Investigation of Basic Concepts of Fuzzy Arithmetic

Isma`il Muhammad Musa

Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the Degree of

Master of Science
in
Applied Mathematics and Computer Science

Eastern Mediterranean University
February 2015
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

Prof. Dr. Serhan Çiftçioğlu
Acting Director

I certify that this thesis satisfies the requirements as a thesis for the degree of Master of Science in Applied Mathematics and Computer Science.

Prof. Dr. Nazim Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Applied Mathematics and Computer Science.

Prof. Dr. Rashad Aliyev
Supervisor

Examining Committee

1. Prof. Dr. Rashad Aliyev
2. Prof. Dr. Agamirza Bashirov
3. Asst. Prof. Dr. Ersin Kuset Bodur

ABSTRACT

Fuzzy arithmetic is an extensively used instrument for dealing with uncertainty in a computationally competent method, recently and much better in the upcoming years.

This thesis aims to investigate the basic properties of fuzzy arithmetic as its title implies. The properties of fuzzy arithmetic via definitions, examples and some figures are discussed. The methods give the picture of how fuzzy arithmetic can be flexible alongside imprecise data thereby making its applications to be handy in the modern world.

This thesis investigates the properties of fuzzy sets, properties of fuzzy number, performing arithmetic operations on fuzzy number, properties of L-R fuzzy number, performing operations on L-R fuzzy number, properties of fuzzy interval and properties of L-R fuzzy interval.

Also, the extension principle and fuzzy arithmetic operations using extension principle are investigated. The fuzzy equation is solved by using the method of α–cut.

Keywords: Fuzzy arithmetic, Fuzzy set, Fuzzy number, Fuzzy interval, Extension principle, Fuzzy equation

ÖZ

Bulanık aritmetik son zamanlarda belirsizlik ile başa çıkmak için yaygın bir araç olarak kullanılan ve önümüzdeki yıllarda da sıkça kullanılacak olan çok iyi bir hesaplama yöntemidir.

Bu tezin amacı isminden de anlaşılacağı gibi bulanık aritmetik temel özelliklerini incelemektir. Tanımlar, örnekler ve bazı figürler üzerinden bulanık aritmetik özellikleri tartışılır. Kullanılan teknikler kesin olmayan bilgilerin, bulanık aritmetik kullanılarak çağdaş dünyaya uyarlılığını gösterir.

Bu tez bulanık kümelerin özelliklerini, bulanık sayı özelliklerini, bulanık sayı üzerinde aritmetik işlemleri, L-R bulanık sayı özelliklerini, L-R bulanık sayı üzerinde aritmetik işlemleri, bulanık aralığı ve bulanık aralık üzerinde işlemleri inceler.

Ayrıca, uzatma ilkesi ve bu ilkeyi kullanarak bulanık aritmetik işlemleri incelenir. Bulanık denklem, α-kesim yöntemi kullanılarak çözülür.

Anahtar Kelimeler: Bulanık aritmetik, Bulanık küme, Bulanık sayı, Bulanık aralık, Uzatma ilkesi, Bulanık denklem

ACKNOWLEDGMENT

I would like to thank Almighty God for giving me the strength, health and dedication to see this thesis through. I would also like to convey my genuine gratitude to my parents Alh. Muhammad Musa and Haj. Hadiza Imam for their financial and emotional support through all these years, their untiring faith and assurance in me and in my abilities is what has made me to be the person I am today. No words can delineate how thankful I am to them. May the Almighty in His infinite mercy reward them with the best of rewards in this life and hereafter.

I would like to extend my gratitude to my supervisor, Prof. Dr. Rashad Aliyev for bestowing me with the opportunity to finish my MSc thesis at this prestigious institute (Eastern Mediterranean University) and for the tremendous help he gave to me. I exceptionally want to thank him for he has been actively interested in my work and has always been obtainable to counsel me. I am extremely grateful for his patience, inspiration, passion, and huge vision in fuzzy arithmetic that, seized jointly, make him an outstanding mentor.

I would also like to thank my friends whom our paths crossed those who have affected me positively and negatively, for the experience really mattered and it is what makes life worth living. Finally, I would also like to thank the whole academic and non-academic staff of Eastern Mediterranean University for it is alongsıde their work and dedication that this outstandıng institution is whereas it is nowadays.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

INTRODUCTION

According to Lofti A. Zadeh, the founder of fuzzy logic, a fuzzy set is a class of objects that has a continuum of membership grades. The set is characterized by membership functions which vary between zero and one [1].

Kosko Bart in his book "Fuzzy thinking: the new science of fuzzy logic", says that consider having an apple in your hand. Is the object you hold is an apple? Yes. He claimed that the object in your hand that you are holding belongs to the clumps of space-time we call the set of apple – all apples anywhere, ever. Then you are asked to take a bite, then chew it and swallow it, and put the digestive track to be part of the apple's molecules. He asks if the object you hold is still an apple. Take bite again. Can we still claim that the object is an apple? Down to void keep taking another bite and so on.

Clearly, the apple is still an apple in the beginning. Gradually it loses its property of being apple as the number of pieces bitten off increases. It will not be a member of class of apple in the end whereby it is completely eaten. Basically, the idea of fuzzy logic is to take each object in particular class and assign it to a number indicating a degree to which it belongs to that same class. As for the apple, the number will be zero at the end and one in the beginning. It will be decreasing in between, from one to zero.

Another example that elaborates more on fuzzy set is to take the set of old people. A person at age 90 is in this set. What about a person at age 50? What about a person at age 70? As the age of a person increases or decreases, so the question of whether a person belongs to the set of old people arises. One can easily say that a person at age 90 is very old, and the person at this age is also old, so belongs to the set, but the same words for the people at ages 50 and 70 cannot be easily said. Also, the degree in which you call the person at age 90 to be in set of old people cannot be the same with the degree in which you call the person at age 70. The set of old people being a fuzzy set will have a membership function that will assign each of its members a grade of degree to which it belongs to the set, ranging from zero to one.

There are a lot of false opinions what the fuzzy logic is, like the assumption of many people that fuzzy logic implies to the imprecise methodology which is only used when there is no much importance or necessity of accuracy required in a situation.

Fuzzy logic as an approach in computer science uses the idea in which human brain thinks and solves problems. It uses the idea in which instead of using quantitative terms, natural language terms are used in order to approximate human decision making. For notions which depend upon their contexts, notions that are not able to be defined precisely, formally define fuzzy logic as a form of knowledge representation. It gives the ability to reason more like human in computerized devices.

In accordance with Zadeh’s theory, fuzzy logic is a logic that may be considered as an effort to formalize two extraordinary human capabilities. First of all, fuzzy logic is functioning in imprecise and uncertain environment where it is dealt with partially true or partially possible cases, and these factors are taken into account to reason and to make rational decisions. Besides that, fuzzy logic has a capability to perform different tasks without making any computations and measurements [2].

The mathematical concept of fuzzy logic is easy to understand. It is so flexible that given any system without starting again from scratch it is easy to layer on more functionality. It is tolerant to imprecise data. With fuzzy logic a fuzzy system can be created to match any data. The fuzzy system does not just replace but it also arguments and simplifies the implementation of a conventional control technique.

Another meaning of fuzzy logic is that it causes the source of confusion. The generalization of multivalued logic, a logical system, is what is called fuzzy logic in a narrow sense. Widely speaking, in nowaday’s use, fuzzy logic is much more than just a logical system has such facets as the logical, fuzzy set, theoretic, epistemic, and the relational facets [2].

In its narrow sense, the logical facet of fuzzy logic may be considered as a multivalued logical system to be described in a generalized form. The main task is to define how similar the above proposition to the vital principle of classical logic is.

The fuzzy set theoretic facet considers the fuzzy sets with vague boundaries. The fuzzy set theory formed the idea of fuzzy logic.

The epistemic facet of fuzzy logic is attentive to semantics of natural languages, information analysis and knowledge.

The relational facet of fuzzy logic concentrates on fuzzy relations and fuzzy dependencies more generally.

Chapter 2

REVIEW OF EXISTING LITERATURE ON FUZZY ARITHMETIC

Zadeh defined a fuzzy set, and extended it to the notion of union, intersection, inclusion, complement, convexity etc., and the various properties of these notions are established in the context of fuzzy sets [1].

The concept of the generalized constraint, granulation, precisiation and graduation are the cornerstones of a fuzzy logic as well as its principal distinct properties in a fuzzy logic are given in [2]. The elevation of the concept of precisiation is used in such fields as economics, law, linguistics etc.

The eight classes of fuzzy numbers are considered in [3]. To add these fuzzy numbers, nine operations are provided, and the investigation is carried out to describe interrelationships between given operations. Some modifications of fuzzy numbers are put into consideration. Some distinct classes of membership functions that are closed under precise addition rules are studied.

In [4] fuzzy numbers are represented which are based on the use of parameterized monotonic functions in order to model the membership function or the α-cut. It is called LU representation, in reference to the Lower and Upper branches of the fuzzy numbers involved in operations. The use of advantages of LU-fuzzy numbers in the principal applications of fuzzy calculus is also investigated. LU-fuzzy numbers applications list is provided to view the computational importance related to their adoption.

In [5] fuzzy numbers ranking is approached in newly way based on α-cuts to eliminate the limitations of the existing studies in order to ease the computational procedures. The method is able to rank every kind of fuzzy numbers that has different membership functions, showing that it eliminates several existing fuzzy ranking approaches. The method in ranking of fuzzy numbers is complete and very strong.

Fuzzy number mean value, the idea of interval function and α lower percentile are considered in [6]. Wide family of fuzzy numbers is defined. Then, a technique in order to rank them is obtained, and relate to the other techniques. A fuzzy mapping by L_p distance is introduced to define αLP from a fuzzy number. Then it obtains the mean function $M_c(x, x_{\alpha_1})\colon F(R) \times F(I) \rightarrow R$, a distance minimization in reference to an αLP from the fuzzy number u. Finally, few theorems and remarks that are vital to the theories and techniques for the fuzzy numbers ranking are investigated.

