Multiplicative Runge-Kutta Methods

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ABSTRACT

In this thesis the multiplicative Runge-Kutta Method is developed employing the idea of the ordinary Runge-Kutta Method to multiplicative calculus. The multiplicative Runge-Kutta Methods for the orders 2,3, and 4 are developed and discussed. The developed algorithms are applied to examples where the solutions of the Ordinary Differential Equations are known. This gives the opportunity to check the relative error of the calculation reliably. The results in the multiplicative case are also compared with the results from the ordinary Runge-Kutta Methods of the corresponding order. We can see that the Multiplicative Runge-Kutta Method is advantageous to the ordinary Runge-Kutta method of the same order if the solution is of exponential nature. Finally for completeness the multiplicative Finite Difference method is also presented.

Keywords: Multiplicative Calculus, Runge-Kutta-Method, Ordinary Differential Equations, Numerical Solution

ÖZ

Bu tezde, Runge-Kutta metodu temel alınarak çarpımsal analiz kurallarına gore 2 , 3 ve 4. dereceden çarpımsal Runge-Kutta yöntemleri bulunmuş ve incelenmiştir. Bulunan yöntemler çözümleri bilinen adi diferansiyel denklemlere örnek olarak uygulanmıştır. Böylece hesaplamalardaki hata oranlarının güvenilir bir şekilde kontrol edilmesi sağlanmıştır. Çarpımsal Runge-Kutta metodundan elde edilen sonuçlar ayni dereceden bilinen Runge-Kutta metodu sonuçlarıyla karşılaştırıldı. Bu sonuçlara göre, cözümü eksponensiyel olan denklemlerde çarpımsal Runge-Kutta metodunu kullanmanın ayni dereceden bilinen Runge-Kutta metoduna göre daha avantajli olduğu görülmüştür. Son olarak da çarpımsal Finite Difference metodu anlatılmıştır.

Anahtar Kelimeler: Çarpımsal Analiz, Runge-Kutta-Yöntemi, Adi Diferensiyel Denklemler, Sayısal Çözümler

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INTRODUCTION

Differential and integral calculus was created independently by Isaac Newton and Gottfried Wilhelm Leibnitz. After that Leonard Euler redirect calculus by giving a central place to the concept of function, and thus found analysis. Differentiation and integration are the basic operations in calculus and analysis. Actually, they are the infinitesimal versions of the subtraction and addition on numbers, respectively. From 1967 to 1970 Michael Grossman and Robert Katz indicated in their work [4] that infinitely many calculi can be generated independently. Later on Grossman introduced the so-called Bigeometric calculus [3], where he defined a new kind of derivative and integral, moving the roles of subtraction and addition to division and multiplication, and thus established a new calculus, called multiplicative calculus. The theoretical background of Multiplicative Calculus was given by Bashirov et al in [2]. Multiplicative calculus is based on multiplication and division. Sometimes, it is called an alternative or non-Newtonian calculus as well. As multiplicative calculus is the taylor-made calculus for growth related problems, that are modeled in science and engineering using the exponential function. It is more than self-evident to use mutiplicative calculus also for numerical approximations. Aniszewska developed in [1] the Multiplicative Runge Kutta Method using a different definition of the derivative

$$\frac{\pi f(x)}{\pi x} = \lim_{\epsilon \to 0} \left(\frac{f(1+\epsilon)x)}{f(x)} \right)^{1/\epsilon}$$

without the notion of a complete theory, especially without a multiplicative Taylor theorem. In contrast to Aniszewska [1] we will develop the multiplicative Runge-Kutta method on the basis of the complete theory of [2]. Apart from the multiplicative

Runge-Kutta Method, also the multiplicative finite difference method was invented by Riza et al in [6], and the multiplicative Adams Bashforth-Moulton methods where developed by Mısırlı and Gürefe [5].

In the first part of this thesis the parts of Multiplicative Calculus that are needed for the understanding of the later chapters are reviewed, i. e. we will give the definitions of the derivatives, Taylor Series and the Chain Rule in multiplicative sense. After that we will define another kind of calculus which is Volterra Calculus which can be expressed easily in terms of multiplicative calculus and give the definition of the derivative in terms of Volterra calculus. Then in chapter 3 the Ordinary Runge-Kutta methods of order 2,3, and 4 are reviewed and using the basic ideas developed by Runge and Kutta the derivations of the methods will be given in this chapter explicitely. After completing the basic knowledge needed for the development of the multiplicative Runge-Kutta method, the ideas developed in the previous chapters for the ordinary case will be combined to develop the multiplicative Runge Kutta methods of the orders 2, 3, and 4. In the next chapter, some examples will be solved by using the ordinary Runge-Kutta methods and the multiplicative Runge-Kutta methods. Then by using the results obtained from these examples we will compare the two methods to see which method gives the best solutions. Finally, in the last chapter multiplicative Finite Difference methods will be reviewed as a different approach.

Chapter 2

MULTIPLICATIVE CALCULUS

2.1 Multiplicative Derivatives

The multiplicative derivative can be defined by the formula

$$\lim_{h\longrightarrow 0}\left(\frac{f(x+h)}{f(x)}\right)^{1/h} \tag{2.1}$$

which shows us the number of times that $f(x)$ changes at the time moment x. By comparing the given definition of multiplicative derivative with the definition of the ordinary derivative, which is given as,

$$f'(x)=\lim_{h\longrightarrow 0}\frac{f(x+h)-f(x)}{h} \tag{2.2}$$

it can be observed that the difference $f(x+h)-f(x)$, in the ordinary derivative, is replaced by $\frac{f(x+h)}{f(x)}$ in the multiplicative derivative and the division by h is replaced by the reciprocal power $\frac{1}{h}$.

The limit (2.1) is called the *multiplicative derivative* or, **derivative* of f at x and it can be denoted by $f^*(x)$. If $f^*(x)$ exists for all x from some open set $A \subseteq \mathbb{R}$ then the function $f^*: A \to \mathbb{R}$ is well-defined. The function f^* is called the *derivative of $f: A \to \mathbb{R}$. The symbol $\frac{d^*f}{dt}$ can also be used for the multiplicative derivative. The *derivative of $f^*(x)$ is called the *second *derivative* of $f(x)$ and it can be denoted by $f^{**}(x)$. By using the same idea, the *nth *derivative* of $f(x)$ can also be defined, which is denoted by $f^{*(n)}(x)$ for $n = 0, 1, ...$ where $f^{*(0)}(x) = f$.

If $f(x)$ is a positive function on A and its derivative at x exists, then we may calculate

$$
\begin{aligned}
f^*(x) &= \lim_{h\longrightarrow 0} \frac{f(x+h)}{f(x)}^{1/h} \\
&= \lim_{h\longrightarrow 0} (1+\frac{f(x+h)-f(x)}{f(x)})^{\frac{f(x)}{f(x+h)-f(x)}\cdot\frac{f(x+h)-f(x)}{h}\cdot\frac{1}{f(x)}} \\
&= e^{\frac{f'(x)}{f(x)}} = e^{(\ln\circ f)'(x)}
\end{aligned}
$$

where $(\ln\circ f)(x) = \ln f(x)$. If, the second derivative of $f(x)$ exists, then by substituting $f^*(x)$, we obtain

$$f^{**}(x) = e^{(\ln\circ f)^{(n)}(x)} = e^{(\ln\circ f)''(x)}.$$

Since $f''(x)$ exists $(\ln\circ f)''(x)$ also exists. Repetition of the procedure n times, gives us that, if $f(x)$ is a positive function and its nth derivative exists, then $f^{*(n)}(x)$ also exists and

$$f^{*(n)}(x) = e^{(\ln\circ f)^{(n)}(x)}, n = 0, 1, ... \tag{2.3}$$

The case $n = 0$ is also included in the formula (2.3) since

$$f(x) = e^{(\ln\circ f)(x)}.$$

Thus we may conclude that, the function $f : A \to \mathbb{R}$ is *differentiable at x or on A if it is positive at x and differentiable on A.

By deriving a similar formula to Newton's binomial formula we can express $f^{(n)}$ in terms of $f^{*(n)}$. By using the $n-th$ multiplicative derivative of f, we have

$$(\ln\circ f^{*(n)})(x) = (\ln\circ f)^{(n)}(x) = ((\ln\circ f)^{(k)})^{(n-k)}(x) = (\ln\circ f^{*(k)})^{(n-k)}(x)$$

Thus by using

$$f'(x) = f(x)(\ln\circ f^*)(x),$$

we can calculate the second derivative in terms of the multiplicative derivative as,

$$f''(x) = f'(x)(\ln\circ f^*)(x) + f(x)(\ln\circ f^{**})(x),$$

and by using the second derivative, we can calculate the third derivative in terms of the multiplicative derivative as,

$$f'''(x) = f''(x)(\ln\circ f^*)(x) + 2f'(x)(\ln\circ f^{**})(x) + f(x)(\ln\circ f^{***})(x),$$

By repeating this procedure n times, we obtain the formula for $n-th$ derivative as follows:

$$f^{(n)}(x)=\sum_{k=0}^{n-1}\frac{(n-1)!}{k!(n-k-1)!}f^{(k)}(x)(\ln\circ f^{*(n-k)})(x) \tag{2.4}$$

For the constant function $f(x)=c>0$ on the interval (a,b), where $a<b$, we have

$$f^{*}(x)=e^{(\ln c)'}=e^{0}=1,\quad x\in(a,b).$$

If $f^{*}(x)$=1 for every $x\in(a,b)$, then by using the first multiplicative derivative

$$f^{*}(x)=e^{(\ln\circ f)'(x)}=1,$$

it can be easily seen that $f(x)$=const.>0 where $x\in(a,b)$. Thus we see that the neutral element 0 of addition appears instead of the neutral element 1 of multiplication.

Here are some rules of *differentiation:

$$\begin{aligned}
(cf)^{*}(x) &= f^{*}(x) && (2.5)\\
(fg)^{*}(x) &= f^{*}(x)g^{*}(x) && (2.6)\\
(\frac{f}{g})^{*}(x) &= \frac{f^{*}(x)}{g^{*}(x)} && (2.7)\\
(f^{h})^{*}(x) &= f^{*}(x)^{h(x)}\cdot f(x)^{h'(x)} && (2.8)\\
(f\circ h)^{*}(x) &= f^{*}(h(x))^{h'(x)} && (2.9)
\end{aligned}$$

where c is a positive constant, f and g are *differentiable, h is differentiable. Equation (2.6) can be proved as follows:

$$\begin{aligned}
(fg)^{*}(x) &= e^{(\ln\circ(fg))'(x)}=e^{(\ln\circ f)'(x)+(\ln\circ g)'(x)}\\
&= e^{(\ln\circ f)'(x)}\cdot e^{(\ln\circ g)'(x)}=f^{*}(x)g^{*}(x)
\end{aligned}$$

While on the other hand, the rules for sum and difference are complicated. Here is the rule for sum:

$$(f+g)^{*}(x)=f^{*}(x)^{\frac{f(x)}{f(x)+g(x)}}\cdot g^{*}(x)^{\frac{g(x)}{f(x)+g(x)}}$$

In the next step we can consider the differential equations involving *derivatives. The multiplicative differential equation which contains the *derivative of y can be shown as follows:

$$y^{*}(x)=f(x,y(x)) \tag{2.10}$$

Theorem 1 (Multiplicative Taylor's Theorem for One Variable). *Let A be an open interval and let $f : A \to \mathbb{R}$ be $n+1$ times *differentiable on A. Then for any x, $x+h \in A$, there exists a number $\theta \in (0,1)$ such that*

$$f(x+h) = \prod_{m=0}^{n} (f^{*(m)}(x))^{\frac{h^m}{m!}} \cdot (f^{*(n+1)}(x+\theta h))^{\frac{h^{n+1}}{(n+1)!}}$$

The partial *derivative of $f(x)$ can be defined, considering y fixed, and it is denoted by f_x^*. The partial *derivative of f in y can be defined in a similar way and denote it by f_y^*. We can also define higher order partial *derivatives of f.

The two results, generalizing the equation (2.9) of *differentiation and Multiplicative Taylor's Theorem for One Variable are as follows.

Theorem 2 (Multiplicative Chain Rule). *Let f be a function of two variables y and z with continuous partial *derivatives. If y and z are differentiable functions on (a,b) such that $f(y(x),z(x))$ is defined for every $x \in (a,b)$, then*

$$\frac{d^* f(y(x),z(x))}{dx} = f_y^*(y(x),z(x))^{y'(x)} f_z^*(y(x),z(x))^{z'(x)}$$

Theorem 3 (Multiplicative Taylor's Theorem for Two Variables). *Let A be an open subset of $\mathbb{R}^2$. Assume that the function $f : A \to \mathbb{R}$ has all partial *derivatives of order $n+1$ on A. Then for every (x,y), $(x+h,y+k) \in A$ so that the line segment connecting these two points belongs to A, there exists a number $\theta \in (0,1)$ such that,*

$$f(x+h,y+k) = \prod_{m=0}^{n} \prod_{i=0}^{m} f^{*(m)}_{x^i y^{m-i}}(x,y)^{\frac{h^i k^{m-i}}{i!(m-i)!}} \cdot \prod_{i=0}^{n+1} f^{*(n+1)}_{x^i y^{n+1-i}}(x+\theta h, y+\theta k)^{\frac{h^i k^{n+1-i}}{i!(n+1-i)!}}$$

2.2 Volterra Calculus

2.2.1 Volterra Differential Equations

Volterracalculus is another kind of multiplicative calculus, having multiplication as its main operation, which was created by Vito Volterra. Volterra calculus was introduced to define the derivative of dimensional functions that could not be done using the derivative in the Newtonian sense. It seems to be evident that multiplicative and Volterra differential calculus can be used more effectively as a mathematical tool instead of ordinary differential calculus for the mathematical representation of many problems in science and engineering that can be easily represented in these calculi. As we said a new kind of derivative was defined within this calculus, where the definition of this kind of derivative can be given as follows:

Definition 4. *Let f be a positive function over the open interval (a,b). If the limit*

$$f^{\pi}(x) = \frac{d^{\pi} f(x)}{dx} = \lim_{h\to 0}\left(\frac{f((1+h)x)}{f(x)}\right)^{\frac{1}{h}} \tag{2.11}$$

exists, then f is said to be Volterra type differentiable at $x \in (a,b)$.

The relationship between the ordinary and the Volterra derivative can be given as

$$f^{\pi}(x) = \exp(x(\ln \circ f)'(x)). \tag{2.12}$$

Thus by using the definition of the multiplicative derivative, the Volterra derivative can be written in terms of multiplicative derivative as follows:

$$f^{\pi}(x) = \frac{d^{\pi} f(x)}{dx} = (f^{*}(x))^{x}. \tag{2.13}$$

Representing a higher order Volterra derivative in terms of the ordinary derivative is complicated. Since the relationship between the Volterra derivative and the multiplicative derivative seems to be easier, representing the Volterra derivatives in terms of the multiplicative derivatives is much more easier than representation of the Volterra derivative in terms of the ordinary derivatives.

2.,3. and 4. order Volterra derivatives can be given in terms of the multiplicative derivatives as

$$\begin{aligned}
f^{\pi\pi}(x) &= (f^{**}(x))^{x^2}(f^{*})^{x}, && (2.14)\\
f^{\pi(3)}(x) &= (f^{*(3)}(x))^{x^3}(f^{**}(x))^{3x^2}(f^{*})^{x}, && (2.15)\\
f^{\pi(4)}(x) &= (f^{*(4)}(x))^{x^4}(f^{*(3)}(x))^{6x^3}(f^{**}(x))^{7x^2}(f^{*})^{x}. && (2.16)
\end{aligned}$$

We may also consider the differential equations involving π-derivatives. The Volterra differential equation containing the π-derivative of y can be given as

$$y^{\pi}(x) = f(x, y(x)) \tag{2.17}$$

where f is a positive function defined in some subset G of $\mathbb{R}^{+} \times \mathbb{R}^{+}$.

