can maximize $\mu$ by choosing $\lambda = \rho(G)$ then

$$\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1 + \rho(A^{-1}N)} < 1 \qquad \square$$

## 3.2 Basic Linear iterative methods

There are several iterative methods on the splitting of the coefficient matrix $A$. In this section we introduce classical iterative schemes, namely Jacobi, Gauss-Seidel and SOR methods. For example, if the diagonal entries of the matrix $A$ as the matrix sum

$$A = D - L - U \tag{3.9}$$

Where $D = diag(A) = \{a_{11}, a_{22}, \dots \dots, a_{nn}\}$ and $L$ and $U$ are respectively strictly lower and upper triangular $n \times n$ matrices. Then the following choices are available based on the matrix splitting equation (3.9) for three classical methods [8].

### 3.2.1 Jacobi and Gauss-Seidel Method

Consider a linear system $Ax = b$ equation (3.1), using equation (3.9), we write equation (3.1) in the form

$$(D - L - U)u = b \tag{3.10}$$

$$u = D^{-1}(L + U)u + D^{-1}b \tag{3.11}$$

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and consider the iteration

$$u^{(k+1)} = D^{-1}(L + U)u^{(k)} + D^{-1}b \qquad (3.12)$$

This has the form of equation (3.6) with

$$G_j = D^{-1}(L + U) \quad , \quad \tilde{b} = D^{-1}b \qquad (3.13)$$

Where $G_j$ is called the iteration matrix or error propagation of Jacobi method and $\tilde{b}$ is constant vector. The scalar form of equation (3.12) is [8][9];

$$u_i^{(k+1)} = -\sum_{j=1,i\neq j}^{n} \frac{a_{ij}}{a_{ii}} u_i^{(k)} + \frac{b_i}{a_{ii}} \quad , \quad i = 1,2,\dots,n. \qquad (3.14)$$

If the equations (3.12) or (3.14) are applied solve the system of finite difference equations for Laplace's equation and Poisson's equation, we obtain the Jacobi iteration formula

$$u_{i,j}^{(k+1)} = \frac{1}{4}\left[u_{i-1,j}^{(k)} + u_{i+1,j}^{(k)} + u_{i,j-1}^{(k)} + u_{i,j+1}^{(k)}\right] \qquad (3.15)$$

and

$$u_{i,j}^{(k+1)} = \frac{1}{4}\left[u_{i-1,j}^{(k)} + u_{i+1,j}^{(k)} + u_{i,j-1}^{(k)} + u_{i,j+1}^{(k)} - h^2 f_{ij}\right], \qquad (3.16)$$

respectively. Updates to the solution at $(i,j)$ are computed as a weighted average of solutions at its four neighboring points. Contrary to solution obtained by direct methods, parallel computational techniques are easily used with Jacobi's method since the solution state at iteration $(k + 1)$ is explicit [9].

Convergence of Jacobi's method is too slow for practical serial computation. Gauss-Seidel iteration uses the latest solution information as soon as it becomes

available [8]. Thus when computing $u_i^{(k+1)}$ according to equation (3.14), we could use latest iterates $u_j^{(k+1)}$ $j = 1,2, \dots, i - 1$ on the right to obtain

$$u_i^{(k+1)} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} u_j^{(k+1)} - \sum_{j=1+1}^{n} \frac{a_{ij}}{a_{ii}} u_j^{(k)} + \frac{b_i}{a_{ii}} \quad , \qquad i = 1,2, \dots, n \tag{3.17}$$

In the matrix form of equation (3.10), this is equivalent to

$$(D - L)u^{(k+1)} = Uu^{(k)} + b \tag{3.18}$$

or

$$u^{(k+1)} = (D - L)^{-1}Uu^{(k)} + (D - L)^{-1}b \tag{3.19}$$

This has the form of equation (3.6) with

$$G_{Gs} = (D - L)^{-1}U \quad , \qquad \tilde{b} = (D - L)^{-1}b \tag{3.20}$$

Where $G_{Gs}$ is called the iteration matrix or error propagation of Gauss-Seidel method and $\tilde{b}$ is constant vector. If the equations (3.17) or (3.19) are applied solve the system of finite difference equations for Laplace's equation and Poisson's equation, we obtain

$$u_{i,j}^{(k+1)} = \frac{1}{4}\left[u_{i-1,j}^{(k+1)} + u_{i,j-1}^{(k+1)} + u_{i+1,j}^{(k)} + u_{i,j+1}^{(k)}\right] \tag{3.21}$$

and

$$u_{i,j}^{(k+1)} = \frac{1}{4}\left[u_{i-1,j}^{(k+1)} + u_{i,j-1}^{(k+1)} + u_{i+1,j}^{(k)} + u_{i,j+1}^{(k)} - h^2 f_{i,j}\right] \tag{3.22}$$

respectively.

In section (3.2), we showed that the matrix $A$ generated from the 5 point finite difference approximation is an M-matrix and irreducible diagonally dominant. According to theorem (3.3) and the following theorems both Jacobi and Gauss-Seidel methods are convergent.

