d-dimensional Position Dependent Mass, Quasi-Free Quantum Particle

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ABSTRACT

This thesis deals with the applications of PDM in one and *d*-dimensions. In the one-dimensional case the classical as well as the quantum mechanical approach are discussed, and the position dependent mass formalism in one dimension is applied to harmonic oscillator and the quasi free particle case, i.e. where $V(\vec{r}) = 0$. Furthermore, the extension to *d*-dimensions is given and applied to the *d*-dimensional Harmonic Oscillator and the *d*-dimensional Coulomb Potential.

Keywords: d-dimensional, position dependent mass (PDM), constant mass (CM), point canonical transformation (PCT), quasi-free particle, harmonic oscillator, Pöschl-Teller potential.

ÖZ

Bu tez konuma bağlı kütlenin d-boyuttaki uygulamalarını ele alıyor. Bir boyutta klasik ve kuantum mekaniksel teori tartışıldı ve konuma bağlı kütle formalizm bir boyutlu harmonik osilatör ve bağımsız gibi parçacık için farklı konuma bağlı kütle fonksyonları için uygulanmıştır. Ayrıca konuma bağlı kütle modeli d-boyuta genişletilmiş ve d-boyutlu harmonik osilatör ve Coulomb potansiyellerine uygulanmıştır.

Anahtar sözcükler: konuma bağlı kütle, d-boyutlu konuma bağlı kütle, sabit kütle, nokta kuralsal dönüşüm, bağımsız gibi parçacık, harmonik osilatör, Pöschl-Teller Potansiyeli.

.

DEDICATION

This work is nicely dedicated to my darling spouse SAERAN and my beloved daughter YARA

ACKNOWLEDGMENT

First and foremost, all my praiseworthiness to ALLAH (the Most Gracious and the Most Merciful), I would like to extend thanks to the many people who so generously contributed to the work presented in this thesis.

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INTRODUCTION

The Schrödinger equation for the position dependent mass (PDM) has found its way into standard textbooks of Quantum Mechanics [Fluegge, Schiff]. One of the obvious application areas of the PDM Schrödinger equation can be found in the field of solid-state physics. PDM approach provides a very useful tool in the study of the semiconductors and inhomogeneous crystals [1-12], quantum dots [8], quantum liquids [16], etc. This theory could be extended to higher dimensions [13-15]. The concept of PDM is known to play an important role in the energy-density functional approach in quantum many-body problems in the context of nonlocal terms of the accompanying potential.

Let us first consider the one dimensional case of a quantum particle with the constant mass (CM). The kinetic energy is given as

$$T = \frac{P^2}{2m_0}, \tag{1.1}$$

where the momentum operator $P = -i\hbar \frac{\partial}{\partial x}$. The kinetic energy operator fails to be written in the form

$$T = \frac{P^2}{2m(x)}, \tag{1.2}$$

as the commutator of momentum and position operator are non-zero.

Therefore, the Hermitian PDM kinetic energy operator T can be written in the one-dimensional case can be written as

$$T = \frac{1}{2} m(x)^{a} P m(x)^{2b} P m(x)^{a}, \tag{1.3}$$

with the condition $a + b = -\frac{1}{2}$. The actual combination of the parameters a and b is not clear. This Hermitian kinetic energy operator can be extended to D-dimensions [9] as

$$T = \frac{1}{2} \vec{\nabla}_{D} \frac{1}{m(\vec{r})} \vec{\nabla}_{D}. \tag{1.4}$$

In Chapter 2, we briefly review the classical (constant mass) CM harmonic oscillator. Then, the Harmonic Oscillator is discussed in the framework of PDM classically and quantum mechanically for different position dependent mass functions. Next, we will show the PDM in classical mechanics by uses PCT and connecting together with the case of CM, and illustrate the formalism by supposing few cases of special mass function, also for the Hamiltonian quantum PDM there is a special ordering in the kinetic term frame which is the closest to the classical picture, and solve the problem by ways of the correspondence between PDM and CM systems, in the framework of this scenario, we will have some application to obtain ground, first and second states wave function and probability density, for each one there is a plot.

In Chapter 3 the quasi-free PDM, yielding the Pöschel-Teller Potential as effective potential, in 1d is discussed. In Chapter 3, we choose a one-dimensional (1d) position dependent mass function that yields a Pöschl-Teller type effective potential in quantum mechanics. Also, we chose the ordering ambiguity parameters as $j = l = -\frac{1}{2},\ k = 0,$ (i.e. $j + k + l = -1$), next, we solve the 1-d Schrödinger equation, and we arrive at the hypergeometric differential equation, also we find the eigen-states and eigenvalues.

In Chapter 4, we study a d-dimensional quasi-free PDM in Schrödinger equation (i.e. free particle $V(r) = 0$), trapped in its PDM barriers, exactly wherever inter-dimensional degeneracies stay unchanged within the particular d-dimensional radial Schrödinger equation, and consequences of a power-law radial mass $m(r) = \beta r^{\gamma}$. Also we calculate the d-dimensional PDM Schrödinger equation in the framework of using point canonical transformation (PCT) approach; in this part we will discuss the harmonic oscillator and Coulomb potential. Finally, the conclusions are given in Chapter 5.

Chapter 2

POSITION DEPENDENT MASS HARMONIC OSCILATORS

As a first simple application, we want to discuss the Harmonic Oscillator in the framework of the PDM. This study will focus on the correlation between the classical and the quantum case. Exemplarily, the calculations will be carried out explicitly for three well-known PDM functions.

2.1 Classical Constant Mass

In this section, we review the classical constant mass harmonic oscillator.

In general the Hamiltonian is given as

$$\mathcal{H} = \frac{P^2}{2m} + V(\mathrm{X}) \tag{2.1}$$

with $V(\mathrm{X}) = \frac{1}{2} m\omega^2 \mathrm{X}^2$. Let $m = \omega = 1$, harmonic oscillator Hamiltonian becomes

$$\mathcal{H} = \frac{P^2}{2m} + \frac{1}{2}\mathrm{X}^2 \tag{2.2}$$

where P and X are the classical momentum and variables of position. The Hamiltonian $\mathcal{H}$ can be factorized like

$$\mathcal{H} = a^+ a^-, \qquad a^\pm = \frac{1}{\sqrt{2}}(\mp iP + \mathrm{X}). \tag{2.3}$$

With $a^\pm$ fulfilling $a^+ = (a^-)^*$, where $(a^-)^*$ denotes the complex conjugate of a^-. Most of these functions close up the particular Heisenberg together with Poisson brackets, i.e.

$$i\{a^-, a^+\} = 1, \qquad i\{\mathcal{H}, a^\pm\} = \pm a^\pm . \tag{2.4}$$

We can develop non-autonomous integrals involving motion in the type

$$Q^{\pm} = a^{\pm} e^{\mp it} \,. \qquad (2.5)$$

Substituting $a^{\pm}$ from Eq. (2.3) into Eq. (2.5), we obtain

$$Q^{\pm} = \frac{1}{\sqrt{2}} (\mp iP + \mathrm{X}) e^{\mp it}. \qquad (2.6)$$

The new variable Q satisfies the relations $Q^{+} = (Q^{-})^{*}$ and $\mathcal{H} = Q^{+}Q^{-}$. As the Hamiltonian is really a conserved amount, so total energy E is conserved. Let $Q^{+} = \sqrt{E} e^{i\phi}$ with ϕ being a constantly phase fixed through the initial conditions, we can obtain the phase trajectories of constant mass as

$$X(t) = A \, cos(\omega t + \phi), \qquad (2.7)$$

where $A = \sqrt{2E},$ is the radius. Eq. (2.7) becomes then

$$X(t) = \sqrt{2E} \, cos(t + \phi), \qquad (2.8)$$

and

$$P(t) = \frac{d}{dx} X(t) = -\sqrt{2E} \, sin(t + \phi). \qquad (2.9)$$

2.2 Classical PDM Harmonic Oscillator

This part is dedicated from investigation of classical PDM harmonic oscillator through ways of PCT joining the problem with all the constant mass (CM) case.

The position dependent mass classical Hamiltonian gets the following type

$$\mathcal{H} = \frac{p^2}{2m(x)} + v(x), \qquad (2.10)$$

where $m(x)$ is definitely an arbitrary PDM, $v(x)$ is the potential, that depends upon the $m(x)$, to get determined. In analogy to the CM harmonic oscillator problem, allow us to assume which the Hamiltonian $\mathcal{H}$ could be factorize again into two functions $A^{\pm}(x, p)$ from the type

$$A^{\pm}(x, p) = \mp i \frac{p}{\sqrt{2m(x)}} + \mathcal{W}(x). \qquad (2.11)$$

In this case and the way, we can say

$$\mathcal{H} = A^{+}A^{-} = A^{-}A^{+} = \left(-i\frac{p}{\sqrt{2m(x)}} + \mathcal{W}(x)\right)\left(i\frac{p}{\sqrt{2m(x)}} + \mathcal{W}(x)\right) \tag{2.12}$$

which can be written as

$$\mathcal{H} = \frac{p^2}{2m(x)} + \mathcal{W}^2(x). \tag{2.13}$$

So, the potential $v(x)$ and also position dependent $\mathcal{W}^2(x)$ are connected through

$$v(x) = \mathcal{W}^2(x). \tag{2.14}$$

In analogy to the CM case, we request the operators $A^{\pm}$, $\mathcal{H}$ closes algebra of Heisenberg with Poisson brackets, like within constant mass case Eq. (2.4)

$$i\{A^{-}, A^{+}\} = \frac{2\mathcal{W}\prime(x)}{\sqrt{2m(x)}} \tag{2.15}$$

and

$$i\{\mathcal{H}, A^{\pm}\} = \pm\frac{2\mathcal{W}\prime(x)}{\sqrt{2m(x)}}A^{\pm}. \tag{2.16}$$

In order to continue the analogy we need condition (2.4) to be fulfilled. Therefore, we get from Eq. (2.15)

$$\mathcal{W}(x) = \frac{1}{\sqrt{2}}\int^{x}\sqrt{m(u)}\,du + X_0. \tag{2.17}$$

Alternatively, we can use the point canonical transformation PCT

$$X' = \sqrt{m(x)}, \qquad\qquad X = \int^{x}\sqrt{m(u)}du + X_0, \tag{2.18}$$

where X_0 is a constant of integration, the potential $v(x)$ equal to half of X^2 as

$$v(x) = \frac{1}{2}\left[\int^{x}\sqrt{m(u)}du + X_0\right]^2. \tag{2.19}$$

Previous condition fixes $\mathcal{W}(x)$ like

$$\mathcal{W}(x) = \frac{1}{\sqrt{2}}(X) = \frac{1}{\sqrt{2}}\left[\int^{x}\sqrt{m(u)}du + X_0\right]. \qquad (2\ .20)$$

Finally we get the classical PDM harmonic oscillator Hamiltonian by substitution of Eq. (2.19) into Eq. (2.10) as

$$\mathcal{H} = \frac{P^2}{2m(x)} + \frac{1}{2}\left[\int^x \sqrt{m(u)}du + X_0\right]^2. \quad (2.21)$$

Obviously, this Hamiltonian has essentially the same structure as the CM Hamiltonian in Eq. (2.2). After applying the PCT we get

$$X(x) = \int^x \sqrt{m(u)}du + X_0 \quad (2.22)$$

and

$$P(x,p) = \frac{d}{dt}X(x) = \frac{p}{\sqrt{m(x)}}. \quad (2.23)$$

We note that the constant X_0 can be set such that

$$v(x=0) = 0. \quad (2.24)$$

Now, the classical PDM harmonic oscillator problem is reduced to the classical CM harmonic oscillator problem, in which the completely new momentum and also position variables usually are connected to the particular mass function be Eq. (2.22) and Eq. (2.23). For example, phase space trajectories $x(t)$ and $p(t)$ are specified

$$x(t) = X^{-1}\left(\sqrt{2E}cos(t+\phi)\right) \quad (2.25)$$

and

$$p(t) = -\sqrt{2E}\sqrt{m\big(x(t)\big)}\,sin(t+\phi). \quad (2.26)$$

Depending on the form of $m(x)$ the motion can be harmonic or not. Additionally, the integrals of motion in analogy to the CM case can be determined as

$$Q^{\pm} = \frac{1}{\sqrt{2}}[\mp iP(x,p) + X(x)]e^{\mp it}. \quad (2.27)$$

Further, using results of Eqs. (2.22), (2.23), and Eq. (2.27) we get

$$Q^{\pm} = \frac{1}{2}\left[\mp i\frac{p}{\sqrt{m(x)}} + \int \sqrt{m(u)}du + X_0\right]e^{\mp it} = A^{\pm}e^{\mp it}\,. \quad (2.28)$$

We focus on the relationship between the PDM and the CM classical harmonic oscillator, Eqs. (2.22) and (2.23) give the point canonical transformations connecting the PDM with the CM classical harmonic oscillator. Observe that for a few decisions of $m(x)$, equation Eq. (2.22) may well not outline this mass characteristic domain $D(m)$ onto the entire genuine line, since, it can be obliged in the event that $X(x)$ representing the position of constant mass CM harmonic oscillator.