Chapter 3

RUNGE KUTTA METHODS

In numerical analysis, the Runge-Kutta methods are important family of implicit and explicit iterative methods for the approximation of solutions of ordinary differential equations. In the following we will follow the ideas of [7] to review the ordinary Runge-Kutta methods. These techniques were developed around 1900 by the German mathematicians C. Runge and M.W. Kutta. These methods are used to find the solutions of the ordinary differential equations of the form:

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{3.1}$$

Since $f(x, y(x))$ is just the slope $y'(x)$ of the desired exact solution $y(x)$ of (3.1), one has for $h \neq 0$ approximately

$$\frac{y(x+h) - y(x)}{h} \approx f(x, y(x)) \tag{3.2}$$

or

$$y(x+h) \approx y(x) + hf(x, y(x)) \tag{3.3}$$

Thus by using the initial condition $y(x_0) = y_0$, the solution of the equation (3.1) takes the form

$$y_{n+1} = y_n + hf(x_n, y_n) \tag{3.4}$$

We know that the Taylor series has desirable features, particularly in its ability to keep the errors small, but it also has the disadvantage of requiring the evaluation of the higher derivatives of the function $f(x, y)$. In the Taylor series method, each of these higher order derivatives is evaluated at the point x_i in order to evaluate $y(x_{i+1})$. The Runge-Kutta approach is to aim for the desirable features of the Taylor series method, but with the replacement of the requirement for the evaluation of higher order derivatives with the requirement to evaluate $f(x, y)$ at some points within the step x_i to x_{i+1}. Since it is not initially known at which points in the interval these evaluations should be done, it is possible to choose these points in such a way that the result is consistent with the Taylor series solution to some particular, which we shall call the order of the Runge-Kutta method.

Let us derive the 2. order Runge-Kutta method for the solution of the differential equation (3.1). Our starting point is the Taylor series expansion for $y(x + h)$, which has the form

$$y(x+h) = y(x) + hf(x,y) + \frac{h^2}{2}y''(x) + \frac{h^3}{3!}y'''(x) + \cdots + \frac{h^p}{p!}y^{(p)}(x+\theta h). \quad (3.5)$$

For the derivation of the 2. order Runge-Kutta method we should consider the second order Taylor series formula. Thus we need to evaluate $y''(x)$. Since we have $y'(x) = f(x, y)$, for $y''(x)$ we take the partial derivatives of $f(x, y)$ with respect to x and y, and get

$$y''(x) = f_x(x,y) + f_y(x,y)y'(x) \quad (3.6)$$

Again since we know that $y'(x) = f(x, y)$, instead of $y'(x)$ we can write $f(x, y)$ and get:

$$y''(x) = f_x(x,y) + f_y(x,y)f(x,y) \quad (3.7)$$

Then by substituting $y''(x)$, the 2. order Taylor series takes the form:

$$y(x+h) = y(x) + hf(x,y) + \frac{h^2}{2}(f_x(x,y) + f_y(x,y)f(x,y)) + O(h^3). \quad (3.8)$$

The Runge-Kutta method assumes that the correct value of the slope over the step can be written as a linear combination of the function $f(x, y)$ evaluated at certain points in the step. In the method of order 2 this results in writing the iteration step in the form:

$$y(x+h) = y(x) + Ahf_0 + Bhf_1 \quad (3.9)$$

where

$$f_0 = f(x,y) \quad (3.10)$$

$$f_1 = f(x+Ph, y+Qhf_0) \quad (3.11)$$

We still need to determine the constants A, B, P and Q, where we can do this by comparing the Runge-Kutta formula with the second order Taylor series given above. In order to do this we must find the Taylor series expansion for f_1, which can be written as:

$$f_1 = f(x,y) + f_x(x,y)Ph + f_y(x,y)Qhf_0 + O(h^2) \tag{3.12}$$

By substituting the Taylor expansion (3.12) of f_1 into the Runge-Kutta formula for $y(x+h)$ we obtain:

$$y(x+h) = y(x) + (A+B)hf(x,y) + Bh^2Pf_x(x,y) + Bh^2Qf_y(x,y)f(x,y) + O(h^3) \tag{3.13}$$

Now we can compare the two Taylor series expansions (3.8) and (3.13) to find relations for the constants A, B, P and Q.

$$A + B = 1, \quad BP = \frac{1}{2}, \quad BQ = \frac{1}{2} \tag{3.14}$$

We thus have three conditions on the four constants such that the direct Taylor series and the Runge-Kutta formula will agree to second order in h.

Since we have three conditions for the constants A, B, P and Q, we have more than one choice. If we choose $A = 0$, we have $B = 1$ and $P = Q = \frac{1}{2}$, which leads to the 2. order Runge-Kutta method:

$$y(x+h) = y(x) + hf_1 \tag{3.15}$$

where

$$f_0 = f(x,y) \tag{3.16}$$

$$f_1 = f\left(x + \frac{h}{2}, y + \frac{h}{2}f_0\right) \tag{3.17}$$

Runge-Kutta methods of order 3 and order 4 can be derived by using the same constructions. For the Runge-Kutta method of order 3 we need to consider the 3. order Taylor series expansion which has the form:

$$y(x+h) = y(x) + hf(x,y) + \frac{h^2}{2}y''(x) + \frac{h^3}{3!}y'''(x) + O(h^4). \tag{3.18}$$

Thus we need to find $y''(x)$ and $y'''(x)$ for the expansion. Since $y''(x)$ was calculated before we just calculate $y'''(x)$ which will take the form:

$$y'''(x) = f_{xx}(x,y) + 2f_{xy}(x,y)f(x,y) + f_{yy}(x,y)f(x,y)^2 + \\ + f_y(x,y)y''(x) \quad (3.19)$$

Then by substituting these derivatives into the 3. order Taylor series expansion we get:

$$y(x+h) = y(x) + hf(x,y) + \frac{h^2}{2}(f_x(x,y) + f_y(x,y)f(x,y)) + \\ + \frac{h^3}{3!}(f_{xx}(x,y) + 2f_{xy}(x,y)f(x,y) + f_{yy}(x,y)f(x,y)^2 + f_y(x,y)y''(x)) \quad (3.20)$$

For the Runge-Kutta method of order 3 we can make the ansatz:

$$y(x+h) = y(x) + Ahf_0 + Bhf_1 + Chf_2 \quad (3.21)$$

where

$$f_0 = f(x,y) \quad (3.22)$$
$$f_1 = f(x + P_1h, y + Q_1hf_0) \quad (3.23)$$
$$f_2 = f(x + P_2h, y + Q_2hf_0 + Q_3hf_1) \quad (3.24)$$

Then we need to find the Taylor series expansions for f_1 and f_2, substitute into $y(x+h)$ function for the Runge-Kutta method and get the Taylor series expansion for $y(x+h)$. After that by comparing the 3. order Taylor series expansion with the Taylor series expansion for $y(x+h)$ of the Runge-Kutta Method, we may find the values for the constants $A, B, C, P_1, P_2, Q_1, Q_2$ and Q_3.Thus we get the 3. order Runge-Kutta method as:

$$y(x+h) = y(x) + \frac{h}{6}(f_0 + 4f_1 + f_2) \quad (3.25)$$

where

$$f_0 = f(x,y) \quad (3.26)$$
$$f_1 = f(x + \frac{h}{2}, y + \frac{h}{2}f_0) \quad (3.27)$$
$$f_2 = hf(x + h, y - hf_0 + 2hf_1) \quad (3.28)$$

Also for the 4. order Runge-Kutta method we do the same constructions. The only difference is that we need to use the 4. order Taylor series expansion which has the form:

$$y(x+h) = y(x) + hf(x,y) + \frac{h^2}{2}y''(x) + \frac{h^3}{3!}y'''(x) + \frac{h^4}{4!}y^{(4)}(x) + O(h^5). \quad (3.29)$$

Since $y''(x)$ and $y'''(x)$ were calculated before, it remains only to calculate $y^{(4)}(x)$.

$$\begin{aligned} y^{(4)}(x) = {} & f_{xxx}(x,y) + 3f_{xxy}(x,y)f(x,y) + 3f_{xyy}(x,y)f(x,y)^2 + \\ & + f_{yyy}(x,y)f(x,y)^3 + 3f_{xy}(x,y)y''(x) + \\ & + 3f_{yy}(x,y)f(x,y)y''(x) + f_y(x,y)(x,y)y'''(x) \end{aligned} \quad (3.30)$$

After substituting the derivatives into the 4. order Taylor series expansion we get:

$$\begin{aligned} y(x+h) = {} & y(x) + hf(x,y) + \frac{h^2}{2}(f_x(x,y) + f_y(x,y)f(x,y)) + \\ & + \frac{h^3}{3!}(f_{xx}(x,y) + 2f_{xy}(x,y)f(x,y) + f_{yy}(x,y)f(x,y)^2 + f_y(x,y)y''(x)) + \\ & + \frac{h^4}{4!}(f_{xxx}(x,y) + 3f_{xxy}(x,y)f(x,y) + 3f_{xyy}(x,y)f(x,y)^2 + f_{yyy}(x,y)f(x,y)^3 + \\ & + 3f_{xy}(x,y)y''(x) + 3f_{yy}(x,y)f(x,y)y''(x) + f_y(x,y)(x,y)y'''(x)) \end{aligned} \quad (3.31)$$

Then we make the following ansatz for the Runge-Kutta method of order 4 :

$$y(x+h) = y(x) + Ahf_0 + Bhf_1 + Chf_2 + Dhf_3 \quad (3.32)$$

where

$$f_0 = f(x,y) \quad (3.33)$$

$$f_1 = f(x + P_1h, y + Q_1hf_0) \quad (3.34)$$

$$f_2 = f(x + P_2h, y + Q_2hf_0 + Q_3hf_1) \quad (3.35)$$

$$f_3 = f(x + P_3h, y + Q_4hf_0 + Q_5hf_1 + Q_6hf_2) \quad (3.36)$$

Then we need to find the Taylor series expansions for f_1, f_2 and f_3, substitute into $y(x+h)$ function for the Runge-Kutta method and get the Taylor series expansion for $y(x+h)$. After that by comparing the 4. order Taylor series expansion with the Taylor series expansion for $y(x+h)$ of the Runge-Kutta Method of order 4, we can find the values for the constants $A, B, C, D, P_1, P_2, P_3, Q_1, Q_2, Q_3, Q_4, Q_5$ and Q_6.Thus we get the 4. order Runge-Kutta method as:

$$y(x+h) = y(x) + \frac{h}{6}(f_0 + 2f_1 + 2f_2 + f_3) \tag{3.37}$$

where

$$\begin{aligned} f_0 &= f(x,y) &(3.38)\\ f_1 &= f(x+\frac{h}{2}, y+\frac{h}{2}f_0) &(3.39)\\ f_2 &= f(x+\frac{h}{2}, y+\frac{h}{2}f_1) &(3.40)\\ f_3 &= f(x+h, y+hf_2) &(3.41)\end{aligned}$$

Chapter 4

MULTIPLICATIVE RUNGE KUTTA METHODS

4.1 Multiplicative Runge Kutta Method of order 2

In the followingwe will derive the multiplicative Runge Kutta Method of order 2 for the solution of the multiplicative differential equation:

$$y^*(x) = f(x, y) \tag{4.1}$$

Analogously to the ordinary Runge Kutta Method our starting point is the Taylor expansion for $y(x + h)$. In this case we will use the multiplicative Taylor expansion as given in Theorem 2 of [2] for the first two terms

$$y(x + h) = y(x) \cdot (y^*(x))^h \cdot (y^{**}(x))^{h^2/2} \cdot \ldots \tag{4.2}$$

using the multiplicative differential equation (4.1) we can substitute $y^*(x)$ by $f(x, y)$ and get:

$$y(x + h) = y(x) \cdot (f(x, y))^h \cdot (y^{**}(x))^{h^2/2} \cdot \ldots \tag{4.3}$$

Since $y^*(x) = f(x, y)$ we can write $y^{**}(x)$ as:

$$y^{**}(x) = (y^*(x))^* = (f(x, y))^* \,. \tag{4.4}$$

Keeping in mind that $y(x)$ is a function in the variable x we have to apply the multiplicative chain rule as denoted in Theorem 3 in [2] to get $f(x, y(x))^*$.

$$\frac{d^*}{dx^*} f(x, y) = f_x^*(x, y) f_y^*(x, y)^{y'(x)} \tag{4.5}$$

inserting the result of equation(4.5) into (4.4) gives

$$y^{**}(x) = (y^*(x))^* = f_x^*(x, y) f_y^*(x, y)^{y'(x)}. \tag{4.6}$$

Inserting $y^{**}(x)$ into the multiplicative Taylor expansion, which is equation (4.3) we get:

$$y(x+h) = y(x) \cdot (f(x,y))^h \cdot \left(f_x^*(x,y) f_y^*(x,y)^{y'(x)}\right)^{h^2/2} \cdot \ldots \tag{4.7}$$

where $f_x^*(x,y)$ and $f_y^*(x,y)$ denote the multiplicative partial derivatives with respect to x and y. In analogy to the ordinary Runge Kutta Method we make now the ansatz:

$$y(x+h) = y(x) \cdot f_0^{ah} \cdot f_1^{bh} \tag{4.8}$$

with

$$f_0 = f(x,y) \tag{4.9}$$

$$f_1 = f(x+ph, y \cdot f_0^{qh}) \tag{4.10}$$

Again by using the multiplicative Taylor Series we need to find the Taylor expansion for f_1. Thus f_1 becomes:

$$f_1 = f(x,y) \cdot f_x^*(x,y)^{ph} \cdot f_y^*(x,y)^{(yf_0^q h)' ph} \tag{4.11}$$

Then by inserting f_0 and the multiplicative Taylor expansion of f_1 into the function $y(x+h)$ of the Runge-Kutta method we get

$$y(x+h) = y(x) \cdot (f(x,y))^{ah} \cdot (f(x,y) \cdot f_x^*(x,y)^{ph} \cdot f_y^*(x,y)^{(yf_0^q h)' ph})^{bh} \tag{4.12}$$

After that we need to rearrange the terms with respect to orders of h and get the Taylor expansion for $y(x+h)$ in the Runge-Kutta method:

$$y(x+h) = y(x) \cdot (f(x,y))^{(a+b)h} \cdot f_x^*(x,y)^{bph^2} \cdot f_y^*(x,y)^{bpf_0^{qh} y' h^2} \cdot \ldots \tag{4.13}$$

Thus the comparison of the two Taylor series expansions which are the equations (4.7) and (4.13) gives:

$$a+b = 1 \tag{4.14}$$

$$bp = \frac{1}{2} \tag{4.15}$$

$$bpf_0^q h = \frac{1}{2} \tag{4.16}$$

Therefore we get $q = 0$. If we select $p = 1$ we get $a = \frac{1}{2}$ and $b = \frac{1}{2}$. Thus the 2. order multiplicative Runge-Kutta method takes the form:

$$y(x+h) = y(x) \cdot f_0^{\frac{h}{2}} \cdot f_1^{\frac{h}{2}} \tag{4.17}$$

where

$$f_0 = f(x, y) \tag{4.18}$$

$$f_1 = f(x+h, y) \tag{4.19}$$

4.2 Multiplicative Runge Kutta Method of order 3

In the following wewill the derive the multiplicative Runge Kutta Method of order 3 for the solution of the multiplicative differential equation:

$$y^*(x) = f(x, y) \tag{4.20}$$

Analogously to the ordinary Runge Kutta Method our starting point is the Taylor expansion for $y(x+h)$. In this case we will use the multiplicative Taylor expansion as given in Theorem 2 of [2] for the first four terms

$$y(x+h) = y(x) \cdot (y^*(x))^h \cdot (y^{**}(x))^{h^2/2} \cdot (y^{***}(x))^{h^3/3!} \cdot \ldots \tag{4.21}$$

using the multiplicative differential equation (4.20) we can substitude $y^*(x)$ by $f(x, y)$ and get:

$$y(x+h) = y(x) \cdot (f(x, y))^h \cdot (y^{**}(x))^{h^2/2} \cdot (y^{***}(x))^{h^3/3!} \cdot \ldots \tag{4.22}$$

