# Schrödinger Equation with Noninteger Dimensions

**Ala Hamd Hssain**

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Prof. Dr. ElvanYılmaz
Director

I certify that this thesis satisfies the requirements as a thesis for the degree of Master of Science in Physics.

Prof. Dr. Mustafa Halilsoy
Chair, Department of Physics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis of the degree of Master of Science in Physics.

Assoc. Prof. S. Habib Mazharimousavi
Supervisor

Examining Committee

1. Prof. Dr. Özay Gurtug
2. Prof. Dr. Mustafa Halilsoy
3. Assoc. Prof. S. Habib Mazharimousavi

# ABSTRACT

Exact solutions in quantum theory play crucial roles in the application areas of the theory. For instance knowing the exact eigenvalues and eigen-functions of the Hamiltonian of the Hydrogen atom helps the Chemists to find, with a high accuracy, the energy levels of more complicated atoms like Helium and Calcium. Therefore any attempt to find an exact solvable system in quantum mechanics is remarkable. For this reason in this thesis we aim to find exactly solvable systems in quantum theory but not in integer dimensions. We consider noninteger dimensional quantum systems. The corresponding Schrödinger equation is introduced. With specific potential, an infinite well, we solve the Schrödinger equation both its angular part and radial part. The angular part admits a solution in terms of Gegenbauer polynomial functions and the radial part gives a solution in terms of the Bessel functions.

**Keywords**: Noninteger dimensions; Schrödinger equation; Gegenbauer polynomial functions; Bessel functions.

# ÖZ

Kuantum Kuram tatbikatında kesin çözümler önemli rol oynamaktadır. Örneğin H-atom Hamilton fonksiyonunun düzgün değer ve fonksiyonlarının bilinmesi kimyacılara Helyum ve Kalsiyum gibi atomların yüksek enerji seviyelerini doğru olarak vermektedir. Bu nedenle kesin çözülebilir yöntemler hep önem arzetmiştir. Bu tezde tam sayılı olmayan boyutlarda kesin çözüm hedeflenmiştir. Kesirli boyutlu kuantum sistemleri ele alınmış olup Schrödinger denklemi yazılmıştır. Özel potansiyel için sonsuz bir kuyu için Schrödinger denkleminin radyal ve açısal kısımlar incelenmiştir. Açısal kısım Gegenbauer, radyal kısım ise Bessel fonksiyonları cinsinden elde edilmiştir.

**Anahtar kelimeler:** Kesirli boyutlar; Schrödinger denklemi; Gegenbauer polinomları; Bessel fonksiyonları.

# DEDICATION

To My Family

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# Chapter 1

# INTRODUCTION

Applications of the fractional calculus have been revealed in different areas of research such as physics [1,2], chemistry [3] and engineering [4,5]. One of the fundamental equations which has been extended in this scenario is the Schrödinger equation [6-17]. In these works the extended Schrödinger equation is introduced as

$$i\hbar\int_0^1 d\omega p(\omega)\frac{\partial^\omega}{\partial t^\omega}\psi(x,t)=-\frac{\hbar^2}{2m}\nabla^2\psi(x,t)+V(x,t)\psi(x,t) \tag{1.1}$$

in which $m$ is the mass, $p(\omega)$ is a distribution function, $x=(r,\theta)$ and

$$\frac{\partial^\omega}{\partial t^\omega}\psi(x,t)=\frac{1}{\Gamma(n-\omega)}\int_0^t d\tau\frac{\psi^n(x,\tau)}{(t-\tau)^{\omega+1-n}} \tag{1.2}$$

and

$$\nabla^2\psi(x,t)=\frac{1}{r^{\alpha-1}}\frac{\partial}{\partial r}\left(r^{\alpha-1}\frac{\partial}{\partial r}\psi(x,t)\right)+\frac{1}{r^2\sin^{\alpha-2}\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin^{\alpha-2}\theta}\frac{\partial}{\partial\theta}\psi(x,t)\right). \tag{1.3}$$

Herein, $n-1<\omega<n$, $\psi^n(x,\tau)=\frac{\partial^n}{\partial t^n}\psi(x,t)$ and $\alpha$ is the noninteger dimensions [18-21]. We note that a specific choice of the distribution function i.e., $p(\omega)=\delta(\omega-1)$, the fractional Schrödinger equation in noninteger dimensions becomes the Schrödinger equation in noninteger dimensions which is the subject of this study.

Hausdorff introduced the concept of fractional dimensions which attracted the attentions of researchers and it has been used widely after the novel works of Mandelbrot where he has shown in different the fractal nature [22]. Upon the recently development of evolution equation and related field of application [23-33] which includes evolution equation to fractional dimensions, the Schrödinger equation became a special case of these development. This is because of its applications in various fields. The investigation of this situation has been done by different ways. Fractional derivative is one of them while the modified spatial operator is the other [22] which we use to solve noninteger Schrödinger equation involving a space derivative of noninteger order N [34] which represents a noninteger dimensions.

In this thesis our aim is first to introduce the Schrödinger equation in a noninteger dimensions. The potential which we shall consider is going to be a radial symmetric field and therefore the angular part of the Schrödinger equation is separable from the radial part. This helps us to find the general angular solution to the Schrödinger equation which is applicable in all such systems with a radial potential. The radial part of the Schrödinger equation depends on the form of the potential and therefore we set a specific system and upon that a general solution will be found.

The Potential which we are interested in is the infinite well which is a very good approximation for the potential in which a nucleus experiences within the nuclei. Although our aim is not going through the deep of the shell model in nuclear physics but the obvious application of our study is seen to be over there.

Since 1979, fractal geometry has been applied not only in mathematics but also in diverse fields as physics, chemistry, biology and computer graphics [43]. One of its applications is what the cosmologists claimed in 1987 which states that the galaxy's structure is highly irregular and self similar but not homogenous [43, 44]. Another application is the use of computer science in the fractal image compression [44]. A relatively newer application of the fractal geometry is in telecommunication by constructing the fractal-shaped antenna [18, 44].

# Chapter 2

# SCHRÖDINGER EQUATION WITH NONINTEGER DIMENSIONS

## 2.1 Introduction

In this chapter, we start with the time-dependent Schrödinger equation with non-integer dimensional space. The equation describes a quantum particle motion which undergoes radially symmetric potential $V(r)$. This differential equation can be written as introduced in Ref. [20],

$$H_N \Psi(\vec{r},t) = i\hbar \frac{\partial \Psi(\vec{r},t)}{\partial t} \tag{2.1}$$

$$H_N = -\frac{\hbar^2}{2\mu}\nabla_N^2 + V(r) \tag{2.2}$$

where $H_N$ is the fractional Hamiltonian operator [20] and modified spatial operator can represent as [39,27]

$$\nabla_N^2 = \frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin^{N-2}\theta}\frac{\partial}{\partial \theta}\left(\sin^{N-2}\theta \frac{\partial}{\partial \theta}\right). \tag{2.3}$$

