Invariants and Bestvina-Brady Groups

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ABSTRACT

The aim of this thesis is to compute the Euler Characteristic of the Bestvina-Brady groups. First, the analytical topology, the point-set topology is introduced. Then, the main purpose of the use of algebraic topology; the homeomorphism problem is stated. To understand the homeomorphism problem, certain well-known topological spaces are defined, and to solve it, the notion of topological invariants is introduced. Two main topological invariants; Euler characteristic and the Fundamental group theories are studied. Finally, by the use of C.T.C. Wall and the Bestvina-Brady papers, under certain homotopic conditions the Euler Characteristic of the Bestvina-Brady groups is computed.

Keywords: Topological invariants, Euler Characteristic, Fundamental Group, Bestvina-Brady Group

ÖZ

Bu tezin amacı Bestvina-Brady grupların Euler karakteristiğini hesaplamaktır. İlk olarak analitik topoloji çeşidi olan nokta-küme topolojisi tanımlanmıştır. Sonrasında, cebirsel topolojinin başlıca amacı olan homeomorfizm problemi tanımlanmıştır. Homeomorfizm problemini anlayabilmek için, iyi bilinen bazı topolojik uzaylar tanımlanmıştır. Topolojik değişmezlerin tanımı verilmiş ve homeomorfizm problemini çözmede nasıl kullanıldıkları anlatılmıştır. İki temel topolojik değişmez konusu çalışılmıştır; bunlar Euler karaktersitik ve Fundamental grup teorileridir. Son olarak, C.T.C. Wall ve Bestvina-Brady çalışmaları referans baz alınarak Bestvina-Brady grupların Euler karakteristiği hesaplanmıltır.

Anahtar kelimeler: Topolojik değişmezler, Euler karakteristik, Fundamental grup, Bestvina-Brady grup.

To my family

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Chapter 1

INTRODUCTION

In my thesis, the Euler characteristic of the Bestvina-Brady groups are studied. To determine the Euler characteristic of the Bestvina-Brady groups, we take the C.T.C Wall's paper [1961] and also the Bestvina-Brady paper as two main references. In C.T.C Wall's paper[8], the Euler characteristic of a group is defined to be the Euler characteristic of its classifying space BG if this exists; $\chi(G) = \chi(BG)$. BG space is a path connected, connected space where the universal cover is contractible. Therefore, spaces and universal covers are defined in my thesis.

In chapter 2, the topics of the point-set topology such as the open and closed sets, interior, exterior, closure,accumulation points and basis for topology, are defined. Also, the basic theorems and some useful examples are given.

Chapter 3 is related with the algebraic topology. As mentioned before, to compute the Euler characteristic of the Bestvina-Brady group, definition of the BG space is used. For this reason spaces are introduced. Also, the aim of the algebraic topology, the homeomorphism problem, is defined in this chapter. Topological equivalence between the topological spaces can be established by using the properties of topology, but sometimes it is not so easy to determine the homeomorphism. Therefore, topological invariants are used. So, the most known topological invariants such as the orientability, nonorientabilirty, the Euler characteristic and the Fundamental group, are defined.

In chapter 4, the Fundamental group which is the most known topological invariant is introduced in detail. First of all, homotopic maps are defined and some important examples are given. Then, construction of the fundamental group is given by a theorem and the proof of the theorem is extended by using a paper.

Another important section in this chapter is the universal covering space (Universal covers will be used in the last chapter) and also the actions on the topological spaces. By using the universal covers and the actions, we can find the fundamental group of spaces and surfaces. So, in the last section of this chapter, fundamental group of spaces and surfaces are computed.

In the last chapter, we first give the definition of a flag complex L. Then, G_L is defined and $\phi : G_L \to \mathbb{Z}$ is composed. The kernel of the map ϕ is called the Bestvina-Brady group(H_L). The aim of my thesis is to compute the Euler characteristic of the H_L groups. First of all, a topological space need to be assigned to the Bestvina-Brady group H_L. Here we use the definition $\chi(G) = \chi(BG)$ providedin C.T.C. Wall's paper. BG space is a path connected space with $\pi_1(BG) = G$. The starting map ϕ, is the map of the fundamental groups. Therefore,

$$l : BG_L \to S^1.$$

Then, l is lifted to the universal covers and we get a new map which is

$$f : EG_L \to \mathbb{R}.$$

Here we refer to the Bestvina-Brady paper which shows that when L is contractible, $f^{-1}(p)/H_L$ is a finite BH_L. Hence, $\chi(H_L) = \chi(BH_L)$. In here, $p \in \mathbb{R}$, so in all examples integer and noninteger cases are considered.

Chapter 2

OVERVIEW ON POINT-SET TOPOLOGY

Topics of point-set topology is related with the topics of analysis. In this section, I will define the topology and subspace topology. Then, some definitions will be given such as open and closed sets. Interior, exterior, closure and accumulation points will be introduced also. In addition, basis will be seen.

2.1 Topology and Open Sets

Definition 2.1.1 *Given a set X, let τ be a set of some subsets of X. If τ satisfies the given conditions then τ is called a topology on X and each element of τ is called an open set.*

$\tau 1)$ *Empty set and X belong to the family τ.*

$\tau 2)$ *Union of any sub-family of τ is an element of τ.*

(J is finite or infinite) For all $i \in J$ there exist $A_i \in \tau \Longrightarrow \bigcup_{i \in J} A_i \in \tau$

$\tau 3)$ *Intersection of finite sub-family of τ is an element of τ.*

(J is finite) For all $i \in J$ there exist $A_i \in \tau \Longrightarrow \bigcap_{i \in J} A_i \in \tau$

Remark 2.1.2 *From the previous definition, X and $\varnothing$ are open sets. Union of finite or infinite number of open sets is open. Intersection of finite number of open sets is also open.*

Definition 2.1.3 *The family τ defined above is said to be a topology on the set X. (X, τ) is called a topological space.*

Example 2.1.4 *Given a nonempty set* X*, let* τ *be the power set of* X*. That means, for all* $x \in X$ *there exist* $\{x\} \in \tau$*. This topology* τ *on* X *is said to be the discrete topology.*

Example 2.1.5 *Given any set* X*, let* $\tau = \{\varnothing, X\}$*. This topology on* X *is said to be indiscrete topology.*

Example 2.1.6 *Given a set* X*, let* $\tau = \{U \subset X \mid U = \varnothing \text{ or } U' \text{ is finite}\}$*. Topology* τ *is called the finite complement topology.*

Theorem 2.1.7 *Let* $(\tau_i)_{i \in I}$ *be a collection of topologies on* X*. Then the intersection of all* τ_i *is also a topology on* X*.*

Proof.

$\tau 1)$ If the empty set and X belong to the family τ_i for all i, then the empty set and X belong to the intersection of all τ_i.

$\tau 2)$ Assume $U_i \in \bigcap \tau_i$. Then, $U_i \in \tau_i$ for all i. Therefore, $\bigcup U_i \in \bigcap \tau_i$.

$\tau 3)$ Let $U_1, U_2 \in \bigcap \tau_i$. That means, $U_1, U_2 \in \tau_i$ for all $i \implies U_1 \cap U_2 \in \bigcap \tau_i$.

■

Definition 2.1.8 *Given a topological space* (X, τ)*, let* $F \subset X$ *. If* $F' \in \tau$ *(* F' *is open) then the set* F *is closed according to the topology* τ *on* X *.*

Remark 2.1.9 *Let* K *be a set of all closed subsets of the topological space* (X, τ)*.* $f : \tau \to K$ *denoted by* $f : A \to X - A$ *is one-to-one and onto.*

$$K = \{F \subset X \text{ such that } X - F = F' \in \tau\}$$

Remark 2.1.10 *When a set is closed with respect to a certain topology* τ *, then it is called a* τ*-closed set.*

Example 2.1.11 *In* $\mathbb{R}$, $[a,b]$, $(-\infty,a]$ *and* $[a,\infty)$ *are closed because their complements are open.*

Example 2.1.12 *In* $\mathbb{R}, (a,b]$ *is neither open nor closed.*

Example 2.1.13 *Each set of the discreate topology on the set X is both open and closed.*

Remark 2.1.14 *In topological spaces, we can't say every subset is either open or closed. Subsets can also be both open and closed or neither open nor closed.*

Theorem 2.1.15 *Given a topological space* (X,K) *. The family K satisfies the following properties:*

1. $\varnothing \in K, X \in K$
2. *Intersection of finite or infinite number of closed sets (family of K) is closed.*
3. *Union of finitely many closed sets (family of K) is also closed.*

Theorem 2.1.16 *Given a topological space* (X,τ)*, let U be an open subset and K be a closed one in* (X,τ)*. Then,*

1. $U-K$ *is open.*
2. $K-U$ *is closed.*

Definition 2.1.17 *(Subspace Topology) Let us have a topological space* (X, τ) *and let* $G \subset Y \subset X$*. G is called open in Y if there exists an open* $U \subset X$ *such that* $G = Y \cap U$*. This induced topology on* Y, τ_y *is said to be the subspace topology.*

Example 2.1.18 *Given a set X, let Y be* $[0,1]$*. Subset* $\left(\frac{1}{2},1\right]$ *is open in the subspace topology on Y but it is not open in* $\mathbb{R}$*.*

2.2 Interior, Exterior, Closure and Accumulation Points

In this section, some basic conceptions of topological spaces are introduced. Definitions of closed set, closure of a set, limit point and Hausdorff space will be given.