Since $y^*(x) = f(x, y)$ we can write $y^{**}(x)$ and $y^{***}(x)$ as:

$$y^{**}(x) = (y^*(x))^* = (f(x, y))^* \tag{4.23}$$

$$y^{***}(x) = (y^*(x))^{**} = (f(x, y))^{**} \tag{4.24}$$

Keeping in mind that $y(x)$ is a function in the variable x we have to apply the multiplicative chain rule as denoted in Theorem 3 in [2] to get $f(x,y(x))^{**}$ and $f(x,y(x))^{***}$.

$$\begin{aligned}
\frac{d^*}{dx}f(x,y) &= f_x^*(x,y)\cdot f_y^*(x,y)^{y'(x)} && (4.25)\\
\frac{d^{**}}{dx}f(x,y) &= \frac{d^*}{dx}\left(\frac{d^*}{dx}f(x,y)\right)\\
&= \frac{d^*}{dx}\left(f_x^*(x,y)\cdot f_y^*(x,y)^{y'(x)}\right)\\
&= f_{xx}^*(x,y)\cdot f_{xy}^*(x,y)^{y'(x)}\cdot f_{yx}^*(x,y)^{y'(x)}\cdot\\
&\quad f_{yy}^*(x,y)^{y'(x)^2}\cdot f_y^*(x,y)^{y''(x)} && (4.26)
\end{aligned}$$

The partial multiplicative derivatives are commutative. This can be shown easily by applying the partial multiplicative derivative in different orders and compare them.

$$f_{xy}^*(x,y)=\partial_y^*(\partial_x^*f(x,y))=\partial_y^*\exp\left\{\frac{f_x(x,y)}{f(x,y)}\right\}=\exp\left\{\partial_y\ln\exp\left\{\frac{f_x(x,y)}{f(x,y)}\right\}\right\}=$$
$$=\exp\left\{\partial_y\left\{\frac{f_x(x,y)}{f(x,y)}\right\}\right\}=\exp\left\{\frac{f_{xy}(x,y)f(x,y)-f_y(x,y)f_x(x,y)}{f(x,y)^2}\right\}\quad(4.27)$$

$$f_{yx}^*(x,y)=\partial_x^*(\partial_y^*f(x,y))=\partial_x^*\exp\left\{\frac{f_y(x,y)}{f(x,y)}\right\}=\exp\left\{\partial_x\ln\exp\left\{\frac{f_y(x,y)}{f(x,y)}\right\}\right\}=$$
$$=\exp\left\{\partial_x\left\{\frac{f_y(x,y)}{f(x,y)}\right\}\right\}=\exp\left\{\frac{f_{yx}(x,y)f(x,y)-f_x(x,y)f_y(x,y)}{f(x,y)^2}\right\}\quad(4.28)$$

With the property that the partial derivatives are commutative in newtonian calculus we can see by comparing the equations (4.27) and (4.28) that also the multiplicative partial differentiation is commutative. So equation (4.26), which defines the second multiplicative derivative, simplifies then to:

$$\frac{d^{**}}{dx}f(x,y)=f_{xx}^*(x,y)\cdot f_{xy}^*(x,y)^{2y'(x)}\cdot f_{yy}^*(x,y)^{y'(x)^2}\cdot f_y^*(x,y)^{y''(x)}\quad(4.29)$$

Inserting the result of equation (4.5) and (4.29) into (4.22) gives the third order multiplicative Taylor series expansion as:

$$y(x+h)=y(x)\cdot(f(x,y))^h\cdot(f_x^*(x,y)\cdot f_y^*(x,y)^{y'(x)})^{h^2/2}\cdot(f_{xx}^*(x,y)\cdot f_{xy}^*(x,y)^{2y'(x)}\cdot$$
$$\cdot f_{yy}^*(x,y)^{y'(x)^2}\cdot f_y^*(x,y)^{y''(x)})^{h^3/3!}\cdot\ldots\quad(4.30)$$

In analogy to the ordinary Runge Kutta Method we make now the ansatz:

$$y(x+h) = y(x) \cdot f_0^{ah} \cdot f_1^{bh} \cdot f_2^{ch} \tag{4.31}$$

with

$$\begin{aligned} f_0 &= f(x,y) & (4.32)\\ f_1 &= f(x+ph, y \cdot f_0^{qh}) & (4.33)\\ f_2 &= f(x+p_1h, y \cdot f_0^{q_1h} \cdot f_1^{q_2h}) & (4.34) \end{aligned}$$

Then by using the multiplicative Taylor Series we can find the Taylor expansions for f_1 and f_2 as

$$f_1 = f(x,y) \cdot f_x^*(x,y)^{ph} \cdot f_y^*(x,y)^{(yf_0^q h)' ph} \tag{4.35}$$

and

$$f_2 = f(x,y) \cdot f_x^*(x,y)^{p_1h} \cdot f_y^*(x,y)^{(yf_0^{q_1h}f_1^{q_2h})' p_1h} \cdot f_{xx}^*(x,y)^{((p_1h)^2)/2} \cdot f_{xy}^*(x,y)^{(p_1h)^2(yf_0^{q_1h}f_1^{q_2h})'} \cdot$$
$$\cdot f_{yy}^*(x,y)^{(p_1h)^2(yf_0^{q_1h}f_1^{q_2h})'^2/2} \cdot f_y^*(x,y)^{(p_1h)^2(yf_0^{q_1h}f_1^{q_2h})''/2} \tag{4.36}$$

Then by inserting the multiplicative Taylor series expansions for f_0, f_1 and f_2, which are the equations (4.9) , (4.35) and (4.36), into the equation (4.31) we get the Taylor series expansion for the 3. order Runge-Kutta method function $y(x+h)$ as:

$$y(x+h) = y(x) \cdot (f(x,y))^{ah} \cdot (f(x,y) \cdot f_x^*(x,y)^{ph} \cdot f_y^*(x,y)^{(yf_0^{qh})' ph})^{bh} \cdot (f(x,y) \cdot f_x^*(x,y)^{p_1h} \cdot$$
$$\cdot f_y^*(x,y)^{(yf_0^{q_1h}f_1^{q_2h})' p_1h} \cdot f_{xx}^*(x,y)^{((p_1h)^2)/2} \cdot f_{xy}^*(x,y)^{(p_1h)^2(yf_0^{q_1h}f_1^{q_2h})'} \cdot$$
$$\cdot f_{yy}^*(x,y)^{(p_1h)^2(yf_0^{q_1h}f_1^{q_2h})'^2/2} \cdot f_y^*(x,y)^{(p_1h)^2(yf_0^{q_1h}f_1^{q_2h})''/2})^{ch} \tag{4.37}$$

After rearranging the terms with respect to orders of h we get:

$$y(x+h) = y(x) \cdot (f(x,y))^{(a+b+c)h} \cdot f_x^*(x,y)^{(pb+cp_1)h^2} \cdot f_y^*(x,y)^{(pbf_0^{qh}+cp_1f_0^{q_1h}f_1^{q_2h})h^2 y'} \cdot$$
$$\cdot f_{xx}^*(x,y)^{cp_1h^3/2} \cdot f_{xy}^*(x,y)^{cp_1^2 f_0^{q_1h}f_1^{q_2h}h^3 y'} \cdot f_{yy}^*(x,y)^{cp_1^2 f_0^{2q_1h}f_1^{2q_2h}h^3(y')^2/2} \cdot f_y^*(x,y)^{cp_1^2 f_0^{q_1h}f_1^{q_2h}h^3 y''/2}$$
$$\tag{4.38}$$

Thus by comparing the 3. order multiplicative Taylor series expansion with the Taylor series expansion of the 3. order Runge-Kutta method we get the equalities:

$$
\begin{aligned}
a + b + c &= 1 && (4.39)\\
bp + cp_1 &= \frac{1}{2} && (4.40)\\
bpf_0^{qh} + cp_1 f_0^{q_1 h} f_1^{q_2 h} &= \frac{1}{2} && (4.41)\\
cp_1^2 &= \frac{1}{3} && (4.42)\\
cp_1^2 f_0^{q_1 h} f_1^{q_2 h} &= \frac{1}{3} && (4.43)\\
cp_1^2 f_0^{2q_1 h} f_1^{2q_2 h} &= \frac{1}{3} && (4.44)\\
cp_1^2 f_0^{q_1 h} f_1^{q_2 h} &= \frac{1}{3} && (4.45)
\end{aligned}
$$

Therefore we get $q = 0$. If we select $p = \frac{1}{2}$ and $p_1 = 1$ we get $q_1 = -q_2 \frac{\ln f_1}{\ln f_0}$, $q_2 = 1$, $a = \frac{1}{3}$, $b = \frac{1}{3}$ and $c = \frac{1}{3}$. Thus the 3. order Runge-Kutta method takes the form:

$$
y(x + h) = y(x) \cdot f_0^{\frac{h}{3}} \cdot f_1^{\frac{h}{3}} \cdot f_2^{\frac{h}{3}} \tag{4.46}
$$

with

$$
\begin{aligned}
f_0 &= f(x, y) && (4.47)\\
f_1 &= f(x + \frac{h}{2}, y) && (4.48)\\
f_2 &= f(x + h, y \cdot f_0^{-\frac{\ln f_1}{\ln f_0} h} \cdot f_1^h) && (4.49)
\end{aligned}
$$

4.3 Multiplicative Runge Kutta Method of order 4

In the following wewill the derive the multiplicative Runge Kutta Method of order 4 for the solution of the multiplicative differential equation:

$$
y^*(x) = f(x, y) \tag{4.50}
$$

Analogously to the ordinary Runge Kutta Method our starting point is the Taylor expansion for $y(x + h)$. In this case we will use the multiplicative Taylor expansion as given in Theorem 2 of [2] for the first five terms

$$
y(x + h) = y(x) \cdot (y^*(x))^h \cdot (y^{**}(x))^{h^2/2} \cdot (y^{***}(x))^{h^3/3!} \cdot y^{*(4)}(x)^{h^4/4!} \cdot \ldots \tag{4.51}
$$

using the multiplicative differential equation (4.50) we can substitude $y^*(x)$ by $f(x,y)$ and get:

$$y(x+h) = y(x) \cdot (f(x,y))^h \cdot (y^{**}(x))^{h^2/2!} \cdot (y^{***}(x))^{h^3/3!} \cdot y^{*(4)}(x)^{h^4/4!} \cdot \ldots \quad (4.52)$$

Since $y^*(x) = f(x,y)$ we can write $y^{**}(x)$, $y^{***}(x)$, $y^{*(4)}(x)$ as:

$$\begin{aligned} y^{**}(x) &= (y^*(x))^* = (f(x,y))^* && (4.53)\\ y^{***}(x) &= (y^*(x))^{**} = (f(x,y))^{**} && (4.54)\\ y^{*(4)}(x) &= (y^*(x))^{***} = (f(x,y))^{***} && (4.55) \end{aligned}$$

Keeping in mind that $y(x)$ is a function in the variable x we have to apply the multiplicative chain rule as denoted in Theorem 3 in [2] to get $f(x,y(x))^{**}$, $f(x,y(x))^{***}$ and $f(x,y(x))^{*(4)}$.

$$\begin{aligned} \frac{d^*}{dx}f(x,y) &= f_x^*(x,y) \cdot f_y^*(x,y)^{y'(x)} && (4.56)\\ \frac{d^{**}}{dx}f(x,y) &= f_{xx}^*(x,y) \cdot f_{xy}^*(x,y)^{2y'(x)} \cdot f_{yy}^*(x,y)^{y'(x)^2} \cdot f_y^*(x,y)^{y''(x)} && (4.57)\\ \frac{d^{***}}{dx}f(x,y) &= f_{xxx}^*(x,y) \cdot f_{xxy}^*(x,y)^{3y'(x)} \cdot f_{xyy}^*(x,y)^{3y'(x)^2} \cdot f_{yyy}^*(x,y)^{y'(x)^3} \cdot \\ &\quad \cdot f_{xy}^*(x,y)^{3y''(x)} \cdot f_{yy}^*(x,y)^{3y'(x)y''(x)} \cdot f_y^*(x,y)^{y'''(x)} && (4.58) \end{aligned}$$

Inserting the results of those second, third and forth multiplicative derivatives into (4.52) gives us the 4. order multiplicative Taylor series expansion as follows:

$$\begin{aligned} y(x+h) &= y(x) \cdot (f(x,y))^h \cdot (f_x^*(x,y) \cdot (f_y^*(x,y)^{y'(x)})^{h^2/2!} \cdot (f_{xx}^*(x,y) \cdot f_{xy}^*(x,y)^{2y'(x)} \cdot \\ &\cdot f_{yy}^*(x,y)^{y'(x)^2} \cdot f_y^*(x,y)^{y''(x)})^{h^3/3!} \cdot (f_{xxx}^*(x,y) \cdot f_{xxy}^*(x,y)^{3y'(x)} \cdot f_{xyy}^*(x,y)^{3y'(x)^2} \cdot \\ &\cdot f_{yyy}^*(x,y)^{y'(x)^3} \cdot f_{xy}^*(x,y)^{3y''(x)} \cdot f_{yy}^*(x,y)^{3y'(x)y''(x)} \cdot f_y^*(x,y)^{y'''(x)})^{h^4/4!} \cdot \ldots \quad (4.59) \end{aligned}$$

In analogy to the ordinary Runge Kutta method we now make the ansatz:

$$y(x+h) = y(x) \cdot f_0^{ah} \cdot f_1^{bh} \cdot f_2^{ch} \cdot f_3^{dh} \quad (4.60)$$

with

$$f_0 = f(x, y) \tag{4.61}$$

$$f_1 = f(x + ph, y \cdot f_0^{qh}) \tag{4.62}$$

$$f_2 = f(x + p_1 h, y \cdot f_0^{q_1 h} \cdot f_1^{q_2 h}) \tag{4.63}$$

$$f_3 = f(x + p_2 h, y \cdot f_0^{q_3 h} \cdot f_1^{q_4 h} \cdot f_2^{q_5 h}) \tag{4.64}$$

By using the multiplicative Taylor Series we can find the Taylor expansions for f_1 , f_2 and f_3 as

$$f_1 = f(x, y) \cdot f_x^*(x, y)^{ph} \cdot f_y^*(x, y)^{(yf_0^q h)' ph} \tag{4.65}$$

$$\begin{aligned} f_2 = f(x, y) \cdot f_x^*(x, y)^{p_1 h} \cdot f_y^*(x, y)^{(y f_0^{q_1 h} f_1^{q_2 h})' p_1 h} \cdot \\ \cdot f_{xx}^*(x, y)^{((p_1 h)^2)/2} \cdot f_{xy}^*(x, y)^{(p_1 h)^2 (y f_0^{q_1 h} f_1^{q_2 h})'} \cdot \\ \cdot f_{yy}^*(x, y)^{(p_1 h)^2 (y f_0^{q_1 h} f_1^{q_2 h})'^2 /2} \cdot f_y^*(x, y)^{(p_1 h)^2 (y f_0^{q_1 h} f_1^{q_2 h})''/2} \end{aligned} \tag{4.66}$$

$$\begin{aligned} f_3 = f(x, y) \cdot f_x^*(x, y)^{p_2 h} \cdot f_y^*(x, y)^{(y f_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})' p_2 h} \cdot f_{xx}^*(x, y)^{((p_2 h)^2)/2} \cdot \\ \cdot f_{xy}^*(x, y)^{(p_2 h)^2 (y f_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})'} \cdot f_{yy}^*(x, y)^{(p_2 h)^2 (y f_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})'^2 /2} \cdot f_y^*(x, y)^{(p_2 h)^2 (y f_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})''/2} \cdot \\ \cdot f_{xxx}^*(x, y)^{(p_2 h)^3 /6} \cdot f_{xxy}^*(x, y)^{(p_2 h)^3 (y f_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})'/2} \cdot f_{xyy}^*(x, y)^{(p_2 h)^3 (y f_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})'^2 /2} \cdot \\ \cdot f_{yyy}^*(x, y)^{(p_2 h)^3 (y f_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})'^3 /6} \cdot f_{xy}^*(x, y)^{(p_2 h)^3 (y f_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})''/2} \cdot \\ \cdot f_{yy}^*(x, y)^{(p_2 h)^3 (y f_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})' (y f_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})''/2} \cdot f_y^*(x, y)^{(p_2 h)^3 (y f_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})'''/6} \end{aligned} \tag{4.67}$$