Herein $N$ represents a noninteger dimensions [39]. The general solution to Eq. (2.1) may be written as

$$\Psi(\vec{r},t) = \psi(r,\theta) e^{-iEt/\hbar} \tag{2.4}$$

in which the spatial wave functions $\psi(r,\theta)$ satisfies the time-independent Schrödinger equation with noninteger dimensions [20]

$$\left[-\frac{\hbar^2}{2\mu}\nabla_N^2+V(r)\right]\psi(r,\theta)=E\psi(r,\theta). \tag{2.5}$$

Herein $V\ (r)$ is the potential function, $\mu$ is the mass and $E$ is the energy of the quantum particle. We start our analysis by substituting Eq. (2.3) in Eq. (2.5)

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}\psi(r,\theta)\right)+\frac{1}{r^2\sin^{N-2}\theta}\frac{\partial}{\partial\theta}\left(\sin^{N-2}\theta\frac{\partial}{\partial\theta}\psi(r,\theta)\right)\right] +V(r)\psi(r,\theta)=E\ \ \psi(r,\theta). \tag{2.6}$$

This equation can be solved by using the separation method in which one can write

$$\psi(r,\theta)=R(r)\Theta(\theta) \tag{2.7}$$

and upon a substitution into Eq. (2.6) yields [20]

$$-\frac{\hbar^2}{2\mu}\left[\frac{\Theta(\theta)}{r^{N-1}}\frac{d}{dr}\left(r^{N-1}\frac{dR(r)}{dr}\right)+\frac{R(r)}{r^2\sin^{N-2}\theta}\frac{d}{d\theta}\left(\sin^{N-2}\theta\frac{d\Theta(\theta)}{d\theta}\right)\right]+ V(r)R(r)\Theta(\theta)=E\ R(r)\Theta(\theta). \tag{2.8}$$

Rearranging Eq. (2.8) it reduces to

$$\frac{r^{3-N}}{R(r)}\frac{d}{dr}\left(r^{N-1}\frac{dR(r)}{dr}\right)+\frac{1}{\Theta(\theta)\sin^{N-2}\theta}\frac{d}{d\theta}\left(\sin^{N-2}\theta\frac{d\Theta(\theta)}{d\theta}\right) +\frac{2\mu r^2}{\hbar^2}\left(E-V(r)\right)=0. \tag{2.9}$$

This equation can be separated into two equations in term of $r$ and $\theta$. The procedures also introduces a constant $\xi^2$,

$$\frac{r^{3-N}}{R(r)}\frac{d}{dr}\left(r^{N-1}\frac{dR(r)}{dr}\right)+\frac{2\mu r^2}{\hbar^2}[E-V(r)]= \\ -\frac{1}{\Theta(\theta)sin^{N-2}}\frac{d}{d\theta}\left(sin^{N-2}\theta\frac{d\Theta(\theta)}{d\theta}\right)=\xi^2 \tag{2.10}$$

where $\xi^2>0$ is a separation constant. From (2.10) it follows that one finds the following radial equation

$$\frac{r^{3-N}}{R(r)}\left((N-1)r^{N-2}\frac{dR(r)}{dr}+r^{N-1}\frac{d^2R}{dr^2}\right)+\frac{2\mu r^2}{\hbar^2}\left[E-V(r)\right]-\xi^2=0 \tag{2.11}$$

which after simplification becomes

$$\frac{d^2R}{dr^2}+\frac{(N-1)}{r}\frac{dR(r)}{dr}+\left[\frac{2\mu}{\hbar^2}E-\frac{\xi^2}{r^2}-\frac{2\mu}{\hbar^2}V(r)\right]R(r)=0. \tag{2.12}$$

The angular part of (2.10) can be written as

$$\frac{1}{\sin^{N-2}\theta}\frac{d}{d\theta}\left(\sin^{N-2}\theta\frac{d\Theta(\theta)}{d\theta}\right)+\xi^2\Theta(\theta)=0. \tag{2.13}$$

Upon rearrangement we can write it as

$$(N-2)\frac{\cos\theta}{\sin\theta}\frac{d\Theta(\theta)}{d\theta}+\frac{d^2\Theta(\theta)}{d\theta^2}+\xi^2\Theta(\theta)=0 \tag{2.14}$$

which finally becomes

$$\frac{d^2\Theta(\theta)}{d\theta^2}+\frac{(N-2)}{\tan\theta}\frac{d\Theta(\theta)}{d\theta}+\xi^2\Theta(\theta)=0. \tag{2.15}$$

## 2.2 Solution to the Angular Differential Equation

Although the solution of radial differential equation (2.12) is potential dependent, but the angular differential equation of variable $\theta$ (2.15) is completely independent of the potential [39]. Therefore, we can find a general solution for all radially symmetric potentials [35]. First, let's solve the angular part. To do so we introduce a new variable $x = \cos\theta$ [39] upon which, one finds

$$\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx}, \qquad \frac{d^2}{d\theta^2} = \frac{d^2x}{d\theta^2}\frac{d}{dx} + \left(\frac{dx}{d\theta}\right)^2 \frac{d^2}{dx^2} \tag{2.16}$$

$$\frac{d}{d\theta} = -\sin\theta \frac{d}{dx}, \qquad \frac{d^2}{d\theta^2} = -\cos\theta \frac{d}{dx} + \sin^2\theta \frac{d^2}{dx^2}. \tag{2.17}$$

A substitution from Eq. (2.17) in Eq. (2.15) we find

$$-\cos\theta \frac{d\Theta(x)}{dx} + \sin^2\theta \frac{d^2\Theta(x)}{dx^2} - \frac{(N-2)\sin\theta}{\tan\theta}\frac{d\Theta(x)}{dx} + \xi^2\Theta(x) = 0 \tag{2.18}$$

and upon $\sin^2\theta = 1 - x^2$ it reduces to

$$\frac{d^2\Theta(x)}{dx^2} - \frac{(N-1)x}{1-x^2}\frac{d\Theta(x)}{dx} + \frac{\xi^2}{1-x^2}\Theta(x) = 0. \tag{2.19}$$

Finally, the latter equation becomes

$$\left(1-x^2\right)\frac{d^2\Theta(x)}{dx^2} - (N-1)x\frac{d\Theta(x)}{dx} + \xi^2\Theta(x) = 0. \tag{2.20}$$

This differential equation is called the Gegenbauer differential equation (GDE) which is given in its standard form as

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - (2\alpha+1)x\frac{dy}{dx} + p(p+2\alpha)y = 0 \tag{2.21}$$

in which $\alpha$ is a real constant and $p$ is a nonnegative integer number. The general regular and convergence solution to GDE is called Gegenbauer polynomial $C_p^{\alpha}(x)$ of order α and degree $p$ where $\alpha > -\frac{1}{2}$ [41]. $C_p^{\alpha}(x)$ is written as

$$C_p^{\alpha}(x) = \frac{(2\alpha)_p}{p!} F\left(-p, 2\alpha + p; \alpha + \frac{1}{2}; \frac{1-x}{2}\right), \quad p = 0,1,2,3,... \tag{2.22}$$

They are given as a Gaussian hypergeometric function which is the special case of the hypergeometric series [38, 40]. Also $(2\alpha)_p$ represents the falling factorial [42] of $(2\alpha)$

$$(2\alpha)_p = \frac{\Gamma(2\alpha+1)}{\Gamma(2\alpha - p + 1)} = \frac{(2\alpha)!}{(2\alpha - p)!}. \tag{2.23}$$