Definition 2.2.1 *The interior of A (a subset of a topological space X) is the union of all open sets in A and it is denoted by int A.*

Definition 2.2.2 *The exterior of a subset A of a topological space X is the* $int(X - A)$ *and is denoted by ext A.*

Definition 2.2.3 *Closure of A is the smallest closed set containing A and is denoted by* $\overline{A}$*. Note that*

$$intA \subseteq A \subseteq \overline{A}$$

The following definition is also commonly used to calculate the closure points.

Definition 2.2.4 *Let X be a topological space and* $A \subset X$*. Then,* $x \in \overline{A}$ *if and only if each open set U containing x intersects A.*

Definition 2.2.5 *(Limit points or accumulation points) Let X be a topological space* *and $A \subseteq X$. If every neighborhood of $x \in X$ intersects A in some point other than x itself, then it is called a limit point of A. The set of all limit points of A is denoted by $\widetilde{A}$.*

Theorem 2.2.6 *Let (X, τ) be a topological spaces and $A \subset X$. Then,*

$$\overline{A} = A \cup \widetilde{A}$$

Proof. From left to right,

$$\overline{A} \subset A \cup \widetilde{A}$$

If $x \in \overline{A}$ and $x \in A$. Then it can be easily seen that $x \in A \cup \widetilde{A}$. If $x \notin A$. That means $x \in \widetilde{A}$. Therefore, $x \in A \cup \widetilde{A}$. So, $\overline{A} \subset A \cup \widetilde{A}$. From right to left, we need to show that

$$A \cup \widetilde{A} \subset \overline{A}$$

Let $x \in \widetilde{A}$. For every $x \in U \in \tau$. There exist $(U \cap A) - \{x\} \neq \varnothing$. By using definition, $x \in \overline{A.}$ Therefore, $\widetilde{A} \subset \overline{A.}$ It is known that $A \subset \overline{A}$. Because of that reason

$$A \cup \widetilde{A} \ \subset \ \overline{A}$$

■

Definition 2.2.7 *(Hausdorff space) If for each two different points, there exists neighbourhoods that do not intersect, then the space is called Hausdorff spaace.*

2.3 Basis for a Topology

Definition 2.3.1 *[7] If X is a set, a basis for a topology on X is a collection $\mathscr{B}$ of subsets of X, such that*

b1) For each $x \in X$, there exists at least one $B \in \mathscr{B}$, containing x.

b2) If x belongs to the intersection of B_1 and B_2, then there exists a basis element B_3, containing x, such that $B_3 \subset B_1 \cap B_2$.

Example 2.3.2 $X = \{x, y, z, w\}$ *and* $\mathscr{B} = \{\{x\}, \{w\}, \{x, y, w\}, \{x, z, w\}\}$.

Example 2.3.3 $\mathscr{B} = \{(x, y) \mid x, y \in \mathbb{R}\}$ *is also a basis for the set* $X = \mathbb{R}$.

Note that if $\mathscr{B}$ is a basis, then for $B_1, B_2 \in \mathscr{B}$, $B_1 \cap B_2$ is not necessarily in $\mathscr{B}$.

Definition 2.3.4 *(Bases define topology) Given a topological space X, let the collection $\mathscr{B}$ be a basis for X.*

$\tau_B = \mathscr{B}^* = \{U \subset X$ *such that for all* $x \in U$*, there exists* $B \in \mathscr{B}$ *such that* $x \in B \subset U\}$

is said to be the topology $\tau_{\mathscr{B}}$ generated by $\mathscr{B}$. Note here that each basis element B is itself an element of $\tau_{\mathscr{B}}$.

Let us now check that definition does in fact produce a topology in X.

$\tau 1)$ $\varnothing \in \tau_{\mathscr{B}}$ and $X \in \tau_{\mathscr{B}}$ as;

$$For\ all\ x \in X,\ there\ exists\ B \in \mathscr{B}\ such\ that\ x \in B \subset X.$$

$\tau 2)$ Let us take U_1 and U_2 of $\tau_{\mathscr{B}}$, and show that $U_1 \cap U_2$ belongs to $\tau_{\mathscr{B}}$.

For all x$\in U_1 \cap U_2$, does there exists $B \in B$ *such thatx* $\in B \subset U_1 \cap U_2$?

x$\in U_1 \Longrightarrow$ *There exists* $B_1 \in \mathscr{B}$ *such that* $x \in B_1 \subset U_1$.

x$\in U_2 \Longrightarrow$ *There exists* $B_2 \in \mathscr{B}$ *such that* $x \in B_2 \subset U_2$.

x$\in B_1 \cap B_2 \Longrightarrow$ By using definition 2.3.1, *there exists* $\mathrm{B}_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$.

Now, by induction, show that any finite intersection $U_1 \cap U_2 \cap \ldots \cap U_n$ of elements of $\tau_{\mathscr{B}}$ is in $\tau_{\mathscr{B}}$. Trivial for $n = 1$. Suppose it's true for $n-1$ and prove it for n. Now,

$$U_1 \cap U_2 \cap \ldots \cap U_n = (U_1 \cap U_2 \cap \ldots \cap U_{n-1}) \cap U_n.$$

By hypothesis,

$$U_1 \cap U_2 \cap \ldots \cap U_{n-1}$$

belongs to $\tau_{\mathscr{B}}$, and by the result just proved

$$(U_1 \cap U_2 \cap \ldots \cap U_{n-1}) \cap U_n$$

also belongs to $\tau_{\mathscr{B}}$. If $U_i \in \tau_B$, then is the union of all U_i in $\tau_{\mathscr{B}}$? So, for all $x \in \bigcup_i U_i$. Does there exist

$$B \in \mathscr{B} \text{ such that } x \in B \subset \bigcup_i U_i \text{ ?}$$

If $x \in \bigcup_i U_i$ then $x \in U_i$ for at least one U_i. Since,

$$U_i \in \tau_B; \text{ there exists } B \in \mathscr{B} \text{ such that } x \in B \subset U_i \subset \bigcup_i U_i.$$

Example 2.3.5 *Let $\mathscr{B}$ be the collection of all circular regions (interiors of circles) in the plane. Then $\mathscr{B}$ satisfies both of conditions for a basis.*

Lemma 2.3.6 *Given a set X, let $\mathscr{B}$ be a basis for a topology τ on X. Then τ is the collection of all unions of elements of $\mathscr{B}$.*

Proof. Let $\mathscr{A}$ be the collection of all unions of elements of B. Show that $\mathscr{A} \subset \tau$; $\mathscr{B} \subset \tau$ by definition. Since τ is a topology, union of elements of $\mathscr{B}$ is also in τ. $\tau \subset \mathscr{A}$; Take $U \in \tau$. For all $x \in U$, there exists $B_x \in B$ such that $x \in B_x \subset U$. Therefore, $U = \bigcup_{x \in U} B_x$. Hence,

$$U \in \mathscr{A}.$$

■

Lemma 2.3.7 *Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for the topologies τ and τ', respectively, on X. Then, the followings are equivalent;*

1) $\tau \subset \tau'$ (τ *is corser than* τ')

2) *For each* $x \in X$ *and for each basis element* $B_x \in \mathscr{B}$ *containing* x, *there exists* $B_x' \in \mathscr{B}$ *such that* $x \in B_x' \subset B_x$.

Proof. $(2 \Longrightarrow 1)$ Let $U \in \tau$. For each $x \in U$, there exists $B_x \in \mathscr{B}$ such that $x \in B_x \subset U$. Since $x \in B_x$, there exists $B_x' \in \mathscr{B}'$. So that $x \in B_x' \subset B_x$ holds.

$\therefore$ For each $x \in U$, *there exists* $B_x' \in \mathscr{B}'$ such that $x \in B_x' \subset U \Longrightarrow U \in \tau'$.

$(1 \Longrightarrow 2)$ Let $B_x \in \tau$ (since $\mathscr{B} \subset \tau_{\mathscr{B}}$). So, by using definition 2.3.1, $B_x \in \tau'$.

$\therefore$ For each $x \in B_x$, *there exists* $B_x' \in \mathscr{B}'$ such that $x \in B_x' \subset B_x$ holds.

Hence (2) is satisfied. ■

Example 2.3.8 *The collection* $\mathscr{B}$ *of all circular regions in the plane generates the same topology as the collection* $\mathscr{B}'$ *of all rectangular regions.*

We now define 3 topologies on the real line $\mathbb{R}$;

Definition 2.3.9 *Let* $\mathscr{B}$ *be the collection of all open intervals in* $\mathbb{R}$; $(a_1, b_1) = \{ x \mid a_1 < x < b_1\}$. *Then the topology generated by* $\mathscr{B}$ *is called the* ***standard topology*** *on* $\mathbb{R}$.

If $\mathscr{B}'$ *is the collection of all half-open intervals of the form* $[a_1, b_1)$, *then the topology generated by* $\mathscr{B}'$ *is called the lower-limit topology on* $\mathbb{R}$. *Lower limit topology will be denoted by* τ_l.

Lemma 2.3.10 *The* $\mathbb{R}_l$ *topology is strictly finer than the standard topology on* $\mathbb{R}$. $(\tau \subseteq \tau_l)$

Proof. *For all* $(a1,b_1)\in\tau$ and for all $x\in(a_1,b_1)$, there exists $[c_1,d_1)\in\tau_l$ such that $x\in[c_1,d_1)\subset(a_1,b_1)$.