2.3 Some Examples of Classical PDM Harmonic Oscillator

Let us consider the following PDM function

$$m_1(x) = m_0 \left[\frac{1+\lambda+x^2}{1+x^2}\right]^2, \tag{2.29}$$

that is a free of singularities within genuine line, (i.e., $D(m_1) = \mathbb{R}$. It will take its greatest worth $m_0(1+\lambda)^2$ at $x = 0$ and also tends to value of constant m_0 as $|x| \to \infty$. The mass function of Eq. (2.29) is represented graphically in Fig. 2.1

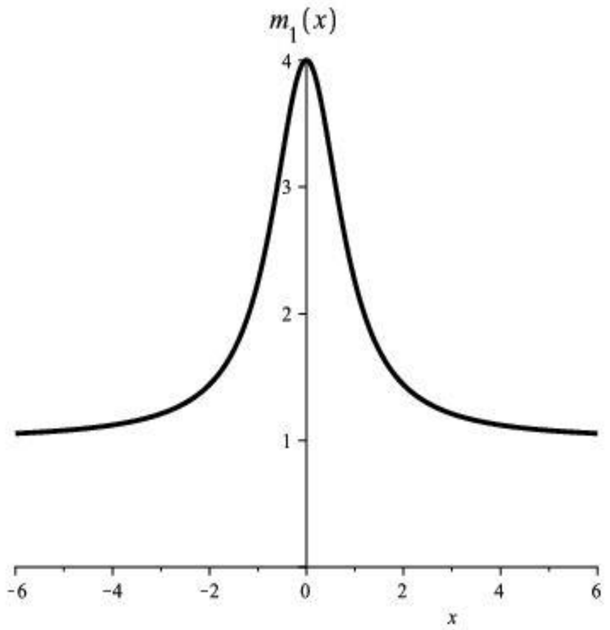

Figure 2.1: Plot of the PDM $m_1(x) = m_0 \left[\frac{1+\lambda+x^2}{1+x^2}\right]^2$ for the parameters $m_0 = \lambda = 1$.

With the harmonic oscillator potential as

$$v_1(x) = \frac{1}{2}\left(X_1(x)\right)^2. \tag{2.30}$$

We get for the position

$$X_1(x) = \int \sqrt{m_1(x)}dx = \int \sqrt{m_0 \left(\frac{1+\lambda+x^2}{1+x^2}\right)^2}. \tag{2.31}$$

Integration, Eq. (2.31) yields to

$$X_1(x) = \sqrt{m_0}[x + \lambda \arctan x]. \tag{2.32}$$

Substituting Eq. (2.32) into Eq. (2.30), we get the potential

$$v_1(x) = \frac{1}{2}m_0[x + \lambda \arctan x]^2. \tag{2.33}$$

Finally, the plot of the potential given in Eq. (2.33), as shown in Fig. 2.2, shows a slight deformation for the constant mass harmonic oscillator

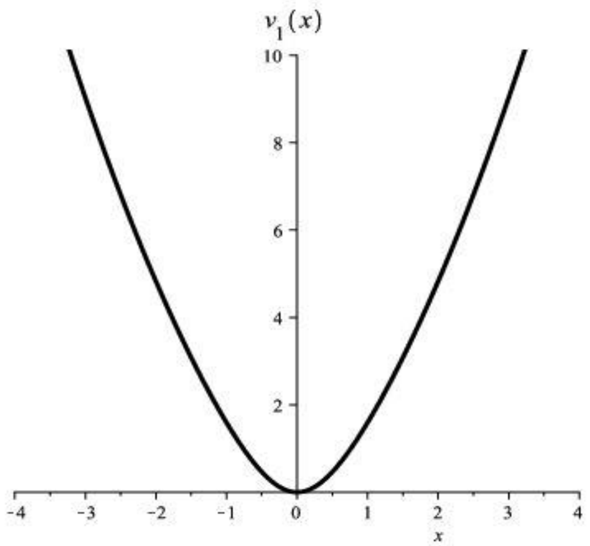

Figure 2.2: Plot of the effective potential $v_1(x)$ for the parameters $m_0 = \lambda = 1$.

The phase trajectories

$$\frac{m_0}{2}[x + \lambda \arctan x]^2 + \frac{1}{2m_0}\left(\frac{1+x^2}{1+\lambda-x^2}\right)^2 p^2 = E \tag{2.34}$$

are presented graphically in Fig. 2.3 for certain values.

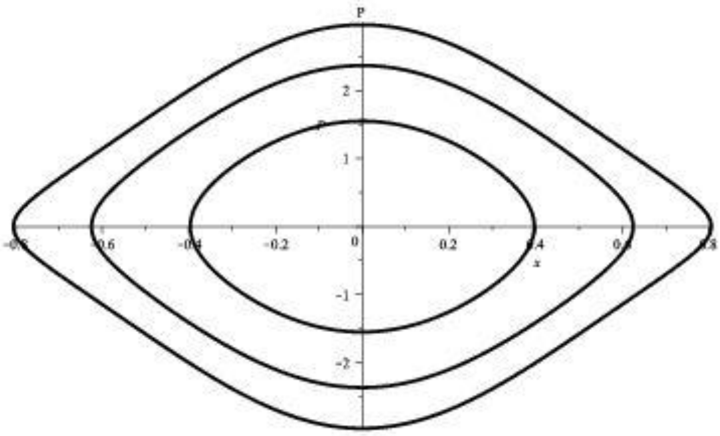

Figure 2.3: Plot of the phase space trajectories, if $m_0 = \lambda = 1, \phi = 0$ and also $E = 1.1, 0.3, 0.7$.

The phase space trajectories look very similar to the ones for the constant mass. Recognize that, on the point of confinement $\lambda \to 0$, the constant mass is recovered.

Another well-known position dependent mass function is

$$m_2(x) = m_0(\tanh(\lambda x))^2 \,. \tag{2.35}$$

The plot of $m_2(x)$ is given in Fig. 2.4. Obviously, $m_2(x)$ has no singularities, i.e. its domain is $D(m_2) = \mathbb{R}$. This function is again nearly constant over a huge interval and looks like the inversion of the previous case. Carrying out the calculations analogously we get for the potential and the position

$$v_2(x) = \frac{1}{2}\left(X_2(x)\right)^2, \tag{2.36}$$

and

$$X_2(x) = \int \sqrt{m_0(\tanh(\lambda x))^2}\, dx. \tag{2.37}$$

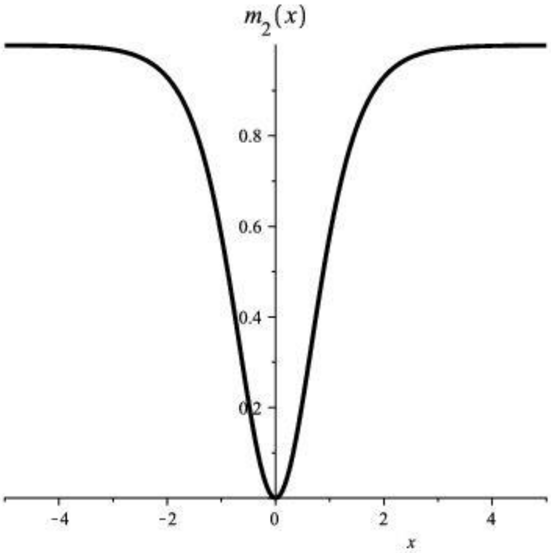

Figure 2.4: Plot of PDM $m_2(x) = m_0(\tanh(\lambda x))^2$ for the parameters $m_0 = \lambda = 1$.

Carrying out the integral in Eq. (2.39), yields to

$$X_2(x) = \frac{1}{\lambda}\sqrt{m_0}\text{sign}(x)\ln(\cosh\lambda x). \tag{2.38}$$

Substitution of Eq. (2.38) into Eq. (2.36) gives the potential as

$$v_2(x) = \frac{m_0}{2\lambda^2}\ln^2(\cosh\lambda x). \tag{2.39}$$

The plot of the Eq. (2.39) is presented in Fig. 2.5.

Also, the phase trajectories has the form

$$\frac{m_0}{2\lambda^2}\ln^2(\cosh\lambda x) + \frac{1}{2m_0}\frac{p^2}{tan^2\lambda x} = E. \tag{2.40}$$

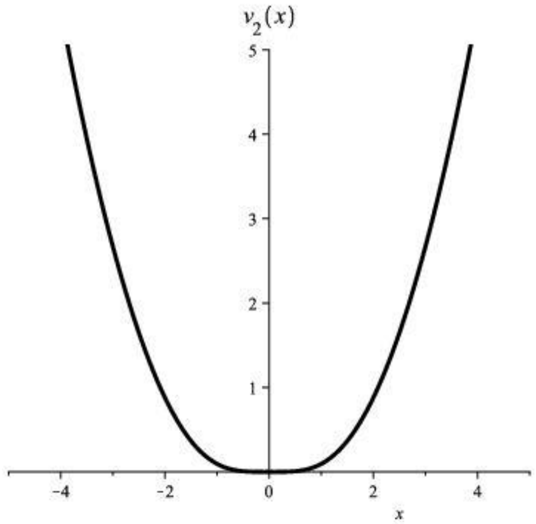

Figure 2.5: Plot of the effective potential $v_2(x) = \frac{m_0}{2\lambda^2} \ln^2(\cosh \lambda x)$ for the parameters $m_0 = \lambda = 1$.

In the third case, we consider the mass function as

$$m_3(x) = \frac{m_0}{1-(\lambda x)^2} \tag{2.41}$$

which is plotted in Fig. 2.6

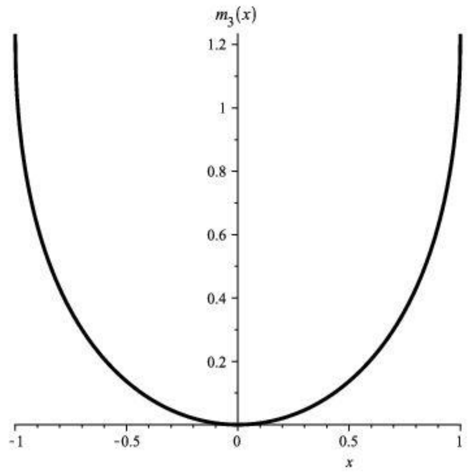

Figure 2.6: Plot of the PDM $m_3(x) = \frac{m_0}{1-(\lambda x)^2}$ for the parameters $m_0 = \lambda = 1$.

The particular mass function in Eq. (2.41) has two singularities, restricting the domain to $D(m_3) = \left(\frac{-1}{\lambda}, \frac{1}{\lambda}\right)$.

The potential of the last case is

$$v_3(x) = \frac{1}{2}\left(X_3(x)\right). \tag{2.42}$$

Also, the variable

$$X_3(x) = \int \sqrt{\frac{m_0}{1-(\lambda x)^2}}\,dx. \tag{2.43}$$

So, after integration, Eq. (2.43) becomes

$$X_3(x) = \frac{\sqrt{m_0}}{\lambda}\arcsin(\lambda x). \tag{2.44}$$

The variable is not divergent, takes values just on the interval $\left[-\frac{\sqrt{m_0}\pi}{2\lambda}, \frac{\sqrt{m_0}\pi}{2\lambda}\right]$, substituting Eq. (2.44) into Eq. (2.42), the limited potential of range like

$$v_3(x) = \frac{m_0}{2\lambda^2}\arcsin^2(\lambda x). \tag{2.45}$$

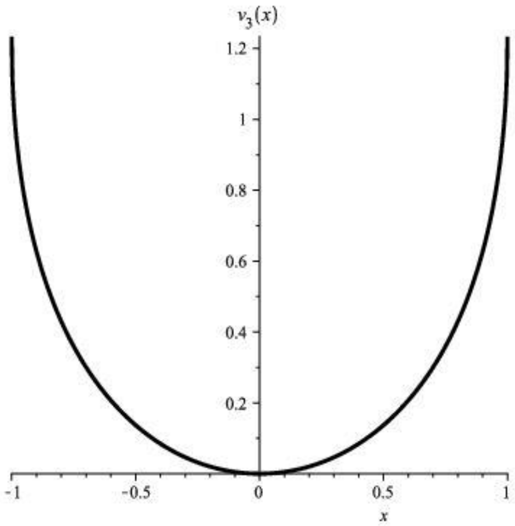

Figure 2.7: Plot of the effective potential $v_3(x) = \frac{m_0}{2\lambda^2}\arcsin^2(\lambda x)$ for the parameters $m_0 = \lambda = 1$.

2.4 Quantum PDM Harmonic Oscillators

Now let us consider the quantum PDM Hamiltonian. We start here again with the Hamiltonian of the Schrödinger equation for one dimension ($\hbar = m = \omega = 1$)

$$H = \left(\frac{P^2}{2m} + \frac{1}{2} m\omega \mathcal{Y}^2\right), \tag{2.46}$$

in which $\mathcal{Y}$ and $P = -i\frac{d}{d\mathcal{Y}}$ are position and also momentum operators respectively, for that coordinate indication. Now Eq. (2.46) becomes

$$H = \frac{1}{2}\left(-\frac{d^2}{d\mathcal{Y}^2} + \mathcal{Y}^2\right) = b^+ b^- + \frac{1}{2}, \tag{2.47}$$

where b^+, b^- are the creation and annihilation operators respectively, and defined as

$$b^- = \frac{1}{\sqrt{2}}\left(\frac{d}{d\mathcal{Y}} + \mathcal{Y}\right), \qquad b^+ = \frac{1}{\sqrt{2}}\left(-\frac{d}{d\mathcal{Y}} + \mathcal{Y}\right). \tag{2.48}$$

Hamiltonian H and Eq. (2.48) together, close the Heisenberg algebra

$$[b^-, b^+] = \frac{1}{2}[\mathcal{Y} + ip, \mathcal{Y} - ip] = \frac{1}{2}([\mathcal{Y}, -ip] + [ip, \mathcal{Y}]), \tag{2.49}$$

or after that

$$[b^-, b^+] = -\frac{i}{2}([\mathcal{Y}, p] + [\mathcal{Y}, p]). \tag{2.50}$$

Finally

$$[b^-, b^+] = 1, \qquad [H, b^\pm] = \pm b^\pm. \tag{2.51}$$

Now, if ϕ_n is the Hamiltonian Eigenstate, and using relations of commutation

$$Hb^+\psi_n = (E_n - \hbar\omega)b^+\phi_n \tag{2.52}$$

and

$$Hb^-\psi_n = (E_n + \hbar\omega)b^-\phi_n. \tag{2.53}$$

From Eq. (2.52) and Eq. (2.53), $b^+\phi_n$ and $b^-\phi_n$ are Eigenstates of Hamiltonian with eigenvalues $E_n + 1$ and $E_n - 1$.