By comparing Eq. (2.20) with Eq. (2.21) one finds

$$\alpha = \frac{N}{2} - 1 \tag{2.24}$$

and

$$\xi^2 = p(p + N - 2) \tag{2.25}$$

where

$$p = -(\frac{N}{2} - 1) + \sqrt{\left(\frac{N}{2} - 1\right)^2 + \xi^2} \tag{2.26}$$

which yields a regular solution to (2.20) given by

$$\Theta(x) = \beta C_p^{\frac{N}{2}-1}(x) = \beta \frac{\Gamma(N-1)}{p!\Gamma(N-p-1)} F\left(-p, p + N - 2; \frac{N}{2} - \frac{1}{2}; \frac{1-x}{2}\right). \tag{2.27}$$

In the latter equation $\beta$ is the normalization constant. Gegenbauer polynomials satisfy the following orthonormality relation [41, 42] as

$$\int_{-1}^{1}\left[C_n^{\alpha}(x)\right]^2(1-x^2)^{\alpha-\frac{1}{2}}dx = \frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha)}{n!(n+\alpha)\left[\Gamma(\alpha)\right]^2} \quad (2.28)$$

hence, by substitution Eq. (2.24) in (2.28) the orthonormality condition is written as

$$\int_{-1}^{1}\left[C_p^{(\frac{N}{2}-1)}(x)\right]^2(1-x^2)^{(\frac{N}{2}-1)-\frac{1}{2}}dx = \frac{\pi 2^{3-N}\Gamma(p+N-2)}{p!\left(p+\frac{N}{2}-1\right)\left[\Gamma\left(\frac{N}{2}-1\right)\right]^2} \quad (2.29)$$

therefore, the normalization constant can be found which may write as following

$$\beta = \frac{\Gamma\left(\frac{N}{2}-1\right)\sqrt{p!\left(p+\frac{N}{2}-1\right)}}{(\pi 2^{3-N})^{\frac{1}{2}}\sqrt{\Gamma(p+N-2)}}. \quad (2.30)$$

Consequently, the normalized solution can be written as

$$\Theta(\cos\theta) = \frac{\Gamma\left(\frac{N}{2}-1\right)\sqrt{p!\left(p+\frac{N}{2}-1\right)}}{(\pi 2^{3-N})^{\frac{1}{2}}\sqrt{\Gamma(p+N-2)}}C_p^{\frac{N}{2}-1}(\cos\theta) \quad (2.31)$$

The first few Gegenbauer polynomials for specific value of $p$ can be found by using the recurrence relations [42] which are represented as

$$C_0^{\alpha}(x) = 1 \quad (2.32)$$

$$C_1^{\alpha}(x) = 2\alpha x \quad (2.33)$$

$$C_p^{\alpha}(x) = \frac{1}{p}\left[2x(p+\alpha-1)C_{p-1}^{\alpha}(x) - (p+2\alpha-2)C_{p-2}^{\alpha}(x)\right], \ \mathrm{p} \geq 2 \quad . \quad (2.34)$$

These yield

$$C_0^{\frac{N}{2}-1}(x) = 1 \qquad (2.35)$$

and subsequently

$$C_1^{\frac{N}{2}-1}(x) = (N-2)x \qquad (2.36)$$

with

$$C_2^{\frac{N}{2}-1}(x) = \left(\frac{N}{2}-1\right)\left(Nx^2-1\right) \qquad (2.37)$$

$$C_3^{\frac{N}{2}-1}(x) = N\left(\frac{N}{2}-1\right)x\left[\frac{2}{3}\left(\frac{N}{2}+1\right)x^2-1\right] \qquad (2.38)$$

$$C_4^{\frac{N}{2}-1}(x) = N\left(\frac{N}{2}-1\right)\left[\left(\frac{N}{2}+1\right)x^2+\frac{2}{3}\left(\frac{N^2}{8}+N+1\right)x^4-\frac{1}{4}\right]. \qquad (2.39)$$

# Chapter 3

# PARTICLE IN A BOX

## 3.1 N-Dimensional Spherical Infinite Well

We begin with solving the radial part of the Schrödinger equation which we found in previous chapter. This equation depends on the potential $V(r)$, and one of the simplest example is the noninteger dimensional spherical infinite spherical well which is written as

$$V(r)=\begin{cases}0 & 0\leq r\leq a\\ \infty & otherwise\end{cases} \tag{3.1}$$

In which $a$is the radius of the well. The radial part of the Schrödinger equation reads

$$\frac{d^2R(r)}{dr^2}+\frac{(N-1)}{r}\frac{dR(r)}{dr}+\left[\frac{2\mu}{\hbar^2}E-\frac{\xi^2}{r^2}\right]R(r)=0\,. \tag{3.2}$$

Introducing the parameter $k$ defined by

$$k^2=\frac{2\mu E}{\hbar^2} \tag{3.3}$$

Eq. (3.2) becomes

$$\frac{d^2R(r)}{dr^2}+\frac{(N-1)}{r}\frac{dR(r)}{dr}+\left[k^2-\frac{\xi^2}{r^2}\right]R(r)=0. \tag{3.4}$$

We shall transform this equation into a convenient form by introducing a new independent variable by

$$z = kr. \tag{3.5}$$

Now, applying the chain rule one finds

$$\frac{d}{dr} = k\,\frac{d}{dz} \tag{3.6}$$

and

$$\frac{d^2}{dr^2} = k^2\,\frac{d^2}{dz^2}. \tag{3.7}$$

Hence, in term of the new variable, Eq. (3.2) becomes

$$\frac{d^2R(z)}{dz^2} + \frac{(N-1)}{z}\frac{dR(z)}{dz} + \left[1 - \frac{\xi^2}{z^2}\right]R(z) = 0. \tag{3.8}$$

Equation (3.8) can be written in a convenient form making use of transformation by

$$R(z) = z^{1-\frac{N}{2}}F(z). \tag{3.9}$$

From the above relation we have

$$\frac{dR(z)}{dz} = \left[\frac{(1-\frac{N}{2})}{z}F(z) + \frac{d^2F(z)}{dz^2}\right]z^{1-\frac{N}{2}} \tag{3.10}$$

and similarly,

$$\frac{d^2R(z)}{dz^2} = \left[\frac{d^2F(z)}{dz^2} + \frac{(2-N)}{z}\frac{dF(z)}{dz} + \left(\left(1-\frac{N}{2}\right)^2 - \left(1-\frac{N}{2}\right)\right)\frac{F(z)}{z^2}\right]z^{1-\frac{N}{2}}. \tag{3.11}$$

Substituting these representations in Eq. (3.8) and dividing the resulting equation by

$(z^{1-\frac{N}{2}})$ we get

$$z^2 \frac{d^2F(z)}{dz^2} + z\frac{dF(z)}{dz} + \left[ z^2 - \frac{1}{4}\left( (N-2)^2 + 4\xi^2 \right) \right] F(z) = 0. \qquad (3.12)$$

Solutions to the Eq. (3.12) can be written in terms of two linearly independent solutions which are $J_\nu(z)$ and $N_\nu(z)$, namely the Bessel functions of the first kind and second kind, i.e.