But the reverse is not true; For all $[a_1,b_1)\in\tau_l$ and for all $y\in[a_1,b_1)$, there does not exist $(c_1,d_1)\in\tau$ *such that* $y\in(c_1,d_1)\subset[a_1,b_1)$ ■

Chapter 3

OVERVIEW ON ALGEBRAIC TOPOLOGY

In this chapter, the basic aim of the algebraic topology will be stated. Algebraic topology studies on the shapes and their properties. Topologically equivalence (homeomorphism) is one of the most important notions of the algebraic topology. To determine the homeomorphism, first of all, we use general properties of topology which are twisting, stretching and extending. For more complicated surfaces topological invariants are used such as orientability, Euler characteristic and the Fundamental group. In addition, homology, cohomology groups, homotopy groups, Connectedness, Hausdorffness are other important topological invariants.

3.1 Surfaces

As mentioned before, topology tries to answer the homeomorphism problem between the topological objects. Before we state the homeomerphism problem some well-known surfaces such as the sphere, torus, Möbius strip, Klein bottle and projective space should be studied. These surfaces can be classified according to their connectedness.

3.1.1 Connected, Path Connected and Simply Connected Spaces

Definition 3.1.1 *Let us have a topological space X. The space X is called a connected space when it can not be defined by the union of two or more disjoint nonempty open subsets.*

Definition 3.1.2 *Let us have a topological space X and let x, y be points of X. There* *exists a continuous map* $g:[a_1,b_1]\to X$ *which we call a path in X from* x_1 *to* y_1 *, such that* $g(a_1)=x_1$ *and* $g(b_1)=y_1$*. If there exists a path between each pair of points in a topological space* X*, then it is called a* $path-connected\ space$*. A path-connected space is also a connected space but the converse is not true.*

Definition 3.1.3 *[6] A topological space* X *is called a* $simply-connected\ space$ *if it is path-connected and if every simple closed curve C in* X *encloses only points in* X*.*

Example 3.1.4 *A sphere is simply connected space. If we draw a loop around the surface and continue to draw a loop then it can be seen that every loop can be contracted to a point.*

Example 3.1.5 *Disc is simply connected. The reason is the same with the previous example, every loop can be reduced to a point.*

Example 3.1.6 *Torus is not simply connected space because it has hole inside. Therefore, it has two different loops. One of them is on the x-axis and the other one is on the y-axis. They can not be identified to each other.*

3.2 Topological Equivalence (Homeomorphism)

When the meaning of topology is studying, example of dougnut and coffee cup is mostly seen. This example is the easiest way to understand the homeomorphism between two topolological objects. For topologists, coffee cup and the dougnut are topologically same. To understand this equivalence, let's think of the coffee cup. When the bottom and the top of the coffee cup are pushed down from the place of the handle, then the final picture of the coffee cup is now the dougnut. Therefore, for topologists, dougnut and coffee cup are same.

Another engrossing example is that, the triangle and the square is topologically equivalent. Both of them are topologically equivalent to the circle. As I mentioned before, twisting and stretching are allowed in topology.

Using the properties of the topology is the easiest way to determine the homeomorphism between the objects. Let's consider other examples of homeomorphism.

Example 3.2.1 *Are tetrahedron and sphere topologically equivalent. If the vertices of the tetrahedron are extended, then the sphere can be found. Thefore, they are topologically equivalent to each other.*

Example 3.2.2 *Now consider the sphere and the torus. It can clearly be seen that they are not topologically equivalent. Torus has a hole inside and has two different generators but, on the sphere every loop can be contracted down to a point.*

There is also a formal definiton of homeomorphism, but sometimes it is not so easy to find a function between the topological objects.

Definition 3.2.3 *If $f : X \to Y$ is one-to-one, onto, continuous and has a continuous inverse , then it is called a homeomorphism and also X and Y are said to be topologically equivalent or homeomorphic spaces.*

As it is written before, it is not easy to find such a function between the topological surfaces. We can use twisting and stretching, but in topology there are several methods to determine the homeomorphism between surfaces. These methods are called topological invariants. In algebraic topology, objects are preserved through deformations, twistings, and strechings. Tearing is not allowed. The general idea of the topology is that, it tries to answer the homeomorphism problem between topological objects but generally it is not so easy ; hence topological invariants are used to determine homeomorphism between the topological objects. A topological invariant of a space X is a property that depends only on the topology of the space. Euler characteristic, orientability, homology, cohomology groups, homotopy groups, Connectedness, Hausdorffness and Fundamental group are some known topological invariants. In my thesis, I analyse the Euler characteristic and the Fundamental group invariants.

3.3 Orientability and Non-Orientability

Definitions of orientable and nonorientable surfaces are related with directions of the tangent and normal vectors. When a smooth closed curve is drawn on the surface and any tangent and normal vector is pushed once around the curve, if the directions of the tangent and normal vectors are same when they come back to the initial point; the surface is called orientble. In the same way; if the direction of the normal vector is reversed when it comes back to the initial point, then the surface is called non-orientable.

Example 3.3.1 *Torus is an orientable surface. If we draw a smooth closed curve on the torus, and start to push any tangent and normal vector once around the curve, then we come back we find the same direction for the tangent and the normal vector. Therefore, torus is orientable.*

Example 3.3.2 *Möbius strip is a non-orientable surface. In the same way; if we draw a smooth closed curve on the Möbius strip, and start to push any tangent and normal vector once around the curve, then the direction of the normal vector is reversed.*

Therefore, the surface is called non-orientable.

Note that; if a surface does not contain a Möbius strip, it is called orientable otherwise it is nonorientable. For intance, the Klein bottle is nonorientable because it contains a Möbius strip. In addition, any surface which doesn't contain one Möbius strip but is the union of two or more Möbius strips, is also called nonorientable.

In addition, we can determine the homeomorphism between two topological spaces by using the definitions of orientability and nonorientability.

Example 3.3.3 *Let's consider torus and Möbius strip. Torus is orientable surface and Möbius strip is not. Therefore, they can not be homeomorphic to each other.*

Example 3.3.4 *Let's consider the cylinder and the Klein bottle. Cylinder is orientable and Klein bottle is not(because when the normal vector comes back to the initial point, it is reversed). That means, they can not be homeomorphic to each other.*

3.4 Euler Characteristic

In algebraic topology, Euler characteristic is a topological invariant which finds a number by using the number of vertices, edges and faces of the topological space. Let S be a surface, then the Euler characteristic $\chi(S)$ is defined by the formula

$$\chi(S) = v - e + f$$

Theorem 3.4.1 *(Euler's Theorem) [1] Let P be a polyhedron which satisfies:*

- *Any two vertices of P can be connected by a chain of edges.*
- *Any loop on P which is made up of straight line segments(not necessarily edges) separates P into two pieces. Then*

$$v - e + f = 2$$

for P.

If P is a polyhedron, then number of vertices, edges and faces can be found. That means, the Euler characteristic of P can be computed easily.

Example 3.4.2 *Let P be a tetrahedron. Then, number of vertices is* 4*, number of edges is* 6 *and number of faces is* 4*. When we compute the Euler characteristic, we find that*

$$\chi(P) = 2.$$

If P is a cube, then number of vertices is 8, number of edges is 12 and number of faces is 6. Then,

$$\chi(P) = 2$$

Let us now the Euler characteristic of some well known surfaces such as the torus, Möbius strip, cyclinder or the projective plane. For these surfaces, it is not possible to compute the Euler characteristic because it is not so easy to determine the number of vertices, edges and faces. Therefore, the planar model is constructed, then the number of vertices, edges and faces can be found.

Example 3.4.3 *Torus can be constructed from a rectangle by gluing both pairs of opposite edges. When the opposite edges are glued, then there exist* 2 *different edges. Also, all vertices are identified with each other. So, in total,* 1 *vertex exist. From the planar model, it can be seen that, the number of face is* 1*. Let us denote the torus by* T*. Then,*

$$\chi(T) = 0$$

Example 3.4.4 *Now, let's consider the Möbius strip. Möbius strip can be created from a rectangle by identifying a single pair of opposite edges in different directions. Then, there exists a half-twist and the ends of the strip are joined together. Therefore, the Euler characteristic of the Möbius strip is* 0*.*

Note that; if two objects are topologically same, they have the same Euler characteristic but objects with the same Euler characteristic need not be topologically equivalent. As an example, torus and Möbius strip are not topologically same because torus is orientable and Möbius strip is not. Similarly, sphere and torus can not be homeomorphic as they have different Euler number.

Chapter 4

THE FUNDAMENTAL GROUP

The fundamental group is one of the techniques used to determine whether two topological spaces are homeomorphic to each other or not. To determine the homeomorphism between two spaces, a continuous map should exist and it should also have continuous inverse.

The closed interval $[0,2]$ can not be homeomorphic to the open interval $(0,2)$. The reason is that, $[0,2]$ is a closed interval so it is compact.(By Heine-Borel theorem [7] every closed interval in $\mathbb{R}^n$ is compact) but $(0,2)$ is not compact.

Compactness is another topological invariant used to determine the homeomorphism between topological spaces. Connectedness, local connectedness are the other topological invariants. For instance, the plane $\mathbb{R}^2$ is not homeomorphic to three-dimensional space $\mathbb{R}^3$. How can we show the homeomorphism by using the compactness or the other topological properties? To answer this question new techniques have been introduced. In this chapter, the fundamental group and its properties are discussed.