Since, difference of energy between two Eigenstates is $\Delta E = \hbar\omega = 1$, if $n = 0$, and

$$b^- b^+ \phi_0 = 0 = \left(h - \frac{1}{2}\right)\phi_0 = \left(E_0 - \frac{1}{2}\right)\phi_0. \tag{2.54}$$

The ground state energy for our case has the form

$$E_0 = \frac{1}{2} \tag{2.55}$$

and the eigenstate functions are given by

$$\phi_n(\mathcal{Y}) = N_n e^{-\frac{1}{2}\mathcal{Y}^2} H_n(\mathcal{Y}), \tag{2.56}$$

where $H_n(\mathcal{Y})$ is the Hermite polynomial. And N_n is the normalization

$$\int_{-\infty}^{\infty} |\ \phi_n(\mathcal{Y})|\ ^2\ d\mathcal{Y} = 1. \tag{2.57}$$

Substituting Eq. (2.56) into Eq. (2.57)

$$\int_{-\infty}^{\infty} N_n e^{-\frac{1}{2}\mathcal{Y}^2} H_n(\mathcal{Y})\, N_n e^{-\frac{1}{2}\mathcal{Y}^2} H_n(\mathcal{Y})\, d\mathcal{Y} = N_n^2 \int_{\mathcal{Y}\, min}^{\mathcal{Y}\, mix} H_n^2(\mathcal{Y}) e^{-\mathcal{Y}^2}. \tag{2.58}$$

If $\mathcal{Y}_{min} = -\infty$ or 0 and $\mathcal{Y}_{max} = \infty$, we can get expression of normalization N_n

$$N_n = \begin{cases} \frac{1}{\sqrt{\pi^{\frac{1}{2}} 2^n n!}}\ , & \mathcal{Y}_{min} = -\infty \\ \frac{1}{\sqrt{\pi^{\frac{1}{2}} 2^{n-1} n!}}\ , & \mathcal{Y}_{max} = \infty \quad . \end{cases} \tag{2.59}$$

From Eq. (2.59) we need that $\phi_n(\mathcal{Y})$ is orthogonal at different n, thus there exist only one option of $\mathcal{Y}_{min} = -\infty$ and Eq. (2.56) becomes

$$\phi_n(\mathcal{Y}) = \frac{1}{\sqrt{\pi^{\frac{1}{2}} 2^n n!}} e^{-\frac{1}{2}\mathcal{Y}^2} H_n(\mathcal{Y}) = \frac{1}{\sqrt{n!}} (b^+)^n \phi_0(\mathcal{Y}). \tag{2.60}$$

After this short review for the quantum harmonic oscillator, allow us to suppose the hermitian PDM kinetic energy

$$T_a = -\frac{1}{2} m^a \frac{d}{dx} m^{2b} \frac{d}{dx} m^a, \tag{2.61}$$

with $a + b = -\frac{1}{2}$. Thus, the PDM Hamiltonian has the form

$$H_a = T_a + V_a(x). \tag{2.62}$$

Substituting kinetic value into Eq. (2.62) one finds

$$H_a = -\frac{1}{2} m^a \frac{d}{dx} m^{2b} \frac{d}{dx} m^a + V_a(x) \tag{2.63}$$

in which $V_a(x)$ is the same as in classical case.

Now, we use the supersymmetric approximation to structure potential $V_a(x)$. Assume H_a can be factorized into terms of linear operators $(A_a^{\pm})$

$$H_a = N + \tfrac{1}{2} = A_a^+ A_a^- + \tfrac{1}{2}, \tag{2.64}$$

with N being the number operator of harmonic oscillator and, A_a^+ and A_a^- have the form

$$\begin{aligned} A_a^- &= \tfrac{1}{\sqrt{2}} m^b \tfrac{d}{dx} m^a + W_a(x) \\ A_a^+ &= -\tfrac{1}{\sqrt{2}} m^a \tfrac{d}{dx} m^b + W_a(x). \end{aligned} \tag{2.65}$$

As we know, $W_a(x)$ is a position dependent function. Also Eq. (2.65) can be written as

$$\begin{aligned} A_a^- &= \tfrac{1}{\sqrt{2}}\left[\tfrac{1}{\sqrt{m}}\tfrac{d}{dx} + \tfrac{a}{\sqrt{m}}(\ln m)'\right] + W_a(x) \\ A_a^+ &= -\tfrac{1}{\sqrt{2}}\left[\tfrac{1}{\sqrt{m}}\tfrac{d}{dx} + \tfrac{a}{\sqrt{m}}(\ln m)'\right] + W_a(x). \end{aligned} \tag{2.66}$$

Let, $F = \frac{1}{\sqrt{m}}\frac{d}{dx}$ be a new differential operator. Then $A_a^{\pm}$ take a simple structure

$$\begin{aligned} A_a^- &= \tfrac{1}{\sqrt{2}}[F + a(\,F \ln m)] + W_a \\ A_a^+ &= -\tfrac{1}{\sqrt{2}}[F + b(\,F \ln m)] + W_a. \end{aligned} \tag{2.67}$$

Eq. (2.64) implies that $W_a(x)$ and $V_a(x)$ are connected in the following form

$$V_a(x) = \frac{1}{\sqrt{2}}\left[\frac{4a+1}{2} W_a(\,F \ln m) - FW_a\right] + {W_a}^2 + \frac{1}{2}\,. \tag{2.68}$$

The commutation of $[A_a^-, A_a^+]$ is

$$[A_a^-, A_a^+] = [A_a^- A_a^+ - A_a^+ A_a^-] = \sqrt{2} F W_a \frac{4a+1}{4}(F^2 \ln m). \tag{2.69}$$

These $(A_a^{\pm})$ must satisfy the Heisenberg algebra, we discover the accompanying interpretation for the $W_a(x)$

$$W_a(x) = \left[\int^x \sqrt{m}du + \mathcal{Y}_0 - \frac{4a+1}{4}(F \ln m)\right], \tag{2.70}$$

where, $\mathcal{Y}_0$ is the constant of integration and must be option (including within case of classical) in this way where the origins in the potential happen at $x = 0$. So, actually the form of the potential is

$$V_a(x) = \frac{1}{2}\left[\left(\int^x \sqrt{m}du + \mathcal{Y}_0\right)^2 + \frac{4a+1}{4}(F^2 \ln m) - \left(\frac{4a+1}{4} F \ln m\right)^2\right]. \quad (2.71)$$

Evidently for $a = -\frac{1}{4}$, the quantum and classical potentials coincide with CM harmonic oscillator. With $b = -\frac{1}{4}$, $a = -\frac{1}{4}$, and Eq. (2.65) we can determine the Hamiltonian PDM harmonic oscillator as following

$$H = -\frac{1}{2}\sqrt[4]{m}\, F^2 \frac{1}{\sqrt[4]{m}} + \frac{1}{2}\left(\int^x \sqrt{m}du + \mathcal{Y}_0\right)^2 = A^+A^- + \frac{1}{2}, \quad (2.72)$$

where

$$\begin{aligned} A^- &= \frac{1}{\sqrt{2}}\left(\sqrt[4]{m}\, F \frac{1}{\sqrt[4]{m}} + \int^x \sqrt{m}du + \mathcal{Y}_0\right), \\ A^+ &= \frac{1}{\sqrt{2}}\left(-\sqrt[4]{m}\, F \frac{1}{\sqrt[4]{m}} + \int^x \sqrt{m}du + \mathcal{Y}_0\right). \end{aligned} \quad (2.73)$$

In our work, there are two algebraic properties for $A^{\pm}$: (i) the spectrum E_n of the Hamiltonian H is the constant case as the constant mass harmonic oscillator, (ii) also, the actual wave functions $\psi_n(x)$ are denoted through

$$\psi_n(x) = \frac{1}{\sqrt{n!}}(A^+)^n\, \psi_0(x). \quad (2.74)$$

We know that, $\psi_0(x)$ can be ground state and defined as the particular wave function annihilated through A^-

$$A^-\psi_0(x) = \frac{1}{\sqrt{2}}\left(\sqrt[4]{m}\, F \frac{1}{\sqrt[4]{m}} + \int^x \sqrt{m}du + \mathcal{Y}_0\right)\psi_0(x) = 0\,. \quad (2.75)$$

Integrate this mathematical expression, we make substitution

$$\psi_0(x) = \sqrt[4]{m}\, \phi_0\left(\int^x \sqrt{m}du + \mathcal{Y}_0\right). \quad (2.76)$$

Then Eq. (2.73) converts into

$$\frac{1}{\sqrt{2}}\left(F+\int^{x}\sqrt{m}du+\mathcal{Y}_0\right)\phi_0\left(\int^{x}\sqrt{m}du+\mathcal{Y}_0\right)=0\ . \tag{2.77}$$

Note, that the last expression is same as the ground state (ϕ_0) to the constant mass. If $\mathcal{Y}$ can be swapped through integral of square root for this function of mass (with the variable change $\mathcal{Y}=\int\sqrt{m}dx+\mathcal{Y}_0$, and $F\equiv\frac{d}{d\mathcal{Y}}$), the rest for our wave functions may be produced through the use of continuously the particular operator A^{+} to the ground state

$$\psi_n(x)=\sqrt[4]{m}\frac{1}{\sqrt{n!}}\left(F+\int^{x}\sqrt{m}du+\mathcal{Y}_0\right)^{n}\phi_0\left(\int^{x}\sqrt{m}du+\mathcal{Y}_0\right). \tag{2.78}$$

By comparing Eq. (2.78) with Eq. (2.60), $\psi_n(x)$ can be wave functions for the position dependent mass are denoted of $\phi_n(x)$ constant mass wave functions, with the following type

$$\psi_n(x)=\sqrt[4]{m}\,\phi_n\left(\int^{x}\sqrt{m}du+\mathcal{Y}_0\right). \tag{2.79}$$

Let

$$\mathcal{Y}(x)=\int\sqrt{m}du+\mathcal{Y}_0, \tag{2.80}$$

where $F(\mathcal{Y})=F(m)$, and the range of $\mathcal{Y}(x)$ is $\mathcal{R}(\mathcal{Y})$. Then

$$\int|\psi_n(x)|^2\,dx=\int|\phi_n(\mathcal{Y})|^2\,d\mathcal{Y}. \tag{2.81}$$

2.5 Some Examples of Quantum PDM Harmonic Oscillator

Here, the cases of some mass functions already used in classical approach. In this division we have also some figures showing the effective potential $V(x)$ and also some wave functions $\psi_n(x)$, also we will have the probability densities $\rho_n(x)=$

$|\psi_n(x)|^2$ for all of them, with $\lambda = 1$. As our first case, consider the mass function $m_1(x)$ given in Eq. (2.29). So, the wave functions has the form

$$\psi_n(x) = \frac{1}{\sqrt{\pi^{\frac{1}{2}} 2^n n!}} \sqrt[4]{m} e^{-\frac{1}{2}\mathcal{Y}^2} H_n(\mathcal{Y}), \tag{2.82}$$

where, $H_n(\mathcal{Y})$ denotes the Hermite polynomials. Now we need to find $\mathcal{Y}$ by use mass function $m_1(x)$

$$\mathcal{Y} = \int \frac{1+\lambda+x^2}{1+x^2} dx. \tag{2.83}$$

Integration, Eq. (2.83) yields to

$$\mathcal{Y} = x + \lambda \arctan x. \tag{2.84}$$

Substituting Eq. (2.84) into Eq. (2.82) we get the general form of the wave functions for the first case

$$\psi_n(x) = \frac{1}{\sqrt{\pi^{\frac{1}{2}} 2^n n!}} \sqrt{\frac{1+\lambda+x^2}{1+x^2}} e^{-\frac{1}{2}[x+\lambda \arctan x]^2} H_n(x+\lambda \arctan x). \tag{2.85}$$

If $n = 0$ and $H_n(\mathcal{Y}) = 1$, we get the ground state wave function (see Fig. 2.8.a)

$$\psi_0(x) = \frac{1}{\pi^{\frac{1}{4}}} \sqrt{\frac{2+x^2}{1+x^2}} e^{-\frac{1}{2}[x+\lambda \arctan x]^2} \tag{2.86}$$

and the probability density for the ground state $\rho_0(x) = |\psi_0(x)|^2$ (see Fig. 2.8.b)

$$\rho_0(x) = \left| \frac{1}{\pi^{\frac{1}{4}}} \sqrt{\frac{2+x^2}{1+x^2}} e^{-\frac{1}{2}[x+\lambda \arctan x]^2} \right|^2. \tag{2.87}$$

