Geodesics on Cosmic Landscapes of Colliding Plane Waves

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ABSTRACT

On a Cosmic Landscape, the metric structure vested with two orthogonal space-like Killing vectors; a class of solutions of the Einstein-Maxwell's field equations, is spotlighted from the global structural viewpoints of the Khan-Penrose and Bell-Szekeres space-time continua or Cosmic Landscapes: a platform for discussing the motion of a test particle. A solution, spring-boarded by the Ferrari-Ibanez hybrid formalism, also provides a launch-pad for discussing the motion of a test particle on a Degenerate Cosmic Landscape. When a particle is placed along the path of two colliding plane waves, it will be forced to follow a geodesic, defined by the properties of the global structure, leading to either a singularity or a horizon. In the null-coordinates,(u, v), the interaction region is bounded, so given the initial conditions the later developments are plotted numerically. The time of fall into the singularity or horizon is also obtained.

Keywords: Cosmic Landscape, gravitational waves, geodesics, horizons/singularities.

ÖZ

Kozmik uzayda birbirine dik iki uzaysal Killing vektörle belirlenen Khan-Penrose ve Bell-Szekeres (Einstein-Maxwell teorisi) uzayları içerisinde test-partikül hareketleri incelenmiştir. Bu yönde karışık (hibrit) bir çözüm uzayı olan Ferrari-Ibanez çözümü örnek alınmıştır. Bir dalga çarpışma uzayında jeodeziler üzerinde hareket eden partiküller tekillik veya ufuk yüzeyine ulaşmaktadır. Işıksal (u, v) koordinat uzayında ilk şartlara bağımlı hareketlerin zaman gelişimi sayısal yöntemlerle çizilmiştir. Aynı yöntemle tekillik/ufuk düzlemine varış zamanı elde edilmiştir.

Anahtar Kelimeler: Kozmik uzay, yerçekim dalgaları, jeodeziler, uzay düzlem ve tekillikleri

DEDICATION

To ALL

Genuine Seekers

Of

Truth

On the workings of the Cosmos

AKNOWLEDGEMENT

I wish to express my profound gratitude to the “Most-High”, the owner of the “Cosmos”, for granting me access to understanding a little about the workings of the visible universe.

I also wish to express my gratitude to my Parents and Family for putting me on the right path to a colorful destiny. I appreciate my friends and well-wishers for believing in me and for granting sweetened and seasoned words of encouragement that kept me going, even when the going got rough and tough. Now, I can say; “Tough times never last but tough people do.”

I wish to express a special thanks to my supervisor, Prof. Dr. Mustafa Halilsoy, who is also the Chair of the department of Physics and Chemistry here at EMU, for a professional touch and finishing. I appreciate the contributions of the faculty members, staff and professional colleagues at EMU.

Seeing myself surrounded by all this caliber of people from all works of life, success is inevitable and failure is not an option. That is why I can confidently say with Sir Isaac Newton: “If I had seen farther, it is by standing on the shoulders of Giants.” A Big Thank you.

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INTRODUCTION

The detection of B-Mode Polarization at Degree angular scales by BICEP2 [1], provides an undeniable proof, (see fig. 1.1) [1], a confirmation of the properties of the gravitational waves produced in the early universe as predicted by the inflationary theory.

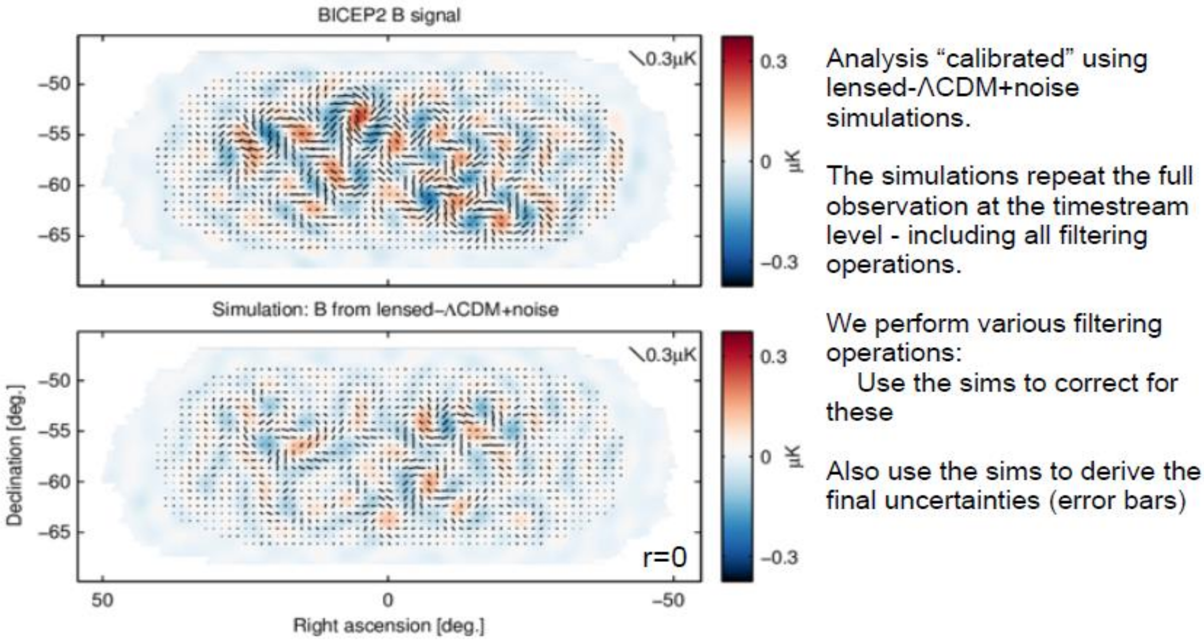

Figure 1.1: The B-Mode Map vs. Simulation [1]

1.1 Research Background

The journey to this milestone in the annals of the history of sciences began a couple of centuries ago. Passing through the corridors of the theoretical minds, and by unraveling the hidden reality [2] of the cosmic mysteries [3] painted in its history [4, 5], where, things that glaringly seemed humanly impossible [6, 7] to the ordinary man on the street, are now made possible [8] through the workings of these theoretical minds; one simply but confirms and affirms that: “what the mind can conceive, it can achieve,” and “the quality of life we live is a function of how we think”. This quest was shouldered-on by a handful theoretical Giants [9], who through the weaponry of thought experiments, formulated some testable theories and principles that seems to govern our life and existence as we walk the sand of times [10, 4].

Gravity is the most elusive physical phenomenon that has overwhelmed the theoretical minds for centuries, of which, the modern theorists see it as a force that is not present in the two dimensional world but materializes along with the emergence of the illusory third and higher dimensions [11, 12, 2].

Newton’s formalism for the Universal Law of Gravitation pictures gravity as an attractive force that acts at a distance. The Law explains how the Moon and the planetary systems move in orbits around their common center of gravity. In his address to his celebrating fans and critics over his famous work on “The Mathematical Principle of Natural Philosophy,” Newton declares; “If I have seen farther, it is by standing on the shoulders of Giants” [9].

However, when confronted with a challenge on how gravity works, he looked at the then visible static Cosmic Landscape, as a visionary founding father of theoretical Physics, through the Telescopic-far-sighted power of his newly born Newtonian formalism; but sadly and regrettably, realizing there is still a long way to completing the cosmic puzzle; he then declared in a cold-hearted low tone:

> I do not know how I may appear to the world, but to myself I seem to have been only like a boy, playing on the sea-shore, and diverting myself, in now and then finding a smoother pebble or prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me [9].

Formulating his theory of general relativity in 1915, Einstein replaced the gravitational force or force of gravity that acts at a distance, with the dynamics of space-time continuum; gravity is seen to arise due to the curvature of the fabric of the space-time continuum or cosmic landscape [13] whenever matter and energy come on stage. This curvature deflects the trajectories or paths of particles, giving rise to a gravitational field; a region of space-time where gravitational influence is experienced. This disturbance on space-time continuum due to gravity is transmitted within the fabric of the cosmos in form of gravitational waves.

Some solutions to the Einstein's field equations enshrined in his theory of general relativity, are centered on the concept of gravitational waves. Among these solutions include the Khan-Penrose global structure [14], and the Bell-Szekeres global structure [15].

1.2 The Basic Concept

1.2.1 What is Gravitational Wave?

The theory of general relativity provides that gravity is the curvature of space-time continuum, produced by mass-energy concentrations in the fabric of space-time. Whenever these mass-energy concentrations move or change shape, they produce distortions in the space-time geometry. These ripples or undulations in the curvature of space-time continuum carry energy and momentum and propagate at the speed of light. In this light, we see a gravitational plane wave as a region of space-time continuum confined between two parallel planes, in which the curvature is a non-zero and propagates at the speed of light through the fabrics of cosmic landscape, in the direction normal to the plane [16 - 19].

1.2.2 Sources of Gravitational Waves

Gravitational waves are said to be produced based on the sizes or masses of the bodies involved over a wide range of time scales. Following [16, 20, 21], we classify gravitational waves based on their sources and waved forms; Periodic, Bursts, and Stochastic waves.

The **Periodic waves** are the sinusoidal kind of waves said to be produced by rotating stars, binary stars, binary black holes and binaries of both stars and black holes. On the other hand, the **Bursts** are waves of short cycles. They are said to be produced by the collisions of stellar systems or black holes, collapse of stellar systems in supernovae to form either neutron stars or black holes, the coalescence of binary stars or neutron stars or black holes or binaries of both stellar systems and black holes, and accretion of stellar systems or small black holes into supermassive black holes at the galactic centers.

The **stochastic waves** are said to be produced by random fluctuations of long durations. The waves are said to be produced by cosmic systems such as; radiating binary stars, deaths of pre-galactic massive stars, vibrations of cosmic strings, and the Big Bang.

1.2.3 Interactions

One of the spear-heading distinctions between electromagnetic waves and the gravitational waves is that, the first ones are oscillations of the electromagnetic field that propagate through space-time. While the latter ones on the other hand, are oscillations of the fabric of space-time itself.

Maxwell’s field equations are said to be linear, since their solutions can be superposed, resulting in the phenomena, that all electromagnetic waves pass through each other without any interaction. On the other hand, Einstein’s field equations are said to be highly non-linear, and their solutions show that, as the waves pass through each other, there will be an emergence of a non-linear interaction through the field equations.

However, whenever two waves of electromagnetic origin pass through each other, they will definitely experience a non-linear interaction between them due to their associated gravitational fields; since, Einstein’s theory provides that; all forms of energy have an associated gravitational field.

1.2.4 Singularities and Horizons

Singularities are said to occur when the mathematical expression that defines and describes the behavior of a continuous function breaks down at some particular point. Following [22], we categorize singularities into three basic types; Quasi-regular, non-scalar curvature, and scalar curvature.

A **scalar curvature singularity** is such that, as the singular point is been approached by some relative observers, some physical quantities diverge, and all observers feel unbounded tidal forces. Examples include the big bang and black holes. While on cosmic landscapes with a **non-scalar curvature singularity**, there is no curvature scalar divergence, yet, some components of the Riemann tensor along an incomplete curve do not tend to finite limits as the singularity is approached. Consequently, all test particles that accrete into this curvature singularity experience infinite tidal forces. However, relative observers can follow geodesics close to this singularity without any effect. On the other hand, in a space-time with **quasi-regular singularity**, the Riemann tensor appears to be completely finite in all reasonable frames. Observers near this singularity, including those that accrete into the singularity itself, do not at any point experience unbounded tidal forces.

However, sometimes, instead of forming singularities in the interaction regions, the impulsive waves form horizons. A horizon in this sense is seen as a smooth, null hyper-surface on which Killing vectors are involved with a one-way membrane [18].

On a general note, these forms of singularities and horizons take the center–stage in discussing any meaningful solutions of colliding plane waves; either electromagnetic plane waves, or gravitational plane waves, or a combination of both. If a test particle is placed on the paths of these two impulsive waves, it will be forced to enter into the region of interaction, following a geodesic that leads to a singularity or a horizon in a finite interval of proper time.

1.3 The Scope

The space-time of colliding plane waves admits two space-like Killing vectors. In the null-coordinates,(u, v), we intend to analyze the behavior of a test particle through geodesic equations. The basic space-time continua or cosmic landscapes such as the Khan-Penrose and Bell-Szekeres will be treated in this work. In the null-coordinates, the interaction region is bounded; so, given the initial conditions, we intend to plot the developments numerically. The time of fall into the singularity will be obtained also, numerically. The prototype space-time for colliding waves is given by

$$dS^2 = 2e^{-M} dudv - e^{-U+V} dx^2 - e^{-U-V} dy^2$$

we do not intend to consider the contribution of the cross polarization of the waves.

As the write-up unfolds, we began by introducing the background that prompted this research, which includes the basic concepts involved. Subsequently, we shall look at the mathematical structure that involves the basic tools and equations of motion regarding the geodesics, Killing vectors, Euler-Lagrange formalism, Newman-Penrose formalism, and Einstein-Maxwell's equations; these will form the second chapter. The third chapter spot-lights the global theoretical structures; Khan-Penrose, Bell-Szekeres and the Ferrari-Ibanez degenerate solutions. These structures provide platforms for discussing the motion of a test particle on the various Cosmic Landscapes that make up the fourth chapter. Finally, we shall summarize and conclude our discussion in the fifth chapter.

Chapter 2

THE MATHEMATICAL STRUCTURE

This chapter intends to provide some mathematical expressions that will play a vital role in our discussions in the subsequent chapters. I often hear my professor and supervisor say affirmatively, as it is acclaimed among the theoretical minds; "Tensor is the language of General Relativity and Cosmology." On this note, therefore, most of the expressions in this work are coded in tensoral notations and connotations.

The chapter begins with the geodesic equations, and ran through; the Killing equations, the Euler-Lagrange formalism, the Newman-Penrose formalism, and the Einstein-Maxwell's field equations.