$$R(z) = C_1 z^{1-\frac{N}{2}} J_\nu(z) + C_2 z^{1-\frac{N}{2}} N_\nu(z). \qquad (3.13)$$

Next, we apply the boundary condition at small $r$ which implies $R(0) = 0$. We know that in the limit $z \to 0$ [37]

$$\lim_{z\to 0} J_\nu(z) = \frac{z^\nu}{2^\nu \nu!}. \qquad (3.14)$$

However, the Bessel functions of the second kind do not remain finite at $z = 0$

$$\lim_{z\to 0} N_\nu(z) = -\frac{(\nu-1)!}{\pi}\left(\frac{2}{z}\right)^\nu \Rightarrow N_\nu(z) = \infty \qquad (3.15)$$

$$R(z) = C_1 z^{1-\frac{N}{2}+\nu} + C_2 z^{1-\frac{N}{2}-\nu} \qquad (3.16)$$

in the limit $z \to 0$, the coefficient of $C_2 = 0$ so that we may take

$$R(z) = C_1 z^{1-\frac{N}{2}} J_\nu(z) \qquad (3.17)$$

which behaves regular at origin, where $C_1$ is normalization constant and $\nu$ is

$$\nu = \frac{1}{2}\sqrt{(N-2)^2 + 4\xi^2}\,\nu = \frac{N}{2} + p - 1. \qquad (3.18)$$

Imposing the second boundary condition i.e. $R(a) = 0$, we see that (3.17) must admit positive roots at $z = z_0$ in which

$$z_0 = ka \tag{3.19}$$

$$R\left(z = z_0\right) = 0 \tag{3.20}$$

$$R(z_0) = C_1 z_0^{1-\frac{N}{2}} J_\nu(z_0) = 0. \tag{3.21}$$

Herein, $z_0$ is given by

$$J_\nu\left(z_0\right) = 0 \tag{3.22}$$

and therefore

$$z_0 = X_{n\nu} \tag{3.23}$$

where $X_{n\nu}$ is the $n^{th}$ root of $J_\nu(x)$ which is given in the form [37]

$$J_\nu\left(X_{n\nu}\right) = 0 \quad n=1,2,3.... \tag{3.24}$$

By using Eq. (3.19) and (3.23)

$$k_n = \frac{X_{n\nu}}{a}. \tag{3.25}$$

From (3.3) and (3.25)

$$\frac{2\mu E_{n\nu}}{\hbar^2} = \frac{X_{n\nu}^2}{a^2} \tag{3.26}$$

which upon this the allowed energies are given by

$$E_{n\nu} = \frac{\hbar^2}{2\mu a^2} X_{n\nu}^2. \tag{3.27}$$

The above equation shows that particle can have only discrete energies i.e., energy of the particle is quantized and $n$ is called quantum number. Now the radial solution can be represented as

$$R_{n\nu}(r) = C_{n\nu} r^{1-\frac{N}{2}} J_\nu\left(\frac{X_{n\nu}}{a} r\right) \tag{3.28}$$

or

$$R_{n\nu}(r) = C_{n\nu} r^{1-\frac{N}{2}} J_\nu(k_{n\nu} r). \tag{3.28}$$

Furthermore, $C_{n\nu}$ is the normalization constant. The radial wave function can be normalized making use of the condition [39]

$$\int_0^a |R_{n\nu}(r)|^2 \, r^{N-1} dr = 1 \tag{3.30}$$

$$|C_{n\nu}|^2 \int_0^a \left( r^{1-\frac{N}{2}} \right)^2 \left[ J_\nu\left(\frac{X_{n\nu}}{a} r\right) \right]^2 r^{N-1} dr = 1 \tag{3.31}$$

which after using orthonormality condition for Bessel function [37]

$$\int_0^a r \left[ J_\nu\left(\frac{X_{n\nu}}{a} r\right) \right]^2 dr = \frac{a^2}{2} \left[ J_{\nu+1}(X_{n\nu}) \right]^2 \tag{3.32}$$

we find

$$C_{n\nu} = \frac{\sqrt{2}}{a J_{\nu+1}(X_{n\nu})}. \tag{3.32}$$

The normalized radial solution to the Eq. (3.2) becomes

$$R_{n\nu}(r) = \frac{\sqrt{2}}{aJ_{\nu+1}(X_{n\nu})} r^{1-\frac{N}{2}} J_\nu(\frac{X_{n\nu}}{a} r). \quad (3.34)$$

Finally, using Eq. (2.27) and the above solution obtained for $R(r)$, the solution to the Eq. (2.6) can be written as

$$\psi_{np}(r,\theta) = \frac{2^{\frac{N}{2}}\Gamma\left(\frac{N}{2}-1\right)\sqrt{p!\left(p+\frac{N}{2}-1\right)}}{a\sqrt{2\pi}\sqrt{\Gamma\left(p+N-2\right)}J_{\nu+1}(X_{n\nu})} r^{1-\frac{N}{2}} J_\nu(\frac{X_{n\nu}}{a} r) C_p^{\frac{N}{2}-1}\left(\cos\theta\right). \quad (3.35)$$

For $p = 0,1,2,...$ the latter wave function satisfies the boundary conditions i.e., at $r = 0$ $and$ $r = a$ it vanishes.

In the Tables 3.1 and 3.2 we present the first four roots of the Bessel functions which appear in our final wave functions and the relative energies of the corresponding particle, respectively.

In Figure 3.1 the radial probability densities of the first four states of the particle confined in a $\frac{5}{2}$D spherical infinite well are depicted. In Figure 3.2 the relative energy levels of the same particle is displayed.

Table 3.1: Zeros of the Bessel functions $J_{\nu}(x)$ [37].

| $n$ | $J_{\frac{1}{4}}(x)$ | $J_{\frac{5}{4}}(x)$ | $J_{\frac{9}{4}}(x)$ | $J_{\frac{13}{4}}(x)$ | $J_{\frac{17}{4}}(x)$ | $J_{\frac{21}{4}}(x)$ |
|---|---|---|---|---|---|---|
| 1 | 2.7809 | 4.1654 | 5.4511 | 6.6850 | 7.8862 | 9.0642 |
| 2 | 5.9061 | 7.3729 | 8.7577 | 10.0902 | 11.3857 | 12.6533 |
| 3 | 9.0424 | 10.5408 | 11.9729 | 13.3577 | 14.7070 | 16.0282 |
| 4 | 12.1813 | 13.6966 | 15.1566 | 16.5746 | 17.9595 | 19.3173 |
| 5 | 15.3214 | 19.9949 | 18.3256 | 19.7666 | 21.1770 | 22.5619 |

Table 3.2: Energy levels proportional to the square of zeros of the Bessel functions $J_{\nu}(x)$ , as given by $\frac{E_{n\nu}}{\varepsilon} = (X_{n\nu})^2, where\ \varepsilon = \frac{\hbar^2}{2\mu a^2}$.