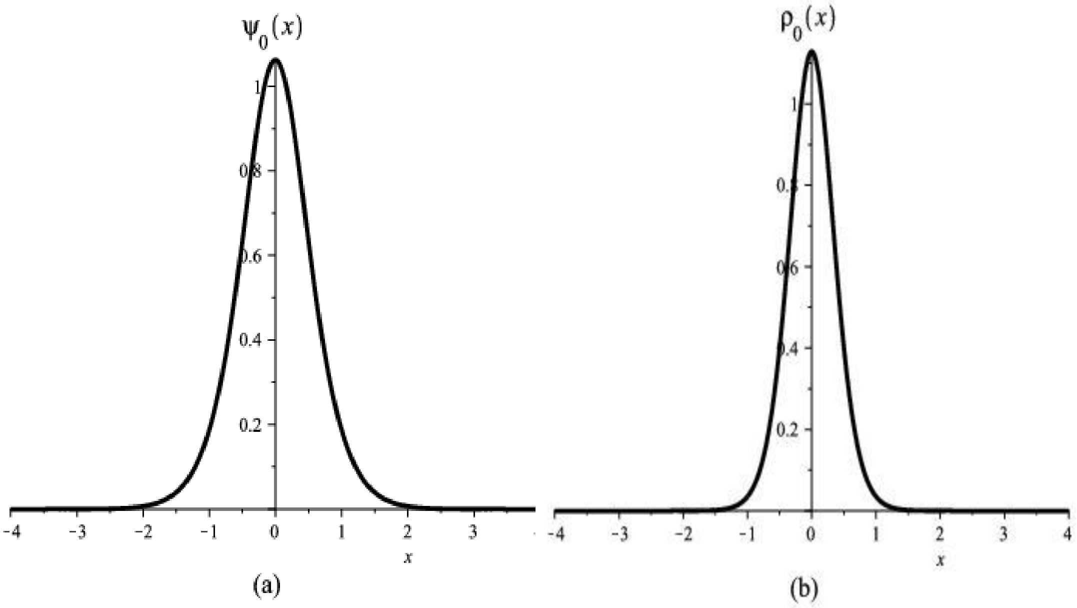

Figure 2.8: (a) Plot of the ground state $\psi_0(x)$, (b) Plot of the probability density $\rho_0(x)$, chosen as $\lambda = 1, n = 0$.

If $n = 1$ and $H_1(\mathcal{Y}) = 2\mathcal{Y}$, we get the first state wave function (see Fig. 2.9.a)

$$\psi_1(x) = \frac{\sqrt{2}}{2}\frac{1}{\pi^{\frac{1}{4}}}\sqrt{\frac{2+x^2}{1+x^2}}\, e^{-\frac{1}{2}[x+\lambda \arctan x]^2} 2(x + \lambda \arctan x) \tag{2.88}$$

and the probability density for the first state $\rho_1(x) = |\ \psi_1(x)|^{\ 2}$ (see Fig. 2.9.b)

$$\rho_1(x) = \left|\ \frac{\sqrt{2}}{2}\frac{1}{\pi^{\frac{1}{4}}}\sqrt{\frac{2+x^2}{1+x^2}}\, e^{-\frac{1}{2}[x+\lambda \arctan x]^2} 2(x + \lambda \arctan x)\right|^2. \tag{2.89}$$

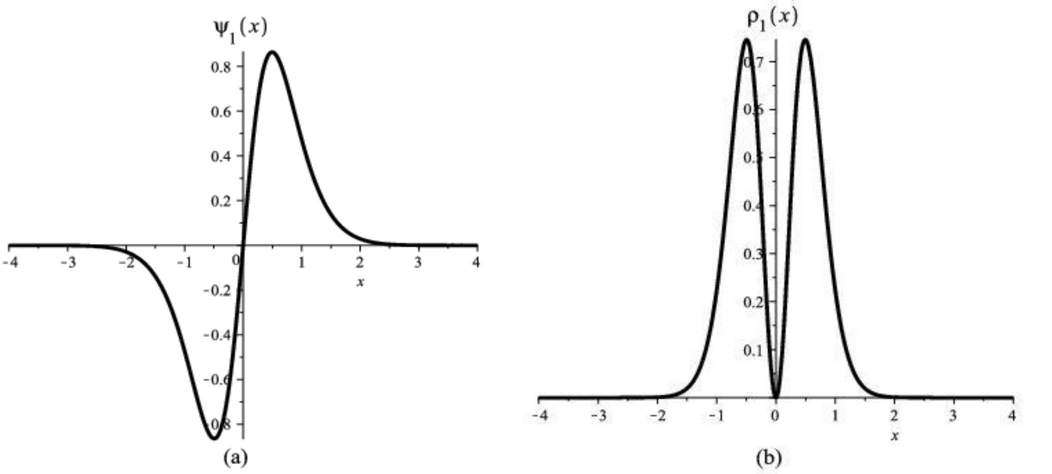

Figure 2.9: (a) Plot of the first state $\psi_1(x)$, (b) Plot of the probability density $\rho_1(x)$, chosen as $\lambda = n = 1$.

If $n = 2$ and $H_2(\mathcal{Y}) = 4\mathcal{Y}^2 - 2$, the second state wave function as (Fig. 2.10.a)

$$\psi_2(x) = \frac{\sqrt{2}}{4}\frac{1}{\pi^{\frac{1}{4}}}\sqrt{\frac{2+x^2}{1+x^2}}\, e^{-\frac{1}{2}[x+\lambda \arctan x]^2} 4(x+\lambda \arctan x)^2 - 2 \qquad (2.90)$$

and the probability density for the second state $\rho_1(x) = |\ \psi_1(x)|\ ^2$ (see Fig. 10.b)

$$\rho_2(x) = \left|\ \frac{\sqrt{2}}{4}\frac{1}{\pi^{\frac{1}{4}}}\sqrt{\frac{2+x^2}{1+x^2}}\, e^{-\frac{1}{2}[x+\lambda \arctan x]^2} 4(x+\lambda \arctan x)^2 - 2\ \right|^2. \qquad (2.91)$$

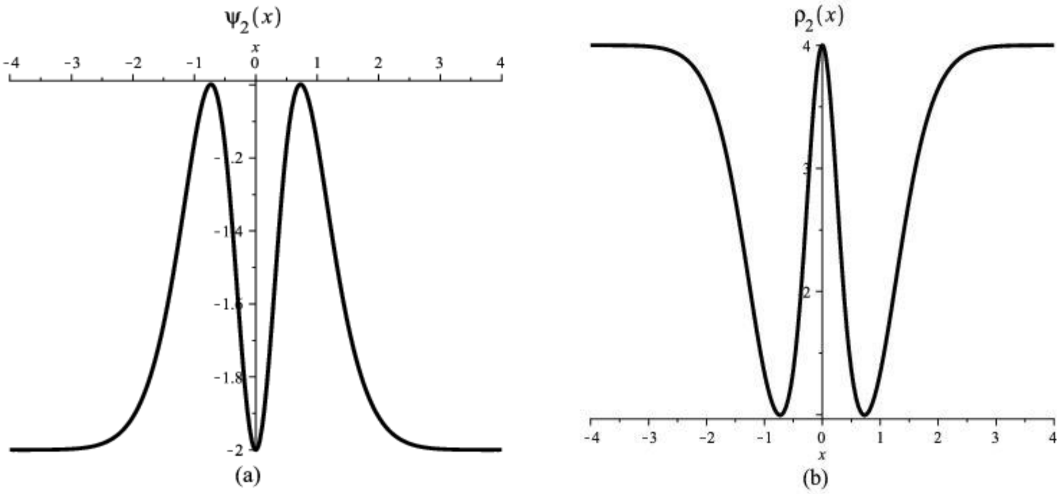

Figure 2.10: (a) Plot of the second state $\psi_2(x)$, (b) Plot of the probability density $\rho_2(x)$, chosen as $\lambda = 1, n = 2$.

In our second case, let's consider the mass function $m_2(x)$ given in Eq. (2.35). So, we need to find $\mathcal{Y}$ by use mass $m_2(x)$

$$\mathcal{Y} = \int \tanh(\lambda x)dx = \int \ln\ \sinh(\lambda x)dx. \qquad (2.92)$$

Integration of Eq. (2.92) yields to

$$\mathcal{Y} = \frac{1}{\lambda} \ln \cosh(\lambda x). \qquad (2.93)$$

So, the general form of the wave functions in the second case is

$$\psi_n(x) = \frac{1}{\sqrt{\pi^{\frac{1}{2}} 2^n n!}} \sqrt{\tanh \lambda x}\, e^{-\frac{1}{2}\left[\frac{1}{\lambda}\ln \cosh(\lambda x)\right]^2} H_n\left(\frac{1}{\lambda} \ln \cosh(\lambda x)\right). \qquad (2.94)$$

If $n = 0$ and $H_n(\mathcal{Y}) = 1$, we get the ground state wave function (see Fig. 2.11.a)

$$\psi_0(x) = \frac{1}{\pi^{\frac{1}{4}}}\sqrt{\tanh(\lambda x)}\, e^{-\frac{1}{2}\left[\frac{1}{\lambda}\ln\cosh(\lambda x)\right]^2} \tag{2.95}$$

and the probability density for the ground state $\rho_0(x) = |\ \psi_0(x)|\ ^2$ (see Fig. 2.11.b)

$$\rho_0(x) = \ |\ \frac{1}{\pi^{\frac{1}{4}}}\sqrt{\tanh(\lambda x)}\, e^{-\frac{1}{2}\left[\frac{1}{\lambda}\ln\cosh(\lambda x)\right]^2}\ |\ ^2 . \tag{2.96}$$

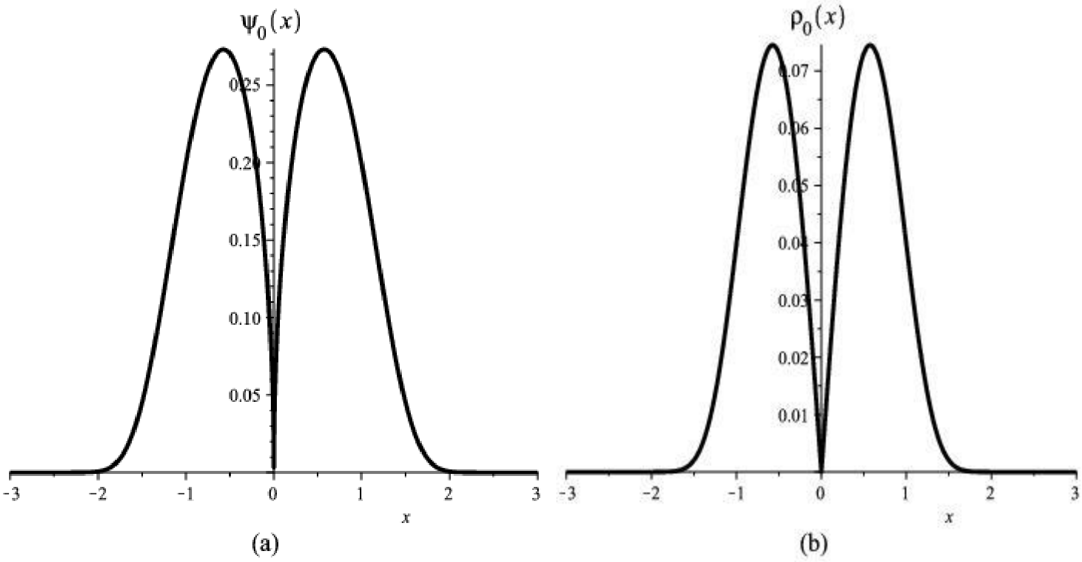

Figure 2.11: (a) Plot of the ground state $\psi_0(x)$, (b) Plot of the probability density $\rho_0(x)$, chosen as $\lambda = 1, n = 0$.