2.1 The Geodesic Equation

Imagine an inertial observer defined by ξ, cruising steadily on a Cosmic Landscape relative to other inertial observers on the same cosmic landscape or space-time continuum. We express the system by

$$\xi^{\Upsilon} = \xi^{\Upsilon}(x^{\sigma}). \tag{2.1}$$

For constant motion, the acceleration of the system is given by

$$\frac{d^2\xi^{\Upsilon}}{d\tau^2} = 0. \tag{2.2}$$

The geodesic equation that defines the system can be expressed as

$$\frac{d^2x^{\rho}}{\partial\tau^2} + \Gamma^{\rho}_{\sigma\eta}\frac{dx^{\sigma}}{d\tau}\frac{dx^{\eta}}{d\tau} = 0, \tag{2.3}$$

where x^{ρ} are the coordinates, $\Gamma^{\rho}_{\sigma\eta}$ is the Christoffel symbol and τ is the proper time.

2.2 The Killing Equation

The Killing equation that defines the motion of a system on a Cosmic Landscape is given by

$$\frac{\partial \xi_\sigma}{\partial x^\rho} + \frac{\partial \xi_\rho}{\partial x^\sigma} - 2\xi_\eta \Gamma^\eta_{\rho\sigma} = 0, \tag{2.4}$$

or

$$\xi_{\sigma,\rho} + \xi_{\rho,\sigma} - 2\Gamma^\eta_{\rho\sigma}\xi_\eta = 0. \tag{2.5}$$

In terms of covariant derivatives, Eqn. (2.4) takes the form

$$\xi_{\sigma;\rho} + \xi_{\rho;\sigma} = 0\,, \tag{2.6}$$

where, ξ_ρ are the Killing vectors and $\rho = (1,2,3,4) = (x, y, u, v)$.

We define the Killing vectors as

$$\xi_\rho = \partial_\rho. \tag{2.7}$$

2.3 The Euler-Lagrange Formalism

2.3.1 The Euler-Lagrange Equations

Consider a mechanical system defined by the action

$$\mathrm{I} = \int \mathcal{L}(q_i, \dot{q}_i, t)dt, \tag{2.8}$$

where,

$\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i)$, is the Lagrangian function,

q_i = Generalized coordinates,

$\dot{q}_i$ = Generalized velocity,

and

i = degrees of freedom.

The Euler-Lagrange Equations of motion corresponding to the integral, I, is defined by the Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0, \tag{2.9}$$

where, $i = 1,2, \dots n$. By variation Principle, we have that

$$\delta I = \int \delta\, \mathcal{L}(q_i, \dot{q}_i) dt = 0\,. \tag{2.10}$$

2.3.2 The shortest path or geodesic

Now, consider the Riemannian metric element

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{2.11}$$

which defines the motion of a system on a flat space-time of infinitesimal length (ds). To transform our coordinates from the Cartesian to the null (u, v) coordinates, we let

$$x = x(u, v), y = y(u, v), and\ z = z(u, v). \tag{2.12}$$

In the Lagrange formalism, the shortest path or geodesics generally is regarded as the minimum arc length defined by the Lagrangian

$$\mathcal{L} = \int_{\sigma}^{\rho} \sqrt{(dx)^2 + (dy)^2 + (dz)^2} \tag{2.13}$$

where

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \Rightarrow dx^2 = \left(\frac{\partial x}{\partial u}\right)^2 du^2 + 2\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} dudv + \left(\frac{\partial x}{\partial v}\right)^2 dv^2,$$

and

$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv \Rightarrow dy^2 = \left(\frac{\partial y}{\partial u}\right)^2 du^2 + 2\frac{\partial y}{\partial u}\frac{\partial y}{\partial v} dudv + \left(\frac{\partial y}{\partial v}\right)^2 dv^2, \tag{2.14}$$

and

$$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv \Rightarrow dz^2 = \left(\frac{\partial z}{\partial u}\right)^2 du^2 + 2\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} dudv + \left(\frac{\partial z}{\partial v}\right)^2 dv^2.$$

Now, let

$$\mathcal{A} \equiv \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$$

$$\mathcal{B} \equiv \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} \tag{2.15}$$

$$\mathcal{C} \equiv \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2,$$

such that the Lagrangian Eq. (2.13) now takes the form

$$\mathcal{L} = \int_{\sigma}^{\rho} \sqrt{\mathcal{A} + 2\mathcal{B}v' + \mathcal{C}v'^2}\,du. \tag{2.16}$$

Now, we take the derivative of the Lagrangian (2.16) with respect to v and v' such that

$$\frac{d\mathcal{L}}{dv} = \frac{1}{2}(\mathcal{A} + 2\mathcal{B}v' + \mathcal{C}v'^2)^{-\frac{1}{2}}\left(\frac{\partial\mathcal{A}}{\partial v} + 2\frac{\partial\mathcal{B}}{\partial v}v' + \frac{\partial\mathcal{C}}{\partial v}v'^2\right), \tag{2.17}$$

and

$$\frac{d\mathcal{L}}{dv'} = \frac{1}{2}(\mathcal{A} + 2\mathcal{B}v' + \mathcal{C}v'^2)^{-\frac{1}{2}}(2\mathcal{B} + 2\mathcal{C}v'). \tag{2.18}$$

Now, by substituting for Eqs. (2.17) and (2.18) into (2.9) we obtain a new Euler-Lagrange equation of motion given by

$$\frac{d}{du}\left[\frac{\mathcal{B} + \mathcal{C}v'}{\sqrt{\mathcal{A} + 2\mathcal{B}v' + \mathcal{C}v'^2}}\right] - \left[\frac{\left(\frac{\partial\mathcal{A}}{\partial v} + 2\frac{\partial\mathcal{B}}{\partial v}v' + \frac{\partial\mathcal{C}}{\partial v}v'^2\right)}{2\sqrt{\mathcal{A} + 2\mathcal{B}v' + \mathcal{C}v'^2}}\right] = 0. \tag{2.19}$$

2.4 The Newman-Penrose Formalism

Here, we intend to look at a handful properties that will form some relevant concepts for building our theoretical structure in the null coordinate. The formalism is structured on four null vectors; l^ρ, n^ρ, m^ρ and $\overline{m}^\rho$, where $\bar{x}$ denotes complex conjugate. Here

$$l = l_\rho dx^\rho, n = n_\rho dx^\rho, m = m_\rho dx^\rho, \tag{2.20}$$

both l^ρ, and n^ρ are real, while m^ρ is complex. Pending on an event, we define l^ρ and n^ρ as the ongoing and the outgoing null normals respectively, while

m^ρ and $\overline{m}^\rho$ assume the role of tangential null vectors. We adopt the two sets of signatures and normalization curvatures

$$(+,-,-,-), \text{ for } l^\rho n_\rho = 1, \text{and } m^\rho \overline{m}_\rho = -1, \tag{2.21}$$

and

$$(-,+,+,+), \text{ for } l^\rho n_\rho = -1, \text{ and } m^\rho \overline{m}_\rho = 1. \tag{2.22}$$

The null vectors satisfy the following conditions, for the signature (+2)

$$\begin{gathered} l^\rho l_\rho = n^\rho n_\rho = m^\rho m_\rho = 0, \\ l^\rho m_\rho = n^\rho m_\rho = 0, \\ l^\rho n_\rho = -1, \\ m^\rho \overline{m}_\rho = +1, \\ l^\rho n^\rho = 1, \\ m^\rho \overline{m}^\rho = -1. \end{gathered} \tag{2.23}$$

For the time-like and space-like unit vectors, $\left(t^\rho, s^\rho, e_\theta^\rho, e_\phi^\rho\right)$, we have

$$\begin{gathered} l^\rho = \frac{1}{\sqrt{2}}(t^\rho + s^\rho), \\ n^\rho = \frac{1}{\sqrt{2}}(t^\rho - s^\rho), \\ m^\rho = \frac{1}{\sqrt{2}}\left(e_\theta^\rho + i e_\phi^\rho\right), \\ t^\rho t_\rho = -1, \\ s^\rho s_\rho = +1, \\ e_\theta^\rho e_{\theta\rho} = +1, \\ e_\phi^\rho e_{\phi\rho} = +1. \end{gathered} \tag{2.24}$$

The global metric in terms of the null vectors now takes the form

$$g_{\rho\sigma} = -l_\rho n_\sigma - n_\rho l_\sigma + m_\rho \overline{m}_\sigma + \overline{m}_\rho m_\sigma \tag{2.25}$$

or

$$g^{\rho\sigma} = -l^{\rho}n^{\sigma} - n^{\rho}l^{\sigma} + m^{\rho}\bar{m}^{\sigma} + \bar{m}^{\rho}m^{\sigma}. \tag{2.26}$$

2.5 The Einstein-Maxwell's Equations

We define the scale-invariant quantities for the electromagnetic waves as

$$\begin{aligned} \phi_0^{\circ} &= \phi_0 \mathrm{B}^{-1}, \\ \phi_1^{\circ} &= \phi_1 (\mathrm{AB})^{-\frac{1}{2}}, \\ \phi_2^{\circ} &= \phi_2 \mathrm{A}^{-1}, \end{aligned} \tag{2.27}$$

where ϕ_n° are scale-invariant quantities of the electromagnetic waves. Using the Szekeres line element [15] defined by

$$ds^2 = 2e^{-M}dudv - e^{-U}(e^{V}coshWdx^2 - 2sinhWdxdy + e^{-V}coshWdy^2), \tag{2.28}$$

and by following some transformations [15, 22, 24], we obtain the Maxwell's Equations as

$$\phi_{1,v}^{\circ} = \left(2\rho^{\circ} - \tfrac{1}{2}M_{,v}\right)\phi_1^{\circ}, \tag{2.29}$$

$$\phi_{2,v}^{\circ} = -\lambda^{\circ}\phi_0 + 4\alpha^{\circ}\phi_1^{\circ} + (\rho^{\circ} - iE^{\circ})\phi_2^{\circ}, \tag{2.30}$$

$$\phi_{0,u}^{\circ} = -(\mu^{\circ} - iG^{\circ})\phi_0^{\circ} - 4\bar{\alpha}^{\circ}\phi_1^{\circ} + \delta^{\circ}\phi_2^{\circ}, \tag{2.31}$$

$$\phi_{1,u}^{\circ} = -\left(2\mu^{\circ} - \tfrac{1}{2}M_{,u}\right)\phi_1^{\circ}, \tag{2.32}$$

$$\phi_{2,v}^{\circ} = \tfrac{1}{2}(U_v + iV_v sinhW)\phi_2^{\circ} - \tfrac{1}{2}(iW_u + V_u coshW)\phi_0^{\circ}, \tag{3.33}$$

$$\phi_{0,u}^{\circ} = \tfrac{1}{2}(U_u - iV_u sinhW)\phi_0^{\circ} + \tfrac{1}{2}(iW_v - V_v coshW)\phi_2^{\circ}, \tag{3.34}$$

where

$$\phi_1 = \frac{1}{2}F_{\rho\sigma}(l^{\rho}n^{\sigma} + m^{\rho}\bar{m}^{\sigma}) = 0 \tag{2.35}$$

throughout the space-time continuum.

Also, by Following [15, 22, 24, 25], the Einstein's field equations in component form can be outlined as

$$U_{uv} = U_u U_v, \tag{2.36}$$

$$2U_{vv} = U_v^2 + W_v^2 + V_v^2 cosh^2 W - 2U_v M_v + 4\phi_0^\circ \phi_0^\circ, \tag{2.37}$$

$$2U_{uu} = U_u^2 + W_u^2 + V_u^2 cosh^2 W - 2U_u M_u + 4\phi_2^\circ \phi_2^\circ, \tag{2.38}$$

$$2V_{uv} = U_u V_v + U_v V_u - 2(V_u W_v + V_v W_u) tanhW + 2(\phi_0^\circ \phi_2^\circ + \phi_2^\circ \phi_0^\circ) sechW, \tag{2.39}$$

$$2W_{uv} = U_u W_v + U_v W_u + 2V_u V_v sinhW coshW + 2i(\phi_0^\circ \phi_2^\circ - \phi_2^\circ \phi_0^\circ) \tag{2.40}$$

and

$$2M_{uv} = U_u V_v + W_u W_v + V_u V_v cosh^2 W. \tag{2.41}$$

Finally, following [22], we obtain the scale-invariant components of the Weyl tensor as

$$\Psi_0^\circ = -\frac{1}{2}[(V_{vv} - U_v V_v + M_v V_v) coshW + 2V_v W_v sinhW]$$

$$+\frac{1}{2} i(W_{vv} - U_v W_v + M_v W_v - V_v^2 coshW sinhW) \tag{2.42}$$

$$\Psi_1^\circ = 0 \tag{2.43}$$

$$\Psi_2^\circ = \frac{1}{2} M_{uv} - \frac{1}{4} i(V_u W_v - V_v W_u) coshW \tag{2.44}$$

$$\Psi_3^\circ = 0 \tag{2.45}$$

$$\Psi_4^\circ = -\frac{1}{2}[(V_{uu} - U_u V_u + M_u V_u) coshW + 2V_u W_u sinhW]$$

$$-\frac{1}{2} i(W_{uu} - U_u W_u + M_u W_u - V_u^2 coshW sinhW). \tag{2.46}$$

It is important to note at this juncture, that whenever the gravitational waves Ψ_0 and Ψ_4 interact, a new Weyl component Ψ_2 emerges.

Chapter 3

THE THEORETICAL GLOBAL STRUCTURES

This chapter embraces the mathematical tools and concepts developed in the preceding chapters to build-up some global structures or space-time continua that will serve as frameworks, within which our subsequent discussion on the particles' motion can be explicitly and conveniently done. Here, we begin with the Khan-Penrose space-time continuum, which I suppose, is the simplest structure to construct so far. Subsequently, we shall discuss the Bell- Szekeres global structure, and then cap-it-up by looking at the Ferrari-Ibanez Degenerate solutions.

3.1 The Khan-Penrose Global Structure

In this structure [14], we consider two approaching plane impulsive gravitational waves by using two metrics to describe them. Firstly, we shall use the Brinkmann-Penrose-Takeno line element [22], to discuss the approaching waves on the flat background. Secondly, we shall use the Rosen's transformed metric [14], to discuss the interactions of the two impulsive waves.

3.1.1 The approaching waves

Here, we shall consider two impulsive waves approaching from the opposing sides of the space-time. We define the approaching wave from the left side of the space-time in figure (3.1) by the line element

$$ds^2 = 2dudr + \delta(u)(X^2 - Y^2)du^2 - dX^2 - dY^2 \tag{3.1}$$

where, $\delta(u)$ is the impulsive wave component, u is the null coordinate and We make $u = 0$ on the hyper-surface.