Then by inserting the results of the equations (4.61) , (4.65) , (4.66) and (4.67) into the function $y(x + h)$ in the 4. order Runge-Kutta method, which is the equation (4.60), we get the Taylor series expansion for $y(x + h)$ as:

$$
\begin{aligned}
y(x+h) = y(x) \cdot (f(x,y))^{ah} \cdot (f(x,y) \cdot f_x^*(x,y)^{ph} \cdot f_y^*(x,y)^{(yf_0^q h)' ph})^{bh} \cdot \\
\cdot (f(x,y) \cdot f_x^*(x,y)^{p_1 h} \cdot f_y^*(x,y)^{(yf_0^{q_1 h} f_1^{q_2 h})' p_1 h} \cdot f_{xx}^*(x,y)^{((p_1 h)^2)/2} \cdot \\
\cdot f_{xy}^*(x,y)^{(p_1 h)^2 (yf_0^{q_1 h} f_1^{q_2 h})'} \cdot f_{yy}^*(x,y)^{(p_1 h)^2 (yf_0^{q_1 h} f_1^{q_2 h})'^2/2} \cdot f_y^*(x,y)^{(p_1 h)^2 (yf_0^{q_1 h} f_1^{q_2 h})''/2})^{ch} \\
\cdot \cdot (f(x,y) \cdot f_x^*(x,y)^{p_2 h} \cdot f_y^*(x,y)^{(yf_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})' p_2 h} \cdot f_{xx}^*(x,y)^{((p_2 h)^2)/2} \cdot \\
\cdot f_{xy}^*(x,y)^{(p_2 h)^2 (yf_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})'} \cdot f_{yy}^*(x,y)^{(p_2 h)^2 (yf_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})'^2/2} \cdot f_y^*(x,y)^{(p_2 h)^2 (yf_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})''/2} \cdot \\
\cdot f_{xxx}^*(x,y)^{(p_2 h)^3/6} \cdot f_{xxy}^*(x,y)^{(p_2 h)^3 (yf_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})'/2} \cdot f_{xyy}^*(x,y)^{(p_2 h)^3 (yf_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})'^2/2} \cdot \\
\cdot f_{yyy}^*(x,y)^{(p_2 h)^3 (yf_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})'^3/6} \cdot f_{xy}^*(x,y)^{(p_2 h)^3 (yf_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})''/2} \cdot \\
\cdot f_{yy}^*(x,y)^{(p_2 h)^3 (yf_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})' (yf_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})''/2} \cdot f_y^*(x,y)^{(p_2 h)^3 (yf_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h})'''/6})^{dh}
\end{aligned}
\tag{4.68}
$$

After rearranging the terms with respect to the orders of h we get:

$$
\begin{aligned}
y(x+h) = y(x) \cdot f(x,y)^{(a+b+c+d)h} \cdot f_x^*(x,y)^{(pb+p_1 c+p_2 d)h^2} \cdot f_y^*(x,y)^{(pbf_0^q h+p_1 c f_0^{q_1 h} f_1^{q_2 h}) y'(x) h^2} \cdot \\
\cdot f_{xx}^*(x,y)^{((p_1^2 c+p_2^2 d)h^3)/2} \cdot f_{xy}^*(x,y)^{(p_1^2 c f_0^{q_1 h} f_1^{q_2 h}+p_2^2 d f_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h}) y'(x) h^3} \cdot \\
\cdot f_{yy}^*(x,y)^{((p_1^2 c f_0^{2q_1 h} f_1^{2q_2 h}+p_2^2 d f_0^{2q_3 h} f_1^{2q_4 h} f_2^{2q_5 h}) y'(x)^2 h^3)/2} \cdot f_y^*(x,y)^{((p_1^2 c f_0^{q_1 h} f_1^{q_2 h}+p_2^2 d f_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h}) y''(x) h^3)/2} \cdot \\
\cdot f_{xxx}^*(x,y)^{(p_2^3 d h^4)/6} \cdot f_{xxy}^*(x,y)^{(p_2^3 d f_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h} y'(x) h^4)/2} \cdot f_{xyy}^*(x,y)^{(p_2^3 d f_0^{2q_3 h} f_1^{2q_4 h} f_2^{2q_5 h} y'(x)^2 h^4)/2} \cdot \\
\cdot f_{yyy}^*(x,y)^{(p_2^3 d f_0^{3q_3 h} f_1^{3q_4 h} f_2^{3q_5 h} y'(x)^3 h^4/6} \cdot f_{xy}^*(x,y)^{(p_2^3 d f_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h} y''(x) h^4)/2} \cdot \\
\cdot f_{yy}^*(x,y)^{(p_2^3 d f_0^{2q_3 h} f_1^{2q_4 h} f_2^{2q_5 h} y'(x) y''(x) h^4)/2} \cdot f_y^*(x,y)^{(p_2^3 d f_0^{q_3 h} f_1^{q_4 h} f_2^{q_5 h} y'''(x) h^4)/6})
\end{aligned}
\tag{4.69}
$$

Then we need to compare the two Taylor series expansions for $y(x+h)$, which are (4.59) and (4.69), in order to find the constants. After those comparisons we get the following equalities:

$$\begin{aligned}
a+b+c+d &= 1 && (4.70)\\
pb+p_1c+p_2d &= \frac{1}{2} && (4.71)\\
pbf_0^{qh}+p_1cf_0^{q_1h}f_1^{q_2h} &= \frac{1}{2} && (4.72)\\
p_1^2c+p_2^2d &= \frac{1}{3} && (4.73)\\
p_1^2cf_0^{q_1h}f_1^{q_2h}+p_2^2df_0^{q_3h}f_1^{q_4h}f_2^{q_5h} &= \frac{1}{3} && (4.74)\\
p_1^2cf_0^{2q_1h}f_1^{2q_2h}+p_2^2df_0^{2q_3h}f_1^{2q_4h}f_2^{2q_5h} &= \frac{1}{3} && (4.75)\\
p_1^2cf_0^{q_1h}f_1^{q_2h}+p_2^2df_0^{q_3h}f_1^{q_4h}f_2^{q_5h} &= \frac{1}{3} && (4.76)\\
p_2^3d &= \frac{1}{4} && (4.77)\\
p_2^3df_0^{q_3h}f_1^{q_4h}f_2^{q_5h} &= \frac{1}{4} && (4.78)\\
p_2^3df_0^{2q_3h}f_1^{2q_4h}f_2^{2q_5h} &= \frac{1}{4} && (4.79)\\
p_2^3df_0^{3q_3h}f_1^{3q_4h}f_2^{3q_5h} &= \frac{1}{4} && (4.80)\\
p_2^3df_0^{q_3h}f_1^{q_4h}f_2^{q_5h} &= \frac{1}{4} && (4.81)\\
p_2^3df_0^{2q_3h}f_1^{2q_4h}f_2^{2q_5h} &= \frac{1}{4} && (4.82)\\
p_2^3df_0^{q_3h}f_1^{q_4h}f_2^{q_5h} &= \frac{1}{4} && (4.83)
\end{aligned}$$

Therefore we get $q = 0$. If we select $p = \frac{1}{4}$, $p_1 = \frac{1}{2}$, $p_2 = 1$, $q_1 = 0$, $q_2 = 0$, $q_3 = 0$, $q_4 = 0$, $q_5 = 0$ we get $a = \frac{1}{12}$, $b = \frac{1}{3}$, $c = \frac{1}{3}$ and $d = \frac{1}{4}$. Thus the 4. order Runge-Kutta method takes the form:

$$y(x+h) = y(x)\cdot f_0^{\frac{h}{12}}\cdot f_1^{\frac{h}{3}}\cdot f_2^{\frac{h}{3}}\cdot f_3^{\frac{h}{4}} \qquad (4.84)$$

with

$$\begin{aligned}
f_0 &= f(x,y) && (4.85)\\
f_1 &= f(x+\frac{h}{4},y) && (4.86)\\
f_2 &= f(x+\frac{h}{2},y) && (4.87)\\
f_3 &= f(x+h,y) && (4.88)
\end{aligned}$$

Chapter 5

COMPARISON OF MULTIPLICATIVE AND ORDINARY RUNGE-KUTTA METHODS

5.1 Comparison of the solutions for $y^* = e^x$ and $y' = xy$

Let us consider the followingfirst order multiplicative differential equation:

$$y^*(x) = e^x \tag{5.1}$$

with the initial condition $y(0) = 1$.

The analytic solution of this multiplicative differential equation is

$$y(x) = e^{\frac{x^2}{2}} \tag{5.2}$$

The corresponding ordinary differential equation to the multiplicative differential equation (5.1) is

$$y'(x) = xy \tag{5.3}$$

with the initial condition $y(0) = 1$ and has the exact solution

$$y'(x) = e^{\frac{x^2}{2}} \tag{5.4}$$

Firstly we will solve the multiplicative differential equation by using the 2. order multiplicative Runge Kutta method and the 3. order multiplicative Runge Kutta method. Thus we can compare these two methods. After that we will solve the corresponding differential equation with the 3. order ordinary Runge Kutta method in order to compare the multiplicative Runge Kutta method with the ordinary Runge Kutta method.

Now let us solve the multiplicative differential equation by using the 2. order multiplicative Runge Kutta method.

Solution of the multiplicative differential equation for $n = 20$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
0.5	1.1331484530668263	1.1331484530668259	4.44089×10^{-16}
1	1.6487212707001282	1.648721270700128	1.11022×10^{-16}
1.5	3.080216848918031	3.080216848918032	2.22045×10^{-16}

Table 5.1: Second order Multiplicative Runge-Kutta method for $y^* = e^x$ for $n = 20$

Then let us solve the multiplicative differential equation by using the 3. order multiplicative Runge Kutta method.

Solution of the multiplicative differential equation for $n = 20$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
0.5	1.1331484530668263	1.1331484530668252	9.99201×10^{-16}
1	1.6487212707001282	1.6487212707001269	7.77156×10^{-16}
1.5	3.080216848918031	3.0802168489180297	4.44089×10^{-16}

Table 5.2: Third order Multiplicative Runge-Kutta method for $y^* = e^x$ for $n = 20$

We will solve the equation (5.3) by using both the 3. order Runge-Kutta method and the equation (5.1) by using the 3. order multiplicative Runge-Kutta method. Then we will compare the two methods.

The solution of the multiplicative differential equation (5.1) and the solution of the ordinary differential equation (5.3) for the stepsize $n = 5$ can be approximated as shown in the tables below. We can summarize the results in tabular form as follows:

Solution of the multiplicative differential equation for $n = 5$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
0.4	1.0832870676749586	1.0832870676749584	1.11022×10^{-16}
1.2	2.054433210643888	2.054433210643888	0
2	7.38905609893065	7.389056098930651	2.22045×10^{-16}

Table 5.3: Third order Multiplicative Runge-Kutta method for $y^* = e^x$ for $n = 5$

The graph below shows the solutions of the multiplicative Runge Kutta method and the exact function, where the dotted graph represents the solutions of the multiplicative Runge Kutta method.

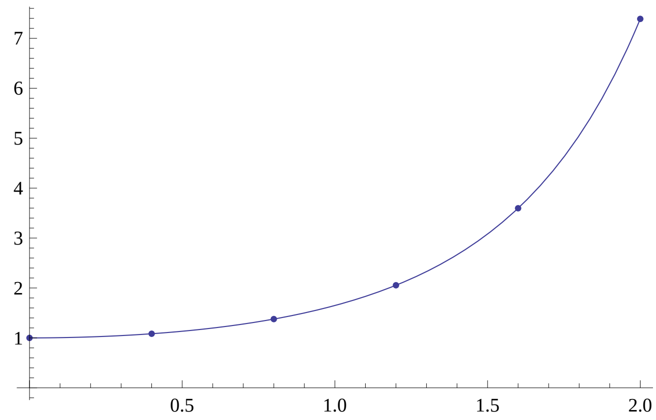

Figure 5.1: Graphs of multiplicative Runge Kutta Method and the exact function for $n = 5$

Solution of the ordinary differential equation for $n = 5$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
0.4	1.0832870676749586	1.0842666666666667	0.000904284
1.2	2.054433210643888	2.056653084150746	0.00108053
2	7.38905609893065	7.333859624050335	0.00747003

Table 5.4: Third order Ordinary Runge-Kutta method for $y' = x \cdot y$ for $n = 5$

The graph below shows the solutions of the ordinary Runge Kutta method and the exact function, where the dotted graph represents the solutions of the ordinary Runge Kutta method.

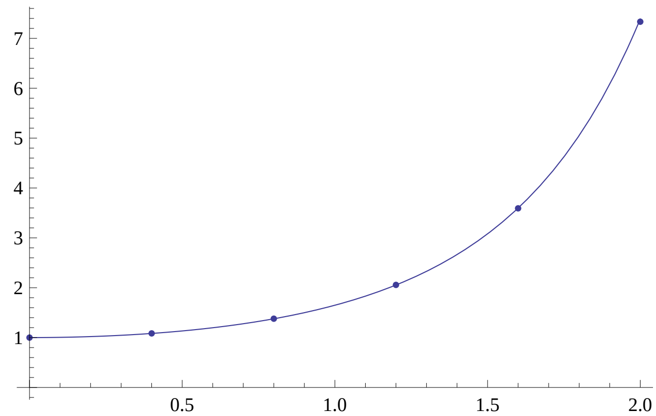

Figure 5.2: Graphs of Ordinary Runge Kutta Method and the exact function for $n = 5$

The solution of the multiplicative differential equation (5.1) and the solution of the ordinary differential equation (5.3) for the stepsize $n = 20$ can be approximated as shown in the tables below. We can summarize the results in tabular form as follows: Solution of the multiplicative differential equation for $n = 20$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
0.5	1.1331484530668263	1.1331484530668252	9.99201×10^{-16}
1	1.6487212707001282	1.6487212707001269	7.77156×10^{-16}
1.5	3.080216848918031	3.0802168489180297	4.44089×10^{-16}

Table 5.5: Third order Multiplicative Runge-Kutta method for $y^* = e^x$ for $n = 20$

The graph below shows the solutions of the multiplicative Runge Kutta method and the exact function, where the dotted graph represents the solutions of the multiplicative Runge Kutta method.

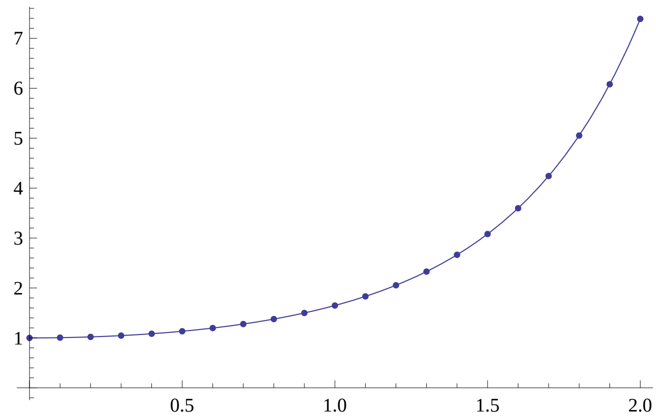

Figure 5.3: Graphs of multiplicative Runge Kutta Method and the exact function for $n = 20$

Solution of the ordinary differential equation for $n = 20$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
0.5	1.1331484530668263	1.133170632741277	0.0000195735
1	1.6487212707001282	1.648770678050206	0.0000299671
1.5	3.080216848918031	3.0802067308712733	3.28485×10^{-6}

Table 5.6: Third order Ordinary Runge-Kutta method for $y' = x \cdot y$ for $n = 20$

The graph below shows the solutions of the ordinary Runge Kutta method and the exact function, where the dotted graph represents the solutions of the ordinary Runge Kutta method.