**Theorem 3.4:** [1]

Let $A$ be an M-matrix. Then both $M_j = D$ and $M_{Gs} = D - L$ result in regular splitting. Furthermore

$$\rho((D - L)^{-1}U) < \rho(D^{-1}(L + U)) < 1. \qquad \text{holds.} \square \quad (3.23)$$

**Theorem 3.5:** [1][10]

If a matrix $A$ is strictly diagonally dominant or irreducibly diagonally dominant, then the associated Jacobi and Gauss-Seidel iterations converge for any $u^{(0)}$. $\square$

**Theorem 3.6:** [8]

Suppose that the matrix $A = D - L - U$ has following property for any $c \in R$

$$det(cD - L - U) = det(cD - \delta L - \delta^{-1}U),$$

for all $\delta \in R \backslash \{0\}$. Then the following property holds

If $\lambda \neq 0$ is an eigenvalue of $G_\omega$ for some $\omega \in (0,2)$ then

$$\mu = \frac{\lambda + \omega - 1}{\omega \lambda^{\frac{1}{2}}}. \qquad \square$$

**Theorem 3.7:** [7]

If $A$ is consistently ordered, then

$$\rho(G_{Gs}) = \left[\rho(G_J)\right]^2.$$

That is, Gauss-Seidel iterations converge twice as fast as Jacobi iterations

**Proof:**

From theorem (3.6)

$$\mu = \frac{\lambda + \omega - 1}{\omega \lambda^{\frac{1}{2}}} \qquad \Rightarrow \qquad \lambda + \omega - 1 = \omega \mu \lambda^{\frac{1}{2}}$$

$$\Rightarrow \quad (\lambda + \omega - 1)^2 = \omega^2 \mu^2 \lambda \quad where \quad \omega = 1 \text{ ( Gauss − Seidel)}$$

$$\Rightarrow \quad \lambda^2 = \mu^2 \lambda \quad \Rightarrow \quad \lambda = \mu^2$$

$$\rho(G_{Gs}) = \left[\rho(G_J)\right]^2 \qquad \square \qquad (3.24)$$

### 3.2.2 Successive Over Relaxation (SOR)

The Gauss Seidel method is adapted to the successive over-relaxation iterative method in which extrapolation parameter omega is introduced and each component is successively changed to the form of a weighted average between both, the former iterate and the computed Gauss-Seidel iterate. By an order of magnitude, the (SOR) iterative method converges more quickly than Gauss Seidel iterative method [8][9].

**Definition 3.7:** [8]

Consider $\tilde{u}$ as an approximation to the solution of the linear system defined by $Au = b$, for this system the residual vector is $r = b - A\tilde{u}$ .$\square$

A residual vector, in the processes of both Jacobi and Gauss-Seidel methods, is tied to all the calculations of an approximate component to the solution vector. The main aim is to generate a sequence of approximation which will result in the residual vectors in order to converge quickly to zero. Consider

$$r_i^{(k+1)} = (r_{1i}^{(k+1)}, r_{2i}^{(k+1)}, r_{3i}^{(k+1)}, \ldots \ldots \ldots \ldots, r_{ni}^{(k+1)})^t$$

Denote the residual vector for the Gauss-Seidel method corresponding to the approximate solution vector $u_i^{(k+1)}$ defined by

$$u_i^{(k+1)} = (u_1^{(k+1)}, u_2^{(k+1)}, \dots, u_{i-1}^{(k+1)}, u_i^{(k)} \dots \dots \dots \dots, u_n^{(k)})^t$$

The $mth$ component of $r_i^{(k+1)}$ is

$$r_{mi}^{(k+1)} = b_m - \sum_{j=1}^{i-1} a_{mj} u_j^{(k+1)} - \sum_{j=i}^{n} a_{mj} u_j^{(k)} \qquad (3.25)$$

or, equivalently,

$$r_{mi}^{(k+1)} = b_m - \sum_{j=1}^{i-1} a_{mj} u_j^{(k+1)} - \sum_{j=i+1}^{n} a_{mj} u_j^{(k)} - a_{mi} u_i^{(k)},$$

$$for\ each\ m = 1,2,\dots,n$$

In particular, the $ith$ component of $r_i^{(k+1)}$ is

$$r_{ii}^{(k+1)} = b_i - \sum_{j=1}^{i-1} a_{ij} u_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} u_j^{(k)} - a_{ii} u_i^{(k)}$$

So

$$a_{ii} u_i^{(k)} + r_{ii}^{(k+1)} = b_i - \sum_{j=1}^{i-1} a_{ij} u_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} u_j^{(k)} \qquad (3.26)$$

Recall, however that in the Gauss-Seidel method, $u_i^{(k+1)}$ is chosen to be

$$u_i^{(k+1)} = \frac{1}{a_{ii}} \left[ b_i - \sum_{j=1}^{i-1} a_{ij} u_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} u_j^{(k)} \right] \qquad (3.27)$$

So, equation (3.26) can be rewritten as

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$$a_{ii}u_i^{(k)} + r_{ii}^{(k+1)} = a_{ii}u_i^{(k+1)}$$

Subsequently, the Gauss Seidel method can be characterized as selecting $u_i^{(k+1)}$. To satisfy

$$u_i^{(k+1)} = u_i^{(k)} + \frac{r_{ii}^{(k+1)}}{a_{ii}} \tag{3.28}$$

Another connection between the residual vectors and the Gauss Seidel technique can be derived. Consider the residual vector $r_{i+1}^{(k+1)}$, associated with the vector $u_{i+1}^{(k+1)} = (u_1^{(k+1)}, \ldots, u_i^{(k+1)}, u_{i+1}^{(k)}, \ldots, u_n^{(k)})^t$. By equation (3.25) the $ith$ component of $r_{i+1}^{(k+1)}$ is

$$r_{i,i+1}^{(k+1)} = b_i - \sum_{j=1}^{i} a_{ij}\, u_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}\, u_j^{(k)}$$

$$r_{i,i+1}^{(k+1)} = b_i - \sum_{j=1}^{i-1} a_{ij}\, u_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}\, u_j^{(k)} - a_{ii}u_i^{(k+1)}$$

By the manner in which $u_i^{(k+1)}$ is defined in equation (3.27) we see that $r_{i,i+1}^{(k+1)} = 0$. In a sense, then, the Gauss-Seidel technique is characterized by choosing each $u_{i+1}^{(k+1)}$ in such a way that the $ith$ component of $r_{i+1}^{(k+1)}$ is zero.