The idea of technique to find a universal order for fuzzy numbers is given in [7], and the fuzzy numbers arbitrary permutation degree in which it is ordered is suggested to be computed. In the context of decision making it deals with the problem of ordering fuzzy numbers. The ability to evaluate that a given fuzzy numbers distribution to be an ordering technique is developed in particular. The used approach is more time demanding in the area of automation with respect to former developed approaches. To efficiently compute orderings, a numerical approach is outlined. Ordering properties such as transitivity are aimed to be investigated by future works when adopting a fuzzy number comparison. The connection with cycle transitivity, not regarded by permutations is interested in particular.

In [8] the fuzziness is considered from two different aspects. First, the properties of fuzzy functions, their fixed point properties, integrals and derivatives are studied. Study of classical real function is the second aspect. Devoted to this aim, that for the real functions theory introducing the methods of fuzzy mathematics can provide some interesting result. Two possible directions of fuzzy infinitesimal calculus are described. Firstly, the one that deals with fuzzy objects. The similar results as in classical case are obtained using reasonable definition. On the other hand it is shown that sometimes the method used to obtain these results differs a lot from the classical ones.

Rather than viewing fuzzy intervals as fuzzy sets, a new way of looking at them is introduced in [9]. They are seen as crisp sets of entities and called gradual real numbers. The fuzzy interval monotonic gradual real numbers are described as a pair of fuzzy thresholds. Without resorting to α-cuts this view enables interval analysis in agreement with Zadeh's extension principle to be fuzzy intervals extension directly.

In [10] the basic ideas regarding interval fuzzy numbers are presented. Related to fuzzy sets many ideas are extended to the interval approach. The arithmetic operations with respect to interval cuts are defined, and the features of the operations are allowed to be analyzed based on the characteristics of the interval arithmetic operations.

In [11] an application of a new fuzzy arithmetic to fuzzy calculus and fuzzy linear equations is discussed. Four computable operations on fuzzy numbers are given.

In [12] the proof of convergence of the arithmetic mean to normal arithmetic mean is given. The fuzzy average approach is examined with fuzzy control rules and defuzzification operators. It is shown that compare to fuzzy control rules, the fuzzy average implementation is simpler.

In [13] proposed α-cut method is used to construct the membership function, and this method is appropriate to deal with different kind of fuzzy arithmetic, and is also applied for such operations as the exponentiation, extraction of nth roots, logarithm etc.

The problems related with subtraction and division arithmetic operations which are useful to represent uncertain information must be solved. A new operation on triangular fuzzy numbers is used to perform the modification process of subtraction and division operations that can be suitable for solving optimization problems [14].

The constrained fuzzy arithmetic is formulated in nonstandard arithmetic form to overcome the disadvantages of standard fuzzy arithmetic that can't deal with constraints with linguistic variables [15]. The basic properties of a proposed fuzzy arithmetic are defined, and the investigation is carried out for some common types of constrains.

The extension problem of Zadeh is an effective tool to develop fuzzy arithmetic. Sometimes the application of the extension principle that is very important to extend classical functions to fuzzy mappings is not convenient for making pessimistic decisions. A new extension principle for such kind of decisions is investigated to define the fuzzy set category [16]. It is proven that the extension principles are rational.

The calculation formulas for t-norm based additions of LR-fuzzy intervals are defined in [17]. The sum of fuzzy intervals based on t-norm for calculating the membership functions is given. It is mentioned that t-norms class is useful in practical calculations.

In [18] some similar characteristics of fuzzy arithmetic and approximate reasoning are investigated. The fuzzy arithmetic uses fuzzy numbers as basic notations, but approximate reasoning uses linguistic rules. One of the main similarities between these theories is using convex fuzzy sets. This similarity is used to apply the main theorems of fuzzy arithmetic in approximate reasoning, and also vice versa. The similarity between above theories allows implementing it on generalized modus ponens problem.

Fuzzy numbers ranking approach in newly way is proposed in [19]. The fuzzy numbers are compared on the base of distance measure. The defined new parametric interval for ranking fuzzy numbers is differed from existing parametric interval by using the statistical principles that provide more advantages in ranking fuzzy numbers. The examples illustrate the advantages of the proposed approach.

[20] aims to consider a working principle of a quality function deployment (QFD) under uncertainty to create the necessary procedures in working with fuzzy data. The fuzzification of input data as well the defuzzification of output data is discussed. The fuzzy numbers measured with degree of fuzziness, and the characteristics of the fuzzy arithmetic and the importance of the proper defuzzification strategy are highlighted. QFD makes it possible to optimally deal with linguistic data.

A decomposition of fuzzy sets or intervals is important in the analysis and studying of properties of fuzzy arithmetic. Some approximations of fuzzy operations are realized by using a decomposition to assure a distributivity of some arithmetic operations, and particularly, multiplication and division operations [21].

Chapter 3

BASIC PROPERTIES OF FUZZY ARITHMETIC

3.1 Properties of fuzzy sets

Let X be the universal set (universe of discourse) which contains all the concerned possible elements in each particular context. A simply set A, classical (crisp) set A, in the universe of discourse X can be defined by list method, in which you list all of its members or by rule method, in which you specify the properties that must be satisfied by the members of the set. The list method is of limited use that can only be applied to finite sets. In the rule method which is more general, a set A is represented as

$$A = \{x \in \mathrm{X} | x \text{ meets some conditions}\}$$

Another way to define a set A is the membership method which introduces a function called characteristic function (also called discrimination function, or indicator function) for A, denoted by $\mu_A(x)$, such that

$$\mu_A(x) = \begin{cases} 1, if\ x \in A \\ 0, if\ x \notin A \end{cases}$$

The membership function $\mu_A(x)$ is mathematically equivalent to the set A, in the sense that knowing $\mu_A(x)$ is the same as knowing A itself.

A fuzzy set A in X (universe of discourse) is defined to be a set of ordered pairs with an element from the universe of discourse as the first coordinate and the second coordinate which is a real number in the interval [0, 1], obtained from a function called membership function, $\mu_A(x)$. The membership function assigns to each element $x \in X$ a degree of membership in the set A.

If there exits at least one point/element $x \epsilon X$ with $\mu_A(x) = 1$, then the fuzzy set A is normal.

If for any two points/elements in the universe, $x_1, x_2 \in X$ and any real number between zero and one, $\lambda \in [0,1]$, the result obtained from membership function of $(\lambda x_1 + (1 - \lambda)x_2)$ is greater than the minimum of membership of either x_1 or x_2 (min $\{\mu_A(x_1), \mu_A(x_2))$, then the fuzzy set A is convex.

An ordinary set of elements with membership strictly greater than α for $0 \leq \alpha \leq 1$ is an α-cut A^α of a fuzzy set A.

Fuzzy interval is a fuzzy set A on X (universe of discourse) in which A is normal, for all real number between zero and one $\{\alpha \in [0,1]\}$, A^α is convex and A^α is closed interval.

Moreover, the fuzzy set A is called a fuzzy number if there exists only one point/element in the universe X $\{x_0 \in X\}$ such that the membership of that element is one $\{\mu_A(x_0) = 1\}$.

Often it is useful in a fuzzy set which is a collection of objects with various degrees of membership, to consider those elements having at least some minimal degree of membership, say alpha (α).

A crisp set A^α yields from a given fuzzy set A with those elements of the universe X who have membership grade in A of at least α, for all $\alpha \in [0,1]$ is called an alpha-level of a fuzzy set (α-cut).

$$A^\alpha = \{x \in X | \mu_A(x) \geq \alpha\}$$

If those elements of the universe X having membership grade in A of strictly greater than α are considered, for all $\alpha \in [0,1]$, then it is called an alpha-cut of a fuzzy set.

$$A^\alpha = \{x \in X | \mu_A(x) > \alpha\}$$

For a fuzzy set A given in the Table 1, an alpha-level at α=0.6 is $A^{0.6} = \{a, b\}$, since $\mu_A(a) = 1.0 \geq 0.6$, $\mu_A(b) = 0.6 \geq 0.6$, but $\mu_A(c) = 0.3 \ngeq 0.6$.

An alpha-cut of A at α=0.6 is $A^{0.6} = \{a\}$, since $\mu_A(a) = 1.0 > 0.6$, but $\mu_A(b) = 0.6 \ngtr 0.6$, and $\mu_A(c) = 0.3 \ngtr 0.6$.

Table 1: A fuzzy set A with its elements and their membership values

x	A
a	1.0
b	0.6
c	0.3

3.2 Properties of fuzzy number. Triangular fuzzy number

A fuzzy number 3 which denotes “about 3” will be given as

$$\mu_3(x) = \begin{cases} \dfrac{x}{3}, 0 \leq x \leq 3 \\ \dfrac{6 - x}{3}, 3 \leq x \leq 6 \\ 0, otherwise \end{cases}$$

Fuzzy number 3 illustrated in Figure 1 is normal convex fuzzy number. Figure 2 describes various kind of fuzzy number, in which A is convex, B is normal non-convex, and C is non-convex and not normal.