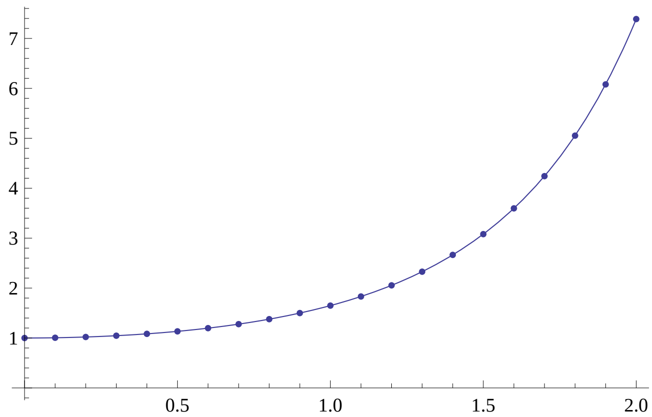

Figure 5.4: Graphs of Ordinary Runge Kutta Method and the exact function for $n = 20$

From the tables above and the graphs below we can see the solutions and the error terms of the Multiplicative Runge Kutta Method and the Ordinary Runge Kutta Method respectively. It can be easily seen that the error terms for the Multiplicative Runge Kutta method are much more smaller than the Ordinary Runge Kutta method. Thus we see that the multiplicative Runge-Kutta method gives us better solutions than the ordinary Runge-Kutta method.

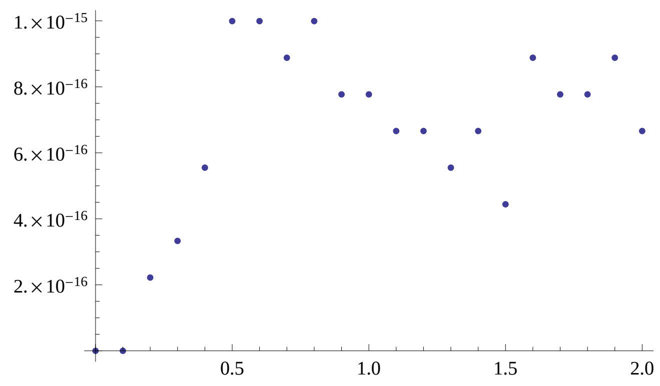

Figure 5.5: Error of Multiplicative Runge Kutta Method for the multiplicative Differential Equation $y^* = e^x$ for $n = 20$

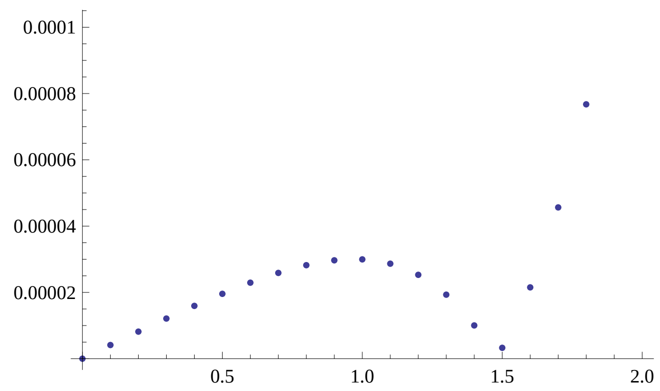

Figure 5.6: Error of Ordinary Runge Kutta Method for the ordinary Differential Equation $y' = y \cdot x$ for $n = 20$

We will solve the equation (5.3) by using both the 4. order Runge-Kutta method and the equation (5.1) by using the 4. order multiplicative Runge-Kutta method. Then we will compare the two methods.

The solution of the multiplicative differential equation (5.1) and the solution of the ordinary differential equation (5.3) for the stepsize $n = 5$ can be approximated as shown in the tables below. We can summarize the results in tabular form as follows:

Solution of the multiplicative differential equation for $n = 5$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
0.4	1.0832870676749586	1.0832870676749584	1.11022×10^{-16}
1.2	2.054433210643888	2.054433210643888	0
2	7.38905609893065	7.389056098930652	4.44089×10^{-16}

Table 5.7: Fourth order Multiplicative Runge-Kutta method for $y^* = e^x$ for $n = 5$

The graph below shows the solutions of the multiplicative Runge Kutta method and the exact function, where the dotted graph represents the solutions of the multiplicative Runge Kutta method.

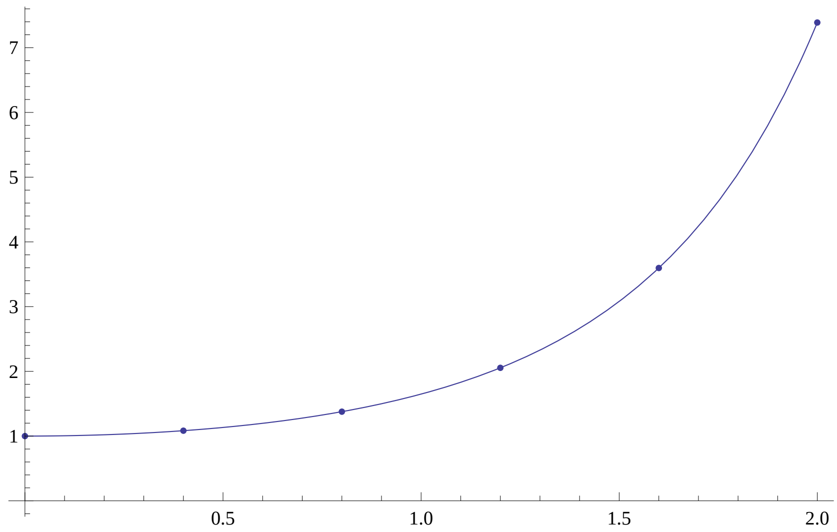

Figure 5.7: Graphs of multiplicative Runge Kutta Method and the exact function for $n = 5$

Solution of the ordinary differential equation for $n = 5$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
0.4	1.0832870676749586	1.0832853333333334	1.601×10^{-6}
1.2	2.054433210643888	2.0542093568120414	0.000108961
2	7.38905609893065	7.378697008955188	0.00140195

Table 5.8: Fourth order Ordinary Runge-Kutta method for $y' = x \cdot y$ for $n = 5$

The graph below shows the solutions of the ordinary Runge Kutta method and the exact function, where the dotted graph represents the solutions of the ordinary Runge Kutta method.

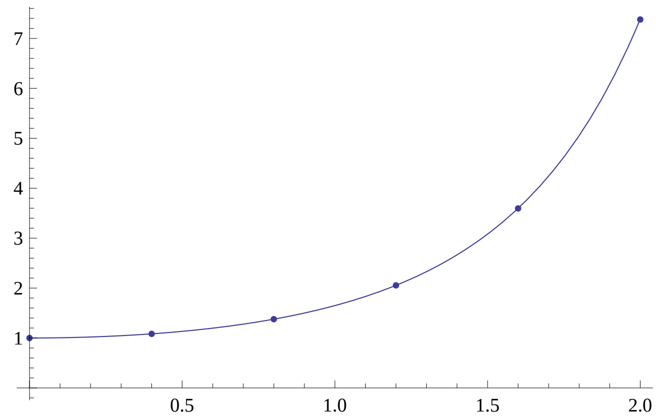

Figure 5.8: Graphs of Ordinary Runge Kutta Method and the exact function for $n = 5$

The solution of the multiplicative differential equation (5.1) and the solution of the ordinary differential equation (5.3) for the stepsize $n = 20$ can be approximated as shown in the tables below. We can summarize the results in tabular form as follows:
Solution of the multiplicative differential equation for $n = 20$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
0.5	1.1331484530668263	1.133148453066826	2.22045×10^{-16}
1	1.6487212707001282	1.648721270700127	6.66134×10^{-16}
1.5	3.080216848918031	3.0802168489180306	2.22045×10^{-16}

Table 5.9: Fourth order Multiplicative Runge-Kutta method for $y^* = e^x$ for $n = 20$

The graph below shows the solutions of the multiplicative Runge Kutta method and the exact function, where the dotted graph represents the solutions of the multiplicative Runge Kutta method.

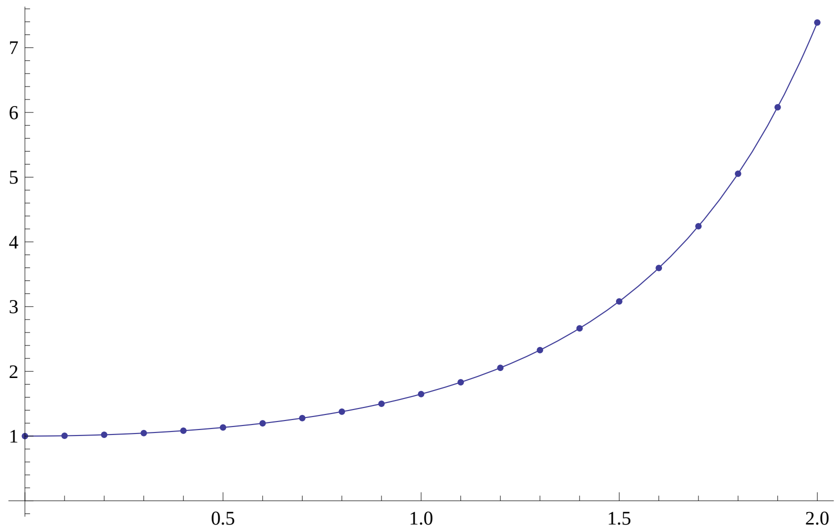

Figure 5.9: Graphs of multiplicative Runge Kutta Method and the exact function for $n = 20$

Solution of the ordinary differential equation for $n = 20$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
0.5	1.1331484530668263	1.1331484461175372	6.13273×10^{-9}
1	1.6487212707001282	1.6487210070533964	1.5991×10^{-7}
1.5	3.080216848918031	3.080212170663906	1.51881×10^{-6}

Table 5.10: Fourth order Ordinary Runge-Kutta method for $y' = x \cdot y$ for $n = 20$

The graph below shows the solutions of the ordinary Runge Kutta method and the exact function, where the dotted graph represents the solutions of the ordinary Runge Kutta method.

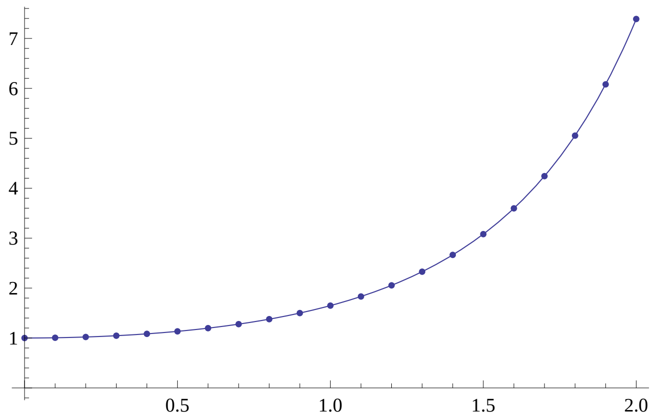

Figure 5.10: Graphs of Ordinary Runge Kutta Method and the exact function for $n = 20$

From the tables above and the graphs below we can see the solutions and the error terms of the Multiplicative Runge Kutta Method and the Ordinary Runge Kutta Method respectively. It can be easily seen that the error terms for the Multiplicative Runge Kutta method are much more smaller than the Ordinary Runge Kutta method. Thus we see that the multiplicative Runge-Kutta method gives us better solutions than the ordinary Runge-Kutta method.

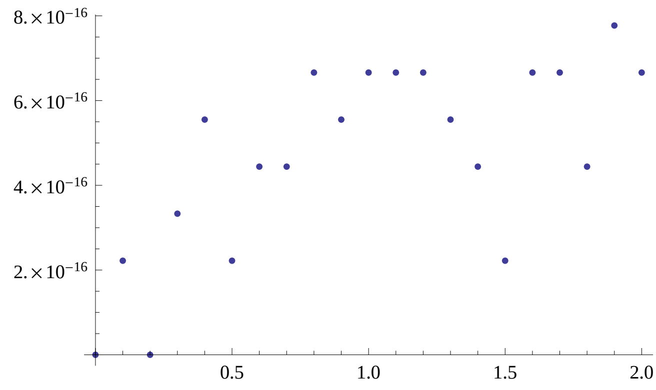

Figure 5.11: Error of Multiplicative Runge Kutta Method for the multiplicative Differential Equation $y^* = e^x$ for $n = 20$

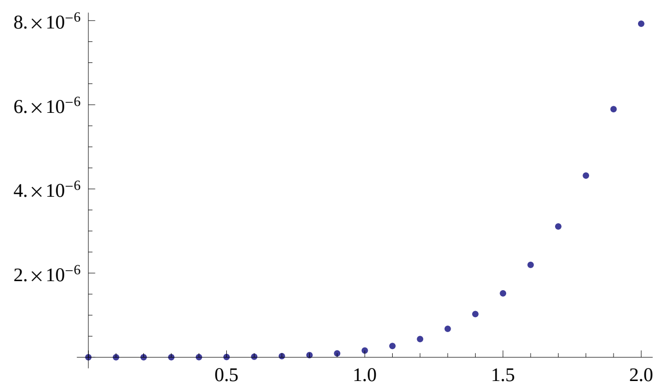

Figure 5.12: Error of Ordinary Runge Kutta Method for the ordinary Differential Equation $y' = y \cdot x$ for $n = 20$

5.2 Comparison of the solutions for $y^*(x) = x$ and $y' = y \ln x$

Let us consider thefollowing first order multiplicative differential equation:

$$y^*(x) = x \tag{5.5}$$

with the initial condition $y(1) = \frac{1}{e}$.

The analytic solution of this multiplicative differential equation is

$$y(x) = e^{-x + x \ln x} \tag{5.6}$$

The corresponding ordinary differential equation to the multiplicative differential equation (5.5) is

$$y'(x) = y \cdot \ln x \tag{5.7}$$

with the with the initial condition $y(1) = \frac{1}{e}$ and has the exact solution

$$y(x) = e^{-x+x\ln x} \tag{5.8}$$

We will solve the equation (5.7) by using both the 3. order Runge-Kutta method and the equation (5.5) by using the 3. order multiplicative Runge-Kutta method. Then we will compare the two methods.

The solution of the multiplicative differential equation (5.5) and the ordinary differential equation (5.7) for the stepsize $n = 5$ can be approximated as shown in the table below. We can summarize the results in tabular form as follows:

Solution of the multiplicative differential equation for $n = 5$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
1.2	0.3748556982078111	0.37475184543817724	0.000277047
1.6	0.428273104098778	0.4280060496647132	0.000623561
2	0.5413411329464507	0.5408909831593566	0.000831546

Table 5.11: Third order Multiplicative Runge-Kutta method for $y^* = x$ for $n = 5$

The graph below shows the solutions of the multiplicative Runge Kutta method and the exact function, where the dotted graph represents the solutions of the multiplicative Runge Kutta method.

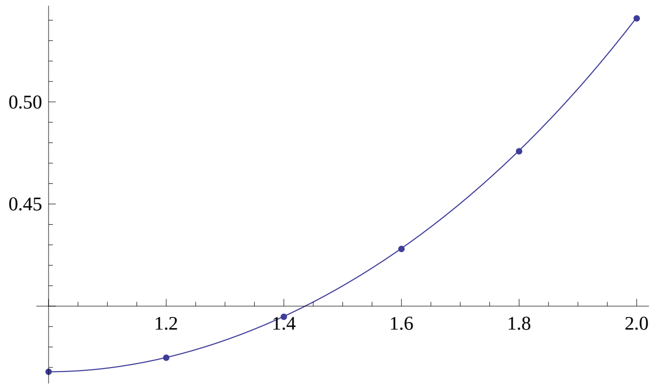

Figure 5.13: Graphs of multiplicative Runge Kutta Method and the exact function for $n = 5$

Solution of the ordinary differential equation for $n = 5$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
1.2	0.3748556982078111	0.3748754427165159	0.0000526723
1.6	0.428273104098778	0.4283203482273432	0.000110313
2	0.5413411329464507	0.5414096914455724	0.000126646

Table 5.12: Third order Ordinary Runge-Kutta method for $y' = y \cdot \ln x$ for $n = 5$

The graph below shows the solutions of the ordinary Runge Kutta method and the exact function, where the dotted graph represents the solutions of the multiplicative Runge Kutta method.