In the same vein, we define the second wave approaching from the right side by the line element

$$ds^2 = 2dvd\rho + \delta(v)(X^2 - Y^2)dv^2 - dX^2 - dY^2 \tag{3.2}$$

where, $\delta(v)$ is the wave component, v is the null coordinate on the hyper-surface where $v = 0$.

For the impulsive wave approaching from the left, we carry out the following transformations

$$u = u,$$

$$r = v - \frac{1}{2}\Theta(u)(1-u)x^2 + \frac{1}{2}\Theta(u)(1+u)y^2,$$

$$X = \big(1 - u\Theta(u)\big)x\,, \tag{3.3}$$

and

$$Y = \big(1 + u\Theta(u)\big)y,$$

where, $\Theta(u)$, is the Heaviside step function. Putting Eq. (3.3) into (3.1) we obtain

$$ds^2 = 2dudv - \big(1 - u\Theta(u)\big)^2 dx^2 - \big(1 + u\Theta(u)\big)^2 dy^2. \tag{3.4}$$

The component describing the gravitational wave here is given by

$$\Psi_4 = \delta(u). \tag{3.5}$$

In the same vein, we wish to carry out a similar transformations for the opposing wave approaching from the right side by letting

$$v = v,$$

$$\rho = u - \frac{1}{2}\Theta(v)(1-v)x^2 + \frac{1}{2}\Theta(v)(1+v)y^2,$$

$$X = \big(1 - v\Theta(v)\big)x, \tag{3.6}$$

and

$$Y = \big(1 + v\Theta(v)\big)y.$$

Putting Eq. (3.6) into (3.2) gives

$$ds^2 = 2dudv - \left(1 - v\Theta(v)\right)^2 dx^2 - \left(1 + v\Theta(v)\right)^2 dy^2. \qquad (3.7)$$

Now, we let the component describing this gravitational wave be defined by

$$\Psi_0 = \delta(v). \qquad (3.8)$$

3.1.2 Regional description

We now split the space-time into four regions and impose some boundary conditions peculiar to the regions that describe our global structure (see Figure. 3.1). Region I is characterized by a flat background, with $u < 0 \; and \; v < 0$ and the line elements in (3.4) and (3.7) now take the form

$$ds^2 = 2dudv - dx^2 - dy^2. \qquad (3.9)$$

Region II is a single u-wave with boundary conditions $u \geq 0\,, v < 0$ and $\Theta(u) = 1$. Here, the line element (3.4) takes the form

$$ds^2 = 2dudv - (1-u)^2 dx^2 - (1+u)^2 dy^2. \qquad (3.10)$$

Region III is a single v-wave with the boundary conditions $v \geq 0 \; and \; u < 0$ and $\Theta(v) = 1$. By imposing these conditions, the line element Eq. (3.7) now takes the form

$$ds^2 = 2dudv - (1-v)^2 dx^2 - (1+v)^2 dy^2. \qquad (3.11)$$

Region IV is the interaction region with the boundary conditions $u \geq 0 \; and \; v \geq 0$. Here, we shall use the Rosen's metric element [14] given by

$$ds^2 = \frac{2t^3 dudv}{rw(pq + rw)^2} - t^2 \left(\frac{r+q}{r-q}\right)\left(\frac{w+p}{w-p}\right) dx^2 - t^2 \left(\frac{r-q}{r+q}\right)\left(\frac{w-p}{w+p}\right) dy^2. \; (3.12)$$

Now, we wish to transform this metric element by letting

$$\Theta(u) = 1,$$

$$\Theta(v) = 1,$$

$$p = u\Theta(u) = u,$$

$$q = v\Theta(v) = v,$$

$$r^2 = 1 - p^2 \Rightarrow (1-p^2)^{\frac{1}{2}} = (1-u^2)^{\frac{1}{2}},$$

$$w^2 = 1 - q^2 \Rightarrow w = (1 - q^2)^{\frac{1}{2}} = (1 - v^2)^{\frac{1}{2}}, \tag{3.13}$$

$$t^2 = 1 - p^2 - q^2 = r^2 - q^2 = w^2 - p^2,$$

and

$$t^2 = 1 - u^2 - v^2 \Rightarrow t = (1 - u^2 - v^2)^{\frac{1}{2}}.$$

Using this transformations, the line element (3.12) now takes the form

$$ds^2 = 2\frac{(1 - u^2 - v^2)^{\frac{3}{2}}}{\sqrt{1-u^2}\sqrt{1-v^2}\left(uv + \sqrt{1-u^2}\sqrt{1-v^2}\right)^2}dudv$$

$$-(1 - u^2 - v^2)\left[\frac{\left(1 - u\sqrt{1-v^2} - v\sqrt{1-u^2}\right)}{\left(1 + u\sqrt{1-v^2} + v\sqrt{1-u^2}\right)}dx^2\right.$$

$$\left.+\frac{\left(1 + u\sqrt{1-v^2} + v\sqrt{1-u^2}\right)}{\left(1 - u\sqrt{1-v^2} - v\sqrt{1-u^2}\right)}dy^2\right]. \tag{3.14}$$

This line element becomes the basic metric element valid for defining and describing the geodesics of particles on the Khan-Penrose global structure (see Figure 3.1).

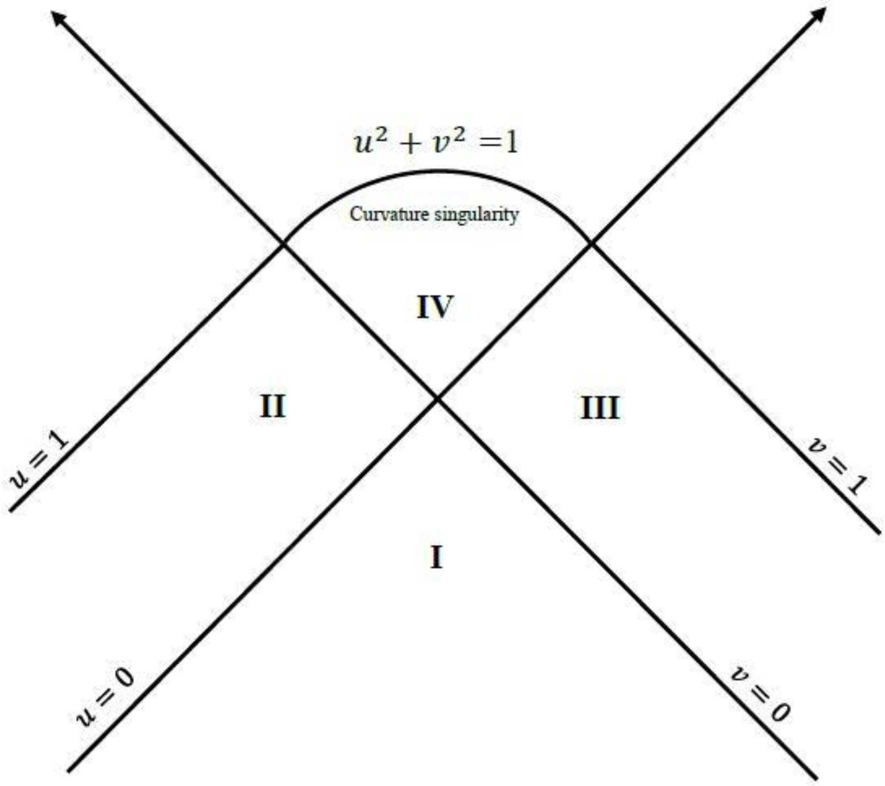

Figure 3.1: The Khan-Penrose Global structure for colliding impulsive gravitational waves in the null(u, v) coordinates. Region I is flat space-time, regions II and III are the single-waves, while region IV is the interaction region.

3.2 The Bell-Szekeres Global Structure

In this structure [15, 22], we x-ray a scenario that describes the collision and subsequent interaction of two step electromagnetic plane waves. We shall split the space-time into four regions as we did in Figure (3.1) as we observe the two impulsive waves from the opposing sides of the space-time. The approaching wave in region II is described by a line element in Brinkmann metric form by

$$ds^2 = 2dudr + a^2\Theta(u)(X^2 + Y^2)du^2 - dX^2 - dY^2 \qquad (3.15)$$

where

$$\Phi_{22} = a^2\Theta(u). \qquad (3.16)$$

The opposing wave in region II is described by the line element in Brinkmann metric form by

$$ds^2 = 2dvd\rho + b^2\Theta(v)(X^2 + Y^2)dv^2 - dX^2 - dY^2 \quad (3.17)$$

where

$$\Phi_{00} = b^2\Theta(v). \quad (3.18)$$

Now, we shall transform our line element such that

$$X = xcosau\theta \;\Rightarrow\; x = \frac{X}{cosau\theta},$$

$$Y = ycosau\theta \;\Rightarrow\; x = \frac{Y}{cosau\theta},$$

and

$$r = v - \frac{1}{2}[cosau\theta sinau\theta(x^2 + y^2)]. \quad (3.19)$$

By imposing some boundary conditions on the various regions, we know that region I is a flat space-time with $u < 0, v < 0$. The line elements in (3.15) and (3.17) now take the form

$$ds^2 = 2dudv - dx^2 - dy^2. \quad (3.20)$$

Region II, is a single u-wave with boundary conditions $u \geq 0, v < 0$. By imposing these conditions, the line element (3.15) now takes the form

$$ds^2 = 2dudv - cos^2au(dx^2 + dy^2). \quad (3.21)$$

Region III is a single v-wave with the boundary conditions $u < 0, v \geq 0$. By imposing these conditions on the line element (3.17) we obtain

$$ds^2 = 2dudv - cos^2bv(dx^2 + dy^2). \quad (3.22)$$

Region IV is considered here as the interaction region, therefore, we intend at this juncture to impose some boundary conditions that will determine the properties of the global structure. We shall begin by integrating Eq. (2.36) to obtain

$$U = -\log\big(f(u) + g(v)\big),$$

$$e^{-U} = e^{\log\left(f(u)+g(v)\right)},$$

therefore

$$e^{-U} = f(u) + g(v)\,. \tag{3.23}$$

Now, we let

$$f = \frac{1}{2} - sin^2 au,$$

$$g = \frac{1}{2} - sin^2 bv\,. \tag{3.24}$$

At, $u = 0, v \geq 0, V = W = M = 0,\ and\ \phi_0 = b$, we find from Eq. (3.24) that

$$f = \frac{1}{2} - sin^2 au, u = 0\ \Rightarrow f = \frac{1}{2}\,,$$

and

$$g = \frac{1}{2} - sin^2 bv, v \neq 0 \Rightarrow g = \frac{1}{2} + cos^2 bv - 1, cos^2 bv - 1 = -sin^2 bv,$$

therefore

$$g = -\frac{1}{2} + cos^2 bv. \tag{3.25}$$

Putting Eq. (3.25) into (3.23) yields

$$U = -\log\left(\frac{1}{2} - \frac{1}{2} + cos^2 bv\right) = -\log(cos^2 bv),$$

or

$$U = -2\log cos^2 bv. \tag{3.26}$$

At $v = 0, u \geq 0, V = W = M = 0\ \ and\ \phi_2 = a$; Eq. (3.24) shows that

$$g = \frac{1}{2} - sin^2 bv, v = 0\ \Rightarrow g = \frac{1}{2},$$

$$f = \frac{1}{2} - sin^2 au, u \neq 0 \Rightarrow g = \frac{1}{2} + cos^2 au - 1, cos^2 au - 1 = -sin^2 au,$$

hence

$$f = -\frac{1}{2} + cos^2 au. \tag{3.27}$$

Putting Eq. (3.27) into (3.23) yields

$$U = -\log\left(-\frac{1}{2} + \frac{1}{2} + cos^2 au\right) = -\log(cos^2 au),$$

or

$$U = -2\log cos^2 au. \qquad (3.28)$$

Now, we let, $W = M = 0$, $\phi_2 = a$, $\phi_0 = b$; from Eqs. (2.29-2.41) we obtain

$$U = -\log\cos(au - bv) - \log\cos(au + bv) \qquad (3.29)$$

and

$$V = \log\cos(au - bv) - \log\cos(au + bv). \qquad (3.30)$$

Therefore, the metric of the interaction region (IV) now takes the form

$$ds^2 = 2dudv - cos^2(au - bv)dx^2 - cos^2(au + bv)dy^2. \qquad (3.31)$$

This is the basic line element (3.31) valid for defining and describing the geodesics of any test particle on the Bell-Szekeres global structure.

3.3 The Ferrari-Ibanez Degenerate Solutions

3.3.1 The metric description

This is a type D class of solutions of Einstein's problems, where two space-like Killing vectors play a vital role in the formation of Cauchy horizons and singularities, in respect to the boundary conditions. The basic idea here is to metal-cast a Schwarzschild black-hole-like solution into the mold of Khan-Penrose Global structure, with the sole aim of describing the nature of the Cauchy horizons and the singularities formed in the interaction region; giving rise to the two degenerate solutions.