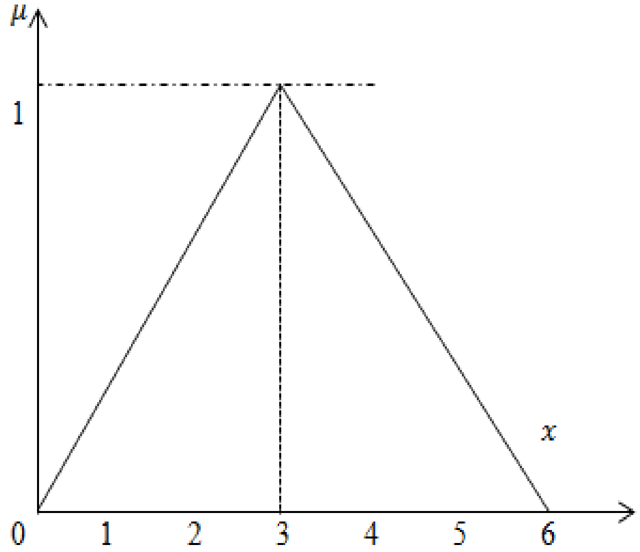

Figure 1: Normal convex fuzzy number 3

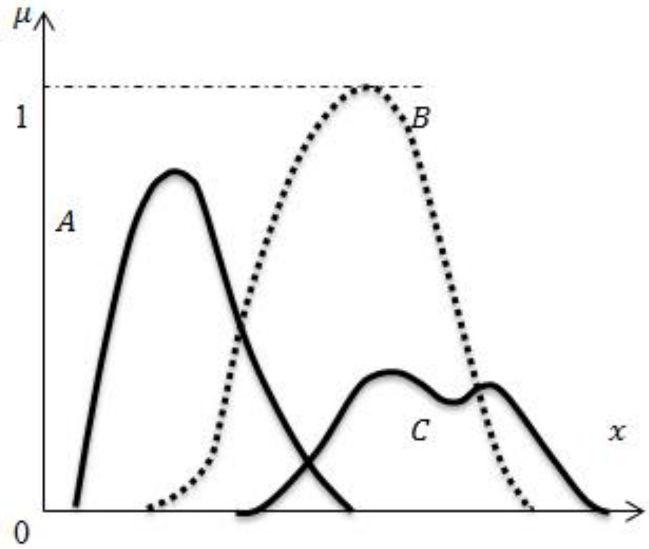

Figure 2: Convex, normal non-convex and non-convex fuzzy numbers

A fuzzy number A represented with three points as $\tilde{A} = (a, b, c)$ with the membership function given below is called a triangular fuzzy number:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a \\ \dfrac{x-a}{b-a}, & a \leq x \leq b \\ \dfrac{c-x}{c-b}, & b \leq x \leq c \\ 0, & x \geq c \end{cases}$$

3.3 Arithmetic operations on fuzzy number

The α-cuts of a fuzzy set and also each fuzzy number fully represent the fuzzy set and the fuzzy number; and for all $\alpha \in [0,1]$, α-cuts of each fuzzy number are considered as closed intervals of real numbers. Based on these properties, with respect to arithmetic operations on their α-cuts, fuzzy numbers arithmetic operations were made.

Before the arithmetic operations on fuzzy numbers, we recall some arithmetic operations on crisp interval.

Crisp interval addition is calculated in the form below with an example and figure 3 illustrates this example.

$$[a, b] + [c, d] = [a + c, b + d]$$

$$[-2, 1] + [0, 5] = [-2, 6]$$

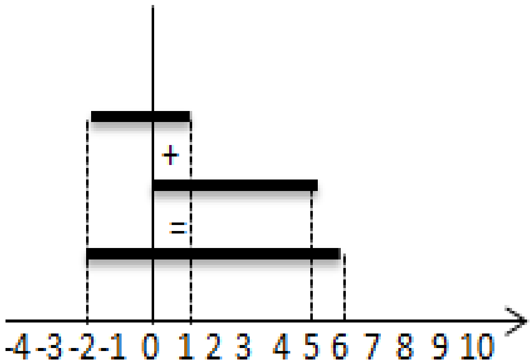

Figure 3: A crisp interval addition

A crisp interval subtraction is calculated in the form below with an example and figure 4 illustrates this example.

$$[a, b] - [c, d] = [a - d, b - c]$$

$$[-2, 1] - [0, 5] = [-7,1]$$

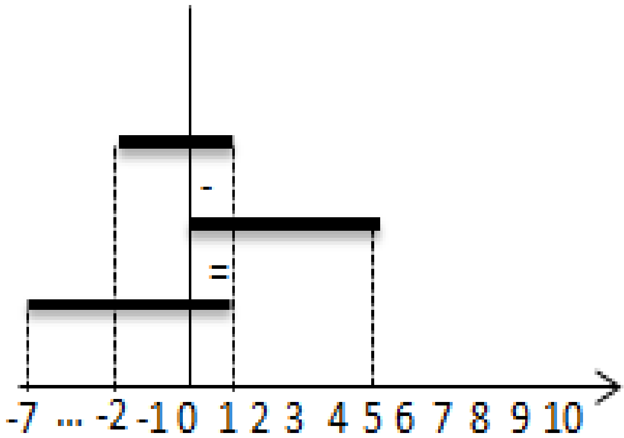

Figure 4: A crisp interval subtraction

Crisp interval multiplication is carried out in the form below with an example given.

$$[a, b] * [c, d] = [\min (ac, ad, bc, bd), \max (ac, ad, bc, bd)]$$

$$[2, 4] \times [3, 2] = [\min(2 \times 3, 2 \times 2, 4 \times 3, 4 \times 2), \max(2 \times 3, 2 \times 2, 4 \times 3, 4 \times 2)]$$

$$= [\min(6,4,12,8), \max(6,4,12,8)] = [4, 12]$$

Crisp interval division is carried out in the form below with an example given.

$$\frac{[a, b]}{[c, d]} = [a, b] \times \left[\frac{1}{d}, \frac{1}{c}\right] = \left[\min\left(\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\right), \max\left(\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\right)\right]$$

$$\frac{[2,4]}{[1,2]} = [2,4] \times \left[\frac{1}{2}, \frac{1}{1}\right] = \left[\min\left(\frac{2}{1}, \frac{2}{2}, \frac{4}{1}, \frac{4}{2}\right), \max\left(\frac{2}{1}, \frac{2}{2}, \frac{4}{1}, \frac{4}{2}\right)\right]$$

$$= [\min(2,1,4,2), \max(2,1,4,2)] = [1,4]$$

3.3.1 Addition of fuzzy numbers using α-cut

Consider

$$\mu_A(x) = \begin{cases} \dfrac{x-a}{b-a}, a \leq\ x \leq b \\ \dfrac{c-x}{c-b}, b \leq x \leq c \\ 0, otherwise \end{cases}$$

and

$$\mu_B(x) = \begin{cases} \dfrac{x-e}{f-e}, e \leq\ x \leq f \\ \dfrac{g-x}{g-f}, f \leq x \leq g \\ 0, otherwise \end{cases}$$

to be membership functions of two fuzzy numbers $A = [a, b, c]$ and $B = [e, f, g]$.

Then, α–cuts of fuzzy numbers A and B are $A^{\alpha} = [(b-a)\alpha + a, c - (c-b)\alpha]$, and $B^{\alpha} = [(f-e)\alpha + e, g - (g-f)\alpha]$, respectively. In order to calculate the addition of fuzzy numbers A and B, firstly, α–cuts of A and B are added using interval arithmetic [13].

$$A^{\alpha} + B^{\alpha} = [(b-a)\alpha + a, c - (c-b)\alpha] + [(f-e)\alpha + e, g - (g-f)\alpha]$$

$$= [(b-a+f-e)\alpha + a + e, c + g - (c-b+g-f)\alpha] \quad (3.1)$$

The membership function $\mu_{A+B}(x)$ is calculated after both components above are taken equal to x, which gives

$$x = (b-a+f-e)\alpha + a + e, and\ x = c + g - (c-b+g-f)\alpha$$

Now, together with the domain of x we get α by expressing α with respect to x and setting $\alpha = 0$ and α =1 in (3.1)

$$\alpha = \frac{x-(a+e)}{(b+f)-(a+e)}, (a+e) \leq x \leq (b+f)$$

and

$$\alpha = \frac{(c+g)-x}{(c+g)-(b+f)}, (b+f) \leq x \leq (c+g)$$

This gives

$$\mu_{A+B}(x) = \begin{cases} \frac{x-(a+e)}{(b+f)-(a+e)}, (a+e) \leq x \leq (b+f) \\ \frac{(c+g)-x}{(c+g)-(b+f)}, (b+f) \leq x \leq (c+g) \\ 0, \quad otherwise \end{cases}$$

For $A = [1,4,5]$ and $B = [2,3,6]$, two fuzzy numbers with membership functions

$$\mu_A(x) = \begin{cases} \frac{x-1}{3}, 1 \leq x \leq 4 \\ 5-x, 4 \leq x \leq 5 \\ 0, otherwise \end{cases} \quad and \quad \mu_B(x) = \begin{cases} x-2, 2 \leq x \leq 3 \\ \frac{6-x}{3}, 3 \leq x \leq 6 \\ 0, otherwise \end{cases}$$

Then, α–cuts of fuzzy numbers A and B respectively are $A^\alpha = [3\alpha + 1, 5 - \alpha]$, and $B^\alpha = [\alpha + 2, 6 - 3\alpha]$. To calculate addition of fuzzy numbers A and B, firstly, the α –cuts of A and B are added using interval arithmetic:

$$A^\alpha + B^\alpha = [3\alpha + 1, 5 - \alpha] + [\alpha + 2, 6 - 3\alpha]$$

$$= [4\alpha + 3, 11 - 4\alpha] \qquad (3.2)$$

To find the membership function $\mu_{A+B}(x)$, $x = 4\alpha + 3$, and $x = 11 - 4\alpha$ are considered.

Now, together with the domain of x we get α by expressing α with respect to x and setting $\alpha = 0$ and α =1 in (3.2):

$$\alpha = \frac{x-3}{4}, 3 \le x \le 7, and\ \alpha = \frac{11-x}{4}, 7 \le x \le 11$$

This gives

$$\mu_{A+B}(x) = \begin{cases} \frac{x-3}{4}, 3 \le x \le 7 \\ \frac{11-x}{4}, 7 \le x \le 11 \\ 0, \quad otherwise \end{cases}$$

3.3.2 Subtraction of fuzzy numbers using α-cut

Consider

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a}, a \le x \le b \\ \frac{c-x}{c-b}, b \le x \le c \\ 0, otherwise \end{cases}$$

and

$$\mu_B(x) = \begin{cases} \frac{x-e}{f-e}, e \le x \le f \\ \frac{g-x}{g-f}, f \le x \le g \\ 0, otherwise \end{cases}$$

to be membership functions of two fuzzy numbers $A = [a, b, c]$ and $B = [e, f, g]$.