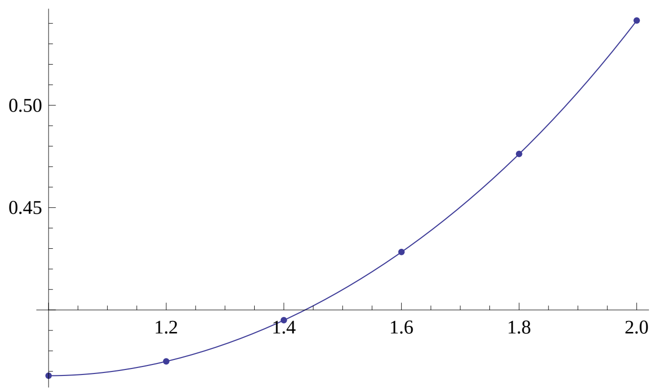

Figure 5.14: Graphs of ordinary Runge Kutta Method and the exact function for $n = 5$

The solution of the multiplicative differential equation (5.5) and the solution of the ordinary differential equation (5.7) for the stepsize $n = 20$ can be approximated as shown in the tables below. We can summarize the results in tabular form as follows:

Solution of the multiplicative differential equation for $n = 20$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
1.2	0.3748556982078111	0.37484919138006406	0.0000173582
1.5	0.40991627894186006	0.4099020478612443	0.000034717
1.8	0.4761682584008	0.4761462166507683	0.0000462898

Table 5.13: Third order Multiplicative Runge-Kutta method for $y^* = x$ for $n = 20$

The graph below shows the solutions of the multiplicative Runge Kutta method and the exact function, where the dotted graph represents the solutions of the multiplicative Runge Kutta method.

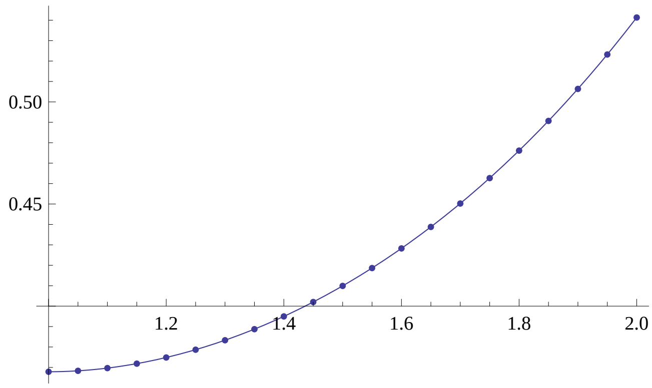

Figure 5.15: Graphs of multiplicative Runge Kutta Method and the exact function for $n = 20$

Solution of the ordinary differential equation for $n = 20$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
1.2	0.37485569820781106	0.3748560197578324	8.57797×10^{-7}
1.5	0.40991627894186006	0.4099169684952071	1.68218×10^{-6}
1.8	0.4761682584008	0.47616927139158094	2.12738×10^{-6}

Table 5.14: Third order Ordinary Runge-Kutta method for $y' = y \cdot \ln x$ for $n = 20$

The graph below shows the solutions of the ordinary Runge Kutta method and the exact function, where the dotted graph represents the solutions of the ordinary Runge Kutta method.

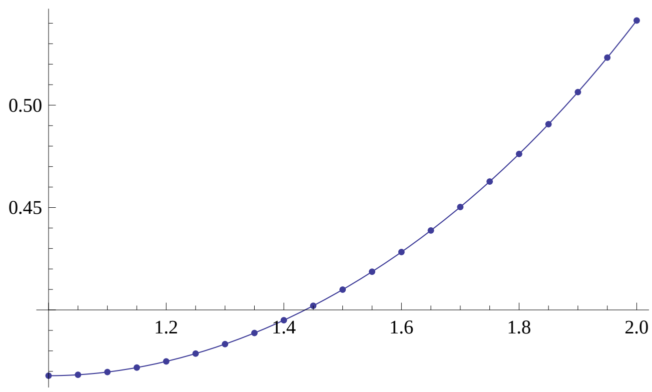

Figure 5.16: Graphs of ordinary Runge Kutta Method and the exact function for $n = 20$

From the tables above and the graphs below we can see the solutions and the error terms of the Multiplicative Runge Kutta Method and the Ordinary Runge Kutta Method respectively. It can be easily seen that the error terms for the Multiplicative Runge Kutta method are bigger than the Ordinary Runge Kutta method. Thus we can say that the ordinary Runge-Kutta method gives us better solutions than the multiplicative Runge-Kutta method.

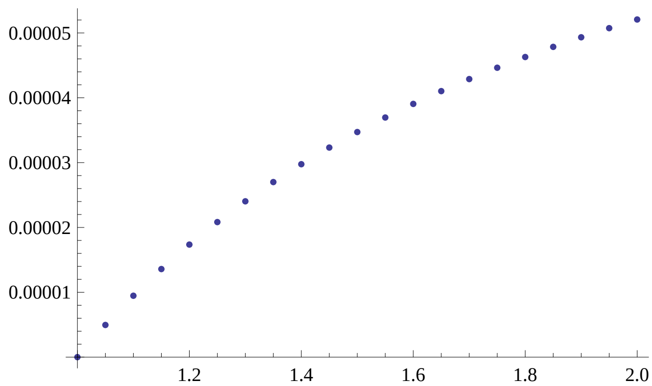

Figure 5.17: Error of Multiplicative Runge Kutta Method for the multiplicative Differential Equation $y^{*} = x$ for $n = 20$

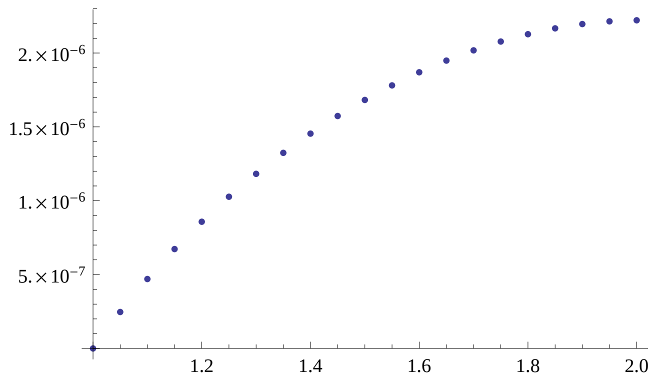

Figure 5.18: Error of Ordinary Runge Kutta Method for the ordinary Differential Equation $y' = y \cdot \ln x$ for $n = 20$

We will solve the equation (5.7) by using both the 4. order Runge-Kutta method and the equation (5.5) by using the 4. order multiplicative Runge-Kutta method. Then we will compare the two methods.

The solution of the multiplicative differential equation (5.5) and the ordinary differential equation (5.7) for the stepsize $n = 5$ can be approximated as shown in the table below. We can summarize the results in tabular form as follows:

Solution of the multiplicative differential equation for $n = 5$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
1.2	0.37485569820781106	0.37483204529157416	0.0000630987
1.6	0.42827310409877	0.42821161528361407	0.000143574
2	0.5413411329464507	0.5412368065244385	0.000192718

Table 5.15: Fourth order Multiplicative Runge-Kutta method for $y^* = x$ for $n = 5$

The graph below shows the solutions of the multiplicative Runge Kutta method and the exact function, where the dotted graph represents the solutions of the multiplicative Runge Kutta method.

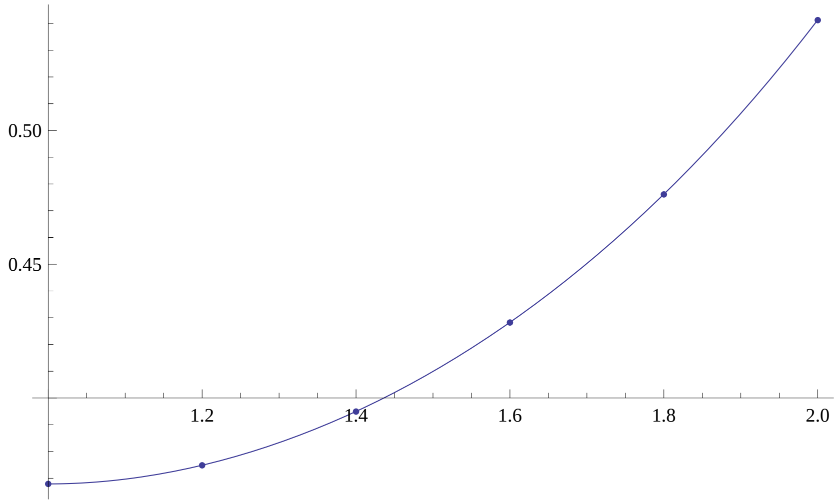

Figure 5.19: Graphs of multiplicative Runge Kutta Method and the exact function for $n = 5$

Solution of the ordinary differential equation for $n = 5$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
1.2	0.37485569820781106	0.37485550990836036	5.02325×10^{-7}
1.6	0.428273104098778	0.4282726718524416	1.00928×10^{-6}
2	0.5413411329464507	0.5413402321041269	1.66409×10^{-6}

Table 5.16: Fourth order Ordinary Runge-Kutta method for $y' = y \cdot \ln x$ for $n = 5$

The graph below shows the solutions of the ordinary Runge Kutta method and the exact function, where the dotted graph represents the solutions of the multiplicative Runge Kutta method.

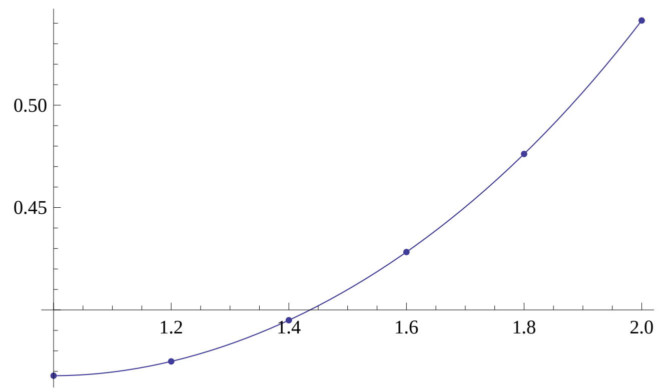

Figure 5.20: Graphs of ordinary Runge Kutta Method and the exact function for $n = 5$

The solution of the multiplicative differential equation (5.5) and the solution of the ordinary differential equation (5.7) for the stepsize $n = 20$ can be approximated as shown in the tables below. We can summarize the results in tabular form as follows:

Solution of the multiplicative differential equation for $n = 20$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
1.2	0.37485569820781106	0.37485410855784085	4.2407×10^{-6}
1.5	0.40991627894186006	0.4099127948624892	8.49949×10^{-6}
1.8	0.4761682584008	0.47616285449291723	0.0000113487

Table 5.17: Fourth order Multiplicative Runge-Kutta method for $y^* = x$ for $n = 20$

The graph below shows the solutions of the multiplicative Runge Kutta method and the exact function, where the dotted graph represents the solutions of the multiplicative Runge Kutta method.

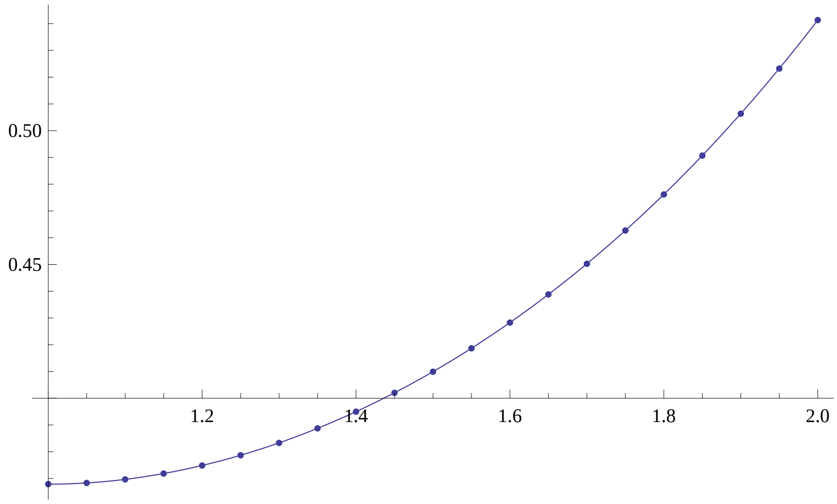

Figure 5.21: Graphs of multiplicative Runge Kutta Method and the exact function for $n = 20$

Solution of the ordinary differential equation for $n = 20$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
1.2	0.37485569820781106	0.3748556975066363	1.87052×10^{-9}
1.5	0.40991627894186006	0.4099162776195715	3.22575×10^{-9}
1.8	0.4761682584008	0.47616825630261755	4.40639×10^{-9}

Table 5.18: Fourth order Ordinary Runge-Kutta method for $y' = y \cdot \ln x$ for $n = 20$

The graph below shows the solutions of the ordinary Runge Kutta method and the exact function, where the dotted graph represents the solutions of the ordinary Runge Kutta method.

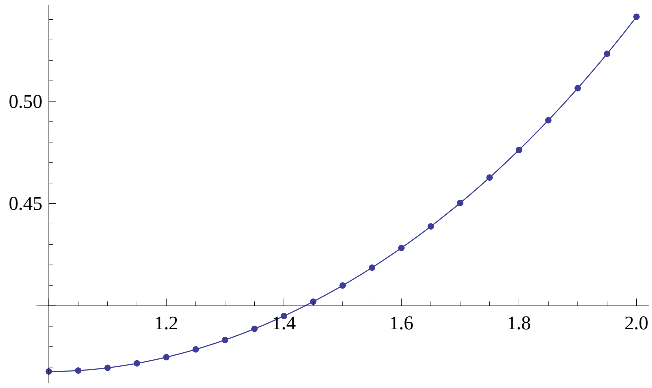

Figure 5.22: Graphs of ordinary Runge Kutta Method and the exact function for $n = 20$

From the tables above and the graphs below we can see the solutions and the error terms of the Multiplicative Runge Kutta Method and the Ordinary Runge Kutta Method respectively. From the tables we can see that the error terms for the Multiplicative Runge Kutta method are bigger than the error termsof the Ordinary Runge Kutta method. Thus we may say that for these differential equations the ordinary Runge-Kutta method gives us better solutions than the multiplicative Runge-Kutta method.