Selecting $u_{i+1}^{(k+1)}$ so that one coordinate of the residual vector is zero, however, is not necessarily the most efficient way to reduce the norm of the vector $r_{i+1}^{(k+1)}$. If we modify the Gauss-Seidel procedure, as given by equation (3.28), to [7].

$$u_i^{(k+1)} = u_i^{(k)} + \omega \frac{r_{ii}^{(k+1)}}{a_{ii}} \tag{3.29}$$

We can reduce the norm of the residual vector for certain choices of positive $\omega$ and obtain faster convergence.

Methods regarding equation (3.29) are known as relaxation methods. The choice of $\omega$ with $0 < \omega < 1$, the procedures are known as under-relaxation methods. We will be interested in choices of $\omega$ with $1 < \omega < 2$, and these are known as over-relaxation methods. They are used to accelerate the convergence for systems that are convergent by the Gauss-Seidel technique. The methods are abbreviated as SOR, to Successive Over-Relaxation, and are especially useful for solving the linear systems that can be seen in the numerical solution of certain partial-differential equations [8][9].

Before instancing the advantages of the SOR method, we note that by using equation (3.26), we can reformulate equation (3.29) for calculation purposes as

$$u_i^{(k+1)} = (1 - \omega)u_i^{(k)} + \frac{\omega}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}\, u_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}u_j^{(k)}\right]$$

$$for\ i = 1,2,\dots,n\,. \tag{3.30}$$

To determine the matrix form of the SOR method, we rewrite this as

$$a_{ii}u_i^{(k+1)} + \omega \sum_{j=1}^{i-1} a_{ij}\, u_j^{(k+1)} = (1 - \omega)a_{ii}u_i^{(k)} - \omega \sum_{j=i+1}^{n} a_{ij}u_j^{(k)} + \omega b_i$$

So, that in vector form, we have

$$(D - \omega L)u^{(k+1)} = [(1 - \omega)D + \omega U]u^{(k)} + \omega b$$

That is

$$u^{(k+1)} = (D - \omega L)^{-1}[(1 - \omega)D + \omega U]u^{(k)} + \omega(D - \omega L)^{-1}b \qquad (3.31)$$

Letting $G_\omega = (D - \omega L)^{-1}[(1 - \omega)D + \omega U]$ and $\tilde{b}_\omega = \omega(D - \omega L)^{-1}b$, gives the SOR method as [7][8];

$$u^{(k+1)} = G_\omega u^{(k)} + \tilde{b}_\omega .$$

If the equation (3.30) is applied in the solving the system of finite difference equations for the Laplace's equation and Poisson's equation results

$$u_{i,j}^{(k+1)} = (1 - \omega)u_{i,j}^{(k)} + \frac{\omega}{4}\left(u_{i-1,j}^{(k+1)} + u_{i,j-1}^{(k+1)} + u_{i,j+1}^{(k)} + u_{i+1,j}^{(k)}\right), \qquad (3.32)$$

$$u_{i,j}^{(k+1)} = (1 - \omega)u_{i,j}^{(k)} + \frac{\omega}{4}\left(u_{i-1,j}^{(k+1)} + u_{i,j-1}^{(k+1)} + u_{i,j+1}^{(k)} + u_{i+1,j}^{(k)} - h^2 f_{ij}\right), \qquad (3.33)$$

respectively.

**Theorem 3.8:** [8]

If $a_{ii} \neq 0$ for each $i = 1,2, ..., n$ then $\rho(G_\omega) \leq | \omega - 1|$ , it means that if $0 < \omega < 2$ the SOR method converges in this period. □

**Theorem 3.9:** [8]

If $A$ is a positive definite matrix and $0 < \omega < 2$, then the SOR method converges for any choice of initial approximate vector, $u^{(0)}$. □

**Theorem 3.10:** [8][10]

Suppose that $A$ is consistently ordered that $G$ has only real eigenvalues and that $\beta = \rho(G) < 1$. Then SOR iteration converges for every $\omega \in (0,2)$ and the spectral radius of the SOR matrix is

$$\rho(G_\omega) = \begin{cases} 1 - \omega + \dfrac{1}{2}\omega^2\beta^2 + \omega\beta\sqrt{1 - \omega + \dfrac{\omega^2\beta^2}{4}} & for \quad 0 < \omega < \omega_{opt} \\ \omega - 1 & for \quad \omega_{opt} \le \omega < 2 \end{cases}$$

Where

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \beta^2}}$$

**Proof:**

From theorem (3.6)

$$\mu = \frac{\lambda + \omega - 1}{\omega\lambda^{\frac{1}{2}}} \Rightarrow (\lambda + \omega - 1)^2 - \omega^2\mu^2\lambda = 0$$

$$\Rightarrow \lambda^2 - (2\omega - \omega^2\mu^2 - 2)\lambda + (\omega - 1)^2 = 0$$

or

$$\lambda^2 - 2\left(1 - \omega + \frac{1}{2}\omega^2\mu^2\right)\lambda + (\omega - 1)^2 = 0$$

$$\lambda = \frac{2\left(1 - \omega + \frac{1}{2}\omega^2\mu^2\right) \pm \sqrt{4\left(1 - \omega + \frac{1}{2}\omega^2\mu^2\right)^2 - 4(\omega - 1)^2}}{2}$$

$$\lambda = \left(1 - \omega + \frac{1}{2}\omega^2\mu^2\right) \pm \sqrt{\left(1 - \omega + \frac{1}{2}\omega^2\mu^2\right)^2 - (\omega - 1)^2}$$

$$\lambda = \left(1 - \omega + \frac{1}{2}\omega^2\mu^2\right) \pm \sqrt{\frac{1}{4}(\omega^2\mu^2)^2 - \omega^3\mu^2 + \omega^2\mu^2}$$