4.1 Homotopic Maps

The main idea here is to manufacture loops in the space which begin and end at some specified point(generally this is called the base point). Let's consider a space X. A loop in this space is defined by a map $\alpha : I \to X$ such that $\alpha(0) = \alpha(1)$. Therefore, we can say that $\alpha(0)$ is the base point of the loop (That means loop starts and finishes at the same point). Product loop $\alpha.\beta$ is defined by

$$\alpha.\beta(s) = \begin{cases} \alpha(2s), & 0 \leq s \leq 1/2 \\ \beta(2s-1), & 1/2 \leq s \leq 1 \end{cases}$$

Here α and β are two loops which start and finish at the same point of X. Unfortunately, this product $\alpha.\beta$ does not form a group structure based at a specific point because of the failure of the associative property of a group. At this point, contiuous deformation will be important for this section.To resolve the failure of group structure we should consider this way: Identify two loops if one of them can continuouly be deformed into the other loop. Then keep the specific point (base point) fixed thoughout the deformation. Continuous deformation will be called a homotopy.

Definition 4.1.1 *[1] Suppose we have the maps $f : X \to Y$ and $g : X \to Y$. If there exists a map $F : X \times I \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$, then f is said to be homotopic to g.*

[4] Homotopy F is said to be a homotopy from f to g and is denoted by $f \underset{F}{\simeq} g$. If, in addition, f and g agree on some subset A of X, we may wish to deform f to g without alterning the values of f on A. In this case we ask a homotopy F from f to g with the additional property that

$$F(a_1,t_1) = f(a_1) \text{ for all } a_1 \in A, \text{ for all } t_1 \in I.$$

When there exists such a homotopy, then f is said to be homotopic to g, relative to A and we write $f \underset{F}{\simeq} g$ *rel* A. We can give a brief example of continuous deformation on a torus. Let's consider two loops on a torus , $\alpha, \beta : I \to X$ which are based at the same point b of X. $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$ are equivalent or homotopic if and only if there exists $F : I \times I \to M$ (F is continuous function and M is torus) such that

$$F(t_1,0) = \alpha(t_1),\ F(t_1,1) = \beta(t_1),\ t_1 \in [0,1]$$

$$F(0,s_1) = \alpha(0) = \beta(0),\ F(1,s_1) = \alpha(1) = \beta(1),\ s_1 \in [0,1].$$

The main point is that, if α can continuously be deformed to β (base point can not change, it should be same), then we can say that α is homotopic to β relative to the subset $\{0,1\}$ of I. Start from the point 0 and end at point 1.

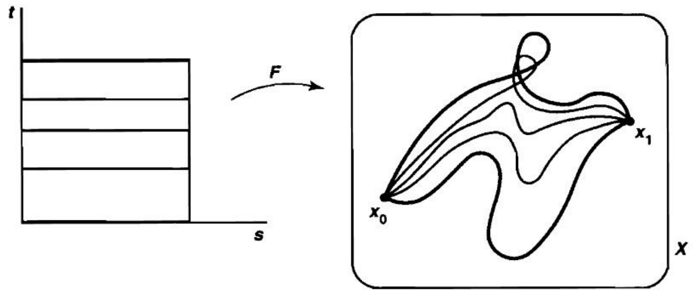

Figure 4.1. Homotopic path

Example 4.1.2 *[1] (Straight-line homotopy) Given a convex subset of a Euclidean space C, let $f, g : X \to C$ be maps (X is an arbitrary topological space). For every point x of X, the straight line joining $f_1(x)$ to $g_1(x)$ stays in C, and a homotopy is defined from f_1 to g_1 simply by sliding f_1 along these straight lines. To be precise, define $F : X \times I \to C$ by*

$$F(x,t) = (1-t)f_1(x) + tg_1(x)$$

To be more precise; we have the function $F : X \times I \to C$ (I is the unit interval $I = [0,1]$) where $f_1(x)$ and $g_1(x)$ here are any two functions. That means they are not necessarily linear. Then from the function

$$F(x,t) = (1-t)f_1(x) + tg_1(x)$$

we can find the equation of a straight-line. We have t variables and functions $f_1(x)$

and $g_1(x)$. Interval for $t_1 = [0,1]$. When the first point(t_1) is 0, $F(x,0) = f_1(x)$. When the second point(t_2) is 1, $F(x,1) = g_1(x)$. Then from the straight-line function

$$y = mt + f_1(x)$$

$$m = \frac{g_1(x) - f_1(x)}{t_2 - t_1} = g_1(x) - f_1(x)$$

Finally, we have

$$y = [g_1(x) - f_1(x)]t + f_1(x)$$

Example 4.1.3 *[1] Suppose we have the maps $h, l : X \to S^n$ which if evaluated on the same point of X never give a pair of antipodal points of S^n. (i.e, $h(x)$ and $l(x)$ are never at opposite ends of a diameter). If S^n is taken to be the unit sphere in E^{n+1} , and we think of h, l as maps into E^{n+1}, then we have a straight line homotopy from h to l.*

$$\text{F(x,t)=}\frac{(1-t)h(x) + tl(x)}{\|(1-t)h(x) + tl(x)\|}$$

$h(x)$ and $l(x)$ are not antipodal points.

Example 4.1.4 *[1] Given the unit circle S^1 in the complex plane, let α and β be two loops on S^1. α and β are defined by*

$$\alpha(s) = \begin{cases} \exp 4\pi is & 0 \leq s \leq 1/2 \\ \exp 4\pi i(2s-1) & 1/2 \leq s \leq 3/4 \\ \exp 8\pi i(1-s) & 3/4 \leq s \leq 1 \end{cases}$$

$$\beta(s) = \exp 2\pi is \qquad 0 \leq s \leq 1$$

α is a loop which runs the intervals $[0,1/2]$, $[1/2,3/4],[3/4,1]$ once round the circle and the first interval and second intervals run in anticlockwise direction. β is also a loop and it is defined by the $[0,1]$ closed interval, and goes once round the circle anticlockwise direction.

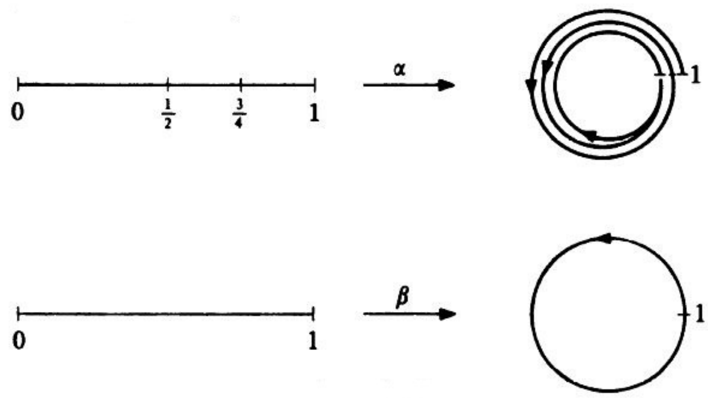

Figure 4.2. Homotopy on the circle

$$F(s,t)=\begin{cases}\exp\frac{4\pi is}{t+1} & 0\le s\le\frac{t+1}{2}\\ \exp 4\pi i(2s-1-t) & \frac{t+1}{2}\le s\le\frac{t+3}{4}\\ \exp 8\pi i(1-s) & \frac{t+3}{4}\le s\le 1\end{cases}$$

is a homotopy from α to β relative to $\{0,1\}$. As mentioned before, α is a loop running between the $[0,1]$ time interval and the 2 opposite directions and they have the same time interval. Therefore, the second and the third rounds cancel one another.Then, finally, α has only one round and it is homotopic to β relative to $\{0,1\}$. (β has only one round between the $[0,1]$ time interval)

Lemma 4.1.5 *[1] On the set of the whole maps from X to Y, the relation of homotopy produces an equivalence relation.*

Proof. The relation of homotopy is an equivalence relation if there exist reflexive, symmetry and transitivity relations. Therefore, first we show that the relation is reflexive. Let α be a path and $I\times I$ be a unit square. F is a continuous map which goes from bottom to top. Choose a function $F(t,s)$ such that it is constant on the vertical lines.

$$F(t,s)=\alpha(t)$$

For each value of s, $\alpha(t)$ is found and it is always the same. In other words, curve is homotopic to itself. So, the first property is satisfied. For the second property, suppose there exists a homotopy between α and β. (α and β are two paths) . Choose the fuctions $\widetilde{F}(t,s) = F(t,1-s)$. Then, we get the symmetry property. For the last property, we should combine the two homotopies.

$$L(t,s) = \begin{cases} F(t,2s) & 0 \leq s \leq \frac{1}{2} \\ K(t,2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Then, transitivity property is satisfied. ■

Lemma 4.1.6 *[1] Homotopy acts well regarding the composition of maps.*

4.2 Construction of the Fundamental Group

In this section we show that homotopy classes of loops based at a point, forms a group under the product operation.