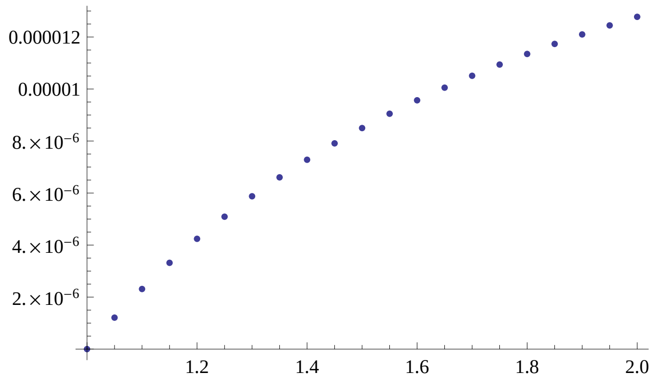

Figure 5.23: Error of Multiplicative Runge Kutta Method for the multiplicative Differential Equation $y^* = x$ for $n = 20$

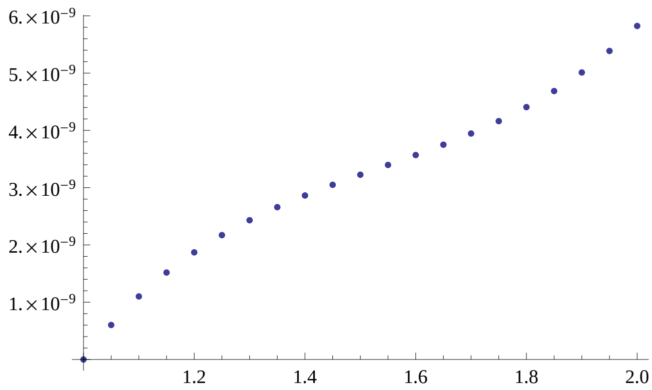

Figure 5.24: Error of Ordinary Runge Kutta Method for the ordinary Differential Equation $y' = y \cdot \ln x$ for $n = 20$

5.3 Comparison of the solutions of $y^*(x) = e^{e^x}$ and $y'(x) = y \cdot e^x$

Let us consider thefollowing first order multiplicative differential equation:

$$y^*(x) = e^{e^x} \tag{5.9}$$

with the initial condition $y(1) = e^e$.

The analytic solution of this multiplicative differential equation is

$$y(x) = e^{e^x} \tag{5.10}$$

The corresponding ordinary differential equation to the multiplicative differential equation (5.9) is

$$y'(x) = y \cdot e^x \tag{5.11}$$

with the with the initial condition $y(1) = e^e$ and has the exact solution

$$y(x) = e^{e^x} \tag{5.12}$$

We will solve the equation (5.11) by using both the 3. order Runge-Kutta method and the equation (5.9) by using the 3. order multiplicative Runge-Kutta method. Then we will compare the two methods.

The solutions of the multiplicative differential equation (5.9) and the ordinary differential equation (5.11) for the stepsize $n = 5$ can be approximated as shown in the tables below. We can summarize the results in tabular form as follows:

Solution of the multiplicative differential equation for $n = 5$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
1.2	27.66358487147661	27.69133311043913	0.00100306
1.6	141.6037160767467	142.13185001915915	0.00372966
2	1618.1779919126539	1630.8177152360943	0.00781108

Table 5.19: Third order Multiplicative Runge-Kutta method for $y^* = e^{e^x}$ for $n = 5$

The graph below shows the solutions of the multiplicative Runge Kutta method and the exact function, where the dotted graph represents the solutions of the multiplicative Runge Kutta method.

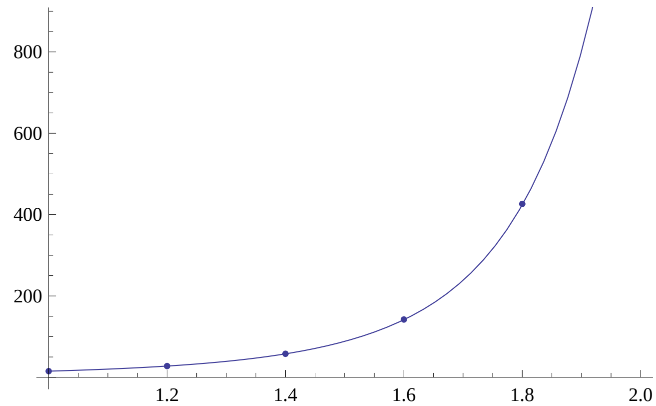

Figure 5.25: Graphs of multiplicative Runge Kutta Method and the exact function for $n = 5$

Solution of the ordinary differential equation for $n = 5$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
1.2	27.66358487147661	27.576062817833574	0.0031638
1.6	141.6037160767467	138.40303324232684	0.0226031
2	1618.1779919126539	1469.4957622022841	0.0918825

Table 5.20: Third order Ordinary Runge-Kutta method for $y' = y \cdot e^x$ for $n = 5$

The graph below shows the solutions of the ordinary Runge Kutta method and the exact function, where the dotted graph represents the solutions of the ordinary Runge Kutta method.

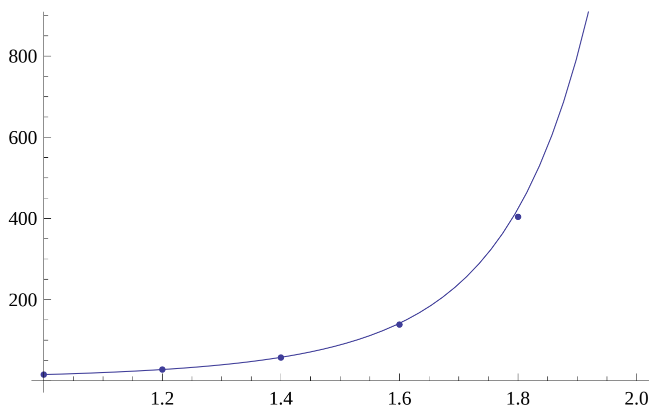

Figure 5.26: Graphs of ordinary Runge Kutta Method and the exact function for $n = 5$

The solutions of the multiplicative differential equation (5.9) and the ordinary differential equation (5.11) for the stepsize $n = 20$ can be approximated as shown in the tables below. We can summarize the results in tabular form as follows:

Solution of the multiplicative differential equation for $n = 20$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
1.2	27.66358487147661	27.665319133748163	0.0000626912
1.5	88.38383317988601	88.40006923569248	0.000183699
1.8	423.96354146031155	424.11068505523303	0.000347067

Table 5.21: Third order Multiplicative Runge-Kutta method for $y^* = e^{e^x}$ for $n = 20$

The graph below shows the solutions of the multiplicative Runge Kutta method and the exact function, where the dotted graph represents the solutions of the multiplicative Runge Kutta method.

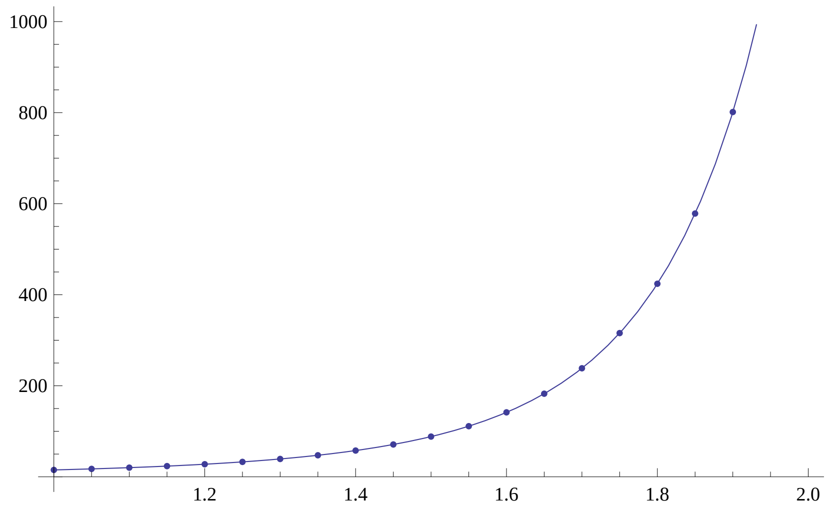

Figure 5.27: Graphs of multiplicative Runge Kutta Method and the exact function for $n = 20$

Solution of the ordinary differential equation for $n = 20$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
1.2	27.663584871476584	27.661661133964927	0.0000695404
1.5	88.38383317988601	88.35158860087355	0.000364824
1.8	423.96354146031155	423.4046262192472	0.00131831

Table 5.22: Third order Ordinary Runge-Kutta method for $y' = y \cdot e^x$ for $n = 20$

The graph below shows the solutions of the ordinary Runge Kutta method and the exact function, where the dotted graph represents the solutions of the ordinary Runge Kutta method.

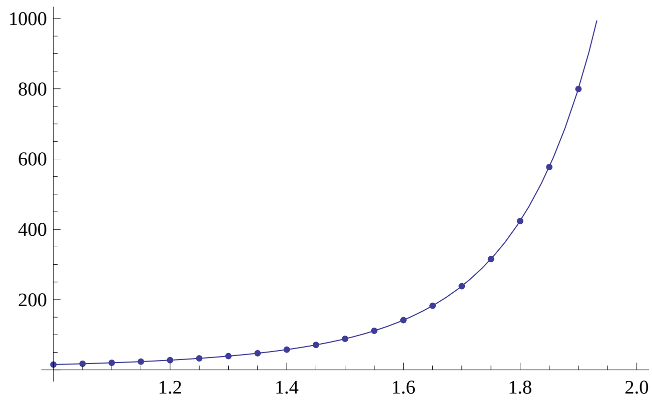

Figure 5.28: Graphs of ordinary Runge Kutta Method and the exact function for $n = 20$

From the tables above and the graphs below we can see the solutions and the error terms of the Multiplicative Runge Kutta Method and the Ordinary Runge Kutta Method respectively. We can see that the error terms for the Multiplicative Runge Kutta method are smaller than the error terms of the ordinary Runge Kutta method, but they do not differ too much. Thus we see that the multiplicative Runge-Kutta method gives us nearly the same solutions with the ordinary Runge-Kutta method.

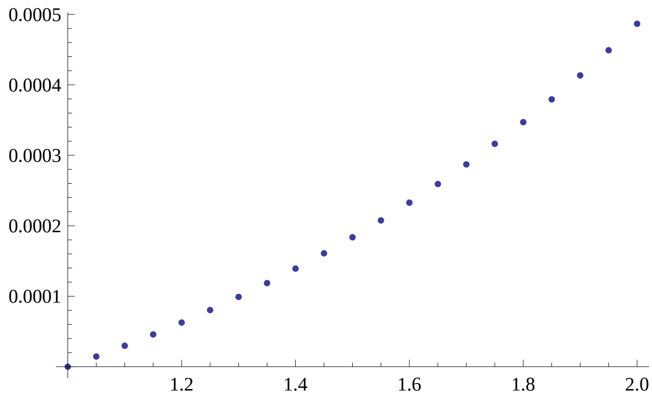

Figure 5.29: Error of Multiplicative Runge Kutta Method for the multiplicative Differential Equation $y^* = e^{e^x}$ for $n = 20$

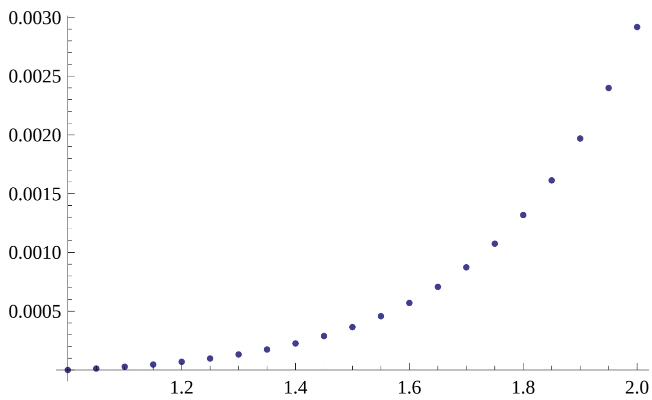

Figure 5.30: Error of Ordinary Runge Kutta Method for the ordinary Differential Equation $y' = y \cdot e^x$ for $n = 20$

We will solve the equation (5.11) by using both the 4. order Runge-Kutta method and the equation (5.9) by using the 4. order multiplicative Runge-Kutta method. Then we will compare the two methods.

The solutions of the multiplicative differential equation (5.9) and the ordinary differential equation (5.11) for the stepsize $n = 5$ can be approximated as shown in the tables below. We can summarize the results in tabular form as follows:

Solution of the multiplicative differential equation for $n = 5$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
1.2	27.663584871476584	27.670868503135402	0.000263293
1.6	141.6037160767467	141.74220684986398	0.000978017
2	1618.1779919126539	1621.4875022572264	0.00204521

Table 5.23: Fourth order Multiplicative Runge-Kutta method for $y^* = e^{e^x}$ for $n = 5$

The graph below shows the solutions of the multiplicative Runge Kutta method and the exact function, where the dotted graph represents the solutions of the multiplicative Runge Kutta method.

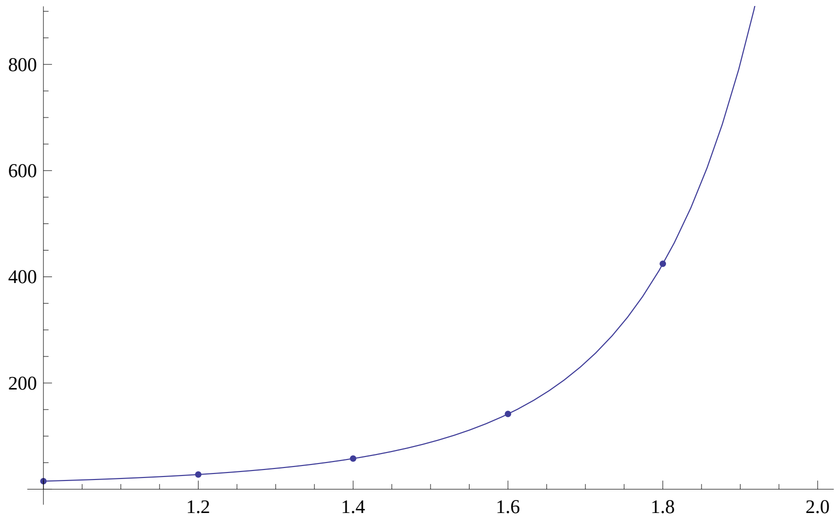

Figure 5.31: Graphs of multiplicative Runge Kutta Method and the exact function for $n = 5$

Solution of the ordinary differential equation for $n = 5$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
1.2	27.663584871476584	27.651967340911646	0.000419958
1.6	141.6037160767467	141.06227795051333	0.00382362
2	1618.1779919126539	1583.5193053670557	0.0214183

Table 5.24: Fourth order Ordinary Runge-Kutta method for $y' = y \cdot e^x$ for $n = 5$

The graph below shows the solutions of the ordinary Runge Kutta method and the exact function, where the dotted graph represents the solutions of the ordinary Runge Kutta method.

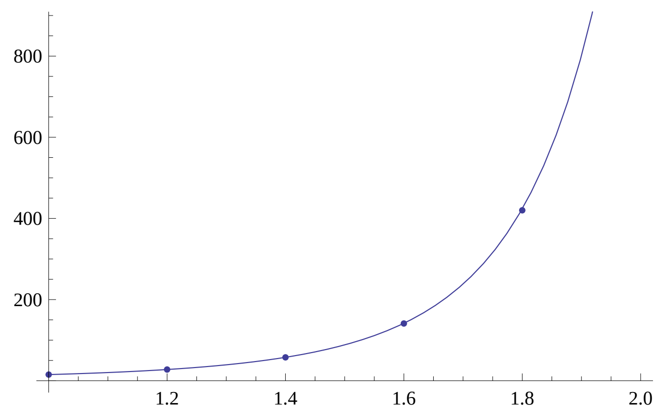

Figure 5.32: Graphs of ordinary Runge Kutta Method and the exact function for $n = 5$

The solutions of the multiplicative differential equation (5.9) and the ordinary differential equation (5.11) for the stepsize $n = 20$ can be approximated as shown in the tables below. We can summarize the results in tabular form as follows:

Solution of the multiplicative differential equation for $n = 20$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
1.2	27.663584871476584	27.664023856314973	0.0000158687
1.5	88.38383317988601	88.38794274302002	0.0000464968
1.8	423.96354146031155	424.000783202918	0.0000878419

Table 5.25: Fourth order Multiplicative Runge-Kutta method for $y^* = e^{e^x}$ for $n = 20$

The graph below shows the solutions of the multiplicative Runge Kutta method and the exact function, where the dotted graph represents the solutions of the multiplicative Runge Kutta method.