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$$\lambda = \left(1 - \omega + \frac{1}{2}\omega^2\mu^2\right) \pm \omega\mu \sqrt{\frac{1}{4}\omega^2\mu^2 - \omega + 1} \qquad (3.34)$$

or

$$\lambda = \frac{1}{4}\left[\omega\mu \pm \sqrt{(\omega\mu)^2 - 4(\omega - 1)}\right]^2 \qquad (3.35)$$

If $\mu$ is an eigenvalue of $G$ then both roots $\lambda$ are eigenvalues of $G_\omega$. Since $\mu$ is real the term inside the square root from $(3.34), (3.35)$ is negative if

$$\frac{1}{4}\omega^2\mu^2 - \omega + 1 < 0$$

or

$$(\omega\mu)^2 - 4(\omega - 1) < 0 \qquad where \qquad (\omega > 1).$$

Then if $\omega^2\mu^2 - 4(\omega - 1) < 0 \quad for \quad (\omega > 1)$ then

$$\tilde{\omega} = \frac{2(1 - \sqrt{1 - \mu^2})}{\mu^2} < \omega < 2, \qquad (3.36)$$

and in this case from $(3.35)$

$$\lambda = \frac{1}{4}\left[\omega\mu \pm i\sqrt{4(\omega - 1) + (\omega\mu)^2}\right]^2$$

$$|\lambda| = \frac{1}{4}[\omega^2\mu^2 + 4(\omega - 1) - \omega^2\mu^2] = \omega - 1 \quad , \quad \omega \in (\tilde{\omega}, 2).$$

If $\omega^2\mu^2 - 4(\omega - 1) \geq 0$ then the two solution $\lambda_\pm$ are real and

$$\max\{|\lambda_+|, |\lambda_-|\} = \left(1 - \omega + \frac{1}{2}\omega^2\mu^2\right) + \omega|\mu|\sqrt{\frac{\omega^2\mu^2}{4} + 1 - \omega},$$

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where

$$\omega^2 \mu^2 - 4(\omega - 1) > 0 \quad for \quad 0 < \omega < \omega_{opt}.$$

For converge we require $\rho(G_\omega)$ to be as small as possible when $\mu = \beta = \rho(G)$ this implies

$$\omega = \frac{2(1 \pm \sqrt{1 - \beta^2})}{\beta^2} = \frac{2}{1 \pm \sqrt{1 - \beta^2}},$$

where

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \beta^2}}. \qquad \qquad \square$$

# Chapter 4

# NUMERICAL RESULTS

In this Chapter, the performance of various iterative methods described in Chapter 3 will be tested for the solution of the model problems listed below.

**Problem 1**

$$u_{xx} + u_{yy} = -2\pi^2 \sin(\pi x) \sin(\pi y) \qquad \Omega = \{(x, y) \mid 0 < x, y < 1\}$$

with
$$u(x, o) = u(x, 1) = u(0, y) = u(1, y) = 0 \in \partial\Omega$$

where
$$u_{exact} = \sin(\pi x) \sin(\pi y)$$

**Problem 2**

$$u_{xx} + u_{yy} = 0 \qquad \Omega = \{(x, y) \mid 0 < x, y < 1\}$$

with
$$u(x, o) = u(x, 1) = 0 , u(0, y) = \sin(\pi y) , u(1, y) = e^\pi \sin(\pi y)$$

where
$$u_{exact} = e^{\pi x} \sin(\pi y)$$

The generated linear system of the above model problems from the five point difference discretization are solved using the following iterative methods.

1. Jacobi Method

2. Gauss Seidel method

3. Successive Over relaxation Method (SOR)

The iterative algorithm of the above methods with different values of mesh size $h = \frac{1}{10}$, $h = \frac{1}{20}$, $h = \frac{1}{40}$ has been computed numerically according to the fixed starting iterative vector and the iteration has been terminated according to the following criteria

$$\left\| u^{(k+1)} - u^{(k)} \right\| < \varepsilon = 10^{-5}$$

To increase the convergence rates of the SOR methods through the application of the model problems, optimum relaxation parameter has been calculated for different values of mesh size $h$ using the following formula

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \beta^2}}$$

One of the considered comparison factors to evaluate the performance of the methods is the maximum error reduction through the iterative procedure. Figures (4.1) and (4.3) illustrate the maximum error reduction for each iteration method to solve problem 1 and problem 2 using the proposed iterative methods with various values of $h = \frac{1}{10}$, $h = \frac{1}{20}$, $h = \frac{1}{40}$ respectively. These figures indicate that SOR iterative method requires less iteration than Jacobi and Gauss Seidel methods.

Exact and approximate solutions of problem 1 and problem 2 with $h = \frac{1}{40}$ are illustrated in Figures (4.2) and (4.4) respectively. We observed from these figures that proposed iterative methods work well and each iterative method produce a reasonable approximate solution.