Then, α–cuts of fuzzy numbers A and B respectively are $A^{\alpha} = [(b-a)\alpha + a, c - (c-b)\alpha]$ and $B^{\alpha} = [(f-e)\alpha + e, g - (g-f)\alpha]$. In order to calculate subtraction of fuzzy numbers A and B, firstly, the α–cuts of A and B were subtracted using interval arithmetic [13].

$$A^{\alpha} - B^{\alpha} = [(b-a)\alpha + a, c - (c-b)\alpha] - [(f-e)\alpha + e, g - (g-f)\alpha]$$

$$= [(b-a)\alpha + a - (g - (g-f)\alpha), c - (c-b)\alpha - \left((f-e)\alpha + e\right)]$$

$$= [(a-g) + (b-a+g-f)\alpha, (c-e) - (c-b+f-e)\alpha] \quad (3.3)$$

Both the first and second components in (3.3) are equated to x to find the membership function $\mu_{A-B}(x)$, which gives

$$x = (a-g) + (b-a+g-f)\alpha, and\ x = (c-e) - (c-b+f-e)\alpha$$

Now, together with the domain of x we get α by expressing α with respect to x and setting $\alpha = 0$ and α =1 in (3.3):

$$\alpha = \frac{x-(a-g)}{(b-f)-(a-g)}, (a-g) \leq x \leq (b-f)$$

and

$$\alpha = \frac{(c-e)-x}{(c-e)-(b-f)}, (b-f) \leq x \leq (c-e)$$

This gives

$$\mu_{A-B}(x) = \begin{cases} \dfrac{x-(a-g)}{(b-f)-(a-g)}, (a-g) \le x \le (b-f) \\ \dfrac{(c-e)-x}{(c-e)-(b-f)}, (b-f) \le x \le (c-e) \\ 0, \quad otherwise \end{cases}$$

Let $A = [1,4,5]$ and $B = [2,3,6]$ be two fuzzy numbers whose membership functions are

$$\mu_A(x) = \begin{cases} \dfrac{x-1}{3}, 1 \le x \le 4 \\ 5-x, 4 \le x \le 5 \\ 0, otherwise \end{cases}$$

and

$$\mu_B(x) = \begin{cases} x-2, 2 \le x \le 3 \\ \dfrac{6-x}{3}, 3 \le x \le 6 \\ 0, otherwise \end{cases}$$

Then, α–cuts of fuzzy numbers A and B respectively are $A^\alpha = [3\alpha+1, 5-\alpha]$, and $B^\alpha = [\alpha+2, 6-3\alpha]$. To calculate the subtraction of fuzzy numbers A and B, firstly, α–cuts of A and B are subtracted using interval arithmetic.

$$A^\alpha - B^\alpha = [3\alpha+1, 5-\alpha] - [\alpha+2, 6-3\alpha]$$

$$= [6\alpha-5, 3-2\alpha] \qquad (3.4)$$

To find the membership function $\mu_{A-B}(x)$, $x = 6\alpha - 5$, and $x = 3 - 2\alpha$ are considered.

Now, together with the domain of x we get α by expressing α with respect to x and setting $\alpha = 0$ and α =1 in (3.4):

$$\alpha = \frac{x+5}{6}, -5 \leq x \leq 1, and\ \alpha = \frac{3-x}{2}, 1 \leq x \leq 3$$

This gives

$$\mu_{A-B}(x) = \begin{cases} \frac{x+5}{6}, -5 \leq x \leq 1 \\ \frac{3-x}{2}, 1 \leq x \leq 3 \\ 0,\ otherwise \end{cases}$$

3.3.3 Multiplication of Fuzzy Numbers using α-cut

Let $A = [1,4,5]$, and $B = [2,3,6]$ be two fuzzy numbers with membership functions

$$\mu_A(x) = \begin{cases} \frac{x-1}{3}, 1 \leq\ x \leq 4 \\ 5-x, 4 \leq x \leq 5 \\ 0,\ otherwise \end{cases}$$

$$\mu_B(x) = \begin{cases} x-2, 2 \leq\ x \leq 3 \\ \frac{6-x}{3}, 3 \leq x \leq 6 \\ 0,\ otherwise \end{cases}$$

Then, α–cuts of fuzzy numbers A and B respectively are $A^\alpha = [3\alpha + 1, 5 - \alpha]$, and $B^\alpha = [\alpha + 2, 6 - 3\alpha]$. To calculate the multiplication of fuzzy numbers A and B, firstly, the α – cuts of A and B are multiplied using interval arithmetic [13].

$$A^\alpha \times B^\alpha = [3\alpha + 1, 5 - \alpha] \times [\alpha + 2, 6 - 3\alpha]$$

$$= [(3\alpha + 1) \times (\alpha + 2), (5 - \alpha) \times (6 - 3\alpha)]$$

$$= [3\alpha^2 + 7\alpha + 2, 3\alpha^2 - 21\alpha + 30] \quad (3.5)$$

To find the membership function $\mu_{AB}(x)$, $x = 3\alpha^2 + 7\alpha + 2$, and $x = 3\alpha^2 - 21\alpha + 30$ are considered.

Now, together with the domain of x we get α by expressing α with respect to x and setting $\alpha = 0$ and α =1 in (3.5)

$$\alpha = \frac{-7 + \sqrt{25 + 12x}}{6}, 2 \leq x \leq 12 \; and \; \alpha = \frac{21 - \sqrt{81 + 12x}}{6}, 12 \leq x \leq 30$$

This gives

$$\mu_{AB}(x) = \begin{cases} \frac{-7 + \sqrt{25 + 12x}}{6}, 2 \leq x \leq 12 \\ \frac{21 - \sqrt{81 + 12x}}{6}, 12 \leq x \leq 30 \\ 0, \quad otherwise \end{cases}$$

3.3.4 Division of fuzzy numbers using α-cut

Let $A = [2,4,12]$ and $B = [1,2,4]$ be two fuzzy numbers with membership functions

$$\mu_A(x) = \begin{cases} \frac{x-2}{2}, 2 \leq x \leq 4 \\ \frac{12-x}{8}, 4 \leq x \leq 12 \\ 0, \; otherwise \end{cases}$$

$$\mu_B(x) = \begin{cases} x - 1, 1 \leq x \leq 2 \\ \frac{4-x}{2}, 2 \leq x \leq 4 \\ 0, \; otherwise \end{cases}$$

Then, α–cuts of fuzzy numbers A and B respectively are $A^\alpha = [2\alpha + 2, 12 - 8\alpha]$, and $B^\alpha = [\alpha + 1, 4 - 2\alpha]$.

To find A^α / B^α, we calculate: A^α / B^α = [$(2\alpha + 2)$ / $(\alpha + 1)$, $(12 - 8\alpha)$ / $4 - 2\alpha)$].

When $\alpha = 0$, we have: $A^0 (/)\ B^0 = [2{,}3]$.

When $\alpha = 1$, we have: $A^1 (/)\ B^1 = [2{,}2] = 2$.

The final result of the division is $A/B = (2{,}2{,}3)$.

3.4 L-R fuzzy number

L-R type fuzzy number is a fuzzy number A described with membership function

$$\mu_A(x) = \begin{cases} L\left(\dfrac{a-x}{\alpha}\right), x \in [a-\alpha, a] \\ 1, \qquad\qquad x \in [a, b] \\ R\left(\dfrac{x-b}{\beta}\right), x \in [b, b+\beta] \\ 0, \qquad otherwise \end{cases}$$

Denoted by $= (a, b, \alpha, \beta)_{LR}$, where A has its peak at [a, b] with a and b as the lower and upper modal values; L and R are the reference functions such that [0, 1] → [0, 1], with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$.

Consider $L(t) = \frac{1}{1+t^2}, R(t) = \frac{1}{1+2|t|}, a = 6, \alpha = 3, \beta = 4$

Then

$$\mu_A(x) = \begin{cases} L\left(\dfrac{a-x}{\alpha}\right), x < 6 \\ 1, \qquad\quad x = 6 \\ R\left(\dfrac{x-a}{\beta}\right), x > 6 \end{cases}$$

$$= \begin{cases} L\left(\frac{6-x}{3}\right) & x < 6 \\ 1, & x = 6 \\ R\left(\frac{x-6}{4}\right), & x > 6 \end{cases}$$

$$= \begin{cases} \frac{1}{1+\left(\frac{6-x}{3}\right)^2}, & x < 6 \\ 1, & x = 6 \\ \frac{1}{1+2\left|\frac{x-6}{4}\right|}, & x > 6 \end{cases}$$

3.4.1 Triangular L-R fuzzy number

$A = (a, \alpha, \beta)$ denotes a triangular fuzzy number A with membership function

$$\mu_A(x) = \begin{cases} 1 - \frac{a-x}{\alpha}, & a - \alpha \leq x \leq a \\ 1, & a \leq x \leq b \\ 1 - \frac{x-b}{\beta}, & b \leq x \leq b + \beta \\ 0, & otherwise \end{cases}$$

where a ∈ R is the center and α > 0 and β > 0 are the left and right spreads of A, respectively. The triangular fuzzy number with α = β is called symmetric triangular fuzzy number, denoted by (a, α).

L-R fuzzy number "around 6" with

$$\mu_A(x) = \frac{1}{1+(x-6)^2}, L(x) = R(x) = \frac{1}{1+x^2}, a = b = 6, \alpha = \beta = 1$$

Figure 5 depicts its graph.