| $n$ | $\left(X_{n\frac{1}{4}}\right)^2$ | $\left(X_{n\frac{3}{4}}\right)^2$ | $\left(X_{n\frac{9}{4}}\right)^2$ | $\left(X_{n\frac{13}{4}}\right)^2$ | $\left(X_{n\frac{17}{4}}\right)^2$ | $\left(X_{n\frac{21}{4}}\right)^2$ |
|---|---|---|---|---|---|---|
| 1 | 7.73 | 17.35 | 29.72 | 44.69 | 62.19 | 82.16 |
| 2 | 34.88 | 54.36 | 76.70 | 101.81 | 129.63 | 160.11 |
| 3 | 81.77 | 111.11 | 143.35 | 178.43 | 216.30 | 256.90 |
| 4 | 148.38 | 187.60 | 229.72 | 274.72 | 322.54 | 373.16 |
| 5 | 234.75 | 399.80 | 335.83 | 390.72 | 448.47 | 509.04 |

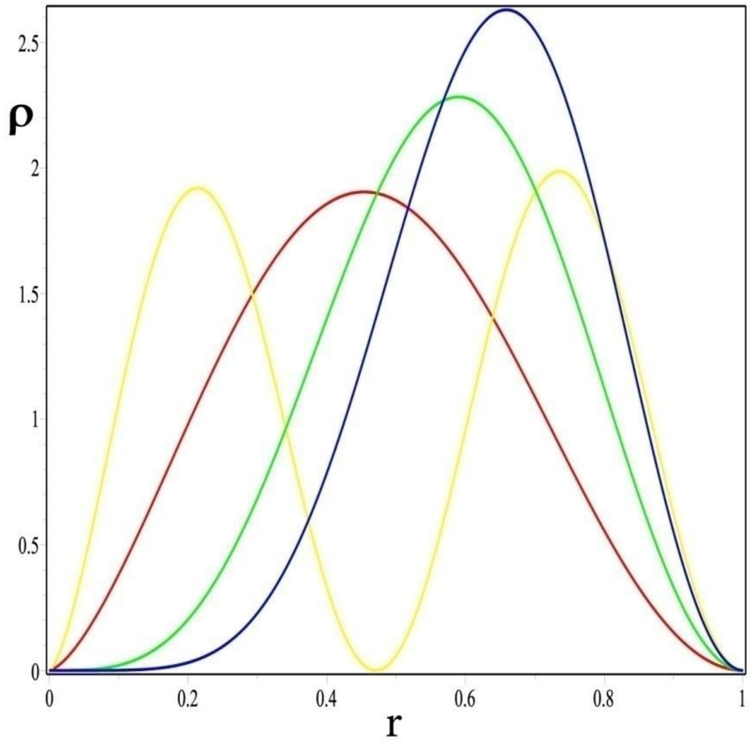


Figure 3.1: A plot of radial probability density function $\rho(r) = r^{2.5}|R_{n\nu}|^2$ of the first four states of a quantum particle in a $2.5-$dimensional infinite well in terms of $r$. Red, green, blue and yellow are the corresponding densities of the first, second, third and fourth states respectively.

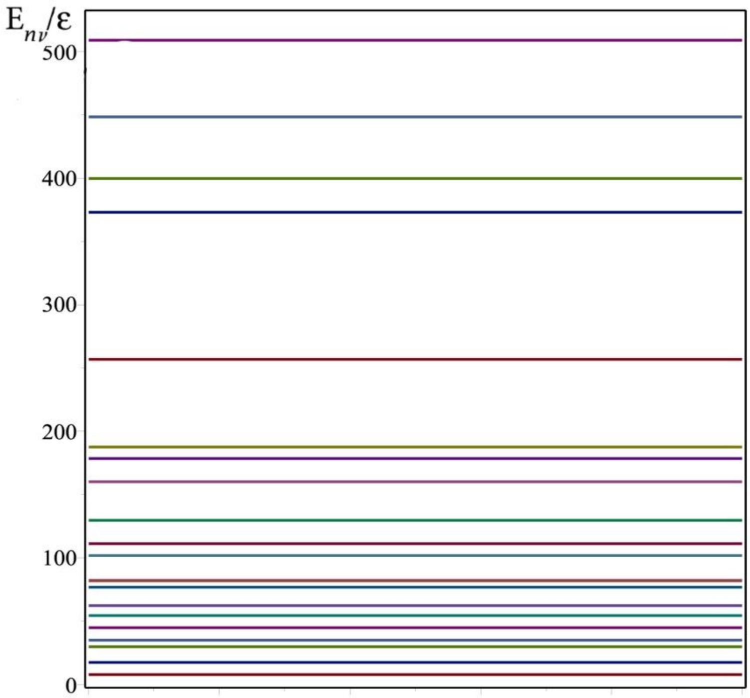


Figure 3.2: The energy levels over $\varepsilon = \frac{\hbar^2}{2\mu a^2}$ of a quantum particle confined in a $2.5-$dimensional infinite well.

## 3.2 Two Dimensional Spherical Potential Well

We have already noted that for planar oscillator $N = 2\ and\ \theta$ is just azimuthal angle but not polar angle [39] and from now we shall call it $\varphi$ so that the equation of (2.13) reduces to

$$\frac{d^2\Phi}{d\varphi^2} + m^2\Phi = 0 \tag{3.36}$$

whose solution reads

$$\Phi(\varphi) = Ae^{\pm i m_l \varphi}. \tag{3.37}$$

For $\Phi$ to be single valued, $\Phi(\varphi) = \Phi(\varphi + 2\pi)$. Therefore;

$$Ae^{\pm im\varphi} = Ae^{\pm im(\varphi+2\pi)} \qquad \text{or} \qquad e^{\pm i2m\pi} = 1 \tag{3.38}$$

this is possible only if $m = 0,1,2,3,...$ which is called magnetic quantum number [36]. The normalized solution is then

$$\Phi_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}, \qquad m = 0, \pm 1, \pm 2, \pm 3, .... \tag{3.39}$$

Now, we can write radial equation for $(N = 2)$ as

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + \left[k^2 - \frac{m^2}{r^2}\right]R(r) = 0. \tag{3.40}$$

Defining $x = kr$,

$$\frac{d}{dr} = k\frac{d}{dx} \qquad , \qquad \frac{d^2}{dr^2} = k^2\frac{d^2}{dx^2}. \tag{3.41}$$

Eq. (3.40) reduced to

$$\frac{d^2R(x)}{dx^2}+\frac{1}{x}\frac{dR(r)}{dx}+\left[1-\frac{m^2}{x^2}\right]R(x)=0. \tag{3.42}$$