Here, the line element that defines this global structure [22, 26, 27], is given by

$$\begin{aligned} ds^2 = {} & \zeta(1 + 2\rho sin\psi + sin^2\psi)(d\psi^2 - d\lambda^2) \\ & - \left(\frac{1 - sin^2\psi}{1 + 2\rho sin\psi + sin^2\psi}\right)(dx - 2\eta sin\lambda y)^2 \\ & - cos^2\lambda(1 + 2\rho sin\psi + sin^2\psi)dy^2, \end{aligned} \qquad (3.32)$$

where

$$X^{\mu} = X^{\mu}(\psi, \lambda), \tag{3.33}$$

while, $\zeta,\ \rho\ and\ \eta$ are constants; satisfying the condition that ρ^2+$\eta^2 = 1$. Now, we let $\zeta = 1, \eta = 0, and\ \rho = \pm 1$, such that the line element (3.32) reduces to

$$ds^2 = (1 + \rho sin\psi)^2(d\psi^2 - d\lambda^2) - \left(\frac{1 - \rho sin\psi}{1 + \rho sin\psi}\right)dx^2$$

$$- cos^2\lambda(1 + \rho sin\psi)^2 dy^2. \tag{3.34}$$

We now carry out some transformations by changing our coordinates. Here, we let

$$\psi = t,$$

$$\lambda = z\,, \tag{3.35}$$

$$X^{\mu} = X^{\mu}(t, z).$$

In the light of this transformation, the line element (3.34) can be expresses as

$$ds^2 = (1 + \rho sint)^2(dt^2 - dz^2) - \left(\frac{1 - \rho sint}{1 + \rho sint}\right)dx^2$$

$$- cos^2 z(1 + \rho sint)^2 dy^2. \tag{3.36}$$

Now, we wish to change the metric signature by invoking the properties of Eqs. (2.21) and (2.22) such that

$$g_{\mu\nu} = (+, -, -, -) \rightarrow g_{\mu\nu} = (-, +, +, +). \tag{3.37}$$

At this point, the line element (3.36) takes the form

$$ds^2 = -(1 + \rho sint)^2(dt^2 - dz^2) + \left(\frac{1 - \rho sint}{1 + \rho sint}\right)dx^2 + cos^2 z(1 + \rho sint)^2 dy^2,$$

and

$$ds^2 = (1 + \rho sint)^2(dz^2 - dt^2) + \left(\frac{1 - \rho sint}{1 + \rho sint}\right)dx^2$$

$$+ cos^2 z(1 + \rho sint)^2 dy^2. \tag{3.38}$$

This metric element defined by Eqn. (3.38) is valid for the formation of Cauchy horizons when $\rho = 1$, and for the formation of singularities when $\rho = -1$.

3.3.2 Metric transformation

Following the Khan-Penrose global structure discussed in section (3.1), region (IV) becomes the interaction region, where horizons and singularities are formed. In order to metal-cast our line element to be valid for defining and imposing the properties of the Khan-Penrose global structure, we wish to carry out the following transformations by letting

$$
\begin{aligned}
u &= \left(\frac{t-z}{2}\right),\\
v &= \left(\frac{t+z}{2}\right),\\
z &= v-u, \qquad\qquad (3.39)\\
t &= u+v,\\
dt &= du+dv,\\
dz &= dv-du,
\end{aligned}
$$

and

$$(dz^2-dt^2) = (dv^2-2dudv+du^2)-(\,du^2+2dudv+dv^2) = -4dudv,$$

therefore

$$(dz^2-dt^2) = -4dudv. \qquad (3.40)$$

In the light of these transformations in Eqs. (3.39) and (3.40), our line element (3.38) now takes the form

$$ds^2 = -4[1+\rho\sin(u+v)]^2dudv + \left[\frac{1-\rho sin(u+v)}{1+\rho sin(u+v)}\right]dx^2$$
$$+\,cos^2(u-v)[1+\rho sin(u+v)]^2dy^2. \qquad (3.41)$$

In order to completely transform the line element (3.41) suitable for the Khan-Penrose structure, we now define the Heaviside step function as function of u and v such that

$$\Theta = \Theta(u)\ and\ \ \Theta = \Theta(v). \qquad (3.42)$$

Now, we let $u \to u\Theta(u)$ and $v \to v\Theta(v)$, such that the line element (3.41) takes the form

$$ds^2 = -4\left[1 + \rho \sin\left(u\Theta(u) + v\Theta(v)\right)\right]^2 d\left(u\Theta(u)\right) d\left(v\Theta(v)\right)$$

$$+ \left[\frac{1 - \rho sin\left(u\Theta(u) + v\Theta(v)\right)}{1 + \rho sin\left(u\Theta(u) + v\Theta(v)\right)}\right] dx^2$$

$$+ cos^2\left(u\Theta(u) - v\Theta(v)\right)\left[1 + \rho sin\left(u\Theta(u) + v\Theta(v)\right)\right]^2 dy^2. \quad (3.43)$$

3.3.3 Regional description

Now, we shall split the space-time continuum into four regions (see Figure 3.2) as we impose some boundary conditions on the line element (3.43). Region I is a flat space-time with $u < 0, v < 0$. Region II is a single u-wave space-time with $0 \leq u < \frac{\pi}{2}, v < 0$. Region III is a single v-wave space-time with $u < 0, 0 \leq v < \frac{\pi}{2}$. Finally, Region IV becomes our interaction region with $0 \leq u, 0 \leq v, u + v < \frac{\pi}{2}$.

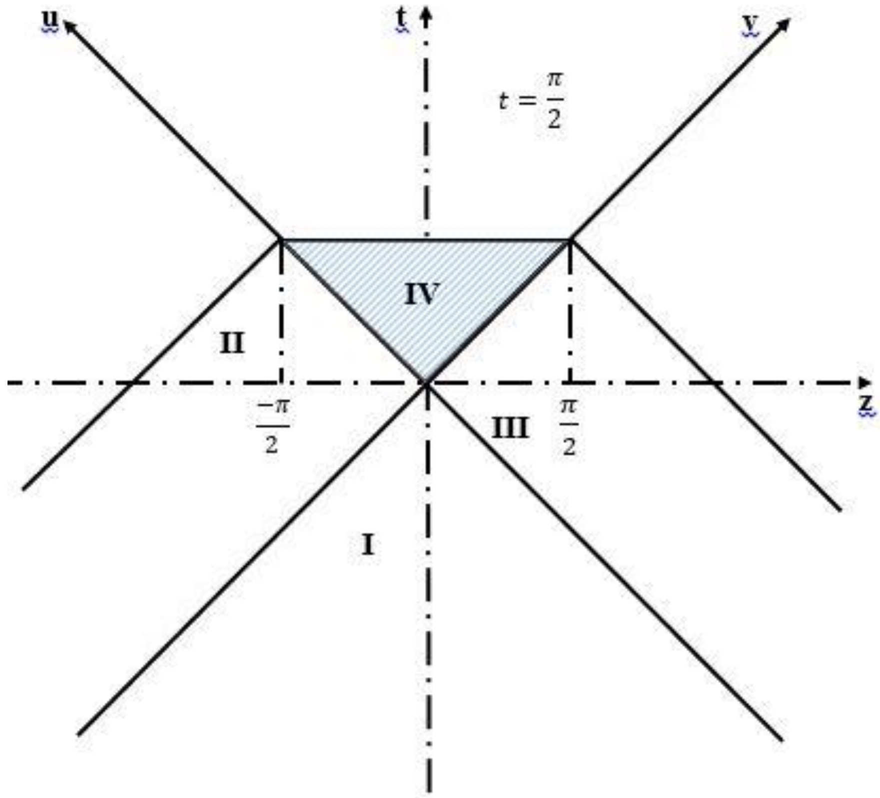

Figure 3.2: The Ferrari-Ibanez Degenerate Global Structure in the null (u, v) coordinates for two impulsive waves. Region I is a flat space-time, Regions II and III are the single wave space-times, while Region IV is the interaction region.

Chapter 4

TIME-LIKE GEODESICS

Here, we spotlight and discuss the motion of a test particle defined by the line element

$$ds^2 = 2e^{-M}dudv - e^{-U+V}dx^2 - e^{-U-V}dy^2. \tag{4.1}$$

We intend to spot-light the prototype space-time element and the particle's motion from the stand points of the two Global structures of colliding gravitational plane waves discussed in sections (3.1) and (3.2); the Khan-Penrose and the Bell-Szekeres Cosmic Landscapes or space-time continua. Subsequently, we shall have a close look at the particle's motion on a Ferrari-Ibanez degenerate Cosmic Landscape.

4.1 Geodesics on the Khan-Penrose Cosmic Landscape

In this section, we wish to spotlight our prototype line element (4.1) on the planform of the Khan-Penrose Cosmic Landscape by deriving suitable equations that will define and describe the motion of our test particle within the confines of the global structure. We shall consider and cross-examine the global structure using the lensing power of two sets of twin-coordinate systems; the null $(\boldsymbol{u}, \boldsymbol{v})$ coordinates and the $(\boldsymbol{x}, \boldsymbol{y})$ coordinates respectively.

4.1.1 Khan-Penrose in $(\boldsymbol{u}, \boldsymbol{v})$ null coordinates

Here we aim at deriving the Equation of motion of the test particle in the null $(\boldsymbol{u}, \boldsymbol{v})$ coordinates. Looking closely at our line element (4.1), it is clear that our Lagrangian can be defined in this context as

$$\mathcal{L} = [2e^{-M}\dot{u}\dot{v} - e^{-U}(e^{V}\dot{x}^2 + e^{-V}\dot{y}^2)]^{\frac{1}{2}}. \tag{4.2}$$

Recall that in Eqs. (2.8) and (2.9), we showed the relationship between the line element and the Lagrangian of a mechanical system with respect to the variation principle, where

$$\int ds = \int \mathcal{L} d\tau$$

and

$$\delta \int ds = \delta \int \mathcal{L} d\tau = 0\,. \tag{4.3}$$

Now, putting Eqs. (4.2) into (4.3) we obtain

$$\delta \int [2e^{-M}\dot{u}\dot{v} - e^{-U}(e^{V}\dot{x}^2 + e^{-V}\dot{y}^2)]^{\frac{1}{2}} d\tau = 0 \tag{4.4}$$

where, v is a function of u ; $v = v(u)$. We now express Eq. (4.4) in terms of u as

$$\delta \int [2e^{-M}v' - e^{-U}(e^{V}x'^2 + e^{-V}y'^2)]^{\frac{1}{2}} du = 0. \tag{4.5}$$

Herein, $' \equiv \frac{d}{du}$ and u is not an affine parameter. From Eq. (4.5), it is clear that our Lagrangian now takes the form

$$\mathcal{L} = [2e^{-M}v' - e^{-U}(e^{V}x'^2 + e^{-V}y'^2)]^{\frac{1}{2}}. \tag{4.6}$$

By imposing Eqs. (2.17) and (2.18), on the Lagrangian (4.6), we obtain

$$\frac{\partial \mathcal{L}}{\partial x'} = -\frac{1}{\mathcal{L}} e^{-U+V} x' = A = \text{constant} \tag{4.7}$$

and

$$\frac{\partial \mathcal{L}}{\partial y'} = -\frac{1}{\mathcal{L}} e^{-U-V} y' = B = c\text{onstant} \tag{4.8}$$

where, A and B are constants.

Now, looking at Eq. (4.7) closely, we see that

$$-\frac{1}{\mathcal{L}} e^{-U+V} x' = A,$$

$$-e^{-U+V} x' = A\mathcal{L},$$

therefore

$$(e^{-U+V}x')^2 = A^2\mathcal{L}^{\,2}. \tag{4.9}$$

Putting our Lagrangian (4.6) into Eq. (4.9), we obtain

$$(e^{-U+V}x')^2 = A^2[2e^{-M}v' - e^{-U+V}x'^2 - e^{-U-V}y'^2]$$

$$\therefore\; e^{-2U+2V}x'^2 = A^2[2e^{-M}v' - e^{-U+V}x'^2 - e^{-U-V}y'^2]. \tag{4.10}$$

In the same vein, looking at Eq. (4.8), we see that

$$-\frac{1}{\mathcal{L}}e^{-U-V}y' = B$$

$$-e^{-U-V}y' = B\mathcal{L}$$

$$\therefore\; (e^{-U-V}y')^2 = B^2\mathcal{L}^{\,2}. \tag{4.11}$$

Putting our Lagrangian (4.6) into Eq. (4.1) shows that

$$(e^{-U-V}y')^2 = B^2[2e^{-M}v' - e^{-U+V}x'^2 - e^{-U-V}y'^2]$$

$$\therefore\; e^{-2U-2V}y'^2 = B^2[2e^{-M}v' - e^{-U+V}x'^2 - e^{-U-V}y'^2]. \tag{4.12}$$

At this juncture, we can solve for x'^2 and y'^2 from Eqs. (4.10) and (4.12), and by doing that we obtain

$$x'^2 = \frac{2A^2e^{-M+U-2V}}{e^{-U} + A^2e^{-V} + B^2e^{V}}v' \tag{4.13}$$

and

$$y'^2 = \frac{2B^2e^{-M+U+2V}}{e^{-U} + A^2e^{-V} + B^2e^{V}}v'. \tag{4.14}$$

Now, substituting for x'^2 and y'^2 as expressed in Eqs. (4.13) and (4.14), our Lagrangian defined in (4.6) now takes the form

$$\mathcal{L} = \Bigg[2e^{-M}v' - e^{-U}\Bigg(e^{V}\frac{2A^2e^{-M+U-2V}}{e^{-U} + A^2e^{-V} + B^2e^{V}}v'$$

$$+ e^{-V}\frac{2B^2e^{-M+U+2V}}{e^{-U} + A^2e^{-V} + B^2e^{V}}v'\Bigg)\Bigg]^{\frac{1}{2}}. \tag{4.15}$$

By expanding the Lagrangian (4.15), we obtain

$$\mathcal{L} = \left(2e^{-M}v' - \frac{2A^2e^{-M-V}}{e^{-U} + A^2e^{-V} + B^2e^{V}}v' - \frac{2B^2e^{-M+V}}{e^{-U} + A^2e^{-V} + B^2e^{V}}v'\right)^{\frac{1}{2}}. \quad (4.16)$$

We now simplify the Lagrangian (4.16) to obtain

$$\mathcal{L} = \left(\frac{2e^{-M-U}v' + 2A^2e^{-M-V}v' + 2B^2e^{-M+V}v' - 2A^2e^{-M-V}v' - 2B^2e^{-M+V}v'}{e^{-U} + A^2e^{-V} + B^2e^{V}}\right)^{\frac{1}{2}}.$$

Hence,

$$\mathcal{L} = \left(\frac{2e^{-M-U}v'}{e^{-U} + A^2e^{-V} + B^2e^{V}}\right)^{\frac{1}{2}}. \quad (4.17)$$

This is our Lagrangian in the (u, v) coordinates. We now wish to spotlight our Lagrangian (4.17) in terms of Eqs. (2.8) and (2.19), such that

$$I = \int \mathcal{L}du = \int \left(\frac{2e^{-M-U}v'}{e^{-U} + A^2e^{-V} + B^2e^{V}}\right)^{\frac{1}{2}} du. \quad (4.18)$$

We now define a function $f(u, v)$ such that

$$f = \left(\frac{2e^{-M-U}v'}{e^{-U} + A^2e^{-V} + B^2e^{V}}\right)^{\frac{1}{2}}, \quad (4.19)$$

where A and B are arbitrary constants. We can express Eq. (4.19) as

$$f = \left(\frac{2e^{-M-U}v'}{e^{-U} + A^2e^{-V} + B^2e^{V}}\right)^{\frac{1}{2}}. \quad (4.20)$$