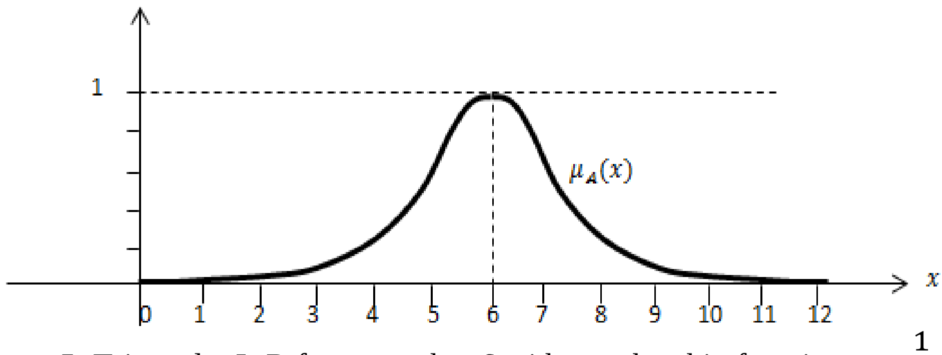

Figure 5: Triangular L-R fuzzy number 6 with membership function $\frac{1}{1+(x-6)^2}$

Also a fuzzy number "around 6" with

$$\mu_A(x) = e^{-(x-6)^2}, L(x) = R(x) = e^{-\left(\frac{x-a}{\alpha}\right)^2}, a = b = 6, \alpha = \beta = 1$$

Figure 6 depicts its graph.

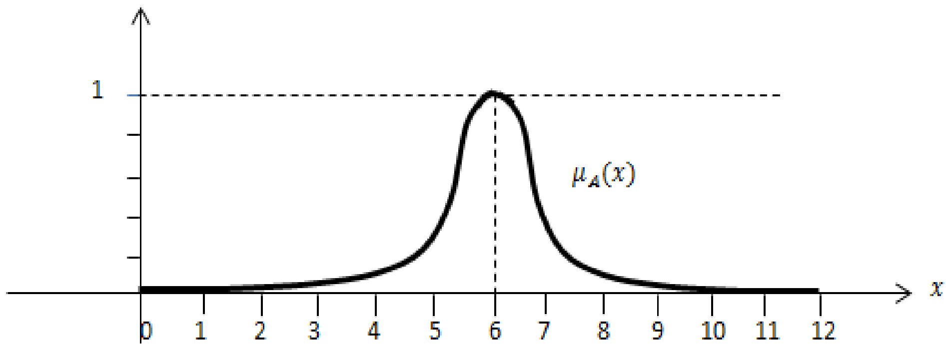

Figure 6: Triangular L-R fuzzy number 6 with membership function $e^{-(x-6)^2}$

Figure 7 depicts a graph showing the difference between fuzzy number 6 with membership functions $\mu_A(x) = \frac{1}{1+(x-6)^2}$, and $\mu_A(x) = e^{-(x-6)^2}$.

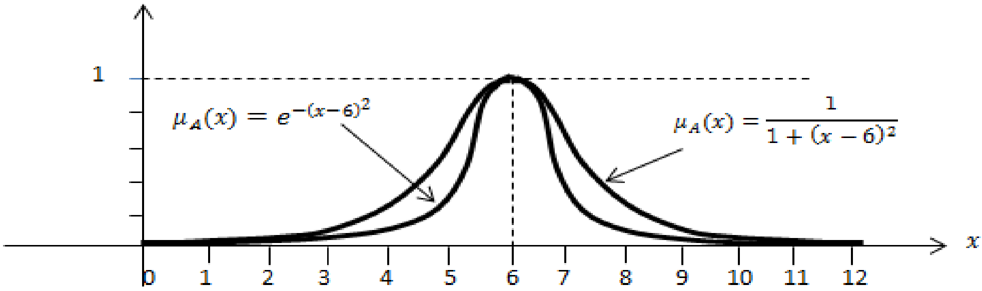

Figure 7: Difference between fuzzy number 6 with membership functions $\mu_A(x) = \frac{1}{1+(x-6)^2}$ and $\mu_A(x) = e^{-(x-6)^2}$

3.4.2 Arithmetic operations on L-R fuzzy number

$A = (a, \alpha, \beta)_{LR}$ and $B = (b, \sigma, \delta)_{LR}$ in $\mathbb{R}$ with non-increasing L-R reference function are considered to be triangular L-R fuzzy numbers.

The following form depicts how L-R fuzzy number addition is carried out

$$A + B = (a, \alpha, \beta)_{LR} + (b, \sigma, \delta)_{LR} = (a + b, \alpha + \sigma, \beta + \delta)_{LR}$$

Considering $= (a, \alpha, \beta)_{LR} = (3,5,4)_{LR}$, and $B = (b, \sigma, \delta)_{LR} = (1,2,7)_{LR}$ being two triangular L-R fuzzy numbers. Then

$$A + B = (3,5,4)_{LR} + (1,2,7)_{LR} = (3 + 1, 5 + 2, 4 + 7)_{LR} = (4,7,11)_{LR}$$

The following form depicts L-R fuzzy number image:

$$\bar{A} = -(a, \alpha, \beta)_{LR} = (-a, \alpha, \beta)_{LR}$$

Consider $A = (a, \alpha, \beta)_{LR} = (1,2,7)_{LR}$ to be a triangular L-R fuzzy number. Then

$$\bar{A} = -(a, \alpha, \beta)_{LR} = -(1,2,7)_{LR} = (-1,2,7)_{LR}$$

The following form shows how L-R fuzzy number subtraction is carried out:

$$A - B = (a, \alpha, \beta)_{LR} - (b, \sigma, \delta)_{LR} = (a - b, \alpha + \delta, \beta + \sigma)_{LR}$$

Considering $A = (a, \alpha, \beta)_{LR} = (3,5,4)_{LR}$ and $B = (b, \sigma, \delta)_{LR} = (1,2,7)_{LR}$ being two triangular L-R fuzzy numbers. Then

$$A - B = (3,5,4)_{LR} - (1,2,7)_{LR} = (3 - 1, 5 + 7, 4 + 2)_{LR} = (2,12,6)_{LR}$$

The following form depicts how L-R fuzzy number multiplication is carried out:

$$A \times B = (a, \alpha, \beta)_{LR} \times (b, \sigma, \delta)_{LR}$$

$$= (a \times b, a \times \sigma + b \times \alpha, a \times \delta + b \times \beta)_{LR}$$

$$k \in \mathbb{R}, k \times A = k \times (a, \alpha, \beta)_{LR}$$

$$= \begin{cases} (ka, k\alpha, k\beta)_{LR} & k > 0 \\ (ka, -k\beta, -k\alpha)_{RL} & k < 0 \end{cases}$$

Considering $A = (a, \alpha, \beta)_{LR} = (3,5,4)_{LR}$ and $B = (b, \sigma, \delta)_{LR} = (1,2,7)_{LR}$ being two triangular L-R fuzzy numbers. Then

$$A \times B = (3,5,4)_{LR} \times (1,2,7)_{LR} = (3 \times 1, 3 \times 2 + 1 \times 5, 3 \times 7 + 1 \times 4)_{LR}$$

$$= (3,11,25)_{LR}$$

3.5 Fuzzy interval

The interval fuzzy subset $A = \{(x, \mu_A(x)) \mid x \in \mathbb{R}\}$ subset A is a set of ordered pairs of the set A of real numbers $\mathbb{R}$, where $\mu_A: \mathbb{R} \to U$ is the membership function of A described in interval-valued form.

$\mu_{A_l}, \mu_{A_u}: \mathbb{R} \rightarrow U$ continuous function exist if the interval-valued membership function μ_A is more continuous, where μ_{A_l} is lower membership function, and $\mu_{A_u}$is upper membership function, such that for all x ∈ ℝ:

$$\mu_A(\mathrm{x}) = [\, \mu_{A_l}(x), \mu_{A_u}(x)]$$

with $\mu_{A_l}(x) \leq \mu_{A_u}(x)$.

The inner support, the outer support and the core of an interval fuzzy subset A of ℝ are defined respectively as follows:

$$isupp_A = \{\mathrm{x} \in \mathbb{R} |\, \mu_{A_l}(x) > 0\}$$

$$osupp_A = \{\mathrm{x} \in \mathbb{R} |\, \mu_{A_u}(x) > 0\}$$

$$core_A = \{\mathrm{x} \in \mathbb{R} |\, \mu_A(x) = [1; 1]\}$$

Fuzzy interval is a fuzzy set A on X (universe of discourse) in which A is normal, for all real number between zero and one $\{\alpha \in [0,1]\}$, A^α is convex and for all A^α is closed interval.

The fuzzy number is the fuzzy set A with exactly one element in the universe X $\{x_0 \in X\}$ such that the membership of that element is one $\{\mu_A(x_0) = 1\}$. The fuzzy set A is called fuzzy interval if there exists more than one element in the universe $\{x \in X\}$ such that the membership of that element is one $\{\mu_A(x) = 1\}$.

3.5.1 Multiplication of a scalar value to the interval

The multiplication of a scalar value to the interval is performed in the following form:

$$a[b_1, b_2] = [ab_1 \wedge ab_2, ab_1 \vee ab_2]$$

Suppose we want to multiply the scalar value 2.5 to the interval [3,7]:

$$2.5 * [3,7] = [2.5 * 3 \wedge 2.5 * 7, 2.5 * 3 \vee 2.5 * 7] = [7.5 \wedge 17.5, 7.5 \vee 17.5]$$

$$= [7.5, 17,5]$$

3.5.2 L-R fuzzy interval

Let $A = (a_1, a_2, \alpha\ , \beta)_{LR}$ denote fuzzy interval with a membership function

$$\mu_A(x) = \begin{cases} L\left(\frac{a_1 - x}{\alpha}\right), & x < a_1, \alpha > 0 \\ 1, & x = [a_1, a_2] \\ R\left(\frac{x - a_2}{\beta}\right), & x > a_2, \beta > 0 \end{cases}$$

$A = (a_1, a_2, \alpha, \beta)_{LR}$, and $B = (b_1, b_2, \sigma, \delta)_{LR}$ in R with non-increasing L-R reference functions are considered to be two L-R fuzzy intervals.