Theorem 4.2.1 *[9] Suppose we have a topological space X and let $\alpha : [0,1] \to X$ be the set of homotopy classes $\langle\alpha\rangle$ based at points p. This set forms a group under the product*

$$\langle\alpha\rangle . \langle\beta\rangle = \langle\alpha.\beta\rangle .$$

Proof. First of all, we check that multiplication is associative. That means, we show that

$$\langle\alpha.\beta\rangle . \langle\gamma\rangle = \langle\alpha\rangle . \langle\beta.\gamma\rangle$$

for any three loops α, β, γ based at p. For associativity property we must show that $(\alpha.\beta).\gamma$ is homotopic to $\alpha.(\beta.\gamma)$ relative to $\{0,1\}$.To show that, first of all, we must compose ϕ with $(\alpha.\beta).\gamma$ where ϕ is the map from I to I defined by

$$\phi(s) = \begin{cases} \frac{1}{2}s, & 0 \le s \le \frac{1}{2} \\ s - \frac{1}{4}, & \frac{1}{2} \le s \le \frac{3}{4} \\ 2s - 1, & \frac{3}{4} \le s \le 1 \end{cases}$$

I is convex and $\phi(0) = 0$, $\phi(1) = 1$, so there is a straight-line homotopy from ϕ to the identity map 1_I relative to $\{0,1\}$. Let us consider

$$\alpha.(\beta.\gamma) = ((\alpha.\beta).\gamma) \circ \phi$$

$$\alpha.(\beta.\gamma)(s) = \begin{cases} \alpha(2s), & 0 \le s \le \frac{1}{2} \\ (\beta.\gamma)(2s-1), & \frac{1}{2} \le s \le 1 \end{cases}$$

If we expand the function $(\beta.\gamma)(2s-1)$, then we find

$$(\beta.\gamma)(2s-1) = \begin{cases} \beta(4s-2), & \frac{1}{2} \le s \le \frac{3}{4} \\ \gamma(4s-3), & \frac{3}{4} \le s \le 1 \end{cases}$$

Finally we get the function,

$$\alpha.(\beta.\gamma)(s) = \begin{cases} \alpha(2s), & 0 \le s \le \frac{1}{2} \\ \beta(4s-2), & \frac{1}{2} \le s \le \frac{3}{4} \\ \gamma(4s-3), & \frac{3}{4} \le s \le 1 \end{cases}$$

Let's consider the first interval which is $[0, \frac{1}{2}]$. For left hand side,

$$\alpha.(\beta.\gamma) = \alpha(2s)$$

For the right hand side,

$$((\alpha.\beta).\gamma)\circ\phi = ((\alpha.\beta).\gamma)(\frac{1}{2}s) = \begin{cases} (\alpha.\beta)(s) & 0 \le s \le 1 \\ \gamma(s-1) & 1 \le s \le 2 \end{cases}$$

$$= \begin{cases} \alpha(2s), & 0 \le s \le \frac{1}{2} \\ \beta(2s-1), & \frac{1}{2} \le s \le 1 \\ \gamma(s-1), & 1 \le s \le 2 \end{cases}$$

By using this table, it is clear that left hand side and right hand side are equal to each other between the interval $[0, \frac{1}{2}]$. Now, look at the next interval which is $[\frac{1}{2}, \frac{3}{4}]$. When we compute the left hand side,

$$\alpha.(\beta.\gamma) = \beta(4s-2)$$

For the right hand side,

$$((\alpha.\beta).\gamma)\circ\phi = ((\alpha.\beta).\gamma)(s - \frac{1}{4})$$

$$((\alpha.\beta).\gamma)(s - \frac{1}{4}) = \begin{cases} (\alpha.\beta)(2s - \frac{1}{2}) & \frac{1}{4} \le s \le \frac{3}{4} \\ \beta(2s - \frac{3}{2}) & \frac{3}{4} \le s \le \frac{5}{4} \end{cases}$$

When we expand $(\alpha.\beta)(2s - \frac{1}{2})$ and write all the intervals we get,

$$((\alpha.\beta).\gamma)(s - \frac{1}{4}) = \begin{cases} \alpha(4s-1) & \frac{1}{4} \le s \le \frac{1}{2} \\ \beta(4s-2) & \frac{1}{2} \le s \le \frac{3}{4} \\ \gamma(2s - \frac{3}{2}) & \frac{3}{4} \le s \le \frac{5}{4} \end{cases}$$

Therefore,

$$\alpha.(\beta.\gamma) = (\alpha.\beta).\gamma\circ\phi = \beta(4s-2)$$

Finally, the last interval is $[\frac{3}{4}, 1]$. For left hand side,

$$\alpha.(\beta.\gamma) = \gamma(4s-3)$$

For right hand side,

$$((\alpha.\beta).\gamma)(2s-1) = \begin{cases} \alpha(8s-4), & \frac{1}{2} \leq s \leq \frac{5}{8} \\ \beta(8s-5), & \frac{5}{8} \leq s \leq \frac{3}{4} \\ \gamma(4s-3), & \frac{3}{4} \leq s \leq 1 \end{cases}$$

$$\alpha.(\beta.\gamma) = (\alpha.\beta).\gamma \circ \phi = \gamma(4s-3)$$

It is clear that

$$\alpha.(\beta.\gamma) = ((\alpha.\beta).\gamma) \circ \phi$$

By using lemma 4.1.6,

$$\begin{aligned} \alpha.(\beta.\gamma) &= ((\alpha.\beta).\gamma) \circ \phi \\ &\backsimeq ((\alpha.\beta).\gamma) \circ 1_I \ rel \ \{0,1\} \\ &= (\alpha.\beta).\gamma \end{aligned}$$

Homotopy class of the constant loop at p, defined by $e(s) = p$ for $0 \leq s \leq 1$ e is the identity element. Similarly with the above argument we should check

$$\langle e \rangle . \langle \alpha \rangle = \langle \alpha \rangle$$

and

$$\langle \alpha \rangle . \langle e \rangle = \langle \alpha \rangle$$

for any loop α based at p. We need a homotopy relative to $\{0,1\}$ from $e.\alpha$ to α. Now $e.\alpha$ is the composition $\alpha \circ \phi$, where $\phi : I \to I$ is defined by

$$\phi(s) = \begin{cases} 0, & 0 \leq s \leq \frac{1}{2} \\ 2s-1, & \frac{1}{2} \leq s \leq 1 \end{cases}$$

The aim is same. Therefore, we should show that

$$e.\alpha = \alpha \circ \phi$$

For that, let's write

$$e.\alpha(s) = \begin{cases} e(2s), & 0 \leq s \leq \frac{1}{2} \\ \alpha(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

First interval is $[0, \frac{1}{2}]$. When we consider

$$\alpha \circ \phi = \alpha(0) = p.$$

The other interval is $[\frac{1}{2}, 1]$.

$$\alpha \circ \phi = \alpha(2s-1)$$

Obviously,

$$e \circ \alpha = \alpha \circ \phi$$

Similarly by using our lemma in previous section

$$e.\alpha = \alpha \circ \phi \cong \alpha \circ 1_I \ rel \ \{0,1\} = \alpha$$

On the other hand,we compose a homotopy relative to $\{0,1\}$ from $\alpha.e$ to α. Now $\alpha.e$ is the composition $\alpha \circ \phi$, where $\phi : I \to I$ is defined by

$$\phi(s) = \begin{cases} 2s, & 0 \leq s \leq \frac{1}{2} \\ 1, & \frac{1}{2} \leq s \leq 1 \end{cases}$$

By using the same method, we should show that

$$\alpha.e = \alpha \circ \phi$$

$\alpha.e(s)$ is composed in the interval $[0, 1]$

$$\alpha.e(s) = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2} \\ e(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

For the interval $[0,\frac{1}{2}]$,

$$\alpha \circ \phi = \alpha(2s)$$

The other interval is $[\frac{1}{2},1]$.

$$\alpha \circ \phi = \alpha(1) = p$$

By using lemma 4.1.6,

$$\alpha.e = \alpha \circ \phi = \alpha \circ 1_I rel\{0,1\} = \alpha$$

Finally, the inverse of the homotopy class C is defined by $\langle \alpha^{-1} \rangle$ where $\alpha^{-1} = \alpha(1-s)$, $0 \leq s \leq 1$. The inverse is well defined.To show $\langle \alpha \rangle . \langle \alpha^{-1} \rangle = \langle e \rangle$ we observe that $\alpha.\alpha^{-1} = \alpha \circ \phi$ where $\phi : I \to I$ is defined by

$$\phi(s) = \begin{cases} 2s, & 0 \leq s \leq \frac{1}{2} \\ 2-2s, & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Since $\phi(0) = \phi(1) = 0$, it is known that $\alpha \simeq g\ rel\ \{0,1\}$, where $g(s) = 0$, $0 \leq s \leq 1$. Now, we should prove that

$$\alpha.\alpha^{-1}(s) = \alpha \circ \phi \simeq \alpha \circ g\ rel\ \{0,1\} = e$$

$$\alpha.\alpha^{-1}(s) = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2} \\ \alpha^{-1}(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

and

$$\alpha \circ \phi = \begin{cases} \alpha(2s), & 0 \leq s \leq \frac{1}{2} \\ \alpha(2-2s), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Also, we know that $\alpha^{-1}(0) = \alpha(1)$ and $\alpha^{-1}(1) = \alpha(0)$. For the other direction, the following function is given.

$$\phi(s) = \begin{cases} 1-2s, & 0 \leq s \leq \frac{1}{2} \\ 2s-1, & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Now, we should check that

$$\alpha^{-1}.\alpha(s) = \alpha \circ \phi \simeq \alpha \circ g \; rel\{0,1\} = e$$

$$\alpha^{-1}.\alpha(s) = \begin{cases} \alpha^{-1}(2s), & 0 \leq s \leq \frac{1}{2} \\ \alpha(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

and

$$\alpha \circ \phi = \begin{cases} \alpha(1-2s), & 0 \leq s \leq \frac{1}{2} \\ \alpha(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

4.3 The Universal Covering Space

Computation of some fundamental groups is not so trivial. One of the most useful parts for this intention is the notion of covering space.