If $n = 1$ and $H_1(\mathcal{Y}) = 2\mathcal{Y}$, we get the first state wave function

$$\psi_1(x) = \frac{\sqrt{2}}{2}\frac{1}{\pi^{\frac{1}{4}}}\sqrt{\tanh(\lambda x)}\, e^{-\frac{1}{2}\left[\frac{1}{\lambda}\ln\cosh(\lambda x)\right]^2} 2\left[\frac{1}{\lambda}\ \ln\cosh(\lambda x)\right] \tag{2.97}$$

and its probability density

$$\rho_1(x) = \ |\ \frac{\sqrt{2}}{2}\frac{1}{\pi^{\frac{1}{4}}}\sqrt{\tanh(\lambda x)}\, e^{-\frac{1}{2}\left[\frac{1}{\lambda}\ln\cosh(\lambda x)\right]^2} 2\left[\frac{1}{\lambda}\ \ln\cosh(\lambda x)\right]\ |\ ^2 . \tag{2.98}$$

If $n = 2$ and $H_2(\mathcal{Y}) = 4\mathcal{Y}^2 - 2$, we get the second state wave function as

$$\psi_2(x) = \frac{\sqrt{2}}{4}\frac{1}{\pi^{\frac{1}{4}}}\sqrt{\tanh(\lambda x)}\, e^{-\frac{1}{2}\left[\frac{1}{\lambda}\ln\cosh(\lambda x)\right]^2} 4\left(\frac{1}{\lambda}\ \ln\cosh(\lambda x)\right)^2 - 2 \tag{2.99}$$

and the probability density for the second state $\rho_1(x) = |\ \psi_1(x)|\ ^2$

$$\rho_2(x) = \left|\ \frac{\sqrt{2}}{4}\frac{1}{\pi^{\frac{1}{4}}}\sqrt{\tanh(\lambda x)}\, e^{-\frac{1}{2}\left[\frac{1}{\lambda}\ln\cosh(\lambda x)\right]^2} 4\left(\frac{1}{\lambda}\ \ln\cosh(\lambda x)\right)^2 - 2\ \right|^2. \quad (2.100)$$

Now let's consider the mass function $m_3(x)$ given in Eq. (2.41). So, we need to find $\mathcal{Y}$ by use mass $m_3(x)$

$$\mathcal{Y} = \frac{1}{\lambda}\arcsin(\lambda x). \quad (2.101)$$

So, the general form of the wave function becomes

$$\psi_n(x) = \frac{1}{\sqrt{\pi^{\frac{1}{2}}2^n n!}}\frac{1}{\sqrt[4]{1-(\lambda x)^2}}e^{-\frac{1}{2}\left[\frac{1}{\lambda}\arcsin(\lambda x)\right]^2} H_n(\mathcal{Y}). \quad (2.102)$$

If $n = 0$ and $H_0(\mathcal{Y}) = 1$, we get the ground state wave function (see Fig. 2.12.a)

$$\psi_0(x) = \frac{1}{\pi^{\frac{1}{4}}}\frac{1}{\sqrt[4]{1-(\lambda x)^2}}e^{-\frac{1}{2}\left[\frac{1}{\lambda}\arcsin(\lambda x)\right]^2} \quad (2.103)$$

and the probability density for the ground state $\rho_0(x) = |\ \psi_0(x)|\ ^2$ (Fig. 2.12.b)

$$\rho_0(x) = \left|\ \frac{1}{\pi^{\frac{1}{4}}}\frac{1}{\sqrt[4]{1-(\lambda x)^2}}e^{-\frac{1}{2}\left[\frac{1}{\lambda}\arcsin(\lambda x)\right]^2}\ \right|^2. \quad (2.104)$$

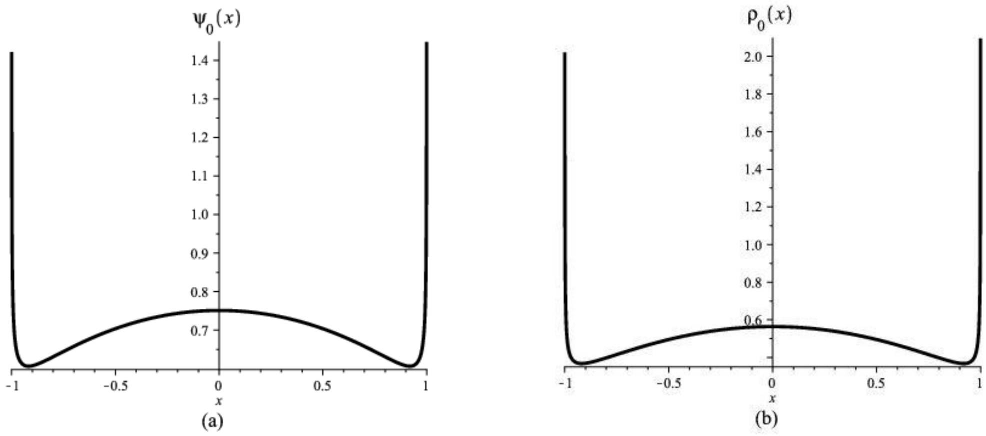

Figure 2.12: (a) Plot of the ground state $\psi_0(x)$, (b) Plot of the probability density $\rho_0(x)$, chosen as $\lambda = 1, n = 0$.

If $n = 1$ and $H_1(\mathcal{Y}) = 2\mathcal{Y}$, we get the first state wave function (see Fig. 2.13.a)

$$\psi_1(x) = \frac{\sqrt{2}}{2}\frac{1}{\pi^{\frac{1}{4}}}\frac{1}{\sqrt[4]{1-(\lambda x)^2}} e^{-\frac{1}{2}\left[\frac{1}{\lambda}\arcsin(\lambda x)\right]^2} 2\left[\frac{1}{\lambda}\arcsin(\lambda x)\right] \tag{2.105}$$

and the probability density for the first state $\rho_1(x) = |\ \psi_1(x)|\ ^2$ (see Fig. 2.13.b)

$$\rho_1(x) = \left|\ \frac{\sqrt{2}}{2}\frac{1}{\pi^{\frac{1}{4}}}\frac{1}{\sqrt[4]{1-(\lambda x)^2}} e^{-\frac{1}{2}\left[\frac{1}{\lambda}\arcsin(\lambda x)\right]^2} 2\left[\frac{1}{\lambda}\arcsin(\lambda x)\right]\ \right|^2 . \tag{2.106}$$

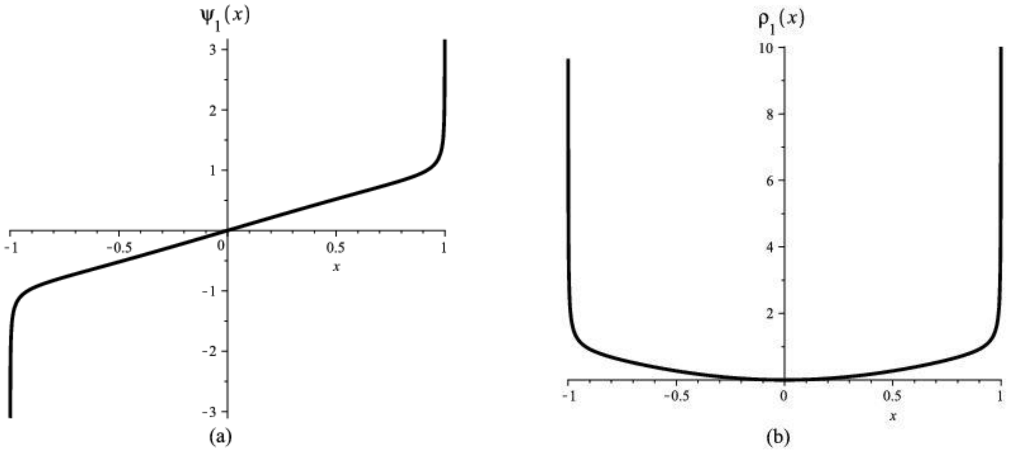

Figure 2.13: (a) Plot of the first state $\psi_1(x)$, (b) Plot of the probability density $\rho_1(x)$, chosen as $\lambda = 1, n = 1$.

If $n = 2$ and $H_2(\mathcal{Y}) = 4\mathcal{Y}^2 - 2$, the second state wave function as (see Fig. 2.14.a)

$$\psi_2(x) = \frac{\sqrt{2}}{4}\frac{1}{\pi^{\frac{1}{4}}}\frac{1}{\sqrt[4]{1-(\lambda x)^2}} e^{-\frac{1}{2}\left[\frac{1}{\lambda}\arcsin(\lambda x)\right]^2} 4\left(\frac{1}{\lambda}\arcsin(\lambda x)\right)^2 - 2 \tag{2.107}$$

and the probability density for the second state $\rho_1(x) = |\ \psi_1(x)|\ ^2$ (see Fig. 2.14.b)

$$\rho_2(x) = \left|\ \frac{\sqrt{2}}{4}\frac{1}{\pi^{\frac{1}{4}}}\frac{1}{\sqrt[4]{1-(\lambda x)^2}} e^{-\frac{1}{2}\left[\frac{1}{\lambda}\arcsin(\lambda x)\right]^2} 4\left(\frac{1}{\lambda}\arcsin(\lambda x)\right)^2 - 2\ \right|^2 . \tag{2.108}$$

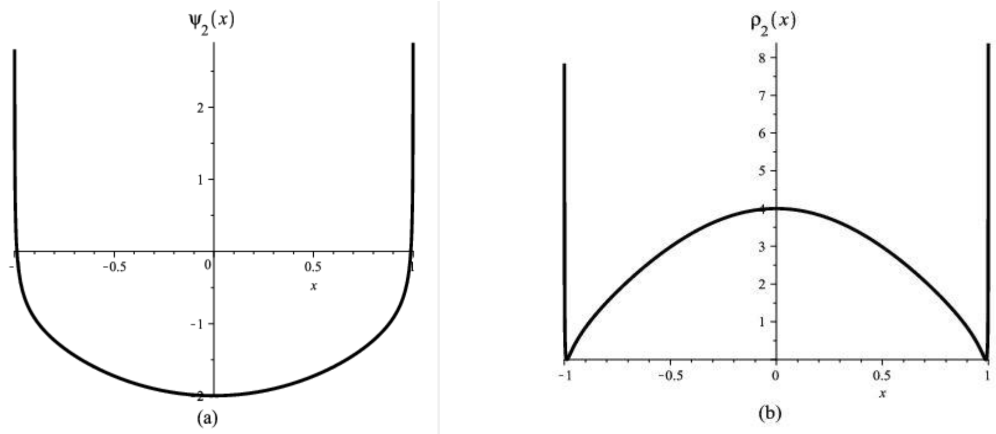

Figure 2.14: (a) Plot of the second state $\psi_2(x)$, (b) Plot of the probability density $\rho_2(x)$, chosen as $\lambda = 1, n = 2$.

Chapter 3

QUANTUM QUASI-FREE PDM

In the position dependent mass PDM Schrödinger equation, the mass and momentum operator no longer commute. The general representation for the kinetic operator was introduce by Von- Roos

$$T = \frac{1}{4}\left[m(x)^j\, Pm(x)^k\, Pm(x)^l + m(x)^l\, Pm(x)^j\, Pm(x)^k\right] \tag{3.1}$$

and the Von- Roos vagueness parameters j, k, l are compelled by the condition

$$j + k + l = -1. \tag{3.2}$$

Also in this chapter, suppose the PDM particle model as

$$m(x) = \frac{m_0}{(1 + A^2x^2)^2}. \tag{3.3}$$

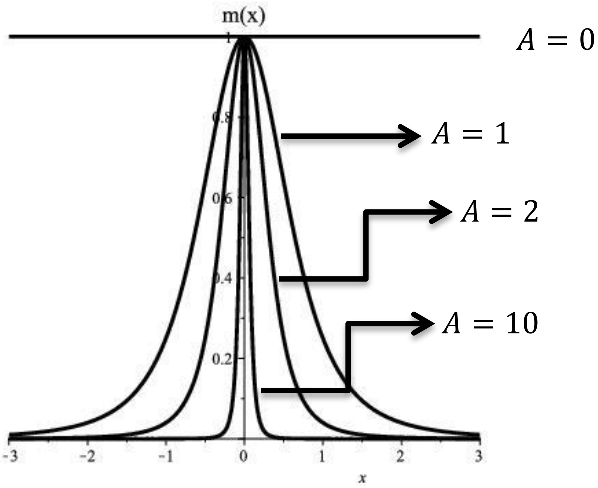

Figure 3.1: Plot of the position dependent mass $m(x) = \frac{m_0}{(1+A^2x^2)^2}$, chosen as $m_0 = 1$, as well as a variety of values with the level parameter A. If $A = 0$ the mass function becomes constant, but if $A = 10$, this mass function becomes very sensitive with respect to position.

Utilizing quantum mechanics Von- Roos Hamiltonian operator $H = T + V(x)$, for the free particle $V(x) = 0$. So Hamiltonian becomes $H = T$, as we know, T is the kinetic and given in Eq. (3.1) and $P = -i\frac{d}{dx} = -i\partial_x$ with $\hbar = 1$. In a clear way, one may demonstrate that the relating Schrödinger equation

$$H\psi(x) = E\psi(x). \tag{3.4}$$

However, PCT is

$$q'(x) = \sqrt{m(x)} \Rightarrow q(x) = \int \sqrt{\frac{m_0}{(1 + A^2x^2)^2}}\, dx. \tag{3.5}$$

Taking integral gives

$$q(x) = \frac{1}{A}\tan^{-1} Ax. \tag{3.6}$$

With the substitution $\psi(x) = m(x)^{\frac{1}{4}}\varphi(q)$ along with the PCT the PDM Schrödinger equation transforms to

$$\frac{1}{2}\left(-\partial_q^2 + a\frac{m''(x)}{m(x)^2} - b\frac{m'(x)^2}{m(x)^3}\right)\varphi(q) = E\varphi(q), \tag{3.7}$$

with

$$a = \frac{1}{4}(1 + 2k), \qquad b = \left[\frac{9}{16} + j(j + k + 1) + k\right]. \tag{3.8}$$

Note that, l is eliminated using the Von Roos constraint in Eq. (3.2). For the first and second derivative of $m(x)$ from Eq. (3.3) we get

$$m'(x) = -\frac{4m_0A^2x}{(1 + A^2x^2)^3}, \tag{3.9}$$

and

$$m''(x) = \frac{24m_0A^2x^2}{(1 + A^2x^2)^4} - \frac{4m_0A^2}{(1 + A^2x^2)^3} \tag{3.10}$$

respectively. Substitution into Eq. (3.7) gives

$$\frac{1}{2}\left(-\partial_q^2 + \frac{4A^2}{m_0}(5a-4b) + tan^2\left(\frac{Aq}{\sqrt{m_0}}\right) - \frac{4a}{m_0}A^2\right)\varphi(q) = E\varphi(q). \quad (3.11)$$

We introduce a variables change of the form $\mathcal{Z} = \frac{Aq}{\sqrt{m_0}}$ to get the one dimensional Schrödinger equation for a Pöschl-Teller like potential [6]

$$\left(-\frac{1}{2m_0}\partial_{\mathcal{Z}}^2 + \frac{2(5a-4b)}{m_0 cos^2(\mathcal{Z})}\right)\varphi(\mathcal{Z}) = \varepsilon\varphi(\mathcal{Z}), \quad (3.12)$$

where

$$\varepsilon = \frac{E}{A^2} + \frac{4}{m_0}(3a-2b) \quad (3.13)$$

and the effective potential has the form

$$V_{eff} = \frac{2(5a-4b)}{m_0 cos^2(\mathcal{Z})} = \frac{1}{2m_0}\frac{\lambda(\lambda-1)}{cos^2(\mathcal{Z})}, \quad (3.14)$$

where $\lambda(\lambda-1) = 4(5a-4b)$.