The other factors of comparison worth to consider is the total number of iteration for each iterative method. Tables (4.1) and (4.2) illustrate the total numbers of iteration for each iterative method with various values of $h$ for the solution of problem 1 and problem 2 respectively. In comparisons between proposed iterative methods, we observed from these tables that SOR method requires less total number of iteration than Jacobi and Gauss Seidel method and also Gauss Seidel method is twice as faster as Jacobi method, (see Theorem 3.7 )
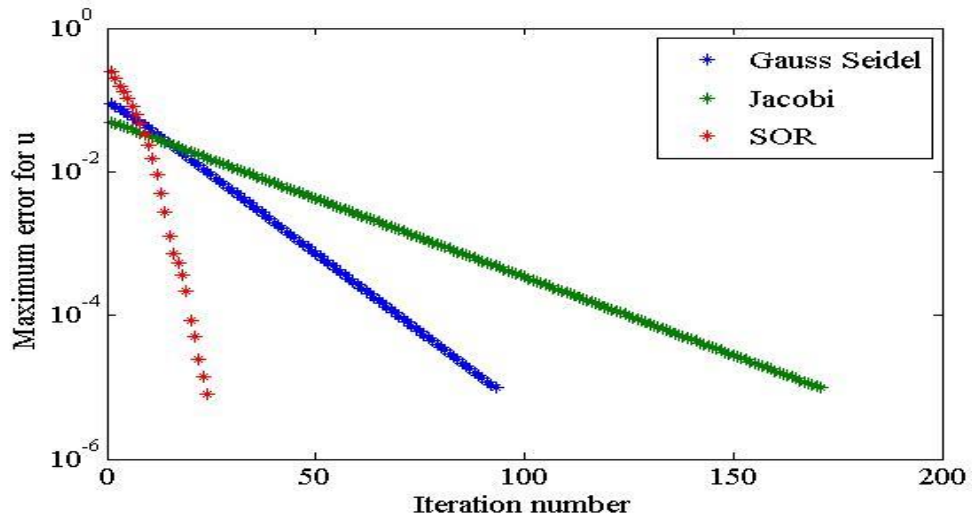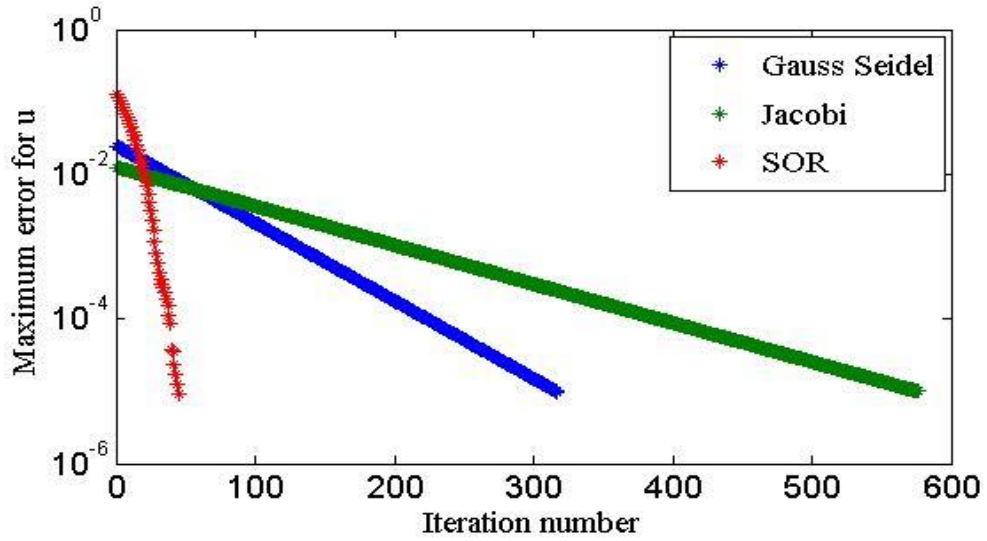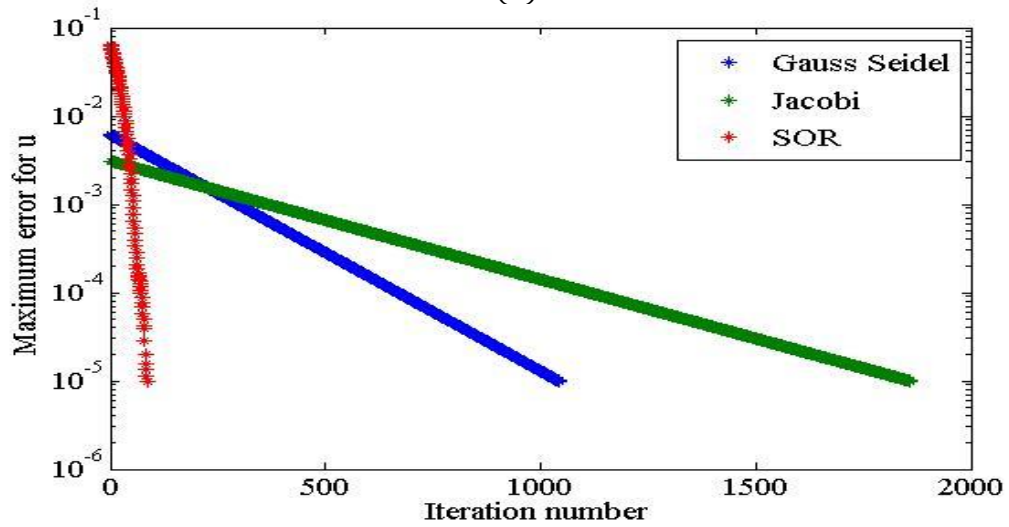
Figure 4.1: The maximum error reductions per each iteration step using Jacobi, Gauss- Seidel and SOR methods. For $(a)$ $h = 1/10, (b)$ $h = 1/20$ and$(c)$ $h = 1/40$ for problem 1
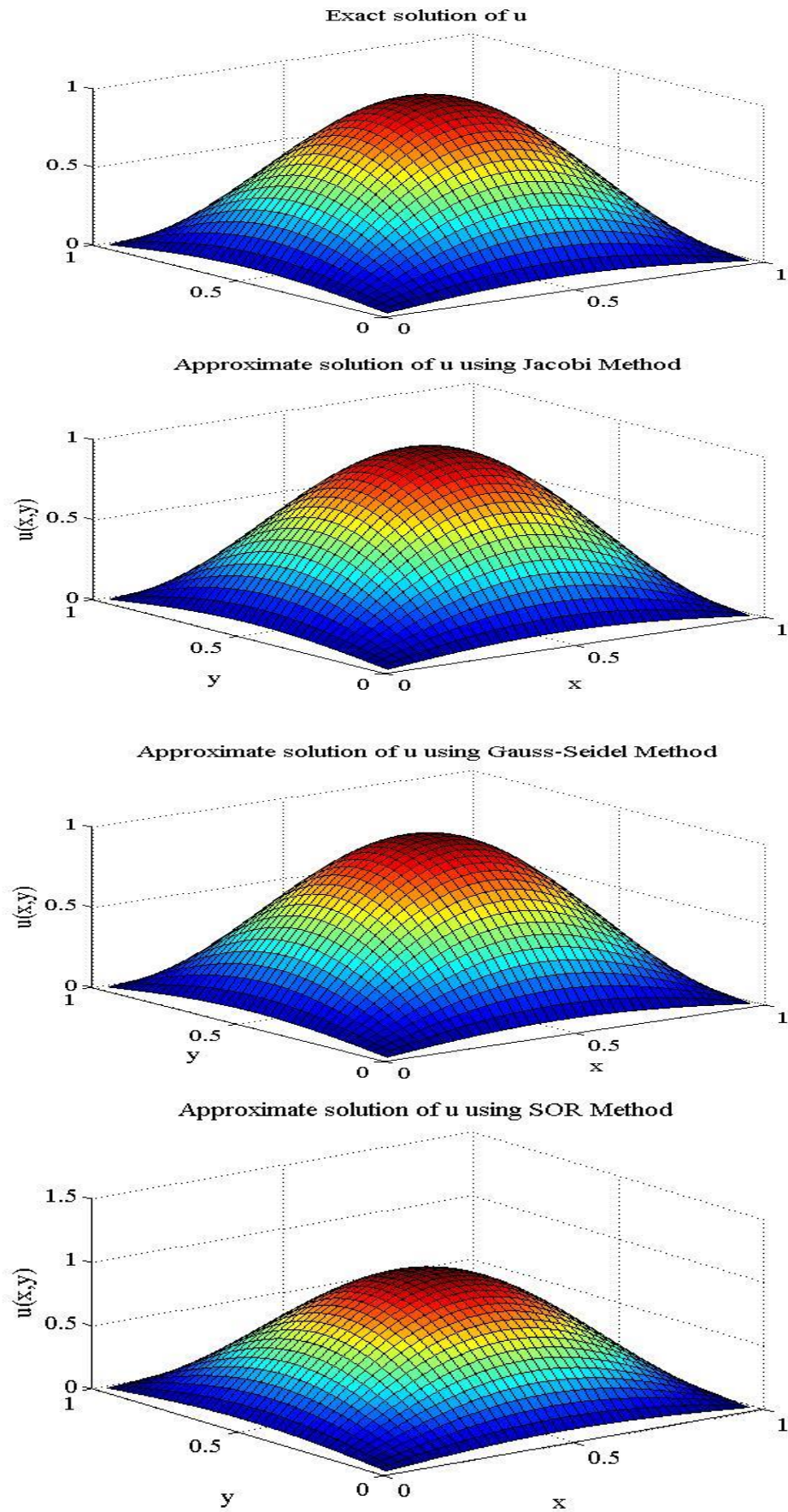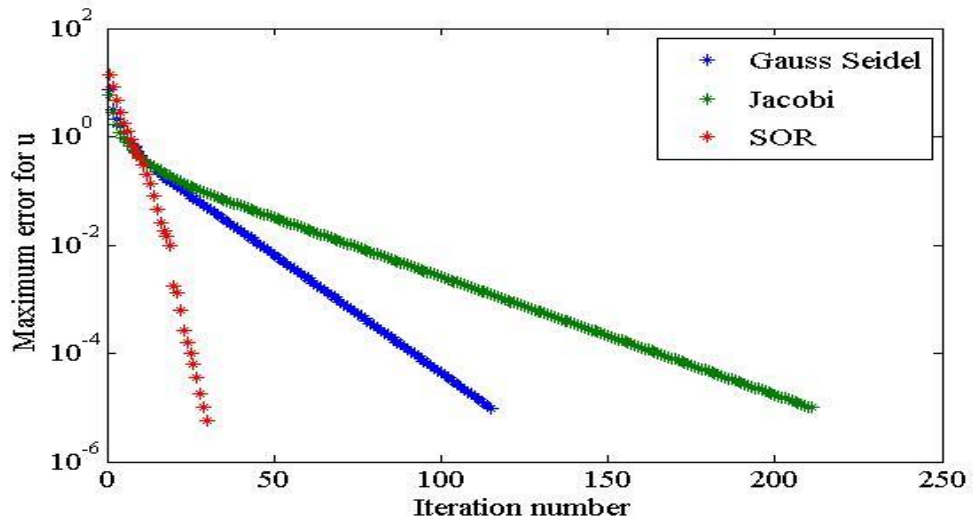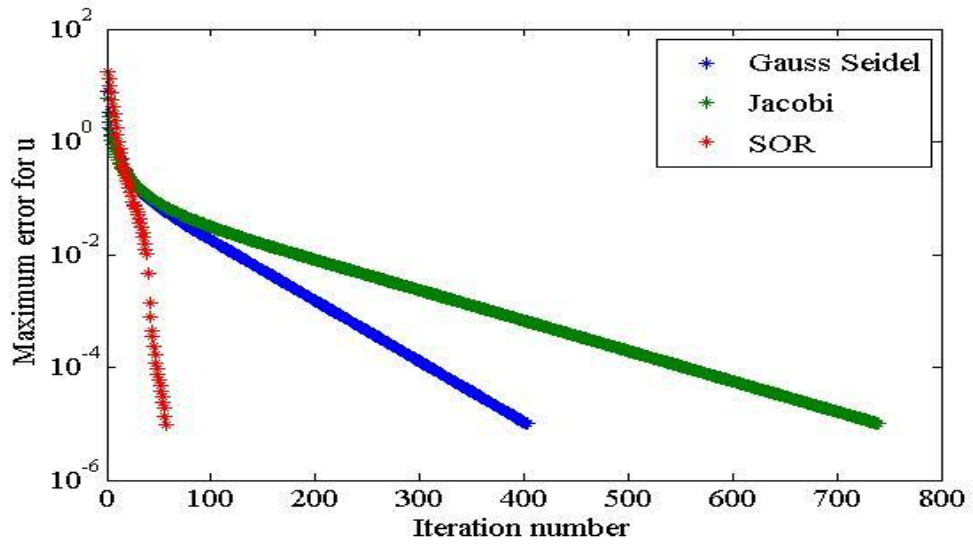
Figure 4.2: Exact and approximate solution of u using Jacobi, Gauss-Seidel and SOR methods with $h = 1/40$ for problem 1