Definition 4.3.1 *[4] Consider $p : E \to B$ which is a continuous surjective map. If the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E, then the open set U of B is called evenly covered by p such that for every α, the restriction of p to V_α is a homeomorphism of V_α onto U. The family $\{V_\alpha\}$ is called a seperation of $p^{-1}(U)$ into slices.*

Definition 4.3.2 *[4] Consider a continuous and surjective map $p : E \to B$. p is said* *to be a covering map and E is called a covering space of B if each point b of B has a neighbourhood U that is evenly covered by p.*

Example 4.3.3 *One of the covering map of circle is* $p:\mathbb{R}\to S^1$ *given by the equation*

$$p(t) = e^{2\pi it}$$

p is a function that wraps the real line $\mathbb{R}$ *around the circle* S^1. *Each interval* $[m, m+1]$ *gets mapped onto* S^1 *and the inverse image of an open arc in the circle is a union of collection of open intervals in the real line.*

Example 4.3.4 $p: S^1 \to S^1$ *is the another covering map of the circle given by the equation*

$$p(t) = t^n \text{ such that } n \geq 1$$

The circle gets wrapped by p around itself n times.

Example 4.3.5 *The space* $T = S^1 \times S^1$ *is called the torus. The covering map of torus is given by*

$$p \times p : \mathbb{R}\times\mathbb{R} \to S^1 \times S^1$$

$p \times p$ *sends each square block of the plane onto the torus.*

4.4 Actions on Topological Spaces

Definition 4.4.1 *[1]A topological group G is said to act as group of homeomorphisms on a space X if each group element induces a homeomorphism of the space in such a way that:*

(a) $hg(x) = h(g(x))$ for all $g, h \in G$ and for all $x \in X$;

(b) $e(x) = x$ for all $x \in X$, where e is the identity element of G;

(c) the function $G \times X \to X$ defined by $(g, x) \to g(x)$ is continuous.

Two elements $x, y \in X$ are in the same orbit $O(x)$ of there exists a group element $g \in G$ sending one to another.

If two elements are in the same orbit, then the relation between these two elements is denoted by $x \sim y$. So two elements are in the same orbit if and only if $x = g(y)$ for some $g \in G$. The space is written $X \diagup G$ and it is called the orbit space.

Theorem 4.4.2 *[1] $\pi_1(X \diagup G)$ is isomorphic to G if G acts as group of homeomorphisms on a simply connected space X, and if every point $x \in X$ has a neighbourhood U satisfying $U \cap g(U) = \varnothing$ for all $g \in G - \{e\}$.*

Example 4.4.3 *As an example we consider the symmetric group S_n acting on the set $\{1, 2, ..n\}$ by permuting its elements. The order(the number of elements) of the symmetric group S_n is $n!$. The symmetric group of the set $x = \{1, 2, ...n\}$ is called the symmetric group of degree n. In addition, $X = \{1, 2, 3, ..n\}$ is not a simply connected space. Therefore, we can not apply the previous theorem, but S_n has subgroups acting on the set $\{1, 2, ...n\}$.*

Example 4.4.4 *As a basic example, let's consider $\mathbb{Z}$ action on $\mathbb{R}$ by additive translations. We have a real line and $\mathbb{Z}$ is acting on $\mathbb{R}$. Then, we are looking for the answer of $\pi_1(\mathbb{R} \diagup \mathbb{Z})$. By using the theorem above, if $\mathbb{Z}$ acts as group of homeomorphisms on a simple connected space $\mathbb{R}$ and if every point $x \in \mathbb{R}$ has a neighbourhood U which satisfies $U \cap g(U) = \varnothing$ for all $g \in \mathbb{Z} - \{e\}$, then $\pi_1(\mathbb{R} \diagup \mathbb{Z})$ is isomorphic to $\mathbb{Z}$.*

Now, consider the real line. Real numbers include all rational numbers ($\mathbb{Z} \subseteq \mathbb{Q}$) and all irrational numbers. Furthermore, any interval of the real line has infinite rational *numbers and infinite irrational numbers. Therefore, first of all, choose an integer number and consider $\mathbb{Z}$ action on $\mathbb{R}$. The non-identity element of $\mathbb{Z}$ sends each integer number to the other integer numbers. That means, all integer numbers can be identified with each other. It is clear that, the difference between integer numbers is always integer. Therefore, all integer numbers are in the same orbit space. As it is known, rational numbers are real numbers also. That is the reason, for example, we can choose* 0.1 *from the real line. The non-identity element of $\mathbb{Z}$ sends* 0.1 *to* $1.1, 2.1, 3.1$ *ext. and also sends to negative numbers such as* $-1.1, -2.1, 3.1$ *ext. It can be written that all* $0.1 + r$ $(r \in \mathbb{Z})$ *numbers are in the same orbit space.It is same for the other rational and irrational numbers too. For instance,* $0.01 + r$, $0.001 + r$, $0.0001 + r$ *ext.*

Finally, the quotient $\mathbb{R}/\mathbb{Z}$ is described with the unit circle $S^1 \subseteq \mathbb{R}^2$. When we consider any closed interval, there is a well-defined bijection of $\mathbb{R}/\mathbb{Z}$ onto S^1.(Every closed interval consists of infinitely many rational and irrational numbers). Also; $\pi_1($ $\mathbb{R}/\mathbb{Z}$ $)$ is isomorphic to $\mathbb{Z}$.

Example 4.4.5 *When we search for more examples in this section, we can consider $\mathbb{Z} \times \mathbb{Z}$ acting on $\mathbb{R} \times \mathbb{R}$. By using the theorem, $G = \mathbb{Z} \times \mathbb{Z}$ and $X = \mathbb{R} \times \mathbb{R}$. For this action we have a plane. Therefore, we need to consider vertices, edges and faces also. First of all, let's start with a point. The non-identity element of $\mathbb{Z} \times \mathbb{Z}$ sends each point to the other points. As a numerical example, if we choose a point* $(1,0)$ *and consider $\mathbb{Z} \times \mathbb{Z}$ acting on this point, then the other points such as* $(2,0)$, $(3,0)$, $(-1,0)$, $(-2,0)$ *are found. If we select a point* $(1,1)$ *, then all points of the form* $m(1,0) + n(0,1)$ *can be found. This means that, every point can be found by using the formula* $m(a,b) + n(c,d)$. *Therefore, every point can be identified with each other.*

Now, let's consider edges. Start with any edge on the x-axis. It can be identified with the other edges which lie on the x-axis. It is not possible to find other edges which lie on the y-axis by using $\mathbb{Z}\times\mathbb{Z}$ *action on* $\mathbb{R}\times\mathbb{R}$. *For example, choose an edge whose endpoints are a* $(0,0)$ *and* $(1,0)$. *As I mentioned above, the first point* $(0,0)$ *is identified to* $(1,0)$ *and the second point* $(1,0)$ *is identified to* $(2,0)$. *It can not be identified to point* $(1,1)$. *Therefore, in total, two different edges exist. All edges which lie on the x-axis can be identified with other edges on the x-axis and all edges which lie on the y-axis be defined to other edges on the y-axis. In addition,each block of the plane has one face. They can be identified with each other too.*

Furthermore, it is more simpler to use planar model instead of the plane. This means that, by using the planar model it can be seen that every point can be identified to each other. Opposite edges can also be identified (x-axis to x-axis and y-axis to y-axis). Each block has only one face so planar model has one face too.From this planar model, we get a torus. Finally, $\pi_1(\mathbb{R}\times\mathbb{R}/\mathbb{Z}\times\mathbb{Z})$ *is isomorphic to* $\mathbb{Z}\times\mathbb{Z}$ *and it is the fundamental group of the torus.*

Definition 4.4.6 $\mathbb{Z}_n$ *is an additive group. In an additive group,zero is called the identity element and* $-a$ *is the inverse of the element* a.

$$\bar{1} = 1 + n\mathbb{Z}$$

$$\mathbb{Z}_n = \left\{ k.\bar{1} \mid k \in \mathbb{Z} \right\}$$

Example 4.4.7 *(Action on the real projective space) Let's consider* $\mathbb{Z}_2$ *acting on* S^2. *So,* $G = \mathbb{Z}_2$ *and* $X = S^2$. $\mathbb{Z}_2$ *is an additive group and it has two elements* $\bar{0}$ *and* $\bar{1}$. *Therefore, it can be written that*

$$\bar{0} = 2k \text{ and } \bar{1} = 1 + 2k$$

As it is given in the definition above, in additive group zero is the identity element. The projective plane has two discs. Identity element sends each point to itself. That means, it doesn't do any change on the projective plane. The non-identity element of the cyclic group sends each point to the other disc, to its antipode. Therefore, these two points can be identified and half-sphere is obtained. Half-sphere means that it is a disk and the antipodal points are boundaries which identified with each other. Therefore, the fundamental group is $\mathbb{Z}_2$.

4.5 Fundamental Group of Spaces and Surfaces

In this section, I will compute the fundamental group of some known spaces and surfaces.

Example 4.5.1 *Let's consider the fundamental group of* $\mathbb{R}^n$. *First of all, consider any loop in* $\mathbb{R}^n$. *This loop can continuously be deformed to another loop circulating randomly to its basepoint. That means,there exists only one homotopy class of loops. Therefore, the fundamental group of* $\mathbb{R}^n$ *is trivial and is denoted by* $\pi_1(\mathbb{R}^n) = 0$.

Note that if a space is path-connected and has trivial fundamental group, then it is called a simply-connected space.