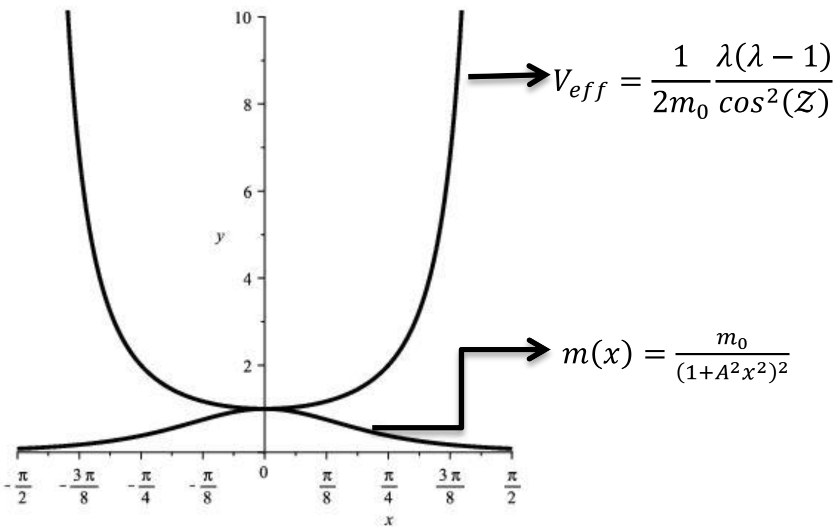

Figure 3.2: Plot of the effective potential $V_{eff} = \frac{1}{2m_0}\frac{\lambda(\lambda-1)}{cos^2(\mathcal{Z})}$ and position dependent mass function $m(x) = \frac{m_0}{(1+A^2x^2)^2}$ together. Chosen as $\lambda = 2, m_0 = A = 1$.

Now, Substituting Eq. (3.13) into Eq. (3.12)

$$\left(-\frac{1}{2m_0}\partial_{\mathcal{Z}}^2+\frac{1}{2m_0}\frac{\lambda(\lambda-1)}{cos^2(\mathcal{Z})}\right)\varphi(\mathcal{Z})=\left[\left(\frac{E}{A^2}+\frac{4}{m_0}\right)(3a-2b)\right]\varphi(\mathcal{Z}). \qquad (3.15)$$

If we set $j=l=-\frac{1}{4}$, and $k=-\frac{1}{2}$, and substituting into Eq. (3.8), we get $a=b=0$.

If we set $j=l=-\frac{1}{2}$, and $k=0$, and substituting into Eq. (3.8), we get a and b as

$$a=\frac{1}{4}, \qquad b=\frac{5}{16}. \qquad (3.16)$$

Substituting Eq. (3.16) into Eq. (3.15)

$$\frac{1}{2m_0}\left(-\partial_{\mathcal{Z}}^2+\frac{\lambda(\lambda-1)}{cos^2(\mathcal{Z})}\right)\varphi(\mathcal{Z})=\left[\left(\frac{E}{A^2}+\frac{4}{m_0}\right)\left(\frac{3}{4}-\frac{10}{16}\right)\right]\varphi(\mathcal{Z}), \qquad (3.17)$$

or

$$\frac{1}{2m_0}\left(-\partial_{\mathcal{Z}}^2+\frac{\lambda(\lambda-1)}{cos^2(\mathcal{Z})}\right)\varphi(\mathcal{Z})=\left(\frac{E}{A^2}+\frac{1}{2m_0}\right)\varphi(\mathcal{Z}). \qquad (3.18)$$

Eq. (3.18) divided by $\frac{1}{2m_0}$ becomes

$$-\partial_{\mathcal{Z}}^2\varphi(\mathcal{Z})+\left(\frac{\lambda(\lambda-1)}{cos^2(\mathcal{Z})}-\frac{2m_0E}{A^2}-1\right)\varphi(\mathcal{Z})=0, \qquad (3.19)$$

or

$$-\partial_{\mathcal{Z}}^2\varphi(\mathcal{Z})+\left(\frac{\lambda(\lambda-1)}{cos^2(\mathcal{Z})}-\frac{2m_0E+A^2}{A^2}\right)\varphi(\mathcal{Z})=0. \qquad (3.20)$$

Finally, we get the Schrödinger equation as

$$\varphi''(\mathcal{Z})+\left(K^2-\frac{\lambda(\lambda-1)}{cos^2(\mathcal{Z})}\right)\varphi(\mathcal{Z})=0, \qquad (3.21)$$

where $K^2=\frac{2m_0E+A^2}{A^2}$. Let us introduce a new variable

$$y=cos^2(\mathcal{Z}). \qquad (3.22)$$

The first and the second derivative of Eq. (3.22) are given by

$$y'=-2\sin(\mathcal{Z})\cos(\mathcal{Z}), \qquad y''=-4y+2\,. \qquad (3.23)$$

As we know, $\partial_{\mathcal{Z}}^2\varphi(\mathcal{Z})=\partial_{\mathcal{Z}}^2\varphi(y)=\partial_{\mathcal{Z}}\big(\partial_{\mathcal{Z}}\varphi(y)\big)=\partial_{\mathcal{Z}}(\varphi'(y)\,\partial_{\mathcal{Z}}\,y)$ we get

$$\partial_{\mathcal{Z}}^2\varphi(\mathcal{Z}) = \varphi''(y)(\partial_{\mathcal{Z}}\, y)^2 + \varphi'(y)\partial_{\mathcal{Z}}^2 y. \tag{3.24}$$

Substituting $y,\ y'$ and y'' into Eq. (3.24) we get

$$\partial_{\mathcal{Z}}^2\varphi(\mathcal{Z}) = \varphi''\left(-2\sqrt{y}\sqrt{1-y}\right)^2 + (-4y+2)\varphi', \tag{3.25}$$

or, Eq. (3.25) finally becomes

$$\varphi''(\mathcal{Z}) = 4y(1-y)\varphi'' + (-4y+2)\varphi'. \tag{3.26}$$

Substituting Eq. (3.26) into Eq. (3.21)

$$4y(1-y)\varphi'' + (-4y+2)\varphi' + \left(K^2 - \frac{\lambda(\lambda-1)}{y}\right)\varphi = 0\,, \tag{3.27}$$

or

$$y(1-y)\varphi'' + \left(\frac{1}{2} - y\right)\varphi' - \left(\frac{\lambda(\lambda-1)}{4y} - \frac{K^2}{4}\right)\varphi = 0\,. \tag{3.28}$$

Splitting off a fitting power of y by setting

$$\varphi = y^{\frac{\lambda}{2}}\, v(y). \tag{3.29}$$

The first and the second derivative of Eq. (3.29) are

$$\varphi' = y^{\frac{\lambda}{2}}\, v' + \frac{\lambda}{2} y^{\frac{\lambda}{2}-1}\, v \tag{3.30}$$

and

$$\varphi'' = y^{\frac{\lambda}{2}}\, v'' + \lambda y^{\frac{\lambda}{2}-1}\, v' + \frac{\lambda}{2}\left(\frac{\lambda}{2} - 1\right) y^{\frac{\lambda}{2}-2}\, v\,. \tag{3.31}$$

Substituting Eqs. (3.29), (3.30) and Eq. (3.31) into Eq. (3.28) one finds

$$y(1-y)\left[y^{\frac{\lambda}{2}}\, v'' + \lambda y^{\frac{\lambda}{2}-1}\, v' + \frac{\lambda}{2}\left(\frac{\lambda}{2} - 1\right) y^{\frac{\lambda}{2}-2}\, v\right] +$$

$$\left(\frac{1}{2} - y\right)\left[y^{\frac{\lambda}{2}}\, v' + \frac{\lambda}{2} y^{\frac{\lambda}{2}-1}\, v\right] - \left(\frac{\lambda(\lambda-1)}{4y} - \frac{K^2}{4}\right)\left[y^{\frac{\lambda}{2}}\, v(y)\right] = 0. \tag{3.32}$$

Divided by $y^{\frac{\lambda}{2}}$, Eq. (3.32) becomes

$$y(1-y)v'' + \lambda(1-y)v' + \frac{\lambda}{2}\left(\frac{\lambda}{2}-1\right)y^{-1}(1-y)v +$$

$$\left(\frac{1}{2}-y\right)v' + \frac{\lambda}{2}\left(\frac{1}{2}-y\right)y^{-1}\,v - \frac{1}{4}\left(\frac{\lambda(\lambda-1)}{y}-K^2\right)v = 0. \qquad (3.33)$$

From the last equation we get

$$\lambda v' - \lambda y v' + \frac{1}{2}v' - yv' = \left[\left(\lambda+\frac{1}{2}\right)-(\lambda+1)y\right]v' \qquad (3.34)$$

and

$$\frac{\lambda}{2}\left(\frac{\lambda}{2}-1\right)y^{-1}(1-y)v + \frac{\lambda}{2}\left(\frac{1}{2}-y\right)y^{-1}\,v\; - \frac{1}{4}\left(\frac{\lambda(\lambda-1)}{y}-K^2\right)v =$$

$$\frac{1}{y}\left[(1-y)\frac{\lambda}{2}\left(\frac{\lambda}{2}-1\right)+\frac{\lambda}{2}\left(\frac{1}{2}-y\right)-\frac{1}{4}(\lambda(\lambda-1)-yK^2)\right]v\;. \qquad (3.35)$$

Also, Eq. (3.35) becomes

$$\frac{1}{y}\left[\frac{\lambda}{2}\left(\frac{\lambda}{2}-1\right)-\frac{\lambda(\lambda-1)}{4}+\frac{\lambda}{2}\left(\frac{1}{2}-y\right)-\frac{y\lambda}{2}\left(\frac{\lambda}{2}-1\right)+\frac{yK^2}{4}\right]v =$$

$$-\frac{1}{4}[\lambda^2-K^2]v\,. \qquad (3.36)$$

Now, substituting Eq. (3.36) and Eq. (3.34) into Eq. (3.33), we get the hypergeometric differential equation, and given by

$$y(1-y)v'' + \left[\left(\lambda+\frac{1}{2}\right)-(\lambda+1)y\right]v' - \frac{1}{4}[\lambda^2-K^2]v\;. \qquad (3.37)$$

Complete solution of Eq. (3.37) may be written

$$v = A\;{}_2F_1\left(a,b,\frac{1}{2}+\lambda,y\right) + By^{\left(\frac{1}{2}-\lambda\right)}\;2F_1\left(\frac{1}{2}-a,\frac{1}{2}-b,\frac{3}{2}-\lambda,y\right), \qquad (3.38)$$

where

$$a = \tfrac{1}{2}(\lambda+K), \qquad b = \tfrac{1}{2}(\lambda-K)\,. \qquad (3.39)$$

As we know, $y = cos^2(\mathcal{Z})$ and $\mathcal{Z} = \frac{Aq}{\sqrt{m_0}}$ and from Eq. (3.6) we have $q(x) = \frac{1}{A}\tan^{-1}Ax$, which all imply

$$y = cos^2\left(\frac{A}{\sqrt{m_0}}\frac{1}{A}\tan^{-1} Ax\right) = cos^2\left(\frac{1}{\sqrt{m_0}}\tan^{-1} Ax\right). \qquad (3.40)$$

Since,

$$cos^2\left(\frac{A}{\sqrt{m_0}}\frac{1}{A}\tan^{-1} Ax\right) = \frac{1}{1+Ax^2} \qquad (3.41)$$

Eq. (3.38) becomes

$$\psi_n(x) = A\ {}_2F_1\left(a, b, \frac{1}{2}+\lambda, \frac{1}{1+Ax^2}\right) +$$

$$y^{\left(\frac{1}{2}-\lambda\right)}\ B\ {}_2F_1\left(\frac{1}{2}-a, \frac{1}{2}-b, \frac{3}{2}-\lambda, \frac{1}{1+Ax^2}\right), \qquad (3.42)$$

such that, $\psi = 0$ at $\cos \mathcal{Z} = 0$.