The following form depicts how L-R fuzzy interval addition is carried out:

$$A + B\ = (a_1, a_2, \alpha, \beta)_{LR} + (b_1, b_2, \sigma, \delta)_{LR} = \ (a_1 + b_1, a_2 + b_2\ , \alpha\ + \sigma, \beta + \delta)_{LR}$$

Considering $A = (a_1, a_2, \alpha, \beta)_{LR} = (3,1,5,4)_{LR}$ and $B = (b_1, b_2, \sigma, \delta)_{LR} = (1,3,2,7)_{LR}$ to be two L-R fuzzy intervals. Then

$$A + B\ = (3,1,5,4)_{LR} + (1,3,2,7)_{LR}$$

$$= \ (3 + 1, 1 + 3, 5\ + 2, 4 + 7)_{LR}$$

$$= (4,4,7,11)_{LR}$$

The following form depicts L-R fuzzy interval image

$$\bar{A} = -(a_1, a_2, \alpha, \beta)_{LR} = (-a_1, -a_2, \alpha, \beta)_{LR}$$

Considering $A = (a_1, a_2, \alpha, \beta)_{LR} = (3{,}1{,}5{,}4)_{LR}$ being L-R fuzzy intervals. Then

$$\bar{A} = -(3{,}1{,}5{,}4)_{LR} = (-3, -1{,}5{,}4)_{LR}$$

The following form depicts how L-R fuzzy interval subtraction is carried out:

$$A - B = (a_1, a_2, \alpha, \beta)_{LR} - (b_1, b_2, \sigma, \delta)_{LR}$$

$$= (a_1 - b_1, a_2 - b_2, \alpha + \delta, \beta + \sigma)_{LR}$$

Considering $A = (a_1, a_2, \alpha, \beta)_{LR} = (3{,}1{,}5{,}4)_{LR}$ and $B = (b_1, b_2, \sigma, \delta)_{LR} = (1{,}3{,}2{,}7)_{LR}$ being two L-R fuzzy intervals. Then

$$A - B = (3{,}1{,}5{,}4)_{LR} - (1{,}3{,}2{,}7)_{LR} = (3 - 1, 1 - 3, 5 + 7, 4 + 2)_{LR}$$

$$= (2, -2{,}12{,}6)_{LR}$$

The following form depicts how L-R fuzzy number multiplication is carried out:

$$k \in R, k \times A = k \times (a_1, a_2, \alpha, \beta)_{LR}$$

$$= \begin{cases} (ka_1, ka_2, k\alpha, k\beta)_{LR} & k > 0 \\ (ka_2, ka_1, |k|\alpha, |k|\beta)_{LR} & k < 0 \end{cases}$$

Chapter 4

EXTENSION PRINCIPLE IN FUZZY ARITHMETIC. FUZZY EQUATIONS

4.1 The basic properties of extension principle

Extension principle is used as a mathematical instrument that takes crisp mathematical notions and procedures, extend them to the fuzzy realm, resulting in computable fuzzy sets by fuzzifying the parameters of a function.

Consider a fuzzy set A with the same inputs of function f.

$$y_1 = f(x_1)$$

$$y_2 = f(x_2)$$

$$.$$

$$.$$

$$.$$

$$y_n = f(x_n)$$

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \ldots + \mu_A(x_n)/x_n$$

Using extension principle, if the input of function f becomes fuzzy, there is a fuzzy output:

$$B = f(A) = \mu_A(x_1)/f(x_1) + \mu_A(x_2)/ f(x_2) + \ldots + \mu_A(x_n)/ f(x_n)$$

Where each image of x_i under function f becomes fuzzy with membership function $\mu_A(x_i)$. Many functions are many-to-one, which means if used, several x might map to a single y. The extension principle says that in such a case, in between the membership values of these several x elements of the fuzzy set A, the maximum should be chosen as the membership of y. In the event where no element x in X is mapped to an output y, zero is the membership value of y element of set B.

$$\mu_B(y) = \max_{f^{-1}(y)} \mu_A(x)$$

Consider fuzzy set $A = 0.2/-2 + 0.4/-1 + 0.7/0 + 0.9/1 + 0.4/2$, and $f(x) = x^2 - 2$.

Using the extension principle results another fuzzy set B as

$$B = 0.2/f(-2) \ + \ 0.4/f(-1) \ + \ 0.7/f(0) + 0.9/f(1) + 0.4/f(2)$$

$B = 0.2/((-2)^2 - 2) \ + \ 0.4/((-1)^2 - 2) \ + \ 0.7/((0)^2 - 2) \ + \ 0.9/((1)^2 - 2) \ + \ 0.4/((2)^2 - 2)$

$$B = 0.2/2 + 0.4/-1 + 0.7/-2 + 0.9/-1 + 0.4/2$$

Two elements of the fuzzy set B has two different membership values, and

$$B = 0.7/-2 + \max(0.4, 0.9)/-1 + \max(0.2, 0.4)/\,2$$

$$B = 0.7/-2 + 0.9/-1 + 0.4/2$$

4.1.1 Monotonic continuous function

In monotonic continuous function the image of the interval for every point is computed, and the membership values are carried through.

Consider the monotonic continuous function $y = f(x) = 0.5 \times x + 3$, and input fuzzy number around 7 $= 0.4/3 + 0.7/5 + 1.0/7 + 0.7/9 + 0.4/11$

The extension principle may be used in the following form and the figure 8 illustrates the graph.

$$\mathrm{f(around\ 7)} = 0.4/f(3) + 0.7/f(5) + 1.0/f(7) + 0.7/f(9) + 0.4/f(11)$$

$\mathrm{f(around\ 7)} = 0.4/(0.5 \times 3 + 3)$ + $0.7/(0.5 \times 5 + 3)$ + $1.0/(0.5 \times 7 + 3)$ + $0.7/(0.5 \times 9 + 3)$ + $0.4/(0.5 \times 11 + 3)$

$$\mathrm{f(around\ 7)} = 0.4/4.5 + 0.7/5.5 + 1.0/6.5 + 0.7/7.5 + 0.4/8.5$$

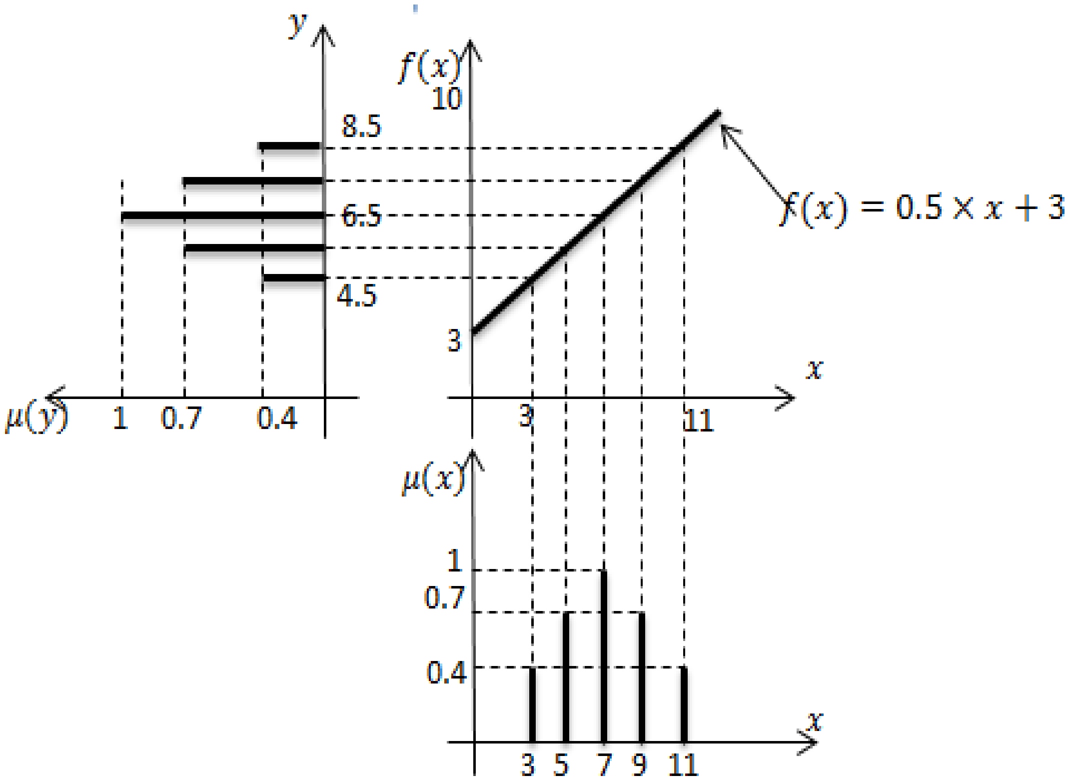

Figure 8: Extension principle using function $y = f(x) = 0.5 \times x + 3$ and input fuzzy number around 7

4.1.2 Non-monotonic continuous function

In non-monotonic continuous function the image of the interval for every point is computed, and the membership values are carried through. The membership values are combined if different inputs maps to single values.