Its solution can be written in terms of the ordinary Bessel functions

$$R(r)=\tilde{C}J_m(kr)+\tilde{D}N_m(kr). \tag{3.43}$$

Similarly, using the first boundary condition and applying asymptotic solution $(x<<1)$ which we used from Eq. (3.14) and (3.15) [37], the Neumann function is not finite at origin so that $\check{D}=0.$ Thus our solution may be written as

$$R(r)=\tilde{C}J_m(kr) \tag{3.44}$$

At $r=a$ one must impose

$$J_m(ka)=0 \tag{3.45}$$

where $X_{nm}$ can be chosen as the $n^{th}$ root of $J_m(X)$. Following (3.46) we find

$$ka=X_{nm} \tag{3.46}$$

$$k_n^2=\frac{2\mu E_{nm}}{\hbar^2}=X_{nm}^2 \tag{3.47}$$

and finding the energy spectrum is determined as

$$E_{nm}=\frac{\hbar^2}{2\mu a^2}X_{nm}^2. \tag{3.48}$$

Now, one can write the radial solution as

$$R_{nm}(r)=\tilde{C}J_m\left(\frac{X_{nm}}{a}r\right) \tag{3.49}$$

and similarly, we use normalization condition to find $\tilde{C}$ . Then we have

$$\int_0^a \left[ J_m\left(\frac{X_{nm}}{a} r\right)\right]^2 rdr = \frac{a^2}{2}\left[J_{m+1}\left(X_{nm}\right)\right]^2 \tag{3.50}$$

with

$$\tilde{C} = \frac{\sqrt{2}}{aJ_{m+1}\left(X_{nm}\right)}. \tag{3.51}$$

Hence, the normalized solution is given by

$$R_{nm}(r) = \frac{\sqrt{2}}{aJ_{m+1}\left(X_{nm}\right)} J_m\left(\frac{X_{nm}}{a} r\right). \tag{3.52}$$

For $m = 0$, we have radial wave function which reads

$$R_{n0}(r) = \frac{\sqrt{2}}{aJ_1\left(X_{n0}\right)} J_0\left(\frac{X_{n0}}{a} r\right) \tag{3.53}$$

and corresponding energy

$$E_{n0} = \frac{\hbar^2}{2\mu a^2} X_{n0}^2. \qquad n = 1, 2, 3, \ldots \tag{3.54}$$

The complete eigenfunctions of the Hamiltonian of the particle in 2D spherical infinite well can be written as

$$\psi_{nm}\left(r,\varphi\right) = \frac{1}{a\ \pi^{\frac{1}{2}} J_{m+1}\left(X_{nm}\right)} J_m\left(\frac{X_{nm}}{a} r\right) e^{im\varphi}. \tag{3.55}$$

In Table 3.3 we give the roots of the Bessel functions which we used to find the energies of the particle in a 2D spherical infinite well. In Table 3.4 we give the relative energies of the particle in 2D well.

In Figures 3.3 and 3.4 we plot the radial probability densities of the first four states and the energy levels of the particle in a 2D infinite well respectively.

Table 3.3: Zeros of the Bessel functions $J_m(x)$[37].

| $n$ | $J_0(x)$ | $J_1(x)$ | $J_2(x)$ | $J_3(x)$ | $J_4(x)$ | $J_5(x)$ |
|---|---|---|---|---|---|---|
| 1 | 2.4048 | 3.8317 | 5.1356 | 6.3802 | 7.5883 | 8.7714 |
| 2 | 5.5201 | 7.0156 | 8.4172 | 9.7610 | 11.0647 | 12.3386 |
| 3 | 8.6537 | 10.1735 | 11.6198 | 13.0152 | 14.3725 | 15.7002 |
| 4 | 11.7915 | 13.3237 | 14.7960 | 16.2235 | 17.6160 | 18.9801 |
| 5 | 14.9309 | 16.4706 | 17.9598 | 19.4049 | 20.8269 | 22.2178 |

Table 3.4 Energy levels proportional to the square of zeros of the Bessel Functions as given by $\frac{E_{nm}}{\varepsilon} = (X_{nm})^2, where\ \varepsilon = \frac{\hbar^2}{2\mu a^2}$.

| $n$ | $(X_{n0})^2$ | $(X_{n1})^2$ | $(X_{n2})^2$ | $(X_{n3})^2$ | $(X_{n4})^2$ | $(X_{n5})^2$ |
|---|---|---|---|---|---|---|
| 1 | 5.78 | 14.68 | 26.37 | 40.71 | 57.58 | 76.94 |
| 2 | 30.47 | 49.22 | 70.85 | 95.28 | 122.43 | 152.24 |
| 3 | 74.89 | 103.5 | 135.02 | 169.40 | 206.57 | 246.50 |
| 4 | 139.04 | 177.52 | 218.92 | 263.20 | 310.32 | 360.24 |
| 5 | 222.93 | 271.25 | 322.55 | 376.72 | 433.76 | 493.63 |

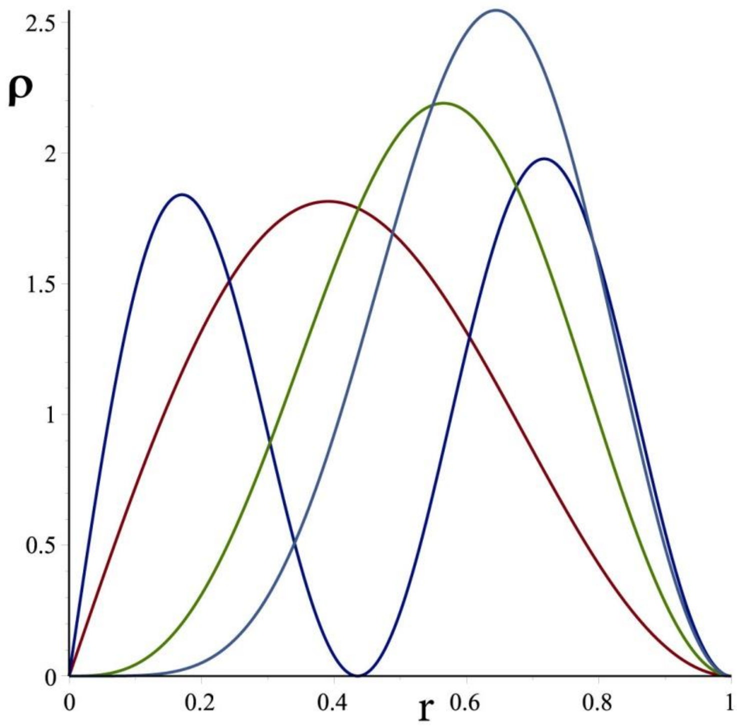


Figure 3.3: A plot of radial probability density function $\rho(r) = r|R_{nm}|^2$of the first four states of a quantum particle in a $2-$dimensional infinite well in terms of $r$. Red, green, light blue and dark blue are the corresponding densities of the first, second, third and fourth states respectively.

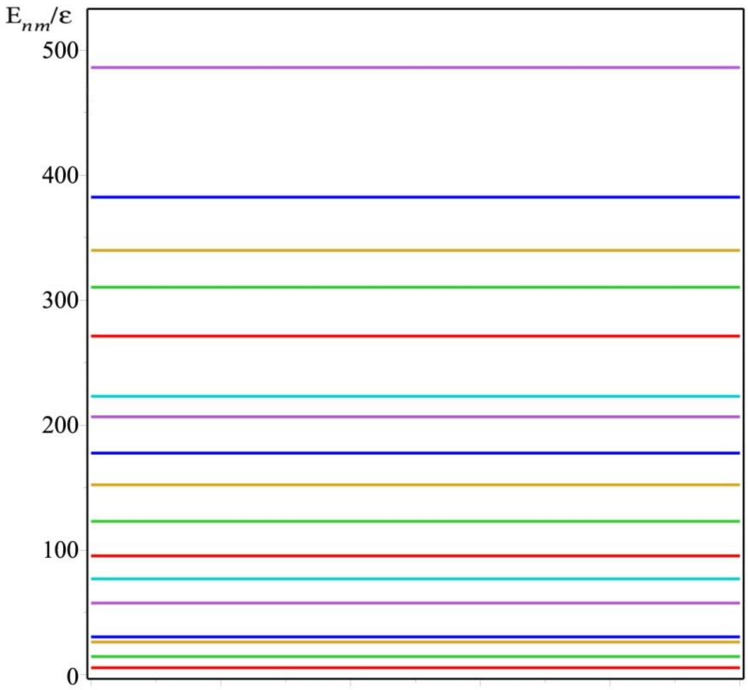


Figure 3.4: The energy levels over $\varepsilon = \frac{\hbar^2}{2\mu a^2}$ of a quantum particle confined in a $2-$dimensional infinite well.