Table 4.1: Total number of iterations for Jacobi, Gauss Seidel and SOR methods with different values of $h = \frac{1}{10}$ , $h = \frac{1}{20}$ , $h = \frac{1}{40}$ for problem 1
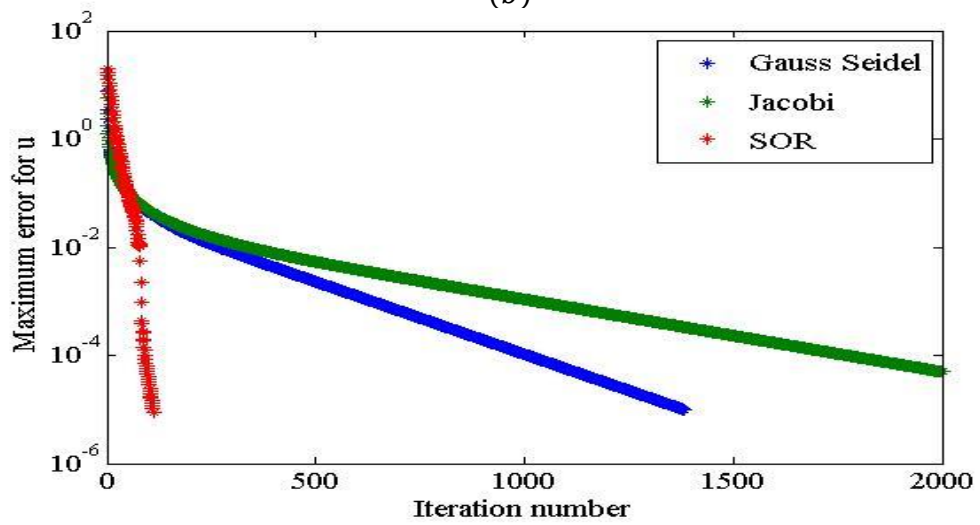
| h | Jacobi | Gauss-Seidel | SOR | |
|---|---|---|---|---|
| | Number of iteration | Number of iteration | Number of iteration | $\omega_{opt}$ |
| $\frac{1}{10}$ | 171 | 93 | 24 | 1.5279 |
| $\frac{1}{20}$ | 576 | 317 | 45 | 1.7295 |
| $\frac{1}{40}$ | 1858 | 1042 | 85 | 1.8545 |