We now define our action (4.18) in terms of this function as

$$I = \int f(u, v)\sqrt{v'}du. \quad (4.21)$$

Here, $v' = \frac{dv}{du}$, $v = v(u)$ and our Lagrangian is given

$$\mathcal{L} = f(u, v)\sqrt{v'}. \quad (4.22)$$

We now impose Eq. (2.9) on the Lagrangian (4.22) to obtain

$$\frac{d}{du}\left(\frac{\partial \mathcal{L}}{\partial v'}\right) = \frac{\partial \mathcal{L}}{\partial v}, \quad (4.23)$$

where

$$\frac{\partial \mathcal{L}}{\partial v} = f_v \sqrt{v'} \tag{4.24}$$

and

$$\frac{d}{du}\left(\frac{\partial \mathcal{L}}{\partial v'}\right) = \frac{d}{du}\left[\frac{f}{2\sqrt{v'}}\right] = \frac{1}{2\sqrt{v'}}(f_u + v' f_v) - \frac{f}{4}\frac{v''}{(v')^{\frac{3}{2}}}\,. \tag{4.25}$$

Eq. (4.23) now takes the form

$$\frac{1}{2\sqrt{v'}}(f_u + v' f_v) - \frac{f}{4}\frac{v''}{(v')^{\frac{3}{2}}} = f_v\sqrt{v'}. \tag{4.26}$$

We now multiply Eq. (4.26) by $2\sqrt{v'}$ to obtain

$$f_u + v' f_v - \frac{f}{2}\frac{v''}{v'} = 2v' f_v, \tag{4.27}$$

such that

$$f_u = \frac{f}{2}\frac{v''}{v'} + v' f_v, \qquad \Rightarrow \frac{f v''}{2v'} = f_u - v' f_v \tag{4.28}$$

$$\therefore \quad v'' = \frac{2v'}{f}\left[f_u - v' f_v\right] \tag{4.29}$$

where

$$\begin{aligned} v &= v(u) \\ v' &= \frac{dv}{du} \\ v'' &= \frac{d^2 v}{du^2} \\ f &= f(u, v) \end{aligned} \tag{4.30}$$

At this point, it is clear that Eq. (4.29) is the equation that defines and describes the geodesic motion of the particle on the Khan-Penrose Cosmic Landscape or space-time continuum in the null (u, v) coordinates.

4.1.2 Khan-Penrose in (x, y)coordinates

Here we intend to derive the equation of motion that defines and describes the geodesics of our test particle as it cruises on the Khan-Penrose Cosmic Landscape in the in $(\boldsymbol{x}, \boldsymbol{y})$ coordinates. To achieve this task, we wish to carry out certain transformations that will guarantee our safe ride to the desired equation of motion in the require coordinates $(\boldsymbol{x}, \boldsymbol{y})$. First of all, we let

$$\tau = u\sqrt{1 - v^2} + v\sqrt{1 - u^2}\,,$$

and

$$\sigma = u\sqrt{1 - v^2} - v\sqrt{1 - u^2} \tag{4.31}$$

Such that our line element (4.1) transforms into

$$ds^2 = (1 - \tau^2)^{-\frac{1}{4}}(1 - \sigma^2)^{-\frac{1}{4}}\, d\tau^2 - (1 - \tau^2)^{\frac{3}{4}}(1 - \sigma^2)^{-\frac{5}{4}}\, d\sigma^2, \tag{4.32}$$

and

$$\int ds = \int \left[(1 - \tau^2)^{-\frac{1}{4}}(1 - \sigma^2)^{-\frac{1}{4}} - (1 - \tau^2)^{\frac{3}{4}}(1 - \sigma^2)^{-\frac{5}{4}}\, \sigma'^2\right]^{\frac{1}{2}} d\tau \tag{4.33}$$

where, $\sigma' = \frac{d\sigma}{d\tau}$. We shall now change our coordinates by carrying out the following transformations, let

$$\tau = sinx,$$

$$\sigma = siny,$$

$$\sigma' = \frac{cosy}{cox}\dot{y}, \tag{4.34}$$

and

$$cos^2x + sin^2x = 1.$$

Following these transformations, the action in Eq. (4.33) now transforms into

$$I = \int \left[(cosx)^{-\frac{1}{2}}\,(cosy)^{-\frac{1}{2}} - (cosx)^{\frac{3}{2}}\,(cosy)^{-\frac{5}{2}}\,\frac{cos^2y}{cos^2x}\dot{y}^2\right]^{\frac{1}{2}} cosxdx. \tag{4.35}$$

But

$$(cosx)^{\frac{3}{2}}\,(cosy)^{-\frac{5}{2}}\,\frac{cos^2y}{cos^2x} = (cosx)^{-\frac{1}{2}}\,(cosy)^{-\frac{1}{2}}, \tag{4.36}$$

the action in (4.35) takes the form

$$I = \int \left\{(cosx)^{-\frac{1}{2}}\,(cosy)^{-\frac{1}{2}} - (cosx)^{-\frac{1}{2}}\,(cosy)^{-\frac{1}{2}}\dot{y}^2\right\}^{\frac{1}{2}} cosxdx \tag{4.37}$$

Simplifying (4.37) we obtain

$$\begin{aligned} I &= \int \left\{(1-\dot{y}^2)(cosx\ cosy)^{-\frac{1}{2}}\right\}^{\frac{1}{2}} cosxdx \\ &= \int \left\{(1-\dot{y}^2)^{\frac{1}{2}}(cosx\ cosy)^{-\frac{1}{4}}\right\} cosxdx\,, \end{aligned} \tag{4.38}$$

therefore

$$I = \int (cosx)^{\frac{3}{4}}\,(cosy)^{-\frac{1}{4}}(1-\dot{y}^2)^{\frac{1}{2}}\,dx\,. \tag{4.39}$$

It is clear from the action in (4.39) that the Lagrangian is given by

$$\mathcal{L} = (cosx)^{\frac{3}{4}}(cosy)^{-\frac{1}{4}}(1-\dot{y}^2)^{\frac{1}{2}} \tag{4.40}$$

where, in this case, the Lagrangian is a function of both x and y

$$\mathcal{L} = \mathcal{L}(x, y, \dot{y}), \qquad y = y(x). \tag{4.41}$$

The equation of motion

In order to obtain our equation of motion using the Lagrangian (4.40), we impose Eqs. (2.9) and (2.19) such that

$$\frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right) - \frac{\partial\mathcal{L}}{\partial y} = 0. \tag{4.42}$$

Here,

$$\frac{\partial\mathcal{L}}{\partial y} = -\frac{1}{4}(cosx)^{\frac{3}{4}}(cosy)^{-\frac{5}{4}}(1-\dot{y}^2)^{\frac{1}{2}}(-siny),$$

$$\therefore \frac{\partial\mathcal{L}}{\partial y} = \frac{1}{4}siny(cosx)^{\frac{3}{4}}(cosy)^{-\frac{5}{4}}(1-\dot{y}^2)^{\frac{1}{2}} \tag{4.43}$$

and

$$\frac{\partial\mathcal{L}}{\partial\dot{y}} = \frac{1}{2}(cosx)^{\frac{3}{4}}(cosy)^{-\frac{1}{4}}(1-\dot{y}^2)^{-\frac{1}{2}}(-2\dot{y})$$

$$\therefore \frac{\partial\mathcal{L}}{\partial\dot{y}} = -\dot{y}\,(1-\dot{y}^2)^{-\frac{1}{2}}(cosx)^{\frac{3}{4}}(cosy)^{-\frac{1}{4}}, \tag{4.44}$$

Also,

$$\frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}}\right) = -\ddot{y}\,(1-\dot{y}^2)^{-\frac{1}{2}}(cosx)^{\frac{3}{4}}(cosy)^{-\frac{1}{4}} - \dot{y}^2\ddot{y}(1-\dot{y}^2)^{-\frac{3}{2}}(cosx)^{\frac{3}{4}}(cosy)^{-\frac{1}{4}}$$

$$-\dot{y}(1-\dot{y}^2)^{-\frac{1}{2}}\Big[-\frac{3}{4}sinx.(cosx)^{-\frac{1}{4}}(cosy)^{-\frac{1}{4}}$$

$$+\frac{1}{4}siny.\dot{y}(cosy)^{-\frac{5}{4}}(cosx)^{\frac{3}{4}}\Big]. \tag{4.45}$$

Now, putting Eqs. (4.43) and (4.45) into (4.42) gives

$$\ddot{y}\,(1-\dot{y}^2)^{-\frac{1}{2}}(cosx)^{\frac{3}{4}}(cosy)^{-\frac{1}{4}} + \dot{y}^2\ddot{y}(1-\dot{y}^2)^{-\frac{3}{2}}(cosx)^{\frac{3}{4}}(cosy)^{-\frac{1}{4}}$$

$$+\frac{\dot{y}}{4}(1-\dot{y}^2)^{-\frac{1}{2}}\Big[-3sinx(cosx)^{-\frac{1}{4}}(cosy)^{-\frac{1}{4}} + siny.\dot{y}(cosy)^{-\frac{5}{4}}(cosx)^{\frac{3}{4}}\Big]$$

$$+\frac{1}{4}siny(cosy)^{-\frac{5}{4}}(cosx)^{\frac{3}{4}}(1-\dot{y}^2)^{\frac{1}{2}} = 0. \tag{4.46}$$

Multiplying Eq. (4.46) by $4(1-\dot{y}^2)^{\frac{1}{2}}(cosx)^{-\frac{3}{4}}(cosy)^{\frac{1}{4}}$ gives

$$4\ddot{y} + \frac{4\dot{y}^2\ddot{y}}{1-\dot{y}^2} + \dot{y}[-3tanx + \dot{y}tany] + tany(1-\dot{y}^2) = 0,$$

which implies that

$$4\ddot{y} + \frac{4\dot{y}^2\ddot{y}}{1-\dot{y}^2} - 3\dot{y}tanx + \dot{y}^2tany + tany - \dot{y}^2tany = 0. \tag{4.47}$$

But,

$$4\ddot{y} + \frac{4\dot{y}^2\ddot{y}}{1-\dot{y}^2} = \frac{4\ddot{y}}{1-\dot{y}^2}(1-\dot{y}^2+\dot{y}^2) = \frac{4\ddot{y}}{1-\dot{y}^2}. \tag{4.48}$$

Putting Eq. (4.48) into (4.47) gives

$$\frac{4\ddot{y}}{1-\dot{y}^2} - 3\dot{y}tanx + tany = 0 \tag{4.49}$$

Hence,

$$4\ddot{y} = (1 - \dot{y}^2)(3\dot{y}tanx - tany), \tag{4.50}$$

or

$$\ddot{y} = \frac{1}{4}(1 - \dot{y}^2)(3\dot{y}tanx - tany). \tag{4.51}$$

It is clear at this point that Eq. (4.51) is the equation that defines and describes the geodesic motion of the particle on the Khan-Penrose Cosmic Landscape or space-time continuum in (x, y) coordinates.

4.2 Geodesics on the Bell-Szekeres Cosmic Landscape

Here, we intend to x-ray and to explore the unique properties of the Bell-Szekeres global structure as derive the equations of motion that define and describe the geodesics of our test particle as it cruises steadily on this Cosmic Landscape. Now, let us take a close look at the line element (4.1) that defines our test particle. In order to discuss the geodesics of the particle on the Bell-Szekeres Cosmic Landscape or space-time continuum, we need to metal-cast our line element (4.1) into the mould-like metric of the form in Eq. (3.31), which is the basic line element valid for defining and describing geodesics on this Cosmic Landscape.

We begin by carrying out the following transformations. We let

$$\begin{aligned} e^{-M} &= 1, \\ e^{-U+V} &= \cos^2(au - bv), \\ e^{-U-V} &= \cos^2(au + bv)\,. \end{aligned} \tag{4.52}$$

In the light of these transformations in (4.52), the line element (4.1) now takes the form

$$ds^2 = 2dudv - \cos^2(au - bv)\,dx^2 - \cos^2(au + bv)\,dy^2, \tag{4.53}$$

where, $a\ and\ b$, are constants. The line element (4.53) now conforms to the basic structure of the Bell-Szekeres Cosmic Landscape and valid for defining and describing the geodesics of any test particle like ours. However, in order to spotlight the geodesics with high degree of clarity and precision, there need for diversified viewpoints. To achieve this, we shall derive the equations of motion fo the particle in different coordinates.

We now carry out the following transformations by changing variables, let

$$\psi = au + bv,$$

$$\theta = au - bv,$$

$$\psi + \theta = 2au,$$

and

$$d\psi + d\theta = 2adu. \tag{4.54}$$

Also, let

$$\psi - \theta = 2bv,$$

$$d\psi - d\theta = 2bdv,$$

and

$$d\psi^2 - d\theta^2 = 4abdudv,$$

hence

$$dudv = \frac{1}{4ab}(d\psi^2 - d\theta^2). \tag{4.55}$$

Following these transformations in Eqs. (4.54) and (4.55), the line element (4.53) now takes the form

$$ds^2 = \frac{1}{2ab}(d\psi^2 - d\theta^2) - \cos^2\theta\, dx^2 - \cos^2\psi\, dy^2. \tag{4.56}$$

The line element (4.56) now becomes our working line element that defines and describes the geodesics of our test particle in the four coordinates we intend to work on. We now define our new coordinates by

$$X^{\mu} = X^{\mu}(\psi, \theta, x, y). \tag{4.57}$$

By imposing section (2.4) on the line element (4.56), we now define the Lagrangian of the system by

$$\mathcal{L} = \frac{1}{2}\left[\frac{1}{2ab}(d\psi^2 - d\theta^2) - \cos^2\theta\, dx^2 - \cos^2\psi\, dy^2\right], \tag{4.58}$$

or

$$\mathcal{L} = \frac{1}{2}\left[\frac{1}{2ab}\left(\dot{\psi}^2 - \dot{\theta}^2\right) - \cos^2\theta\, \dot{x}^2 - \cos^2\psi\, d\dot{y}^2\right], \tag{4.59}$$

where, $\left(\cdot \equiv \frac{d}{ds}\right)$.