## 3.3 Three Dimensional Spherical Infinite Well

In the previous section, we discussed the Schrödinger equation for describing a particle in 2D spherical infinite well. Now we solve time-independent Schrödinger equation of a quantum particle which is confined in a 3D spherical infinite well. Firstly, we can write the wave equation in spherical coordinates as [36]

$$\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\psi(r,\theta,\varphi)- \frac{2\mu r^2}{\hbar^2}(V(r)-E)\psi(r,\theta,\varphi)=0 \tag{3.56}$$

where, its solution is known from separation of variables,

$$\psi(r,\theta,\varphi)=R(r)Y(\theta,\varphi) \tag{3.57}$$

or even

$$\psi(r,\theta,\varphi)=R(r)\Theta(\theta)\Phi(\varphi). \tag{3.58}$$

Let's consider an infinite spherical well. The equations for all variables are given by

$$\frac{d^2\Phi}{d\varphi^2}=-m^2\Phi \tag{3.59}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)+l(l+1)\Theta(\theta)=m^2\Theta(\theta), \tag{3.60}$$

and

$$\frac{d^2R(r)}{dr^2}+\frac{2}{r}\frac{dR(r)}{dr}+\left[k^2-\frac{l(l+1)}{r^2}\right]R(r)=0, \tag{3.61}$$

in which, $m^2$ and $l(l+1)$ are separation constants, and

$$k^2=\frac{2\mu E}{\hbar^2}. \tag{3.62}$$

Now, let's assume $x=\cos\theta$, then the equation (3.60) will change to the form of generalized Legendre differential equation [36]

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{d\Theta(x)}{dx}\right)+\left[l(l+1)-m^2\right]\Theta(x)=0. \tag{3.63}$$

The solution to the angular equations with spherically symmetric boundary conditions are

$$\Phi_m(\varphi)=\frac{1}{\sqrt{2\pi}}e^{im\varphi}, \tag{3.64}$$

and

$$\Theta(\theta)=A_{lm}P_l^m(\cos\theta) \tag{3.65}$$

where $A_{lm}$ is constant with, $m=0,\pm1,\pm2,\pm3,...,\pm l$ and, $l=0,1,2,3,4.....$ By using orthogonality of associate Legendre functions, the normalization constant has a form

$$A_{lm}=\sqrt{\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}} \tag{3.66}$$

Which upon that, the solution can be written as

$$\Theta_l^m(\theta)=\sqrt{\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta). \tag{3.67}$$

It is customary to combine the two angular factors in terms of known functions

which are spherical Harmonics given by

$$Y_l^m(\theta,\varphi)=\in\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\varphi} \tag{3.68}$$

where $\in=(-1)^m$ for $m>0$ and $\in=1$ for $m<0$. Eq. (3.68) is the normalized angular part of the wave function and the spherical Harmonics are orthonormal i.e. [23]

$$\int_0^{\pi}\int_0^{2\pi} Y_l^m(\theta,\varphi) Y_{l'}^{m'}(\theta,\varphi)\, d\Omega = \delta_{ll'}\ \delta_{mm'}. \tag{3.69}$$

In which $d\Omega = \sin\theta\, d\theta d\varphi$ . While the angular part of wave function is $Y_l^m(\theta,\varphi)$ for all spherically symmetric situation, the radial part varies. To solve the radial equation Eq. (3.61) with $V(r) = 0$ one finds

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + \left[k^2 - \frac{l(l+1)}{r^2}\right]R(r) = 0. \tag{3.70}$$

A transformation of the form $x = kr$ yields

$$\frac{d^2R(x)}{dx^2} + \frac{2}{x}\frac{dR(x)}{dx} + \left[1 - \frac{l(l+1)}{x^2}\right]R(x) = 0 \tag{3.71}$$

which is Bessel's equation. Its general solution is

$$R(x) = \tilde{A}j_l(x) + \tilde{B}n_l(x) \tag{3.72}$$

where $\tilde{A}$ and $\tilde{B}$ are integration constants, $j_l(x)$ and $n_l(x)$ are spherical Bessel functions and spherical Neumann functions respectively. Whose behavior for $r = 0$ and $r = a$ are given by [22, 24]

$$j_l(x) \underset{x\to 0}{\approx} \frac{2^l l!}{(2l+1)!} x^l, \tag{3.73}$$

$$n_l(x) \underset{x\to 0}{\approx} -\frac{(2l)!}{2^l l!} x^{-(l+1)}. \tag{3.74}$$

The point is that the Bessel function are finite at the origin, but the Neumann functions blow up at the origin. Accordingly, we must have $\tilde{B} = 0,$ which implies

$$R_l(x) = \tilde{A}\, j_l\left(x\right) = \tilde{A}\sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x). \qquad (3.75)$$

The boundary condition fixes the $k$ and the energy i.e.,

$$j_l\left(ka\right) = 0 \qquad (3.76)$$

that is, $ka$ is a zero of $l^{th}$ order spherical Bessel function. The Bessel functions are oscillatory and therefore each one possesses infinite number of zeros. At any rate, the boundary condition requires

$$k_n a = X_{nl} \qquad (3.77)$$

where, $X_{nl}$ is the $n^{th}$ zero of the $l^{th}$ spherical Bessel function. Thus,

$$k_n^2 = \frac{X_{nl}^2}{a^2} \qquad (3.74)$$

and the allowed energies can be written as

$$E_{nl} = \frac{\hbar^2}{2\mu a^2} X_{nl}^2. \qquad (3.75)$$

The normalized radial solution finally is given by

$$R_{nl}(r) = \frac{\sqrt{2}}{a^{3/2} j_{l+1}\left(X_{nl}\right)} j_l\left(\frac{X_{nl}}{a} r\right) \qquad (3.76)$$

In which $\tilde{A}$ is the normalization constant is found to be

$$\tilde{A} = \frac{\sqrt{2}}{a j_{l+1}\left(X_{nl}\right)}. \qquad (3.77)$$

Explicit expressions for the first few $j_l(x)$ are

$$j_0(x) = \frac{\sin x}{x} \tag{3.78}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \tag{3.79}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - \frac{3}{x^2}\cos x. \tag{3.80}$$

The well-behaved solution for the particle in the 3D infinite spherical well can be represented by

$$\psi_{nlm}(r,\theta,\varphi) = \frac{\sqrt{2}}{a^{3/2} j_{l+1}(X_{nl})} j_l\left(\frac{X_{nl}}{a} r\right) Y_l^m(\theta,\varphi). \tag{3.80}$$

From the beginning we have the case of $m = 0$ which is azimuthal symmetry, so that the solution (3.80) reads

$$\psi_{nl}(r,\theta) = \frac{\sqrt{2}}{a^{3/2} j_{l+1}(X_{nl})} j_l\left(\frac{X_{nl}}{a} r\right) P_l\ (\theta) \tag{3.81}$$

In Table 3.5 we present the first five roots of the Bessel functions and following that in Table 3.6 we find the relative energies of the particle in the 3D spherical infinite well.