Example 4.5.2 *Now, let's consider the fundamental group of the circle. As we know from before, group action is one of the methods to find the fundamental group of spaces. The quotient space* $\mathbb{R}/\mathbb{Z}$ *is defined by the unit circle* $S^1 \subseteq \mathbb{R}^2$. *When we consider any closed interval, there is a well-defined bijection of* $\mathbb{R}/\mathbb{Z}$ *onto* S^1.*(Every closed* *interval consists of infinitely many rational and irrational numbers) Also;* $\pi_1(\mathbb{R}/\mathbb{Z})$ *is isomorphic to* $\mathbb{Z}$.

Example 4.5.3 *Let's think of the fundamental group of the sphere. Each loop on the sphere, can continuously be deformed into a point. So, the homotopy class is trivial. Therefore,* $\pi_1(S^2) = 0$.

Example 4.5.4 *The other basic surface in the topology is a disc. It can be seen that all loops on a disc can continuously be deformed to a point which is the basepoint. So, it also has trivial fundamental group.*

Example 4.5.5 *It is known that before, the projective space can be written by* $P^n = S^2/\mathbb{Z}_2$. *Remember that* $\mathbb{Z}_2$ *has two generators. One of them is the identity element so it sends each point to itself. Other element is the non-identity element and sends each point to its antipode on the other disc. Two discs are homotopic to each other. Therefore, there exists a half-sphere. Fundamental group is* $\mathbb{Z}_2$.

Example 4.5.6 *Fundamental group of Möbius strip is* $\mathbb{Z}$.

If the space can be written as a product of two spaces, then the following theorem will be useful to compute the fundamental group of spaces.

Theorem 4.5.7 $\pi_1(X \times Y)$ *is isomorphic to* $\pi_1(X) \times \pi_1(Y)$*, if* X *and* Y *are path connected spaces.*

Example 4.5.8 *Let's consider the torus. The torus T^2 is represented by $S^1 \times S^1$. The fundamental group of S^1 is known, and it is given by $\pi_1(S^1) = \mathbb{Z}$. By using the theorem,*

$$\pi_1(T^2) = \pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$$

Note that T^n(the n-torus) can be written by direct product of n copies of S^1. So, the fundamental group of T^n

$$\pi_1(T^n) = \mathbb{Z}^n$$

Example 4.5.9 *Let's consider the fundamental group of the cylinder. By using the above theorem, cyclinder is defined by $S^1 \times I$. So,*

$$\pi_1(S^1 \times I) \cong \pi_1(S^1) \times \pi_1(I) \cong \mathbb{Z}$$

Example 4.5.10 *For other applications of this theorem the infinite cyclinder can be given. It is represented by $S^1 \times \mathbb{R}$. Therefore,*

$$\pi_1(S^1 \times \mathbb{R}) \cong \pi_1(S^1) \times \pi_1(\mathbb{R}) \cong \mathbb{Z}$$

Chapter 5

BESTVINA-BRADY GROUPS

Before we define Bestvina-Brady groups, we give the definitions of flag complex and right angled Artin group.

Definition 5.0.11 *[5] Let L be a simplicial complex. If every finite family of vertices of L which are pairwise adjacent, spans a simplex in L, then L is said to be a flag complex.*

Definition 5.0.12 *The right angled Artin group G_L has generating set $\{g_1, g_2, .., g_n\}$ in one to one correspondence with the vertex set $\{v_1, v_2, .., v_n\}$ of L and hence has presentation*

$$G_L = \langle g_{1,} g_2, .., g_n \mid g_i g_j = g_j g_i \ \left\{V_{i,} V_j\right\} \ in\ L\rangle$$

Note that there is a surjective map from G_L to $\mathbb{Z}^n$. In addition, this map takes all the generators g_i, to the standard basis elements of $\mathbb{Z}^n$.

Let's compose this map with the map g

$$g : \mathbb{Z}^n \to \mathbb{Z}$$

g is defined by $(x_1, x_2, .., x_n) = x_1 + x_2 + ... + x_n$. Hence, we get the map

$$\phi : G_L \to \mathbb{Z}$$

sending all the generators to $1 \in \mathbb{Z}$. The kernel of this map is said to be the Bestvina-Brady group and is represented by H_L. In my thesis, the Euler characteristic of the

Bestvina-Brady group H_L will be computed. For this, a topological space should be assigned to the Bestvina-Brady group H_L. To compute the Euler Characteristic of groups, the definition provided in C.T.C. Wall's paper[8] " Rational Euler Characteristics" is used. In this paper, Euler characteristic of the group is defined to be the Euler characteristic of its classifying space.

Definition 5.0.13 *[8] In C.T.C. Wall's paper, the Euler characteristic of a group G is defined to be the Euler characteristic of BG if this exists; $\chi(G) = \chi(BG)$. BG space is a path connected space with $\pi_1(BG) = G$. Universal cover of BG is contractible.*

As I mentioned before, L is a finite flag complex, G_L is the right angled Artin group, and $\dot{\phi} : G_L \to \mathbb{Z}$ is the map which sends all the generators to 1. Then there exists a compact, nonpositively curved BG space BG_L. ϕ above is the map of the fundamental groups. Therefore,

$$l : BG_L \to S^1$$

is the lift map to topological spaces. In C.T.C. Wall's paper, $\pi_1(BG_L) = G_L$ and as known from before $\pi_1(S^1) = \mathbb{Z}$. When l lifts to the universal covers there exists a new map which is

$$f : EG_L \to \mathbb{R}$$

EG_L is the universal cover of BG_L (EG_L is contractible) and $\mathbb{R}$ is the universal cover of S^1. There is an action of G_L on X, with orbit space BG_L. H_L acts on EG_L in the same way and Bestvina-Brady paper shows that [2] when L is contractible, $f^{-1}(p)/H_L$ is a finite BH_L. Hence, $\chi(H_L) = \chi(BH_L)$.

Let's consider examples about the Euler Characteristic of Bestvina-Brady groups.

Example 5.0.14 *By the above definitions, we should start with a flag complex. For the simplest and the basic example, let L be Figure 5.1.*

Figure 5.1. Flag complex line

We will denote the right angled Artin group by G_L.

$$G_L = \langle\, a,b \mid ab = ba \,\rangle$$

Let's consider

$$\phi : G_L \to \mathbb{Z}$$

It is known that ϕ *takes all the generators to 1.* G_L *has two generators which we shall call* a,b*. That means,*

$$\phi(a) = 1$$

$$\phi(b) = 1$$

$\mathbb{Z}$ *is an additive group. Identity element of* $\mathbb{Z}$ *is* 0*. So,*

$$\phi(ab^{-1}) = \phi(a)\phi(b)^{-1} = \phi(a) - \phi(b) = 0.$$

Kernel of ϕ *is the set of elements in* G_L *which are mapped to the identity of* $\mathbb{Z}$*.* G_L *here will be* $\mathbb{Z}\times\mathbb{Z}$*. By using the C.T.C Wall's paper,*

$$\pi_1(BG_L) = G_L,$$

G_L *is* $\mathbb{Z}\times\mathbb{Z}$ *and it is the fundamental group of the torus.* BG_L *is the torus and it is path connected. The universal cover of* BG_L *is* EG_L *and it is also contractible. The universal cover of torus*(BG_L) *is the plane*(EG_L)*. By using the map which is given before,*

$$f : EG_L \to \mathbb{R}$$

where $\mathbb{R}$ *is the universal cover of* S^1*. As I mentioned before, there is an action of* $\mathbb{Z}\times\mathbb{Z}$

on the universal cover $\mathbb{R}\times\mathbb{R}$, *with orbit space* BG_L, *which is the torus. Also,* H_L *acts on* $\mathbb{R}\times\mathbb{R}$ *in the same way ; Bestvina-Brady paper shows that when L is contractible,* $f^{-1}(p)/H_L$ *is a finite* BH_L. *Hence,* $\chi(H_L)=\chi(BH_L)$. *The function* f *is a map from plane to the real line. Kernel of* ϕ, H_L *acts on* $f^{-1}(p)$. *Point p belongs to the real line, so it could be an integer number or a non-integer one. If* $p\in\mathbb{Z}$; H_L *acts on* $f^{-1}(p)$. *In each block of the plane, there exists two different edges which are a and b. Also, there exist four different vertices. When* H_L *acts on* $f^{-1}(p)$; *that means the line cuts each block of the plane once from the integer points. First of all, let's start with vertices. Choose any point on the line and consider the* H_L *action on* $f^{-1}(p)$. *That means,* ab^{-1} *can be applied to each point on this line. When the* ab^{-1} *is applied to every point, then all the other points on the same real line can be found. Therefore, all points are identified with each other. So, they are in the same orbit space. That means, there exists* 1 H_L *orbit of* $0-cells$. *Now, let's consider edges. Start with any edge on the line. It can be identified with the other edges which lie on the same line. As a numerical example, choose an edge whose endpoints are* $(0,0)$ *and* $(1,-1)$. *When we apply* ab^{-1} *to* $(0,0)$, *we first find the point* $(1,0)$ *and then* $(1,-1)$. *In the same way ; apply* ab^{-1} *to* $(1,-1)$, *first we find* $(2,-1)$ *and then* $(2,-2)$. *All vertices on the same line can be identified with each other, so they are in the same orbit space. Therefore, there exists* 1 H_L *orbit of* $1-cells$. *In addition,* H_L *orbit of* $2-cells$ *is* 0, *because L is* 1 *dimensional.*

$$\chi(H_L)=1-1=0$$

Now let $p\notin\mathbb{Z}$ *and consider the* H_L *action on* $f^{-1}(p)$. *Now, each block of the plane is cut twice from the edges or it can be said that there exist two types of edges on the same line; one shorter than the other one. Choose any edge and apply* ab^{-1} *to each of the endpoints. For instance, start with the left endpoint. First, apply a and turn back* *with b^{-1}. Then we find the left endpoint of the other edge. In the same way, choose right endpoint of the edge and apply a and b^{-1}. Then we find right endpoint of the other edge. Therefore, there exist two different edges and vertices in total. So, number of H_L orbit of $0-cells$ and $1-cells$ is 2.*

$$\chi(H_L) = 2 - 2 = 0.$$

Note that, the planar model of the given L can be used to determine the number of vertices and edges In this example, L is not a complicated flag complex but in the other examples more complicated flag complexes will be seen. Therefore, it would be easier to determine $\chi(f^{-1}(p)/H_L)$ by using the planar model. For instance, for the previous example, consider only one block of the plane which is the planar model of torus. If $p \in \mathbb{Z}$, then $f^{-1}(p)$ cuts the planar model only one time from the integer points. By using the planar model, all vertices are identified with each other and only one edge cuts the planar model. Therefore, number of the $0-cells$ and $1-cells$ is 1.