The Equations of motion

At this juncture, we shall fully utilize the Lagrangian formalism discussed in section (2.4) in order to obtain the equations of motion that define and describe the geodesics of our test particle on this Cosmic Landscape in terms of the four coordinates (ψ, θ, x, y).

4.2.1 Motion along the $x-$coordinate

We now impose section (2.4) on the Langrangian (4.59) by taking derivatives with respect to x such that

$$\frac{\partial \mathcal{L}}{\partial x} = 0, \tag{4.60}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = -\frac{1}{2}(2cos^2\theta\ \dot{x}) = -\ cos^2\theta\ \dot{x}. \tag{4.61}$$

and

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0, \tag{4.62}$$

which implies that

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \alpha_o\ , \alpha_o = \text{constant} \tag{4.63}$$

Comparing Eqs. (4.61) and (4.63) shows that

$$cos^2\theta\ \dot{x} = \alpha_o. \tag{4.64}$$

Hence,

$$\dot{x} = \frac{\alpha_o}{cos^2\theta}\ . \tag{4.65}$$

This is the equation of motion along the x-coordinate.

4.2.2 Motion along the $y -$ coordinate

Here, we take the derivative of the Lagrangian (4.59) with respect to y, such that

$$\frac{\partial \mathcal{L}}{\partial y} = 0, \tag{4.66}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{1}{2}(-2)cos^2\psi\ \dot{y} = -\ cos^2\psi\ \dot{y}, \tag{4.67}$$

and

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = 0 \tag{4.68}$$

which implies that

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} = \beta_o\ , \beta_o = \text{constant.} \tag{4.69}$$

Comparing Eqs. (4.67) with (4.69) shows that

$$cos^2\psi\ \dot{y} = \beta_o. \tag{4.70}$$

Hence,

$$\dot{y} = \frac{\beta_o}{cos^2\psi}\quad . \tag{4.71}$$

This is the equation of motion that defines and describes the motion of our test particle on this Cosmic Landscape along the y–coordinate.

4.2.3 Motion along the $\psi -$ coordinate

Here, we take the derivative of the Lagrangian (4.59) with respect to ψ, such that

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{1}{2}\ (2cos\psi sin\ \psi \dot{y}^2), \tag{4.72}$$

which implies that

$$\frac{\partial \mathcal{L}}{\partial \psi} = \ \ cos\psi sin\psi \dot{y}^2, \tag{4.73}$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \frac{1}{2}\left(\frac{1}{2ab}.\,2\dot{\psi}\right), \tag{4.74}$$

which implies that

$$\frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \ \ \frac{1}{2ab}.\,\dot{\psi}. \tag{4.75}$$

Also,

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}}\right) = \ \ \frac{1}{2ab}\ddot{\psi}. \tag{4.76}$$

By imposing Eq. (2.8), we obtain

$$\frac{1}{2ab}\ddot{\psi} - cos\ \psi sin\ \psi \dot{y}^2 = 0. \tag{4.77}$$

But we know from Eq. (4.71) that

$$\dot{y} = \frac{\beta_o}{cos^2\psi},$$

and

$$\dot{y}^2 = \frac{{\beta_o}^2}{cos^4\psi}\ . \tag{4.78}$$

Putting (4.78) into (4.77), gives

$$\frac{1}{2ab}\ddot{\psi} - cos\ \psi sin\ \psi \frac{{\beta_o}^2}{cos^4\psi} = 0, \tag{4.79}$$

and

$$\frac{1}{2ab}\ddot{\psi} - {\beta_o}^2 \frac{sin\ \psi}{cos^3\psi} = 0\ . \tag{4.80}$$

Hence,

$$\ddot{\psi} = 2ab{\beta_o}^2 \frac{sin\,\psi}{cos^3\psi}\,. \qquad (4.81)$$

This is the equation of motion that defines and describes the motion of our test particle along the ψ-coordinate as it cruises steadily on this Cosmic Landscape.

4.2.4 Motion along the $\boldsymbol{\theta}-$ coordinate

Here, we take the derivatives of Lagrangian (4.59) with respect to θ, such that

$$\frac{\partial\mathcal{L}}{\partial\theta} = -\frac{1}{2}\,(2cos\,\theta)(-sin\,\theta)\dot{x}^2,$$

which implies that

$$\frac{\partial\mathcal{L}}{\partial\theta} = \;\; cos\,\theta sin\,\theta\dot{x}^2, \qquad (4.82)$$

also

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = -\frac{1}{2}\left(\frac{1}{2ab}.\,2\dot{\theta}\right),$$

which implies that

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \; -\frac{1}{2ab}.\,\dot{\theta}, \qquad (4.83)$$

and

$$\frac{d}{ds}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \; -\frac{1}{2ab}\ddot{\theta}. \qquad (4.84)$$

Putting Eq. (4.82) and (4.84) into (2.8), we obtain

$$-\frac{1}{2ab}\ddot{\theta} - cos\,\psi sin\,\theta\dot{x}^2 = 0. \qquad (4.85)$$

Recall from Eqn. (4.36) that

$$\dot{x} = \frac{\alpha_o}{cos^2\theta},$$

and

$$\dot{x}^2 = \frac{{\alpha_o}^2}{cos^4\theta}\,. \qquad (4.86)$$

Putting Eq. (4.86) into (4.85) yields

$$\frac{1}{2ab}\ddot{\theta} + cos\,\theta sin\,\theta\,\frac{{\alpha_o}^2}{cos^4\theta} = 0, \qquad (4.87)$$

and

$$\frac{1}{2ab}\ddot{\theta} + {\alpha_o}^2\,\frac{sin\,\theta}{cos^3\theta} = 0. \qquad (4.89)$$

Hence,

$$\ddot{\theta} = -2ab{\alpha_o}^2\,\frac{sin\,\theta}{cos^3\theta}\,. \qquad (4.90)$$

This is the equation of motion that defines and describes the geodesics of our test particle along the θ-coordinate.

At this juncture, we have concluded the derivation of required equations of motion. On a general note, there seems to be four equations that define and describe the motion of our test particle here, as it moves steadily within the Bell-Szekeres Cosmic Landscape or space-time continuum. The four equations are Eqs. (4.65), (4.71),(4.81) and (4.90)

$$\dot{x} = \frac{\alpha_o}{cos^2\theta}\,, \qquad (4.65$$

$$\dot{y} = \frac{\beta_o}{cos^2\psi}\,, \qquad (4.71)$$

$$\ddot{\psi} = 2ab{\beta_o}^2\,\frac{sin\,\psi}{cos^3\psi}\,, \qquad (4.81)$$

and

$$\ddot{\theta} = -2ab{\alpha_o}^2\,\frac{sin\,\theta}{cos^3\theta}. \qquad (4.90)$$

4.3 Geodesics on a Degenerate Cosmic Landscape

Here, we take a close look at the basic properties of this global structure and the roles that Killing vectors along with their associated constants play in the formation of horizons and singularities as we spotlight the geodesics of our test particle on this Cosmic Landscape.

In order to discuss the geodesics of our test particle within this framework, we need to transform the metric line element (4.1) into the form in Eq. (3.38), which is valid for defining and describing the geodesics of any test particle on the Degenerate Cosmic Landscape. We now begin with some suitable transformations as we let

$$e^{-M} = \frac{1}{2}(1 + \rho sint)^2,$$

$$e^{-U+V} = -\left(\frac{1 - \rho sint}{1 + \rho sint}\right),$$

$$e^{-U-V} = -cos^2 z(1 + \rho sint)^2, \tag{4.91}$$

and

$$dudv = dz^2 - dt^2.$$

In the light of these transformations (4.91), our line element (4.1) now takes the form

$$ds^2 = (1 + \rho sint)^2(dz^2 - dt^2) + \left(\frac{1 - \rho sint}{1 + \rho sint}\right) dx^2$$

$$+ cos^2 z(1 + \rho sint)^2 dy^2. \tag{4.92}$$

This line element (4.92) becomes our working metric element for defining and describing the geodesics of our test particle as it moves steadily on this Cosmic Landscape.

By imposing Eq. (2.7) n the line element (4.92), we obtain the translational Killing vectors for regions II anod III as

$$\xi_{(1)} = \partial_x\,, \tag{4.93}$$

and

$$\xi_{(2)} = \partial_y\,. \tag{4.94}$$

By imposing Eqs. (2.4) - (2.6) on the line element (4.92), we obtain the Killing vectors fully operational in region IV as

$$\xi^{I}_{(3)} = cosy\,\partial_z + siny\,tanz\,\partial_y\,, \tag{4.95}$$

and

$$\xi^{I}_{(4)} = -siny\,\partial_z + cosy\,tanz\,\partial_y\,. \tag{4.96}$$

It is clear from the line element (4.92) that the Lagrangian of this mechanic system is defined by

$$\mathcal{L} = \frac{1}{2}w_1^2(\dot{z}^2 - \dot{t}^2) + \frac{w_2}{2w_1}\dot{x}^2 + \frac{cos^2 z}{2}w_1^2\dot{y}^2 = -\frac{\varepsilon}{2}\,, \tag{4.97}$$

where,

$$w_1 = 1 + \rho sin\,t,$$

$$w_2 = 1 - \rho sin\,t, \tag{4.98}$$

and

$$\epsilon = 1 \text{ for time} - \text{like geodesic},$$

$$\epsilon = 0 \text{ for null geodesic}, \tag{4.99}$$

$$\epsilon = -1 \text{ for space} - \text{like geodesic}.$$

Since we are considering a time-like geodesic of a test particle on a given space-time continuum, we shall take $\epsilon = 1$. Based on this, the metric condition for the geodesic that defines and describes the trajectory of our test particle on the Degenerate Cosmic Landscape now takes the form

$$w_1^2(\dot{z}^2 - \dot{t}^2) + \frac{w_2}{w_1}\dot{x}^2 + cos^2 z w_1^2\dot{y}^2 = -1. \tag{4.100}$$

We now impose the Lagrangian formalism of section (2.4) on our Lagrangian (4.97) in order to derive the equations of motion that define and describe the geodesics of our test particle. First, we consider motion along the x-coordinate. Here, we obtain

$$\frac{d\mathcal{L}}{d\dot{x}} = K_x = \text{constant}, \qquad (4.101)$$

and

$$\frac{\mathcal{w}_2}{\mathcal{w}_1}\dot{x}^2 = K_x\,, \qquad (4.102)$$

therefore

$$\dot{x} = \frac{\mathcal{w}_1}{\mathcal{w}_2}K_x. \qquad (4.103)$$

This is equation of motion of our test particle along the x-coordinate. Also, by applying the same method for the motion along the y-coordinate we obtain

$$\frac{d\mathcal{L}}{d\dot{y}} = K_y = \text{constant}, \qquad (4.104)$$

and

$$cos^2 z\mathcal{w}_1^2\dot{y}^2 = K_y, \qquad (4.105)$$

therefore

$$\dot{y} = \frac{K_y}{cos^2 z\mathcal{w}_1^2}\,. \qquad (4.106)$$

This is the equation of motion that defines and describes the geodesics of our test particle along the y-coordinate. Also, by applying the same procedure, we obtain equation of motion along the z-coordinate as

$$\frac{d\mathcal{L}}{d\dot{z}} = \mathcal{w}_1^2\dot{z}, \qquad (4.107)$$

and

$$\frac{d\mathcal{L}}{dz} = -cos\, z\, sin\, z\, \mathcal{w}_1^2\dot{y}^2, \qquad (4.108)$$

which implies that

$$\frac{d\mathcal{L}}{dz} = -cos\, z\, sin\, z\, \mathcal{w}_1^2\,\frac{K_y^2}{cos^4 z\mathcal{w}_1^4}\,, \qquad (4.109)$$

therefore

$$\frac{d\mathcal{L}}{dz} = -\frac{sinz}{cos^3 z}\frac{K_y^2}{w_1^2}\,. \tag{4.110}$$

By imposing Eq. (2.9), we obtain

$$\left(\frac{d\mathcal{L}}{d\dot{z}}\right) - \frac{d\mathcal{L}}{dz} = 0, \tag{4.111}$$

which implies that$\frac{d}{d\tau}$

$$\frac{d}{d\tau}(w_1^2\dot{z}\,) = -\frac{sinz}{cos^3 z}\frac{K_y^2}{w_1^2}\,. \tag{4.112}$$

We now define a function, $\Im$, such that

$$\dot{\Im} = w_1^2\dot{z}\,, \tag{4.113}$$

and

$$\frac{d}{d\tau}(\dot{\Im}\,) = -\frac{sinz}{cos^3 z}\frac{K_y^2}{w_1^2}\,. \tag{4.114}$$

Now, we multiply (4.114) by $2\dot{\Im}d\tau$ to obtain

$$2\dot{\Im}d\dot{\Im} = -\frac{sinz}{cos^3 z}\frac{K_y^2}{w_1^2}2\dot{\Im}d\tau, \tag{4.115}$$

and

$$\dot{\Im}^2 = -2\frac{sinz}{cos^3 z}\frac{K_y^2}{w_1^2}\;w_1^2 d\tau, \tag{4.116}$$

therefore

$$\dot{\Im}^2 = -2\frac{sinz}{cos^3 z}dzK_y^2. \tag{4.117}$$

Now, we let

$$u = cosz,$$

and

$$du = -sinzdz, \tag{4.118}$$

this implies that

$$\dot{\mathfrak{J}}^2 = +2\int \frac{du}{u^3} K_y^2 + K_z^2\,, \tag{4.119}$$

$$\dot{\mathfrak{J}}^2 = \frac{2U^{-2}}{-2} K_y^2 + K_z^2, \tag{4.120}$$

$$\dot{\mathfrak{J}}^2 = K_z^2 - \frac{K_y^2}{cos^2 z}, \tag{4.121}$$

and

$$\dot{\mathfrak{J}} = \sqrt{K_z^2 - \frac{K_y^2}{cos^2 z}}\,. \tag{4.122}$$

Substituting for Eq. (4.113), we obtain

$$w_1^2 \dot{z} = \sqrt{K_z^2 - \frac{K_y^2}{cos^2 z}}\,, \tag{4.123}$$

therefore

$$\dot{z} = \frac{1}{w_1^2}\sqrt{K_z^2 - \frac{K_y^2}{cos^2 z}}\,. \tag{4.124}$$

This is the equation of motion that defines and describes the geodesics of our test particle along the z-coordinate on this Cosmic Landscape. However, since the associated constant for the Killing vector along the z-coordinate in this case could be a function of both y and z, we now let

$$K_z^2 = K_y^2 + K_z^2. \tag{4.125}$$

In the light of this transformation, Eq. (4.124) now takes the form

$$\dot{z} = \frac{1}{w_1^2}\sqrt{K_y^2 + K_z^2 - \frac{K_y^2}{cos^2 z}}, \tag{4.126}$$

or

$$\dot{z} = \frac{1}{w_1^2}\sqrt{K_y^2\left(1 - \frac{1}{cos^2 z}\right) + K_z^2}, \tag{4.127}$$

where

$$cos^2z + sin^2z = 1 \Rightarrow \; cos^2z - 1 = -sin^2z, \tag{4.128}$$

and

$$\dot{z} = \frac{1}{w_1^2}\sqrt{K_y^2\left(\frac{cos^2z - 1}{cos^2z}\right) + K_z^2}\,, \tag{4.129}$$

where

$$\frac{cos^2z - 1}{cos^2z} = \frac{-sin^2z}{cos^2z} = -tan^2z, \tag{4.130}$$

therefore

$$\dot{z} = \frac{1}{w_1^2}\sqrt{K_z^2 - K_y^2tan^2z}\,. \tag{4.131}$$

This becomes equation of motion that defines and describes the geodesics of our tests particle along the z-coordinate in terms of Eq. (4.125).