Figure 3.5 shows the radial probability densities of the particle in a 3D spherical well for the first four states. In Figure 3.6 the relative energy levels of the particle is shown.

Table 3.5 Zeros of the spherical of the Bessel functions $J_l(x)$ [45].

| $n$ | $j_0(x)$ | $j_1(x)$ | $j_2(x)$ | $j_3(x)$ | $j_4(x)$ | $j_5(x)$ |
|---|---|---|---|---|---|---|
| 1 | 3.1416 | 4.4934 | 5.7635 | 6.9879 | 8.1826 | 9.3558 |
| 2 | 6.2832 | 7.7253 | 9.0950 | 10.4171 | 11.7049 | 12.9665 |
| 3 | 9.4248 | 10.9041 | 12.3229 | 13.6980 | 15.0397 | 16.3547 |
| 4 | 12.5664 | 14.0662 | 15.5146 | 16.9236 | 18.3013 | 19.6532 |
| 5 | 15.7080 | 17.2208 | 18.6890 | 20.1218 | 21.5254 | 22.9046 |

Table 3.6: Energy levels proportional to the square of zeros of the spherical Bessel Functions as given by $\frac{E_{nl}}{\varepsilon} = (X_{nl})^2, where\ \varepsilon = \frac{\hbar^2}{2\mu a^2}$.

| $n$ | $(X_{n0})^2$ | $(X_{n1})^2$ | $(X_{n2})^2$ | $(X_{n3})^2$ | $(X_{n4})^2$ | $(X_{n5})^2$ |
|---|---|---|---|---|---|---|
| 1 | 9.87 | 20.19 | 33.22 | 48.83 | 66.95 | 87.53 |
| 2 | 39.48 | 59.68 | 82.72 | 108.52 | 137.01 | 168.13 |
| 3 | 88.83 | 118.90 | 151.85 | 187.64 | 226.19 | 267.48 |
| 4 | 157.91 | 197.86 | 240.70 | 286.41 | 334.94 | 386.25 |
| 5 | 246.74 | 202.23 | 349.28 | 404.89 | 463.34 | 524.62 |

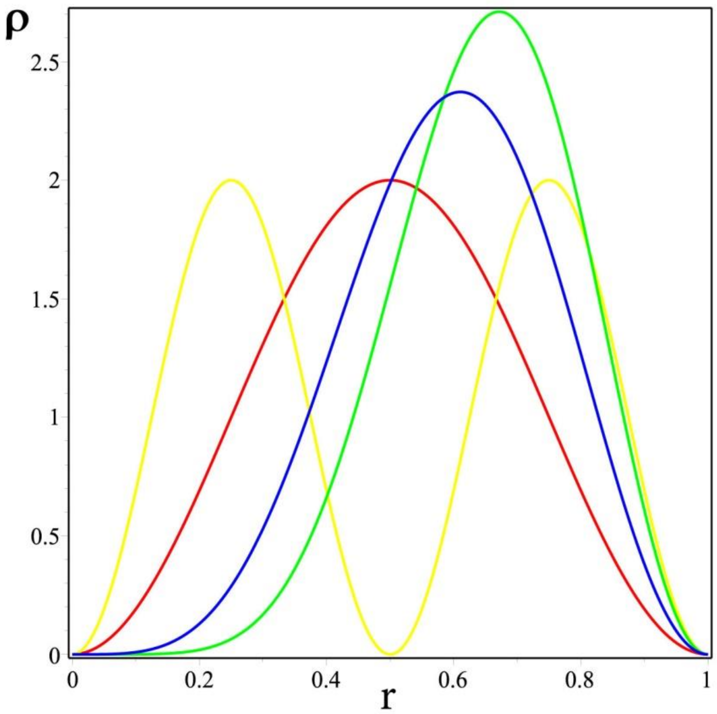


Figure 3.5: A plot of radial probability density function $\rho(r) = r^2|R_{nl}|^2$ of the first four states of a quantum particle in a 3-dimensional infinite well in terms of $r$. Red, blue, green and yellow are the corresponding densities of the first, second, third and fourth states respectively.

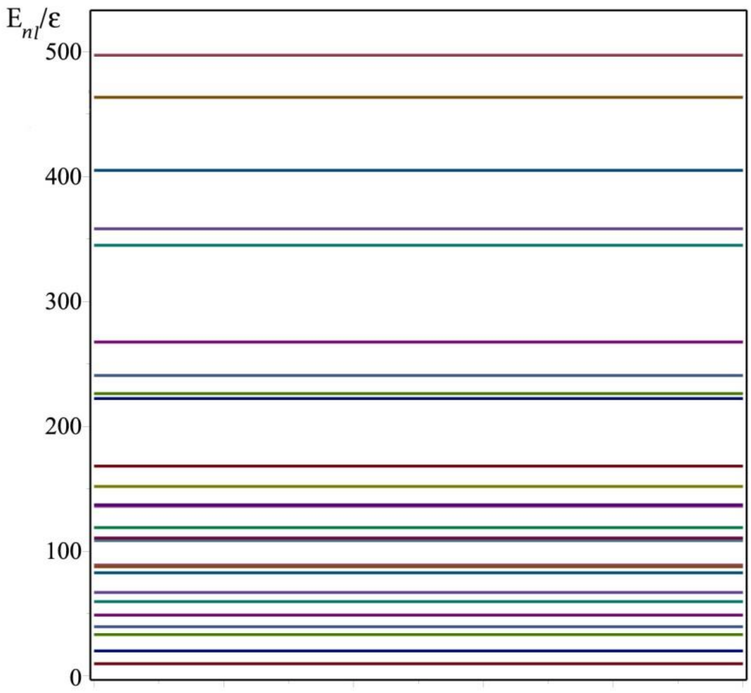


Figure 3.6: The energy levels over $\varepsilon = \frac{\hbar^2}{2\mu a^2}$ of a quantum particle confined in a $3-$dimensional infinite well.

# Chapter 4

# CONCLUSION

We have studied the Schrödinger equation in noninteger dimensions. The general form of the equation is introduced and for specific potential well is solved. Energy eigenvalues and energy eigenkets are found. The general solutions are given in terms of Bessel functions and Gegenbauer polynomial functions. Our work shows that noninteger dimensional systems are solvable exactly. This is the main purpose of this study. Having exactly solvable systems open the gates to the more complicated systems with practical applications. Mathematically speaking, noninteger dimensions means a new type of differential equations but in physics they represent some corners in our nature which are not seen in the integer dimensional frames. We believe that this work can be extended to the more realistic potential and upon that the applications will be manifested.

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