Recall, the original Pöschl-teller potential has the form [6]

$$V = \frac{\hbar^2\alpha^2}{2m}\left[\frac{v(v-1)}{sin^2\alpha(r-r_0)} + \frac{\mu(\mu-1)}{cos^2\alpha(r-r_0)}\right], \qquad \left(0 \le \alpha(r-r_0) \le \frac{\pi}{2}\right), \qquad (3.43)$$

with α being a reciprocal length, $v,\ \mu > 1$. As the potential goes to infinity at $\alpha(r-r_0) = 0$ and $\alpha(r-r_0) = \frac{\pi}{2}$ we get the boundary condition $\psi = 0$ at these points. It can be shown that the Schrödinger equation

$$\frac{\partial\psi^2}{\partial r^2} + \frac{2m}{\hbar^2}\ (E-V)\psi = 0, \qquad (3.44)$$

can be integrated in closed form. So let as write the eigenfunctions as

$$\psi = sin^v\ \alpha\ (r-r_0)cos^u\ \alpha\ (r-r_0)z \qquad (3.45)$$

$z \neq 0$ at the boundaries. Defining the independent variable

$$\mathcal{Y} = \sin\alpha\ (r-r_0) \qquad (3.46)$$

and with z in the form

$$z = \sum_k a_k\mathcal{Y}^k\ . \qquad (3.47)$$

From Eqs. (3.43), (3.44),(3.45) and (3.46) we obtain the recurrence relation

$$a_{k+2}[(k+v+2)(k+v+1)-v(v-1)]+$$

$$a_k\left[-(\mu+v+k)^2+\frac{2m}{\alpha^2\hbar^2}E\right]=0\,. \tag{3.48}$$

From the condition of stopping the recurrence relation, we obtain the Energy eigenvalue

$$E_n=\frac{\alpha^2\hbar^2}{2m}(\mu+v+2n)^2. \tag{3.49}$$

So we can easily convert this original process to obtain the eigenenergy in our case by taking $\alpha=1,\ \mu=\lambda$ and $\lim_{v\to 1}E_n=1$, our eigenvalue has the form

$$E_n=\frac{1}{2m}(\lambda+1+2n)^2\,, n=0,1,2,3\ldots\quad . \tag{3.50}$$

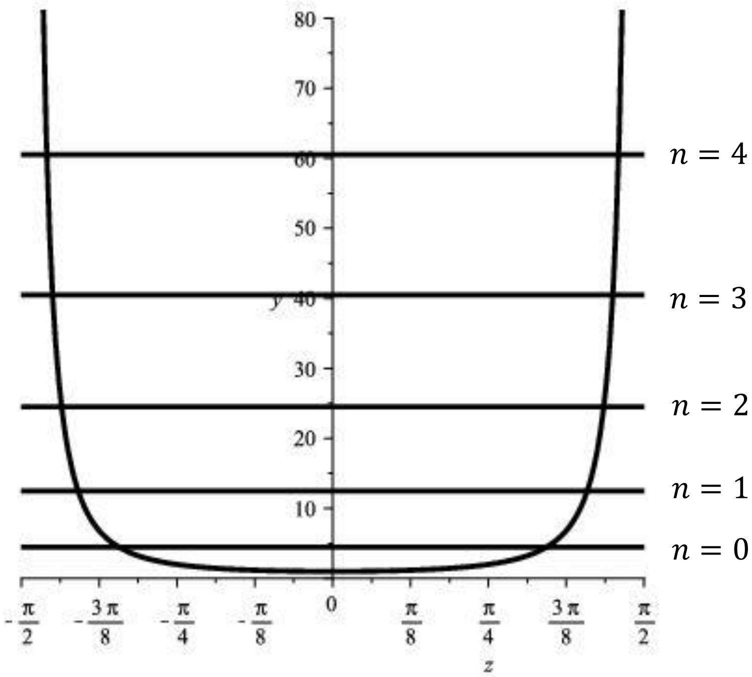

Figure 3.3: Plot of the effective potential $V_{eff}=\frac{1}{2m_0}\frac{\lambda(\lambda-1)}{cos^2(Z)}$ and the Energy eigenvalue $E_n=\frac{1}{2m}(\lambda+1+2n)^2$ together. Chosen as $\lambda=2, m_0=1, n=0\ldots 4$

Chapter 4

SOLUTIONS OF PDM SCHRODINGER EQUATION IN d-DIMENSIONS

4.1 d-Dimensional Quasi-free PDM Schrödinger Equation

All of us look at the free particle (i.e., $V(r) = 0$), having a PDM $m(r) = \frac{m_0}{(1+A^2r^2)^2}$ within the d-dimensional Schrödinger equation. The PDM $M(r) = m_0\, m(r)$, and $\alpha = \gamma = 0$, and $\beta = -1$ in Eq. (1.1) of Tanaka [21]. The Schrödinger Hamiltonian would likely study (using atomic units $m_0 = \hbar = 1$)

$$H = \frac{1}{2}\left(\vec{\mathrm{p}}\frac{1}{M(r)}\right).\vec{\mathrm{p}} = -\frac{\hbar^2}{2\,m_0}\left(\vec{\nabla}\frac{1}{m(r)}.\vec{\nabla}\right) \tag{4.1}$$

and also supposing the actual d-dimensional spherical symmetric formula, together with

$$\psi(\vec{r}) = r^{-(d-1)/2}\, R_{n_r,\ell_d}(r) Y^{L}_{\ell_d,m_d}(\theta,\varphi). \tag{4.2}$$

From Hamiltonian Eq. (4.1) would certainly result in the subsequent time independent d-dimensional for the radial Schrödinger equation form

$$\left\{\frac{d^2}{dr^2} - \frac{\ell_d(\ell_d+1)}{r^2} + \frac{m'(r)}{m(r)}\left(\frac{d-1}{2r} - \frac{d}{dr}\right) + 2m(r)E\right\} R_{n_r,\ell}(r) = 0 \tag{4.3}$$

where $\ell_d = \ell + (d-3)/2$ if $d \geq 2$ also, ℓ can be the angular momentum quantum number, and $n_r = 0,1,2,3 \ldots$ is actually the radial quantum number, and also $m'(r) = \frac{d}{dr} m(r)$. However, the radial part of Schrödinger equation together with CM and angular momentum ℓ_d is actually written as

$$\left\{\frac{d^2}{dr^2} - \frac{\ell_d(\ell_d+1)}{r^2} + 2\mathcal{E}_{n_r}\right\} R(r) = 0, \tag{4.4}$$

where $\mathcal{E}_{n_r}$ is the energy eigenvalue.

Now, we apply to the Eq. (4.3) the following transformation

$$r = q(r)\,, \qquad R(r) = g(r)\phi\big(q(r)\big) \tag{4.5}$$

would certainly result in $m(r) = g(r)^2 q'(r)$, manifested by the requirement of a disappearing coefficient of the first-order derivative of $\phi\big(q(r)\big)$ (hence a one-dimensional form of the Schrödinger equation is achieved), and also $q'(r)^2 = m(r)$ to avoid position dependent energies-multiplicity (i.e., $2Em(r)/q'(r)^2 \Longrightarrow 2E$). So, we get the following condition on transformation Eq. (4.5) to be a PCT

$$q(r) = \int^r \sqrt{m(t)}dt \Longrightarrow g(r) = m(r)^{\frac{1}{4}}\,. \tag{4.6}$$

This in effect indicates

$$\left\{-\frac{1}{2}\frac{d^2}{dq^2} + V_{eff}\big(q(r)\big)\right\} \phi_{n_r,\ell_d}\big(q(r)\big) = E_d \phi_{n_r,\ell_d}\big(q(r)\big), \tag{4.7}$$

and with an effective potential written as

$$V_{eff}\big(q(r)\big) = \frac{\ell_d(\ell_d+1)}{2r^2 m(r)} - U_d(r), \tag{4.8}$$

where

$$U_d(r) = \frac{m''(r)}{8m(r)^2} - \frac{7m'(r)^2}{32m(r)^3} + \frac{m'(r)(d-1)}{4rm(r)^2}. \tag{4.9}$$

Now, consequences associated with asymptotically disappearing mass setting r goes to infinity. A free particle (i.e.,$V(r) = 0$) together with an asymptotically disappearing PDM $m(r) = \frac{m_0}{(1+A^2r^2)^2}$ would experience an effective potential

$$V_{eff}\big(q(r)\big) = \frac{A^2}{2}\left[\frac{x(x-1)}{sin^2(Aq)} + \frac{\lambda(\lambda-1)}{cos^2(Aq)}\right] - \frac{A^2}{2}. \tag{4.10}$$

Now, from the our PDM $m(r)=\frac{m_0}{(1+A^2r^2)^2}$ and which under PCT Eq. (4.6)

$$q(r)=\int^{r}\sqrt{\frac{1}{(1+A^2r^2)^2}}\;dr\,. \qquad (4.11)$$

Integration of Eq. (4.11) implies

$$q(r)=\frac{1}{A}\arctan(Ar)\Longrightarrow Ar=\tan(Aq). \qquad (4.12)$$

The first and the second derivative of $m(r)=\frac{m_0}{(1+A^2r^2)^2}$ as

$$m'(r)=-\frac{4A^2r}{(1+A^2r^2)^3}, \qquad m''(r)=\frac{24A^4r^2}{(1+A^2r^2)^4}-\frac{4A^2r}{(1+A^2r^2)^3}. \qquad (4.13)$$

Substitution Eq. (4.13) with Eq. (4.12) into Eq. (4.9)

$$U_d(r)=-(A^2d)\,tan^{\,2}(Aq)+\frac{A^2}{2}(1-2d). \qquad (4.14)$$

Now, substitution Eq. (4.10) into Eq. (4.7) and such settings, says

$$\left\{-\frac{1}{2}\frac{d^2}{dq^2}+\frac{A^2}{2}\left[\frac{x(x-1)}{sin^2(Aq)}+\frac{\lambda(\lambda-1)}{cos^2(Aq)}\right]\right\}\phi_{n_r,\ell_d}(q)=\varepsilon\phi_{n_r,\ell_d}(q), \qquad (4.15)$$

where

$$x(x-1)=\ell_d(\ell_d+1), \quad \lambda(\lambda-1)=\ell_d(\ell_d+1)+2d, \qquad \varepsilon=E+\frac{1}{2}A^2\,. \quad (4.16)$$

Eq. (4.15) is actually a standard one-dimensional type of the Schrödinger equation having a generalized trigonometric Pöschl-Teller effective potential.

We will apply the previous processes to the particular trigonometric Pöschl-Teller effective potential, and it has the form

$$V_{eff}=\frac{A^2}{2}\left[\frac{x(x-1)}{sin^2(Aq)}+\frac{\lambda(\lambda-1)}{cos^2(Aq)}\right]. \qquad (4.17)$$

If $A=1$, Eq. (4.17) written as

$$V_{eff}=\left[\frac{x(x-1)}{2sin^2(Aq)}+\frac{\lambda(\lambda-1)}{2cos^2(Aq)}\right] \qquad x,\lambda>1. \qquad (4.18)$$

Now, using the general solution of the Schrödinger equation $H\phi_{n_r,\ell_d}(q) = \varepsilon\phi_{n_r,\ell_d}(q)$ as Eq. (4.15), which reads

$$\phi_{n_r,\ell_d}(q) = sin^x(Aq)cos^\lambda(Aq)\left\{C\ {}_2F_1\left[\frac{\mu}{2}+\sqrt{\frac{\varepsilon}{2}},\frac{\mu}{2}-\sqrt{\frac{\varepsilon}{2}},\lambda+\frac{1}{2};sin^2(Aq)\right]+\right.$$

$$\left.sin^{1-2x}(Aq)B\ {}_2F_1\left[\frac{1+\lambda-x}{2}+\sqrt{\frac{\varepsilon}{2}},\frac{1+\lambda-x}{2}-\sqrt{\frac{\varepsilon}{2}},\frac{3}{2}-\lambda;sin^2(Aq)\right]\right\}, \quad (4.19)$$

where $\mu = x+\lambda$. We can find now the Radial wavefunction $R_{n_r,\ell_d}(r)$ of H, which satisfy boundary conditions $R_{n_r,\ell_d}(0) = R_{n_r,\ell_d}\left(\frac{\pi}{2A}\right) = 0$. Since, $R_{n_r,\ell_d}(0) = 0$, it turns out that $B = 0$. So, Eq. (4.19) becomes

$$\phi_{n_r,\ell_d}(q) = C\ sin^x(Aq)cos^\lambda(Aq)\times$$

$${}_2F_1\left(-n_r, x+\lambda+n_r, x+\frac{1}{2}; sin^2(Aq)\right) \quad (4.20)$$

where $\frac{\mu}{2}\pm\sqrt{\frac{\epsilon_{n_r}}{2}} = -n_r$,

and

$$\epsilon_{n_r} = \frac{A^2}{2}(\mu+2n_r)^2 = \frac{A^2(x+\lambda+2n_r)^2}{2}, \quad n = 0,1,2,3,\dots \quad . \quad (4.21)$$

Now, from the boundary condition $R_{n_r,\ell_d}(0) = R_{n_r,\ell_d}\left(\frac{\pi}{2A}\right) = 0$, seeing that described by simply Salem and Montemayor (look Eq. (4.7) in [22]). Therefore would certainly produce

$$E_{n_r,\ell_d} = \frac{A^2}{2}\left(\left(c+\frac{1}{2}\Delta+2n_r\right)^2-1\right), \quad (4.22)$$

where $\Delta = \sqrt{(2\ell_d+1)^2+8d}$.

Also, the radial Schrödinger equation has the form

$$R_{n_r,\ell_d}(r) = \widetilde{C}\,\rho^{\ell_d+1}(1+\rho^2)^{-\frac{1}{4}(2\ell_d+5+\Delta)}\times$$

$${}_2F_1\left(-n_r, c+\frac{\Delta}{2}+n_r, c; \frac{\rho^2}{1+\rho^2}\right) \quad (4.23)$$

where $\rho = Ar$, and $c = \ell_d + \frac{3}{2}$.