To obtain the equation of motion along the t-coordinate, we wish to substitute for the other parameters into the metric condition (4.100). Now, let

$$w_1^2\dot{z}^2 - w_1^2\dot{t}^2 + \frac{w_2}{w_1}\dot{x}^2 + cos^2zw_1^2\dot{y}^2 = -1, \tag{4.132}$$

and

$$w_1^2\dot{t}^2 = 1 + w_1^2\dot{z}^2 + \frac{w_2}{w_1}\dot{x}^2 + cos^2zw_1^2\dot{y}^2,$$

$$w_1^2\dot{t}^2 = 1 + w_1^2\left(\frac{K_z^2 - \frac{K_y^2}{cos^2z}}{w_1^4}\right) + \frac{w_2}{w_1}\frac{w_1^2}{w_2^2}K_x^2 + cos^2zw_1^2\frac{K_y^2}{cos^4zw_1^4},$$

$$w_1^2\dot{t}^2 = 1 + \frac{K_z^2 - \frac{K_y^2}{cos^2z}}{w_1^2} + \frac{w_1}{w_2}K_x^2 + \frac{K_y^2}{cos^2zw_1^2}\,,$$

$$w_1^2\dot{t}^2 = 1 + \frac{K_z^2}{w_1^2} + \frac{w_1}{w_2}K_x^2 - \frac{K_y^2}{w_1^2cos^2z} + \frac{K_y^2}{cos^2zw_1^2}\,,$$

hence

$$w_1^2\dot{t}^2 = 1 + \frac{K_z^2}{w_1^2} + \frac{w_1}{w_2}K_x^2. \tag{4.133}$$

Solving for $\dot{t}$ from Eq. (4.133), we obtain

$$\dot{t}^2 = \frac{1}{w_1^2}\left(1 + \frac{K_z^2}{w_1^2} + \frac{w_1}{w_2}K_x^2\right), \tag{4.134}$$

and

$$\dot{t} = \frac{1}{w_1}\sqrt{1 + \frac{w_1}{w_2}K_x^2 + \frac{K_z^2}{w_1^2}}\,. \tag{4.135}$$

This is the equation of motion that defines and describes the geodesics of our test particle on this Cosmic Landscape along the t-coordinate. However, in terms of Eq. (4.125) we obtain

$$\dot{t} = \frac{1}{w_1}\sqrt{1 + \frac{w_1}{w_2}K_x^2 + \frac{K_y^2 + K_z^2}{w_1^2}}\,. \tag{4.136}$$

To conclude this section, we need to note that there are four equations that define and describe the motion of the particle here, as it moves steadily on a Degenerate Cosmic Landscape or space-time continuum. The four equations are

$$1.\ \dot{x} = \frac{w_1}{w_2}K_x\,, \tag{4.103}$$

$$2.\ \dot{y} = \frac{K_y}{cos^2 z w_1^2}\,, \tag{4.106}$$

$$3a.\ \dot{z} = \frac{1}{w_1^2}\sqrt{K_z^2 - \frac{K_y^2}{cos^2 z}}\,, \tag{4.124}$$

$$3b.\ \dot{z} = \frac{1}{w_1^2}\sqrt{K_z^2 - K_y^2 tan^2 z}\,, \tag{4.131}$$

$$4a.\ \dot{t} = \frac{1}{w_1}\sqrt{1 + \frac{w_1}{w_2}K_x^2 + \frac{K_z^2}{w_1^2}}\,, \tag{4.135}$$

and

$$4b.\ \dot{t} = \frac{1}{w_1}\sqrt{1 + \frac{w_1}{w_2}K_x^2 + \frac{K_y^2 + K_z^2}{w_1^2}}. \quad (4.136)$$

4.4 Discussion

Here, we seek to find solutions to the equations of motion that we obtained in sections (4.1) – ((4.3) as we discuss how they define and describe the geodesics of our test particle on the various Cosmic Landscapes. We shall begin by solving the equations of motion on the Bell-Szekeres Cosmic Landscape, since it appears to be the simplest. We shall proceed to solve and discuss the highly non-linear equation of the particle's motion on the Khan-Penrose Comic Landscape. We shall proceed to spot-light the the geodesics of the particle on the Degenerate Cosmic Landscape; the Ferrari-Ibanez space-time continuum.

4.4.1 The Bell-Szekeres Cosmic Landscape solutions

Here, we intend to reduce the four equations of motion for our test particle obtained in section (4.2) to a simple and manageable two-dimensional equation along the ψ and θ coordinates. Recall that our working line element (4.55) valid for the geodesics of our test particle on the Bell-Szekeres Cosmic Landscape is given by

$$ds^2 = \frac{1}{2ab}(d\psi^2 - d\theta^2) - \cos^2\theta\, dx^2 - \cos^2\psi\, dy^2. \quad (4.137)$$

We now divide the line element (4.137) by ds^2 to obtain

$$1 = \frac{1}{2ab}\left(\dot{\psi}^2 - \dot{\theta}^2\right) - \cos^2\theta\, \dot{x}^2 - \cos^2\psi\, \dot{y}^2, \quad (4.138)$$

where, $\left(\cdot \equiv \frac{d}{ds}\right)$. To collapse this line element into an equation of motion along the ψ and θ coordinates only, we fix both x and y as constants. Recall also, that Eq. (4.65) defines and describes the geodesics of our test particle on this Cosmic Landscape along the x-coordinate, such that

$$\cos^2\theta\, \dot{x}^2 = \alpha_o = \text{constant}. \quad (4.139)$$

In the same vein, Eq. (4.71) provides the geodesic equation for our test particle along the y-coordinate, such that

$$\cos^2\psi\,\dot{y}^2 = \beta_o = \text{constant}. \tag{4.140}$$

Substituting for Eqs. (4.139) and (4.140) into the line element (4.138), makes it collapse drastically into

$$1 = \frac{1}{2ab}\left(\dot{\psi}^2 - \dot{\theta}^2\right). \tag{4.141}$$

The Lagrangian valid for this transformed system is now defined by

$$\mathcal{L} = \frac{1}{2ab}\left(\dot{\psi}^2 - \dot{\theta}^2\right). \tag{4.142}$$

We now impose the properties of section (2.4) on Lagrangian (4.142) to obtain

$$\dot{\psi} = \text{constant}, \tag{4.143}$$

and

$$\dot{\theta} = c\text{onstant}. \tag{4.144}$$

Dividing (4.143) by (4.144), we obtain

$$\frac{\dot{\psi}}{\dot{\theta}} = \text{constant}, \tag{4.155}$$

this implies that

$$\frac{d\psi}{d\theta} = \text{constant}. \tag{4.156}$$

Hence,

$$\psi = \kappa\theta + \ell, \tag{4.157}$$

where, $\kappa\ and\ \ell$, are constants, ψ is a function of θ. Eq. (4.157) is now our simplified equation of motion that defines and describes the geodesic of our test particle as it moves steadily along the ψ and θ coordinates on this Cosmic Landscape. A numerical solution to Eq. (4.157) is obtained for $0\ \leq \psi \leq 1$ and $0 \leq \theta \leq 1$, which gives a straight line graph (see Figure 4.1). The line defines and describes the path or geodesic of our test particle as it moves straight into the horizon on this Cosmic Landscape.

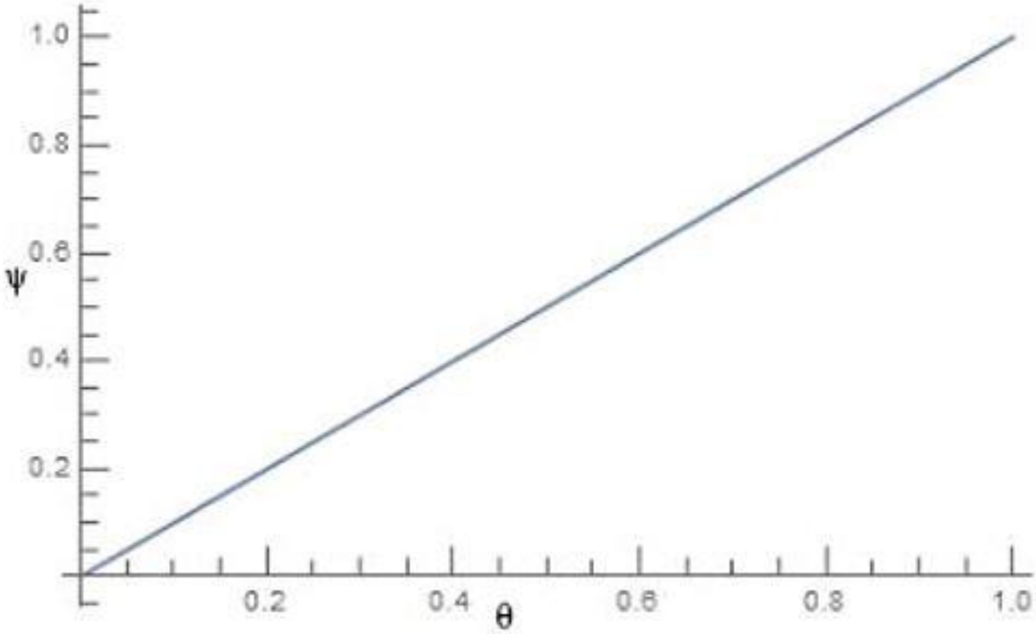

Figure 4.1: Geodesic of a test particle on Bell-Szekeres Cosmic Landscape along the θ and ψ coordinates for $0 \leq \psi \leq 1$ and $0 \leq \theta \leq 1$.

4.4.2 The Khan-Penrose Cosmic Landscape Solutions

Here, we seek to solve the equations of motion for our test particle obtained in section (4.1). However, since the equations are highly non-linear, we intend to evaluate the equations numerically as we discuss the geodesics of our test particle cruising steadily on the Khan-Penrose Cosmic Landscape.

Recall that, Eqns. (4.29) and (4.51) are the geodesic equations that define and describe the paths or the particle's motion on this Cosmic Landscape in terms of the null coordinates (u, v) and the Cartesian coordinates (x, y) respectively. However, since the impulsive waves in this structure are best described in the null coordinates(u, v), we shall discuss the geodesic of our test particle using Eqn. (4.29).

Now, recall that Eq. (4.29), is given as

$$v'' = \frac{2v'}{f}[f_u - v' f_v], \tag{4.158}$$

where, is a f is a function defined by

$$f = \left[\frac{2e^{-M-U}}{e^{-U} + A^2 e^{-V} + B^2 e^{V}}\right]^{\frac{1}{2}}, \tag{4.159}$$

A and B are both canstants. To simplify further, we let $A = B = 0$. The function (4.159) reduces into

$$f = \sqrt{2}\,(e^{-M})^{\frac{1}{2}}. \tag{4.160}$$

Now, comparing the line element (4.1) that defines our test particle with the line element (3.14) shows that

$$e^{-M} = \frac{(1 - u^2 - v^2)^{\frac{3}{2}}}{\sqrt{1-u^2}\sqrt{1-v^2}\left(uv + \sqrt{1-u^2}\sqrt{1-v^2}\right)^2}, \tag{4.161}$$

where,

$$f = f(u, v), \text{ and } v = v(u) \tag{4.162}$$

Now, setting $0 \le u < 1, 0 \le v < 1$, and initial conditions $u = 0, v = 0$ and $v' = 0.1$. Our equation of motion (4.158) is solved numerically using Maple (see Figure 4.2 and figure 4.3 respectively). In our plots, we used v_0 for the initial speed of our test particle and we considered initial speed range of $v_0 = 0.1n,\ n = 0 \dots N$.