Figure 4.3: The maximum error reductions per each iteration step using Jacobi, Gauss-Seidel and SOR methods for $(a)$ $h = 1/10,$ $(b)$ $h = 1/20$ and$(c)$ $h = 1/40$ for problem 2
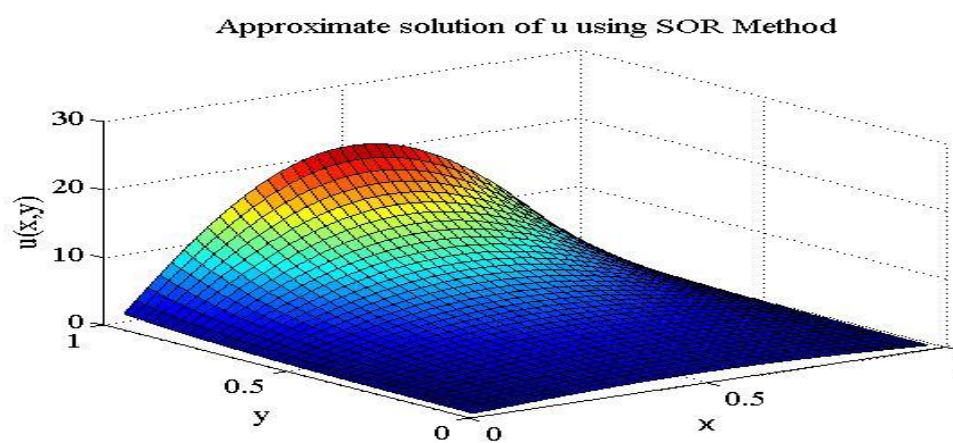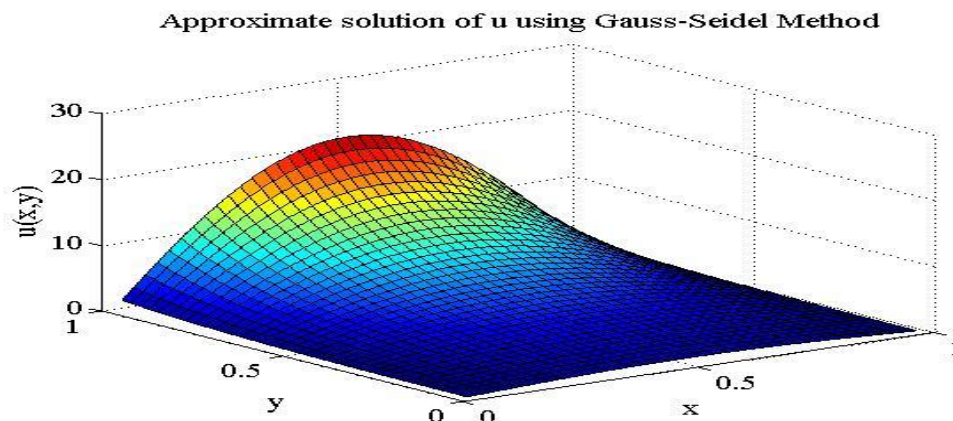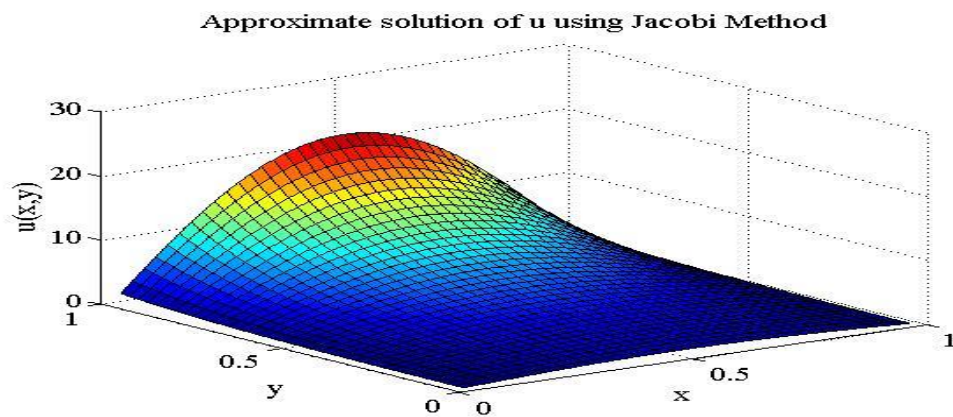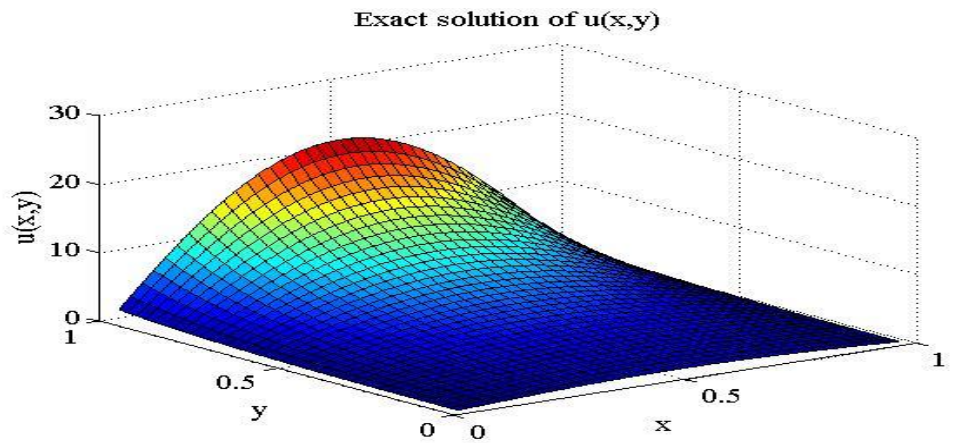
Figure 4.4: Exact and approximate solution of u using Jacobi, Gauss-Seidel and SOR methods with $h = 1/40$ for problem 2