However, if $x = 0, 1$ (the requirement proposed through relation Eq. (4.16) whenever $\ell_d = 0, -1$) the particular effective potential in Eq. (4.15) collapses directly into

$$V_{eff}\big(q(r)\big) = \frac{A^2}{2}\frac{\lambda(\lambda-1)}{cos^2(Aq)}, \tag{4.24}$$

which in turn admits a defined solution

$$E_{n_r} = 2A^2\left(n_r + \frac{\lambda}{2}\right)^2 - \frac{A^2}{2}, \tag{4.25}$$

and

$$\phi_{n_r,\ell_d}(q) = F cos^{\lambda}(Aq) \times {}_2F_1\left(-n_r, n_r + \lambda, \frac{1}{2}; sin^2(Aq)\right). \tag{2.26}$$

Consequently is exactly the same as Eq. (4.25)

$$E_{n_{r,0}} = 2A^2\left(n_r + \frac{\lambda}{2}\right)^2 - \frac{A^2}{2}, \tag{4.27}$$

also, the radial Schrödinger equation as

$$R_{n_r}(r) = \widetilde{F}\,(1+\rho^2)^{-\frac{1}{4}(2\ell_d+5+\Delta)} \times$$
$${}_2F_1\left(-n_r, c + \frac{\Delta}{2} + n_r, c; \frac{\rho^2}{1+\rho^2}\right), \tag{4.28}$$

where $\lambda = \frac{(1+\Delta)}{2}$.

4.1.1 Consequences of a power-law mass $m(r) = \beta r^{\gamma}$

The radial PDM $m(r) = \beta r^{\gamma}$, the actual PCT function in Eq. (4.6) implies

$$q(r) = \int^{r} \sqrt{\beta r^{\gamma}}\, dr = \sqrt{\beta}\int^{r} t^{\gamma/2}\, dt = \frac{2\sqrt{\beta}}{(\gamma+2)} r^{(\gamma+2)/2)}$$

or

$$\Rightarrow \frac{(\gamma+2)}{2} q(r) = r\sqrt{m(r),} \tag{4.29}$$

and also Eq. (4.9) gives

$$U_d(r) = -\frac{1}{16}\left(\frac{\gamma(3\gamma+12-8d)}{2r^2 m(r)}\right) \equiv -\frac{1}{4}\left(\frac{\gamma(3\gamma+12-8d)}{2(\gamma+2)^2\, q^2(r)}\right). \quad (4.30)$$

Now, the particular d-dimensional position dependent effective mass Schrödinger equation is usually written by [10]

$$\vec{\nabla}_d\left(\frac{1}{m}\vec{\nabla}_d\psi(\vec{r})\right) + 2[E - V(r)]\psi(\vec{r}) = 0, \quad (4.31)$$

where $m = m(r)$ is pertaining to d-dimensional spherical symmetry. Now, $\psi(\vec{r})$ is the wave function, it's presented an angular momentum L may be prepared as [13-17]

$$\psi(\vec{r}) = r^{-(d-1)/2}\, R(r) Y^L_{Ld-2,\dots L_1}(\theta_1, \theta_2, \dots, \theta_{d-1}). \quad (4.32)$$

On the other hand, we have

$$\vec{\nabla}_d \frac{1}{m}\vec{\nabla}_d\psi(\vec{r}) = \left(\vec{\nabla}_d \frac{1}{m}\right).\left[\vec{\nabla}_d\psi(\vec{r})\right] + \frac{1}{m}\nabla^2_d\, \psi(\vec{r})\,. \quad (4.33)$$

Consider

$$\left(\vec{\nabla}_d \frac{1}{m}\right).\left[\vec{\nabla}_d\psi(\vec{r})\right] = -\frac{Y^L_{Ld-2,\dots L_1}(\theta_1, \theta_2, \dots, \theta_{d-1})}{m}\left[\frac{m'}{m}\left(\frac{d-1}{2}\,\frac{1}{r}\,\frac{dR(r)}{dr}\right)\right], \quad (4.34)$$

and

$$\frac{1}{m}\nabla^2_d\, \psi(\vec{r}) = \frac{Y^L_{Ld-2,\dots L_1}(\theta_1, \theta_2, \dots, \theta_{d-1})}{m} \times$$

$$\left[\frac{d^2}{dr^2} - \frac{L(L+d-2)+(d-1)(d-3)/4}{r^2}\right] R(r). \quad (4.35)$$

Substitution of Eq. (4.4) along with Eq. (4.5) into Eq. (4.1) we are able to get the adopting the d-dimensional radial PDM Schrödinger equation

$$\left\{\frac{d^2}{dr^2} + \frac{m'}{m}\left(\frac{d-1}{2}\,\frac{1}{r} - \frac{d}{dr}\right) - \frac{L(L+d-2)+(d-1)(d-3)/4}{r^2} +\right.$$

$$\left. 2m[E - V(r)]\right\}R(r) = 0, \quad (4.36)$$

where $m' = \frac{dm(r)}{dr}$ and E is the energy eigenvalue. However, this d-dimensional radial Schrödinger equation together with CM, the particular potential function $U(\rho)$, the energy spectrum ε and angular momentum Λ have the form

$$\left\{\frac{d^2}{d\rho^2} - \frac{\Lambda(\Lambda + d - 2) + (d-1)(d-3)/4}{\rho^2} + 2[\varepsilon - U(\rho)]\right\}\psi(\rho) = 0. \quad (4.37)$$

Invoking a transformation $\rho \rightarrow r$ through a mapping function $\rho = q(r)$, and rewriting the wave function in the form of

$$\psi(\rho) = g(r)\,R(r). \quad (4.38)$$

We obtain a transformed Schrödinger equation with constant mass

$$\left\{\frac{d^2}{dr^2} + \left(2\frac{g'}{g} - \frac{q''}{q'}\right)\frac{d}{dr} + \left(\frac{g''}{g} - \frac{q''}{q}\frac{g'}{g}\right) - \right.$$

$$\left[\Lambda(\Lambda + \mathrm{d} - 2) + \frac{(d-1)(d-3)}{4}\right]\left(\frac{q'}{q}\right)^2 +$$

$$\left. 2(q')^2\left[\varepsilon - U\big(q(r)\big)\right]\right\}R(r) = 0\,. \quad (4.39)$$

By comparing Eq. (4.39) with Eq. (4.36), we can identify the next conditions within the Eq. (4.38), to become a point canonical transformation PCT

$$g(r) = \left(\frac{q'}{m}\right)^{\frac{1}{2}} \quad (4.40)$$

and

$$V(r) - E + \frac{L(L + d - 2) + (d-1)(d-3)/4}{2mr^2} =$$

$$\frac{(q')^2}{m}\left[U\big(q(r)\big) - \varepsilon\right] + \frac{(d-1)m'}{4m^2 r} + \frac{1}{2m} \times$$

$$\left[\Lambda(\Lambda + \mathrm{d} - 2) + \frac{(d-1)(d-3)}{4}\right]\left(\frac{q'}{q}\right)^2 + \frac{1}{4m}[F(m) - F(q')] \quad (4.41)$$

where

$$F(\mathcal{Z}) = \frac{\mathcal{Z}''}{\mathcal{Z}} - \frac{3}{2}\left(\frac{\mathcal{Z}'}{\mathcal{Z}}\right)^2. \tag{4.42}$$

4.2 Application to the d-Dimensional Harmonic Oscillator

The particular d-dimensional harmonic oscillator while using the wave functions and energy spectrum is actually written by [18]

$$\varepsilon_n = \left(2n + \Lambda + \frac{d}{2}\right)\omega^2, \tag{4.43}$$

and

$$U(\rho) = \frac{1}{2}\omega^4\rho^2. \tag{4.44}$$

The wave functions

$$\psi_n(\rho) = A_n\left(\omega^4\rho\right)^{\Lambda+(d-1)/2}\ \exp\left[-\frac{\omega^4}{2}\rho^2\right] F\left(-n, \Lambda + \frac{d}{2}, \omega^4\rho^2\right), \tag{4.45}$$

where $F\left(-n, \Lambda + \frac{d}{2}, \omega^4\rho^2\right)$ means confluent hypergeometric functions. Now we assume the PDM $m(r) = \alpha r^\lambda$, take PCT function $q(r) = r^v$, where $(v, \alpha, \lambda\ ) \neq 0$ real parameters. If $v = 1 + \frac{\lambda}{2}$ with $\lambda \neq 2$, i.e., $\frac{(q\prime)^2}{m}$ =const, we discussed in this Chapter. Now substituting them into Eq. (4.41)

$$V(r) = \frac{v^2}{\alpha}U\big(q(r)\big) = \frac{\alpha}{2}C^2 r^{\lambda+2} \tag{4.46}$$

and energy spectrum is

$$E_n = \frac{v^2}{\alpha}\varepsilon_n = \frac{\lambda+2}{2}C\left(2n + \Lambda(L) + \frac{d}{2}\right) \tag{4.47}$$

and the wave functions has the form

$$R(r) = a_n(\eta r)^{(1+\lambda/2)(\Lambda(L)+(d-1)/2)+\lambda/4} \times$$

$$\exp\left[-\frac{\eta}{2}r^{2+\lambda}\right] F\left(-n, \Lambda(L) + \frac{d}{2}, \eta r^{2+\lambda}\right) \tag{4.48}$$

where C can be a real potential parameter, $\eta = 2\alpha C/(2+\lambda)$, and $\Lambda(L)$ fulfills with situation, and that is proven as

$$16L(L+d-2)+4(d-1)(d-3)=$$

$$(2+\lambda)^2[4\Lambda(\Lambda+\mathrm{d}-2)+(d-1)(d-3)]-3\lambda^2+(8d-12)\lambda\,. \quad (4.49)$$

Especially, we acquire $\lambda = 0$, such as, m is position independent and next get $\Lambda = \mathrm{L}$ in terms of Eq. (4.49). It is straight to obtain out that Eqs. (4.46)-(4.48) agree with Eqs. (4.43)- (4.45). For $\lambda = 2$ we get for the potential

$$V(r)=\frac{\alpha}{2}C^2r^4, \quad (4.50)$$

the energy spectrum

$$E_n=2C\left(2n+\Lambda_1(L)+\frac{d}{2}\right), \quad (4.51)$$

the wave function

$$R(r)=a_n\left(\frac{\alpha C}{2}r\right)^{2\Lambda_1(L)+d+1/2}\times$$

$$\exp\left[-\frac{\alpha C}{4}r^4\right]F\left(-n,\Lambda_1(L)+\frac{d}{2},\frac{\alpha C}{2}r^4\right), \quad (4.52)$$

where $\Lambda_1(L)$ has the form

$$16\Lambda_1(\Lambda_1+d-2)+3(d-1)(d-3)+$$

$$4d-9-4L(L+d-2)=0. \quad (4.53)$$

4.3 Application to the d-Dimensional Coulomb potential

The Coulomb potential d-dimensional together with the wave functions and energy spectrum is actually written by [19]

$$U(\rho)=-\frac{z}{\rho} \quad (4.54)$$

and the energy spectrum

$$\varepsilon_n=\frac{z^2}{2(\Lambda+n+(d-1)/2)^2}\,, \quad (4.55)$$

for the Coulomb wave functions

$$\psi_n(\rho)=B_n(\beta\rho)^{\Lambda+(d-1)/2}\ \exp[-\beta\rho]\,F(-n,2\,\Lambda+\mathrm{d}-2,2\beta\rho), \quad (4.56)$$

where $\beta = \frac{z}{\Lambda+\text{n}+(d-1)/2}$. The PDM and PCT function for this case are the same as harmonic oscillator $(m(r) = \alpha r^{\lambda}, q(r) = r^{v}, v = 1 + \frac{\lambda}{2})$ and $\lambda \neq 2$. Substituting them into Eq. (4.40) and Eq. (4.41),

$$V(r) = -\frac{P}{r^{1+\lambda/2}} \tag{4.57}$$

and the energy spectrum has the form

$$E_n = -\frac{2\alpha P^2}{(2+\lambda)^2}\frac{1}{(\Lambda(L)+n+(d-1)/2)^2}\ . \tag{4.58}$$

The Coulomb wave function as

$$R_n(r) = b_n r^{\left(\Lambda+\frac{(d-1)}{2}\right)(1+\lambda/2)+\lambda/4}\,\text{exp}\left[-\gamma r^{1+\lambda/2}\right] \times$$
$$F\left(-n, 2\Lambda(L)+d-2, 2\gamma r^{1+\lambda/2}\right). \tag{4.59}$$

where P can be real potential parameter, and $\Lambda(L)$ satisfies together with Eq. (4.49), as well as

$$\gamma = \frac{4\alpha\, P}{(2+\lambda)^2}\frac{1}{\Lambda+\text{n}+(d-1)/2}. \tag{4.60}$$

Chapter 5

CONCLUSION

The problem of the PDM is discussed in this thesis exhaustingly. The starting point of the discussion of the PDM was selected as the classical harmonic oscillator under the PDM. The classical harmonic oscillator was solved for several PDM functions. Then this approach was also expanded to the quantum domain, also calculating the wave functions and the probability densities in for different PDM functions. In this quantum domain, obviously the behavior is of PDM is significantly different. In Chapter 3, the problem of a quasi-free PDM is reduced to the solution of the Schrödinger equation for a Pöschl-Teller like potential. A generalization to d dimensions is given in Chapter 4. Furthermore, we close Chapter 4 with the application of the d-dimensional potential mass problem to the d-dimensional harmonic oscillator. Chapter 5, closes this thesis with the conclusions.

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