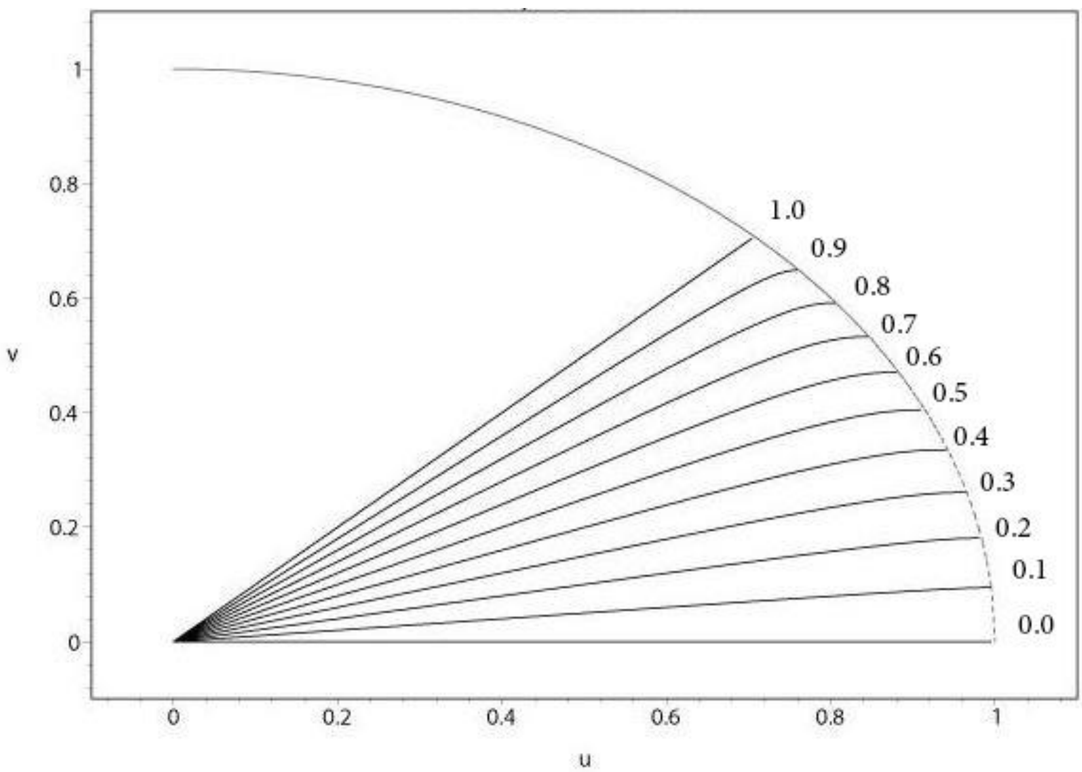

Figure 4.1 the geodesics of a test particle on the Khan-Penrose Cosmic Landscape for initial speeds $v_0 = 0.1n, n = 0 \ldots 10$. The geodesics curved towards right of the path with $v_0 = 1$, as they hit the curved singularity.

Figure (4.2) shows geodesics of our test particles with initial speed ranging from $v_0 = 0.1$ to 1.0. It is clear that the trajectories tend to curve at the tail end of their journey as they approach their touch line; the curvature singularity. However, the geodesic for which $v_0 = 1.0$, appears to straighten up given rise to a straight line trajectory into the curvature singularity.

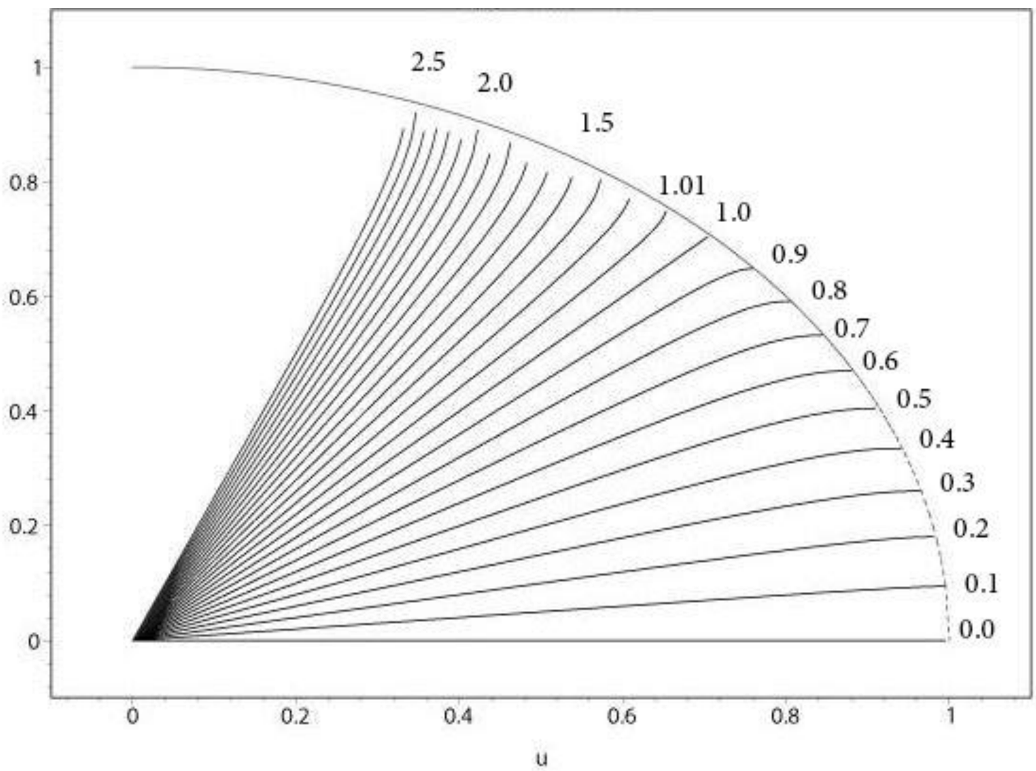

Figure 4.3: the geodesics of a test particle on the Khan-Penrose Cosmic Landscape for initial speeds $v_0 = 0.1n, n = 0 \dots 25$. The geodesics curved towards right or left of a path along which $v_0 = 1$, as they hit the curved singularity.

Figure (4.3) shows a wider spectrum of geodesics of our test particles with initial speed range from $v_0 = 0.1$ to 2.5. It is clear that the trajectories tend to curve at the tail end of their journey towards the right side as they approach their touch line; the curvature singularity, while at $v_0 = 1.0$ gave rise to a straight line trajectory into the singularity just as it is reflected in Figure (4.2). However, as we increase the initial speed above 1.0, the trajectories begin to curve again, but this time, towards the left side. As the initial speed increases, the geodesics seems to vanish before reaching the singularity.

4.4.3 The Degenerate Cosmic Landscape solutions

Here, we consider the solutions to the equations of motion developed in section (4.3) for our test particle on a Degenerate Cosmic Landscape. We intend to have a close look at how horizons and singularities are formed on this Cosmic Landscape.

4.4.3.1 The Horizons

Following the condition for the formation of horizons stated in section (3.3), we know that as our test particle approaches $t = \frac{\pi}{2}\ or\ u + v = \frac{\pi}{2}$; horizons are formed along the geodesic when $\rho = +1$ as a basic condition imposed on our working line element (4.92).

Now, we carry out some transformations on geodesic equations developed in section (4.3) as we let

$$w_1 = 2 \text{ and } w_2 = 0\,. \tag{4.163}$$

We define a function $\mathfrak{z}$ such that

$$\mathfrak{z} = \frac{1}{w_1}\sqrt{1 + \frac{w_1}{w_2}K_x^2 + \frac{K_z^2}{w_1^2}}\,, \tag{4.164}$$

$$\dot{t} = \frac{dt}{d\tau} = \mathfrak{z}\,, \tag{4.165}$$

$$dt = \mathfrak{z}d\tau \Rightarrow \frac{\partial}{\partial\tau} = \mathfrak{z}\frac{\partial}{\partial t}\,, \tag{4.166}$$

and

$$\frac{dt}{d\tau} = \mathfrak{z}\frac{dz}{dt} \tag{4.167}$$

this implies that

$$\frac{dz}{dt} = \frac{1}{\mathfrak{z}}\frac{dz}{d\tau} = \frac{1}{\mathfrak{z}}\mathfrak{x} \tag{4.168}$$

where,

$$\mathfrak{x} = \frac{1}{w_1^2}\sqrt{K_z^2 - \frac{K_y^2}{cos^2 z}}\,. \tag{4.169}$$

By Eq. (4.163) it is clear that (4.169) takes the form

$$\mathfrak{x} = \frac{1}{4}\sqrt{K_z^2 - \frac{K_y^2}{cos^2 z}}\,. \tag{4.170}$$

Now, we consider a situation where $w_2 = 0$.

Case I: $K_x \neq 0 \Rightarrow \mathfrak{z} \to \infty$ (4.171)

and

$$\frac{dz}{dt}\,\Big|_{t\to\frac{\pi}{2}, w_1=2, w_2=0} = \frac{\mathfrak{x}}{\mathfrak{z}}\,\Big|_{\mathfrak{z}\to\infty} = 0 \tag{4.172}$$

Case II: $K_x = 0 \Rightarrow \mathfrak{z} = \frac{1}{2}\sqrt{1+\frac{K_z^2}{4}} = \frac{1}{4}\sqrt{4+K_z^2}$ (4.173)

and

$$\frac{dz}{dt} = \frac{\mathfrak{x}}{\mathfrak{z}} = \frac{\sqrt{K_z^2 - \frac{K_y^2}{cos^2 z}}}{\sqrt{4+K_z^2}} = \frac{1}{cosz}\sqrt{\frac{K_z^2 cos^2 z - K_y^2}{4+K_z^2+K_y^2}}. \tag{4.174}$$

In terms of $K_z^2 = K_z^2 + K_y^2$, we obtain

$$\frac{dz}{dt} = \frac{\mathfrak{x}}{\mathfrak{z}} = \frac{\sqrt{K_z^2 + K_y^2 - \frac{K_y^2}{cos^2 z}}}{\sqrt{4+K_z^2+K_y^2}} = \frac{\sqrt{K_z^2 + \left(1-\frac{1}{cos^2 z}\right)K_y^2}}{\sqrt{4+K_z^2+K_y^2}}\,. \tag{4.175}$$

But

$$1 - \frac{1}{cos^2 z} = \frac{cos^2 z - 1}{cos^2 z} = \frac{-sin^2 z}{cos^2 z} = -tan^2 z, \tag{4.176}$$

this implies that

$$\frac{dz}{dt}\,\Big|_{t\to\frac{\pi}{2}} = \sqrt{\frac{K_z^2 - K_y^2 tan^2 z}{4+K_z^2+K_y^2}}. \tag{4.177}$$

4.4.3.2 The Singularities

In the same way, following the basic conditions for the formation of singularities discussed in section (3.3), we know that as our test particle approaches $t = \frac{\pi}{2}\ or\ u +$

$v = \frac{\pi}{2}$, singularities are formed along the geodesic when $\rho = -1$ as a condition we imposed on our working line element (4.92).

Now, recall that

$$\frac{dz}{dt} = \frac{\mathfrak{x}}{\mathfrak{z}} = \frac{\frac{1}{w_1^2}\sqrt{K_z^2 - \frac{K_y^2}{cos^2 z}}}{\frac{1}{w_1^2}\sqrt{w_1^2 + \frac{w_1^2}{w_2}K_x^2 + K_z^2}} = \frac{\sqrt{K_z^2 - \frac{K_y^2}{cos^2 z}}}{\sqrt{w_1^2 + \frac{w_1^2}{w_2}K_x^2 + K_z^2}} \quad (4.178)$$

Following the imposed conditions where

$$\rho = -1 \, , t = \frac{\pi}{2}, w_1 = 0, and \; w_2 = 2, \quad (4.179)$$

we obtain

$$\frac{dz}{dt} = \frac{\sqrt{K_z^2 - \frac{K_y^2}{cos^2 z}}}{\sqrt{K_z^2}} \, , \quad (4.180)$$

and in terms of $K_z^2 = K_z^2 + K_y^2$, Eq. (4.180) takes the form

$$\frac{dz}{dt} = \frac{\sqrt{K_z^2 + K_y^2 - \frac{K_y^2}{cos^2 z}}}{\sqrt{K_z^2 + K_y^2}} = \frac{\sqrt{K_z^2 + \left(1 - \frac{1}{cos^2 z}\right)K_y^2}}{\sqrt{K_z^2 + K_y^2}} \quad (4.181)$$

where

$$1 - \frac{1}{cos^2 z} = \frac{cos^2 z - 1}{cos^2 z} = \frac{-sin^2 z}{cos^2 z} = -tan^2 z \quad (4.182)$$

therefore

$$\frac{dz}{dt} = \sqrt{\frac{K_z^2 - K_y^2 tan^2 z}{K_z^2 + K_y^2}}. \quad (4.183))$$

It is clear from Eq. (4.183) that for both cases I and II where $K_x = 0 \; and \; K_x \neq 0 \, , for \; t = \frac{\pi}{2}, and \; \rho = -1$, the slope of the trajectory that leads to the formation of singularities remains the same; it does not depend on K_x, hence

$$\frac{dz}{dt}\Big|_{t\to\frac{\pi}{2},w_1=0,w_2=2} = \sqrt{\frac{K_z^2 - K_y^2 tan^2 z}{K_z^2 + K_y^2}}. \qquad (4.184)$$

Chapter 5

SUMMARY AND CONCLUSION

5 Summary and conclusion

We have gradually come to our proposed destination, of which, I can confidently and affirmatively say with the wise men of great renown; "a thousand miles' journey begins with a step." To wrap it up, it is expedient for us to recount some of the mile-stones that ear-marked our trajectory.

We began by looking at the BICEP2 Report on the newly detected B-Mode Polarization of the gravitational waves formed in the Baby Universe, which prompted this research, after which, we looked at the background and the basic concept of Gravitational wave. We proceeded by hand-picking some vital tools and equations to build-up a mathematical structure, with which we constructed some theoretical Global Structures: the Khan-Penrose, Bell-Szekeres and the Ferrari-Ibanez global structures that served as the Cosmic Landscapes or space-time continua, upon which a test particle is considered as an inertial observer. We analyzed and discussed the time-like geodesic of the particle on the various Cosmic Landscapes or space-time continua.

When electromagnetic plane waves collide, gravitational waves are always generated. These impulsive waves occur along the boundaries of region IV only or may appear throughout the interaction region. When a particle is placed along the path of two colliding plane waves, it will be forced to follow a geodesic, defined by the properties of the Global structure, leading to either curvature singularities or horizons pending on the initial conditions. In the null coordinates,(u, v), the interaction region is bounded; so given the initial conditions the later developments are plotted numerically.

The time-like geodesic of the particles on a Bell-Szekeres Cosmic Landscape appears to be steady, defined by a straight line trajectory that leads into the horizon. The gradient of the path is constant. On the other hand, the geodesics on the Khan-Penrose Cosmic Landscape appear to curve towards either sides away from the trajectory on which the initial speed of the test particle is given as $v_0 = 1.0$. It is clearly and evidently seen that all the geodesics vanish or appear to vanish at the end of their journey as they approach the touch-line; the **curvature singularity** within the interaction region. Finally, the Degenerate Cosmic Landscape gave rise to the formation of both Horizons and singularities. Horizons are formed when $\rho = +1$, while Singularities are formed when $\rho = -1$.

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