Example 5.0.15 *Let L be a Figure 5.2.*

Figure 5.2. Two lines with a common vertex

G_L is the right angled Artin group with generating set $\{a,b,c\}$. That means,

$$G_L = \langle a,b,c \mid ab = ba,\ bc = cb \rangle .$$

ϕ is the map

$$\dot{\phi} : G_L \to \mathbb{Z}$$

which is the fundamental group map of l. That means, $\pi_1(BG_L) = G_L$ and $\pi_1(S^1) = \mathbb{Z}$.

$$l : BG_L \to S^1$$

And also, f is the map of the universal covers,

$$f : EG_L \to \mathbb{R}.$$

If $p \in \mathbb{Z}$; let's consider the H_L action on $f^{-1}(p)$. As it is written before, it would be easier to determine $\chi(f^{-1}(p)/H_L)$ by using the planar model. When p is an integer number, the inverse image of p cut the planar model from six different vertices. All vertices can be identified with each other. That means, they are in the same orbit space. When we consider the edges, the planar model is cut by two different edges. As these edges are in different G_L orbits, they can not be identified with one another.

$$\chi(H_L) = -1$$

Now, consider the noninteger case. When p is not an integer, it cuts the planer model from three different sides but in total we get four different edges. Also, three vertices exist. When we find the difference between vertices and edges,

$$\chi(H_L) = -1.$$

Example 5.0.16 *L be a Figure 5.3.*

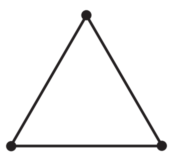

Figure 5.3. Flag complex triangle

Then,

$$G_L = \ < a, b, c \mid ab = ba,\ ac = ca,\ bc = cb >$$

When H_L is acting on $f^{-1}(p)$, we should consider vertices, edges and faces. The *planar model of the universal cover of L is a cube. p is an element of* $\mathbb{R}$*, so consider integer and noninteger cases. If* $p \in \mathbb{Z}$*, all vertices can be identified with one another. They are in the same orbit space. Therefore, number of the* $0-cells$ *is* 1. 3 *different edges cut the planar model so they can not be identified. Hence, number of* $1-cells$ *is* 3. *Also, two triangles cut the planar model. That means, two faces exist, in total. So,*

$$\chi(H_L) = 0.$$

For the non integer case, we have in total 3 *different vertices which can not be identified. That means,* 3 H_L *orbits of* $0-cells$ *exist. Number of the edges are* 6 (6 H_L *orbits of* $1-cells$ *exist*) *and* 3 *different triangles cut the cube*(3 H_L *orbits of* $2-cells$ *exist*)*. So,*

$$\chi(H_L) = 0.$$

When the previous examples are considered, some cases can be generalized. In addition, if the dimension of L is greater than 3 or if it has a surface, the planar model of EG_L can not be imagined or is more complicated. Therefore, simplicial complex L is enough to compute the $\chi(H_L)$. First of all, if $p \in \mathbb{Z}$ each of the vertices can be identified with each other. That means, they are in the same orbit space and number of $0-cells$ of H_L is always 1. Furthermore, in the first example, $f^{-1}(p)$ cuts the planar model only one time and in the second example $f^{-1}(p)$ cuts the planar model two times from different places. Therefore, we can generalize the formula and it can be written that number of $H_L - orbit$ of $1-cell$ is equal to the number of $1-cells$ of L. Also, in the first and second examples dimension of L is 1, so the universal cover doesn't have any 2-dimensional faces. So,

$$\chi(H_L) = 1 - s_1$$

For the non integer case, in the first example, $f^{-1}(p)$, cuts the planar model two times. That means, there exist 2 different vertices and 2 different edges. L has 2 vertices so number of vertices of $f^{-1}(p)/H_L$ is also 2. L has 1 edge, that means, number of edges of $f^{-1}(p)/H_L$ is 2 times the number of edges of L. Therefore, it can be generalized in this way;

$$\chi(H_L) = s_0 - 2s_1$$

where

$$\begin{aligned}
s_0 &= \text{number of } 0-\text{dimension cells of } L \text{ (number of vertices of } L) \\
s_1 &= \text{number of } 1-\text{dimension cells of } L \text{ (number of edges of } L) \\
&\quad \ldots \\
s_n &= \text{number of } \ n-\text{dimension cells of } L
\end{aligned}$$

Now, consider the L's in two dimensional space as in example 5.0.15. For the integer case, the difference between the example 5.0.14 and example 5.0.16 is that L has a surface because it is in $\mathbb{R}^2$. For instance, in the example 5.0.16, L is a triangle. That is to say, number of $2-cells$ of L is 1. When we consider the H_L action on $f^{-1}(p)$, 2 different triangles cut the cube. That means, number of $2-cells$ of H_L is 2 times the number of $2-cells$ of L.

$$\chi(H_L) = 1 - s_0 + 2s_2$$

For the non integer case, number of $0-cells$ of BH_L and number of $1-cells$ of L are same which is given before. To determine the number of $2-cells$ of H_L, let's consider the action. 3 different triangles cut the cube. That means, 3 times the number of $2-cells$ of L gives the number of $2-cells$ of H_L.

$$\chi(H_L) = s_0 - 2s_1 + 3s_2$$

In the previous examples, L was given in two and three dimensional spaces. Now, we need to generalize this to higher dimensional flag complexes. We obtain the general formulas below.

For the integer case,

$$\chi(H_L) = 1 - s_1 + 2s_2 - 3s_3 ... + (-1)^n s_n$$

For the non integer case,

$$\chi(H_L) = s_0 - 2s_1 + 3s_2 ... + (-1)^n (n+1) s_n$$

Lemma 5.0.17 *If L is contractible then the Euler characteristic of the integer case of H_L is equal to the Euler characteristic of the non integer case of H_L. That means,*

$$1 - s_1 + 2s_2 - 3s_3 ... + (-1)^n s_n = s_0 - 2s_1 + 3s_2 ... + (-1)^n (n+1) s_n$$

Proof. L is contractible means it can be continuously deformed into the one point. Hence,

$$\chi(H_L) = \sum_{i=0}^{n} (-1)^i . s_i = s_0 - s_1 + s_2 ... (-1)^n s_n = 1.$$

If we assume that Euler characteristic of two cases are equal to each other. Try to show that this equalty is true.

$$1 - s_1 + 2s_2 - 3s_3 ... + (-1)^n s_n \stackrel{?}{=} s_0 - 2s_1 + 3s_2 ... + (-1)^n (n+1) s_n$$

$$1 - s_1 + 2s_2 - 3s_3 ... + (-1)^n s_n - [s_0 - 2s_1 + 3s_2 ... + (-1)^n (n+1) s_n] \stackrel{?}{=} 0$$

$$1 - s_0 + s_1 - s_2 + s_3 - ... (-1)^n s_n \stackrel{?}{=} 0$$

$$1 - [s_0 - s_1 + s_2 - s_3 - ... (-1)^n s_n] \stackrel{?}{=} 0$$

L is contractible, so

$$1 - [s_0 - s_1 + s_2 - s_3 - ... (-1)^n s_n] = 0$$

Chapter 6

CONCLUSION

In algebraic topology a topological invariant is a property of the topological space that is invariant under homeomorphism. So, topological invariants become very handy when one tries to answer the homeomorphism problem between any two topological spaces. To show that two spaces are not homeomorphic it would be enough to find a topological property not shared by them. Cardinality, countability, connectedness, compactness, Euler characteristic, homotopy groups, homology and cohomology are some well-known examples for topological invariants. As this thesis gives the computation for the Euler characteristic of the Bestvina-Brady groups, (for the case when the flag complex associated is contractible) it especially focuses on the Euler characteristic and the Fundamental group invariants.

Bestvina-Brady groups appear as the kernel of the surjective map from the right angled Artin group, to the set of integers, taking generators to the generators of the latter. These groups are denoted by H_L throughout this thesis. Referring to the C.T.C. Wall's paper [8], to compute the Euler characteristic of a group, a finite classifying space, BG_L space, must be assigned to this group. Bestvina-Brady paper[2] shows that when the flag complex L is contractible, one can obtain a finite model for the BH_L, and hence we can work out $\chi(BH_L)$. So, as we see here the Euler characteristic of the Bestvina-Brady group depends on the homotopy type of L.

One future work here could be to try to calculate the Euler characteristic of these groups when L is non-contractible (which is still an open question as far as we know).

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