Table 4.2: Total number of iterations for Jacobi, Gauss Seidel and SOR methods with different values of $h = \frac{1}{10}$ , $h = \frac{1}{20}$ , $h = \frac{1}{40}$ for problem 2

| h | Jacobi | Gauss-Seidel | SOR | |
|---|---|---|---|---|
| | Number of iteration | Number of iteration | Number of iteration | $\omega_{opt}$ |
| $\frac{1}{10}$ | 211 | 115 | 30 | 1.5279 |
| $\frac{1}{20}$ | 740 | 403 | 57 | 1.7295 |
| $\frac{1}{40}$ | 2518 | 1381 | 111 | 1.8545 |

# Chapter 5

# CONCLUSION

In this thesis, we emphasized on the numerical solution of Elliptic problem using 5 points finite difference discretization. The generated linear system is then solved by basic iterative methods namely; Jacobi, Gauss-Seidel and Successive Over Relaxation methods. The structures of the matrices generated from the 5 point's finite difference discretization are sparse and symmetric positive definite. For that aspect related theorems and definitions are described for the solution of linear system of equations.

Two practical problems were solved for various values of $h$ in order to compare the efficiency of the basic iterative methods. The analyses of results show that the Gauss-Seidel method converges faster than the Jacobi method because it uses more recent numbers to make it guess. Due to this, the eigenvalues of Gauss-Seidel method will always be lower than the Jacobi method according to Theorem(3.7). An even faster method is Successive Over Relaxation method. A numerical result shows that SOR method for a suitable chosen value of optimum relaxation parameter.

We also observed that, Jacobi, Gauss-Seidel and SOR methods are easy to implement. But, impractical for problems with large number of grids and also Successive Over Relaxation requires the optimum value of relaxation parameter for

fast convergence, which needs on extra computation. Thus, the SOR could be considered more efficient of three methods for small grids.

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