Consider the non-monotonic continuous function $y = f(x) = x^2 - 9x + 18$, and input fuzzy number around 5 $= 0.4/1 + 0.7/3 + 1.0/5 + 0.7/7 + 0.4/9.$

The extension principle may be used in the following form and the figure 9 illustrates the graph.

$$\text{f(around 5)} = 0.4/f(1) + 0.7/f(3) + 1.0/f(5) + 0.7/f(7) + 0.4/f(9)$$

$$= 0.4/((1)^2 - 9(1) + 18) + 0.7/((3)^2 - 9(3) + 18) + 1.0/((5)^2 - 9(5)$$

$$+ 18) + 0.7/((7)^2 - 9(7) + 18) + 0.4/((9)^2 - 9(9) + 18)$$

$$= 0.4/(1 - 9 + 18) + 0.7/(9 - 27 + 18) + 1.0/(25 - 45 + 18) + 0.7/(49$$

$$- 63 + 18) + 0.4/(81 - 81 + 18$$

$$= 0.4/10 + 0.7/0 + 1.0/{-2} + 0.7/4 + 0.4/18$$

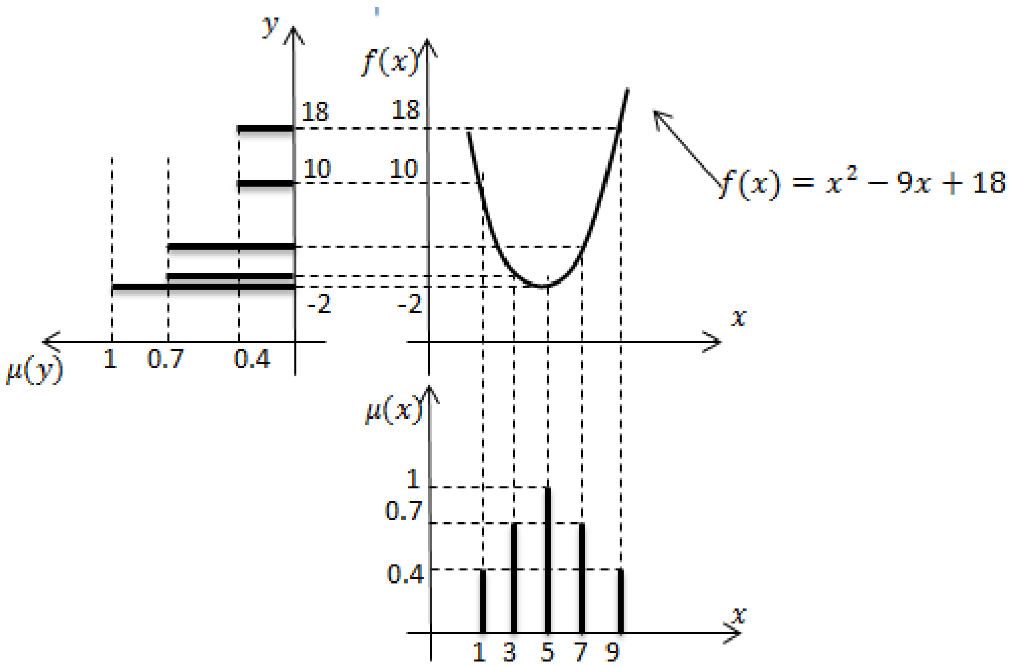

Figure 9. Extension principle with non-monotonic continuous function $y = f(x) = x^2 - 9x + 18$

4.2 Fuzzy arithmetic operations using extension principle

4.2.1 Fuzzy addition using extension principle

In order to execute the fuzzy addition using the extension principle, the below formula is used to obtain the membership value of every $z \in (A + B)$ such that $x + y = z, x \in A \text{ and } y \in B$.

$$\mu_{A+B}(z) = max(\min(\mu_A(x), \mu_B(y)))$$

Consider

$A = 0.4/3 + 0.7/4 + 1/5 + 0.7/6 + 0.4/7$, and $B = 0.5/10 + 1/11 + 0.5/12$

Then

$A + B = min(0.4, 0.5)/(3 + 10) + min(0.7,0.5)/(4 + 10) + min(1,0.5)/(5 + 10) + min(0.7,0.5)/(6 + 10) + min(0.4,0.5)/(7 + 10) + min(0.4,1)/(3 + 11) + min(0.7,1)/(4 + 11) + min(1,1)/(5 + 11) + min(0.7,1)/(6 + 11) + min(0.4,1)/(7 + 11) + min(0.4,0.5)/(3 + 12) + min(0.7,0.5)/(4 + 12) + min(1,0.5)/(5 + 12) + min(0.7,0.5)/(6 + 12) + min(0.4,0.5)/(7 + 12)$

After getting the minimum of the membership values and addition of the elements, we have:

$A + B = 0.4/13 + 0.5/14 + 0.5/15 + 0.5/16 + 0.4/17 + 0.4/14 + 0.7/15 + 1/16 + 0.7/17 + 0.4/18 + 0.4/15 + 0.5/16 + 0.5/17 + 0.5/18 + 0.4/19$

Then

$A + B =$

$0.4/13 + \max(0.5, 0.4)/14 + \max(0.5,0.7,0.4)/15 + \max(0.5,1,0.5)/16 + max(0.4,0.7,0.5)/17 + max(0.4,0.5)/18 + 0.4/19$

This gives

$A + B = 0.4/13+0.5/14+0.7/15+1/16+0.7/17+0.5/18+0.4/19$

4.2.2 Fuzzy subtraction using extension principle

In order to execute fuzzy subtraction using the extension principle, the below formula is used to obtain the membership value of every $z \in (A - B)$ such that $x - y = z, x \in A \, and \, y \in B$.

$$\mu_{A-B}(z) = max(\min(\mu_A(x), \mu_B(y)))$$

Consider

$A = 0.4/9 + 0.7/10 + 1/11 + 0.7/12 + 0.4/13$, and $B = 0.5/4 + 1/5 + 0.5/6$

Then

$A - B = min(0.4, 0.5)/(9 - 4) + min(0.7,0.5)/(10 - 4) + min(1,0.5)/(11 - 4) + min(0.7,0.5)/(12 - 4) + min(0.4,0.5)/(13 - 4) + min(0.4,1)/(9 - 5) + min(0.7,1)/(10 - 5) + min(1,1)/(11 - 5) + min(0.7,1)/(12 - 5) + min(0.4,1)/(13 - 5) + min(0.4,0.5)/(9 - 6) + min(0.7,0.5)/(10 - 6) + min(1,0.5)/(11 - 6) + min(0.7,0.5)/(12 - 6) + min(0.4,0.5)/(13 - 6)$

After getting the minimum of the membership values and subtraction of the elements, we have:

$A - B = 0.4/5 + 0.5/6 + 0.5/7 + 0.5/8 + 0.4/9 + 0.4/4 + 0.7/5 + 1/6 + 0.7/7 + 0.4/8 + 0.4/3 + 0.5/4 + 0.5/5 + 0.5/6 + 0.4/7$

Then

$$A - B = 0.4/3 \; + \; max(0.5, 0.4)\,/4 \; + \; max(0.5,0.7,0.4)\,/5$$
$$+ \; max(0.5,1,0.5)\,/6 + max(0.4,0.7,0.5)/7 + max(0.4, 0.5)\,/8$$
$$+ \; 0.4/9$$

This gives

$$A - B = \; 0.4/3 \; + \; 0.5/4 \; + \; 0.7/5 \; + \; 1/6 \; + \; 0.7/7 + 0.5/8 + \; 0.4/9$$

4.2.3 Fuzzy multiplication using extension principle

In order to execute the fuzzy multiplication using the extension principle, the below formula is used to obtain the membership value of every $z \in (A \times B)$ such that $x \times y = z, x \in A \; and \; y \in B$.

$$\mu_{A \times B}(z) = max(\min(\mu_A(x), \mu_B(y)))$$

Consider

$A \; = \; 0.4/3 + 0.7/4 + 1/5 + 0.7/6 + 0.4/7$, and $B \; = \; 0.5/10 \; + \; 1/11 \; + \; 0.5/12$

Then

$A \times B = \; min(0.4, 0.5)/(3 \times 10)+ min(0.7,0.5)/(4 \times 10)+ min(1,0.5)/(5 \times 10)+ min(0.7,0.5)/(6 \times 10)+ min(0.4,0.5)/(7 \times 10)+ min(0.4,1)/(3 \times 11)+ min(0.7,1)/(4 \times 11)+ min(1,1)/(5 \times 11)+ min(0.7,1)/(6 \times 11)+ min(0.4,1)/(7 \times 11)+ min(0.4,0.5)/(3 \times 12)+ min(0.7,0.5)/(4 \times 12)+ min(1,0.5)/(5 \times 12)+ min(0.7,0.5)/(6 \times 12)+ min(0.4,0.5)/(7 \times 12)$

After getting the minimum of the membership values and multiplication of the elements, we have:

$A \times B = 0.4/30 \ + 0.5/40 \ + 0.5/50 \ + 0.5/60 + 0.4/70 \ + 0.4/33 \ + 0.7/44 \ + 1/55 \ + 0.7/66 \ + 0.4/77 \ + 0.4/36 \ + 0.5/48 \ + 0.5/60 \ + 0.5/72 \ + 0.4/84$

Then

$A \times B =$

0.4/30+0.4/33+0.4/36+0.5/40+0.7/44+0.5/48+0.5/50+1/55+max(0.5,0.5)/60+0.7/66+0.4/70+0.5/72+0.4/77 + 0.4/84

This gives

$A \times B =$

0.4/30+0.4/33+0.4/36+0.5/40+0.7/44+0.5/48+0.5/50+1/55+0.5/60+0.7/66+0.4/70+0.5/72+0.4/84

4.2.4 Fuzzy division using extension principle

In order to execute the fuzzy division using the extension principle, the below formula is used to obtain the membership value of every $z \in (A \div B)$ such that $x \div y = z, x \in A \; and \; y \in B$.

$$\mu_{A \div B}(z) = max(\min(\mu_A(x), \mu_B(y)))$$

Consider

$A = 0.4/8 + 0.7/10 + 1/12 + 0.7/14 + 0.4/16$ and $B = 0.5/2 \ + 1/4 \ + 0.5/6$

Then

$A \div B = \ min(0.4, 0.5)/(8 \div 2)+ min(0.7, 0.5)/(10 \div 2)+ min(1, 0.5)/(12 \div 2)+ min(0.7, 0.5)/(14 \div 2)+ min(0.4, 0.5)/(16 \div 2)+ min(0.4, 1)/(8 \div 4)+ min(0.7, 1)/(10 \div 4)+ min(1, 1)/(12 \div 4)+ min(0.7, 1)/(14 \div 4)+ min(0.4, 1)/(16 \div 4)+ min(0.4, 0.5)/(8 \div 6)+ min(0.7, 0.5)/(10 \div 6)+ min(1, 0.5)/(12 \div 6)+ min(0.7, 0.5)/(14 \div 6)+ min(0.4, 0.5)/(16 \div 6)$

After getting the minimum of the membership values and division of the elements, we have:

$A \div B = 0.4/4 \ + \ 0.5/5 \ + \ 0.5/6 \ + \ 0.5/7 + \ 0.4/8 \ + \ 0.4/2 \ + \ 0.7/2.5 \ + \ 1/3 \ + \ 0.7/3.5 \ + \ 0.4/4 \ + \ 0.4/1.3 \ + 0.5/1.7 \ + \ 0.5/2 \ + \ 0.5/2.3 \ + \ 0.4/2.7$

Then getting the maximum of the duplicated values as

$A \div B =$ 0.4/1.3+0.5/1.7+ max(0.4,0.5)/ 2+0.5/2.3+0.7/2. 5+0.4/2. 7+1/3+0.7/3.5+ max(0.4,0.4)/4+0.5/5+0.5/6+0.5/7+0.4/8

This gives

$A \div B =$ 0.4/1.3+0.5/1.7+ 0.5/ 2+0.5/2.3+0.7/2. 5+0.4/2. 7+1/3+0.7/3.5+0.4/4+0.5/5+0.5/6+ 0.5/7+0.4/8

4.3 Fuzzy equations

4.3.1 Equi-fuzzy

A fuzzy set $A^{(\lambda)}$ over the support A is called equi-fuzzy if for all $a \in A^{(\lambda)}$, $\mu_{A^{(\lambda)}}(a) = \lambda$ (membership of all the elements of $A^{(\lambda)}$ is λ) where $\lambda \in [0,1]$.

4.3.2 Superimposition of Equi-fuzzy

The following formula defines the operation of superimposition S over two equi-fuzzy sets $A^{(\lambda)}$and $B^{(\delta)}$

$$A^{(\lambda)} S B^{(\delta)} = (A - A \cap B)^{(\lambda)} + (A \cap B)^{(\lambda+\delta)}(A - A \cap B)^{(\delta)}$$

where $\lambda, \delta \geq 0, \lambda + \delta \leq 1$ and '+' denote the operation for union of disjoint fuzzy sets or otherwise.

Given two fuzzy numbers A and B, the formula below defines the arithmetic operation using the method of α-cut.

$$(A * B)^{\alpha} = A^{\alpha} * B^{\alpha}$$

where A^{α}, B^{α} are α -cuts of A and B, $\alpha \in (0,1]$ and the '*' sign represents the arithmetic operation over A and B. In the situation of division where $0 \notin B^{\alpha}$ for any $\alpha \in (0,1]$. The subsequent fuzzy number $A * B$ is represented as

$$A * B = \cup \, (A * B)^{\alpha} \cdot \alpha \qquad (4.1)$$

4.3.3 Solution of the fuzzy equation $\boldsymbol{A + X = B}$ by using the method of $\boldsymbol{\alpha}$ -cut

Given fuzzy equation $\boldsymbol{A + X = B}$. Consider $A^{\alpha} = [a_1^{\alpha}, a_2^{\alpha}]$, $B^{\alpha} = [b_1^{\alpha}, b_2^{\alpha}]$ and $X^{\alpha} = [x_1^{\alpha}, x_2^{\alpha}]$, to denote, respectively, the α-cuts of A, B and X [22]. Then the equation $\boldsymbol{A + X = B}$ has solution if and only if the following properties hold:

1) for every $\alpha \in (0,1]$, $b_1^{\alpha} - a_1^{\alpha} = b_2^{\alpha} - a_2^{\alpha}$

2) $\alpha < \gamma \Rightarrow b_1^\alpha - a_1^\alpha \leq b_1^\gamma - a_1^\gamma \leq b_2^\gamma - a_2^\gamma \leq b_1^\alpha - a_1^\alpha$

The first property ensures the existence of the solution of the interval equation

$$A^\alpha + X^\alpha = B^\alpha$$

which is

$$X^\alpha = [b_1^\alpha - a_1^\alpha, b_2^\alpha - a_2^\alpha]$$

The second property ensures that for α and γ the solution of the interval equations are nested, i.e. if $\alpha \leq \gamma$ then $X^\alpha \subseteq X^\gamma$. If for every $\alpha \in (0,1]$ X^α has a solution and the second property is satisfied, then by (4.1) the fuzzy equation has solution X as

$$X = \bigcup_{\alpha \in (0,1]} X^\alpha \cdot \alpha$$

Chapter 5

CONCLUSION

Due to the benefits of membership grading in a fuzzy set, fuzzy arithmetic is recognized to have much influence than interval arithmetic. The linguistic terms can be used in the expression of fuzzy number making it feasible to compute with words instead of numbers. A significant property of fuzzy number is a closure under linear combination. This property eases computation with fuzzy numbers, and indicates the fuzzy number in a small number of parameters and leads to L-R fuzzy number.

In this thesis, we consider the properties of fuzzy numbers to be used to execute the arithmetic operations on them with reference to arithmetic operations on their α-cuts.

Some figures are used to depict the differences in using deferent membership functions yet obtaining same result. The extension principle is used as a mathematical instrument that takes crisp mathematical notions and procedures, extend them to the fuzzy realm, resulting in computable fuzzy sets by fuzzifying the parameters of a function, and using the extension principle arithmetic operations on fuzzy numbers are performed. Some properties of fuzzy equation are discussed.

REFERENCES

[1] Lofti A. Zadeh. (1965). “Fuzzy Sets”. *Information and Control, 8,* pp. 338-353.

[2] Lofti A. Zadeh. (2008). “Is there need for fuzzy logic?”. *Information Sciences 178,* pp. 2751–2779.

[3] J.G. Dijkman, H. Van Haeringen, and S. J. De Lange (1983). “Fuzzy Numbers”. *Journal of Mathematical Analysis and Applications 92,* pp. 301-341.

[4] Luciano Stefanini, Laerte Sorini and Maria Letizia Guerra. (2006). “Parametric representation of fuzzy numbers and application to fuzzy calculus”. *Fuzzy Sets and Systems 157,* pp. 2423 – 2455.

[5] Roohollah Abbasi Shureshjani, Majid Darehmiraki. (2013). “A new parametric method for ranking fuzzy numbers”. *Indagationes Mathematicae 24,* pp. 518–529.

[6] B. Asady, M. Akbari and M. A. Keramati. (2011). “Ranking of fuzzy numbers by fuzzy mapping”. *International Journal of Computer Mathematics, vol.88, no. 8,* pp. 1603-1618.

[7] Gerard Canfora, Luigi Troiano. (2004). “Fuzzy Ordering of Fuzzy Numbers”. *Proceedings of IEEE Conference of Fuzzy Systems,* pp. 669-674.

[8] Vladimir Janis. (1998). “Fuzzy mappings and fuzzy methods for crisp mappings”. *Acta Univ. M. Belli. Math no. 6,* pp. 31-47.

[9] Jerome Fortin, Didier Dubois, Helene Fargier (2008). “Gradual numbers and their application to fuzzy interval analysis”. *IEEE Transactions on Fuzzy Systems*, Vol. 16, No. 2, pp. 388-402.

[10] Gracaliz Pereira Dimuro. (2011). “On Interval Fuzzy Numbers”. *WEIT '11 Proceedings of the 2011 Workshop-School on Theoretical Computer Science*,pp.3-8.

[11] Ming Ma, Menahem Friedman, Abraham Kandel. (1999). “A new fuzzy arithmetic”. *Fuzzy Sets and Systems 108,* pp. 83-90.

[12] Lun Shan Gao. (1999). “The fuzzy arithmetic mean”. *Fuzzy Sets and Systems 107,* pp. 335-348.

[13] Palash Dutta, Hrishikesh Boruah, Tazid Ali. (2011). “Fuzzy Arithmetic with and without using α-cut method: A Comparative Study”. *International Journal of Latest Trends in Computing, Volume 2, Issue 1,* pp. 99-107.

[14] A. Nagoor Gani. (2012). “A New Operation on Triangular Fuzzy Number for Solving Fuzzy Linear Programming Problem”. *Applied Mathematical Sciences, vol. 6, no.11,* pp. 525 – 532.

[15] G.J. Klir, Y.Pan. (1998). “Constrained fuzzy arithmetic: Basic questions and some answers”. *Soft Computing, Volume 2, Issue 2*, pp. 100-108.

[16] Qing He, Hong-Xing Li, C. L. P. Chen, E. S. Lee. (2000). “Extension Principles and Fuzzy Set Categories”. *Computers and Mathematics with Applications 39*, pp. 45-53.

[17] Robert Fuller, Tibor Keresztfalvi. (1992). “t-Norm-based addition of fuzzy intervals”. *Fuzzy Sets and Systems 51*, pp. 155-159.

[18] Hans Hellendoorn. (1991). Fuzzy numbers and approximate reasoning. *Annales Univ. Sci. Budapest, Sect. Comp. 12*, pp. 113-119.

[19] Rahim Saneifard, Mohammad Farrokhy. (2014). “Comparison of Fuzzy Numbers by Using a Statistical Index”. *International Journal of Natural and Engineering Sciences 8 (1)*, pp. 23-26.

[20] X.X.Shen, K.C.Tan, and M. Xie. (2001). “The implementation of quality function deployment based on linguistic data”. *Journal of Intelligent Manufacturing 12*, pp. 65-75.

[21] Luciano Stefanini, Maria Letizia Guerra.(2007). “On fuzzy arithmetic operations: Some properties and distributive approximations”. *Working Papers Series in Economics, Mathematics and Statistics, WP-EMS 2007/03.*

[22] Fokrul Alom Mazarbhuiya, Anjana Kakoti Mahanta, Hemanta K. Baruah (2011). “Solution of the Fuzzy Equation A + X = B Using the Method of Superimposition”. *Applied Mathematics, 2,* pp. 1039-1045.