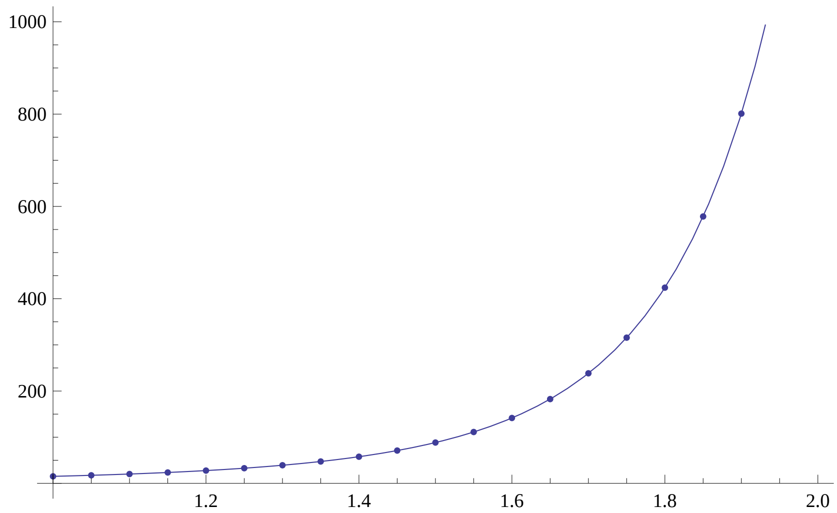

Figure 5.33: Graphs of multiplicative Runge Kutta Method and the exact function for $n = 20$

Solution of the ordinary differential equation for $n = 20$;

x	$y(x)$	$y_{app}(x)$	$relativeError$
1.2	27.663584871476584	27.663519449581884	2.36491×10^{-6}
1.5	88.38383317988601	88.38252069967844	0.0000148498
1.8	423.96354146031155	423.93491979511384	0.0000675097

Table 5.26: Fourth order Ordinary Runge-Kutta method for $y' = y \cdot e^x$ for $n = 20$

The graph below shows the solutions of the ordinary Runge Kutta method and the exact function, where the dotted graph represents the solutions of the ordinary Runge Kutta method.

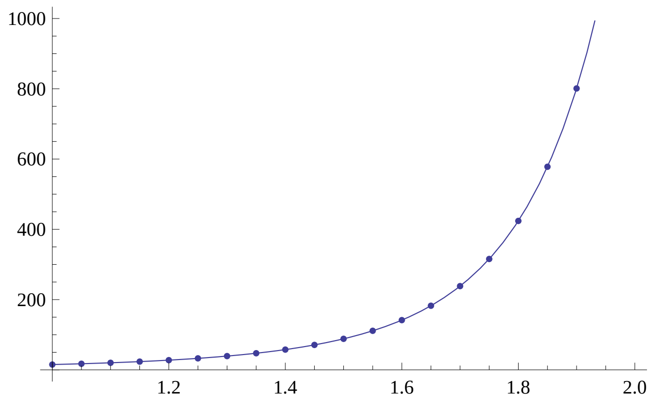

Figure 5.34: Graphs of ordinary Runge Kutta Method and the exact function for $n = 20$

From the tables above and the graphs below we can see the solutions and the error terms of the multiplicative Runge Kutta Method and the ordinary Runge Kutta Method respectively. It can be easily seen that for $n = 5$ the error terms for the Multiplicative Runge Kutta method are better than the error terms of the ordinary Runge Kutta method. For $n = 20$ the error terms are nearly the same. Thus we see that the solutions of the multiplicative Runge-Kutta method are the same with the solutions of the ordinary Runge-Kutta method.

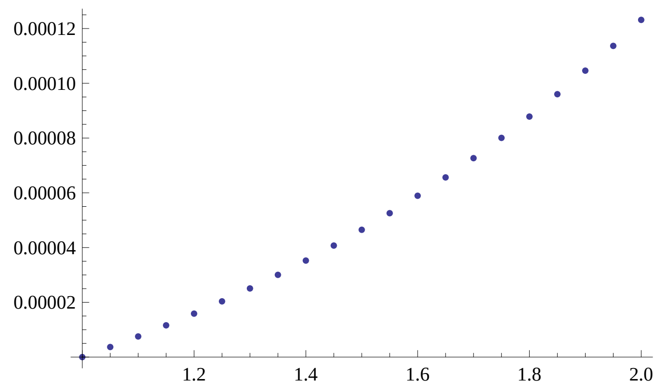

Figure 5.35: Error of Multiplicative Runge Kutta Method for the multiplicative Differential Equation $y^* = e^{e^x}$ for $n = 20$

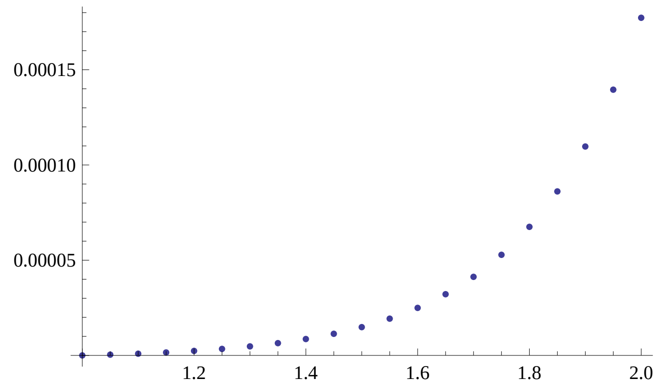

Figure 5.36: Error of Ordinary Runge Kutta Method for the ordinary Differential Equation $y' = y \cdot e^x$ for $n = 20$

Chapter 6

MULTIPLICATIVE FINITE DIFFERENCE METHODS

The starting point for the multiplicative finite difference method is the multiplicative Taylor Theorem. Multiplicative Taylor theorem can be given as follows:

Theorem 5 (Multiplicative Taylor Theorem)**.** *Let f be a (n+1)-times *differentiable function on (a,b). Assume that $x_0 \in [a,b]$. Then, for every $x \in [a,b]$, $x \neq x_0$, there exists a point x_1 between x and x_0 such that*

$$f(x) = f(x_0)\prod_{k=1}^{n}(f^{*(k)}(x_0))^{\frac{(x-x_0)^k}{(k)!}}(f^{*(n+1)}(x_1))^{\frac{(x-x_0)^(n+1)}{(n+1)!}} \tag{6.1}$$

By using the equation (6.1), the multiplicative Taylor expansion of $f(x+h)$ can be found as follows:

$$f(x+h) = \prod_{n=0}^{\infty}[f^{*(n)}(x)]^{\frac{h^n}{n!}} \tag{6.2}$$

By using the equation (6.2), the first few terms of forward and backward expansion with h will be:

$$f(x \pm h) = f(x) \cdot [f^{*}(x)]^{\pm h} \cdot [f^{**}(x)]^{\frac{h^2}{2!}} \cdot [f^{***}(x)]^{\pm\frac{h^3}{3!}} \cdot [f^{*(4)}(x)]^{\frac{h^4}{4!}} \cdot \ldots \tag{6.3}$$

In order to find the first order multiplicative derivative we need to divide the forward expansion to the backward expansion and then reorder the terms. Thus the first order multiplicative derivative will be:

$$f^{*}(x) = \left(\frac{f(x+h)}{f(x-h)}\right)^{\frac{1}{2h}} \cdot \prod_{n=1}^{\infty}(f^{*(2n+1)}(x))^{-}\frac{h^{2n}}{(2n+1)!} \tag{6.4}$$

The second order multiplicative derivative will be found by multiplying the forward expansion with the backward expansion, which is given as follows:

$$f^{**}(x) = \left(\frac{f(x+h)f(x-h)}{f(x)^2}\right)^{\frac{1}{h^2}} \cdot \prod_{n=1}^{\infty}(f^{*(2n+2)}(x))^{-}\frac{2h^{2n}}{(2n+2)!} \tag{6.5}$$

By deleting the remainder terms from the first order and the second order multiplicative derivative formulas, we can obtain the multiplicative finite difference method for the second order multiplicative differential equation in the form:

$$f^{**}(x) = g(x, f, f^*), \quad f(a) = \alpha, f(b) = \beta \tag{6.6}$$

Suppose that we have an interval $[a, b]$, and it is partitioned by the points $a = x_0, x_1, x_2, \ldots, x_{n+1} = b$. The points need not to be equally spaced, for simplicity we assume that they are equally spaced such that

$$x_i = a + ih, \quad 0 \leq i \leq n+1, \quad h = \frac{b-a}{n+1} \tag{6.7}$$

Equation (6.6) in discrete case will be:

$$\begin{aligned} \left[\frac{f_{i+1}f_{i-1}}{f_i^2}\right]^{\frac{1}{h^2}} &= g\left(x_i, f_i, \left(\frac{f_{i+1}}{f_{i-1}}\right)^{\frac{1}{2h}}\right) \\ f_0 &= \alpha \\ f_{n+1} &= \beta. \end{aligned}$$

In order to simplify the method we can take the natural logarithm and get:

$$\begin{aligned} \frac{1}{h^2}[k_{i+1} + k_{i-1} - 2k_i] &= \phi\left(x_i, k_i, \frac{1}{2h}(k_{i+1} - k_{i-1})\right) \\ k_0 &= \ln\alpha \\ k_{n+1} &= \ln\beta. \end{aligned}$$

where

$$k_i = \ln f_i \quad and \quad \phi\left(x_i, k_i, \frac{1}{2h}(k_{i+1} - k_{i-1})\right) = \ln\left[g\left(x_i, f_i, \left(\frac{f_{i+1}}{f_{i-1}}\right)^{\frac{1}{2h}}\right)\right] \tag{6.8}$$

6.1 Error Analysis for the Multiplicative Finite Difference Method

Let us assume that themain contribution to the error comes from the lowest order multiplicative derivative in the error term, then the error term of the first multiplicative derivative can be approximated as follows:

$$E(f^*(x)) \approx \left(|f^{*(3)}(x)|^*\right)^{-\frac{h^2}{3!}} \tag{6.9}$$

where $|\cdot|^*$ represents the multiplicative absolute value function which can be given as

$$|x|^* = \begin{cases} x & \text{for } x \geq 1 \\ \\ \frac{1}{x} & \text{for } x < 1 \end{cases} \tag{6.10}$$

Analogously the error term of the second order multiplicative derivative can be approximated by applying the same idea, such that

$$E(f^{**}(x)) \approx \left(|f^{*(4)}(x)|^*\right)^{-\frac{h^2}{12}} \tag{6.11}$$

As we discussed before $f(x)$ is a positive function. Since we know that the exponential function is a positive function, by using the relation between the multiplicative derivative and the ordinary derivative (2.3), and recalling that the invariant function under the multiplicative derivative is $\exp(\exp(x))$, we can make the ansatz $f(x) = \exp(y(x))$, where $y(x)$ is a real-valued function. Thus the multiplicative derivatives of f can be written in terms of ordinary derivatives of y as:

$$f^{*(n)}(x) = \exp\left\{(\ln \circ \exp(y(x)))^n(x)\right\} = \exp\left\{y(x)^n(x)\right\} \tag{6.12}$$

Thus the error terms for the first and second order multiplicative derivatives can be simplified as:

$$E(f^*(x)) \approx (f^{*(3)}(x))^{-\frac{h^2}{3!}} = \exp\Big\{-\frac{h^2}{3!}y^{(3)}(x)\Big\} \tag{6.13}$$

$$E(f^{**}(x)) \approx (f^{*(4)}(x))^{-\frac{h^2}{12}} = \exp\Big\{-\frac{h^2}{12}y^{(4)}(x)\Big\} \tag{6.14}$$

According to those error terms, we can say that if $y(x)$ is a polynomial of degree n, for $n < 3$ we get an exact approximation for the first multiplicative derivative and for $n < 4$ we get an exact approximation for the second multiplicative derivative.

Chapter 7

MATHEMATICA PROGRAMS FOR RUNGE KUTTA METHODS

7.1 Ordinary Runge Kutta Order 2

```
Runge[a0_, b0_, α_, m0_] :=
 Module[{a = a0, b = b0, j, m = m0},
   h = (b - a)/m;
   Y = X = Table[0, {m + 1}];
   X[[1]] = a;
   Y[[1]] = α;
   For[ j = 1, j ≤ m, j ++,
    k1 = N[h f[X[[j]], Y[[j]]]];
    k2 = N[h f[X[[j]] + h/2, Y[[j]] + k1/2]];
    Y[[j+1]] = N[Y[[j]] + k2];
    X[[j+1]] = N[a + h j]; ];
   Return[Transpose[{X, Y}]] ]

f[x_, y_] = x × y;
Print["Find numerical solutions to the D.E."];
Print["y′ = ", f[x, y] ];
(n = 20;) (pts1 = Runge[0, 2, 1, n];) (Y1 = Y;)
 (Print["The Runge-Kutta solution for  y′ = ", f[x, y]];)
 (Print["Using  n = ", n + 1, "  points."];) (Print[pts1];) (Print[""];)
 (Print["The final value is  y(20) = ", yn+1, " = ", Y[[n + 1]]];)
```

Figure 7.1: 2. order Ordinary Runge Kutta Method Program

7.2 Ordinary Runge Kutta Order 3

```
Runge[a0_, b0_, α_, m0_] :=
 Module[{a = a0, b = b0, j, m = m0},
   h = (b - a)/m;
   Y = X = Table[0, {m + 1}];
   X[[1]] = a;
   Y[[1]] = α;
   For[ j = 1, j ≤ m, j++,
    k1 = N[ h f[X[[j]], Y[[j]]]];
    k2 = N[h f[X[[j]] + h/2, Y[[j]] + k1/2]];
    k3 = N[h f[X[[j]] + h, Y[[j]] - k1 + 2 k2]];
    Y[[j+1]] = N[Y[[j]] + 1/6 (k1 + 4 k2 + k3)];
    X[[j+1]] = N[a + h j]; ];
   Return[Transpose[{X, Y}]] ]

f[x_, y_] = x × y;
Print["Find numerical solutions to the D.E."];
Print["y′ = ", f[x, y] ];
(n = 20;) (pts1 = Runge[0, 2, 1, n];) (Y1 = Y;)
 (Print["The Runge-Kutta solution for  y′ = ", f[x, y]];)
 (Print["Using  n = ", n + 1, "  points."];) (Print[pts1];) (Print[""];)
 (Print["The final value is  y(20) = ", y_{n+1}, " = ", Y[[n + 1]]];)
```

Figure 7.2: 3. order Ordinary Runge Kutta Method Program

7.3 Ordinary Runge Kutta Order 4

```
Runge[a0_, b0_, α_, m0_] :=
 Module[{a = a0, b = b0, j, m = m0},
   h = (b - a)/m;
   Y = X = Table[0, {m + 1}];
   X[[1]] = a;
   Y[[1]] = α;
   For[ j = 1, j ≤ m, j++,
    k1 = N[ h f[X[[j]], Y[[j]]]];
    k2 = N[h f[X[[j]] + h/2, Y[[j]] + k1/2]];
    k3 = N[h f[X[[j]] + h/2, Y[[j]] + k2/2]];
    k4 = N[h f[X[[j]] + h, Y[[j]] + k3]];
    Y[[j+1]] = N[Y[[j]] + 1/6 (k1 + 2 k2 + 2 k3 + k4)];
    X[[j+1]] = N[ a + h j]; ];
   Return[Transpose[{X, Y}]] ]

f[x_, y_] = x × y;
Print["Find numerical solutions to the D.E."];
Print["y′ = ", f[x, y] ];
(n = 20;) (pts1 = Runge[0, 2, 1, n];) (Y1 = Y;)
 (Print["The Runge-Kutta solution for  y′ = ", f[x, y]];)
 (Print["Using  n = ", n + 1, "  points."];) (Print[pts1];) (Print[""];)
 (Print["The final value is  y(20) = ", yn+1, " = ", Y[[n + 1]]];)
```

Figure 7.3: 4. order Ordinary Runge Kutta Method Program

7.4 Multiplicative Runge Kutta Order 2

```
Runge[a0_, b0_, α_, m0_] :=
 Module[{a = a0, b = b0, k = α, m = m0},
   h = (b - a)/m;
   Y = X = Table[0, {m + 1}];
   X[[1]] = a;
   Y[[1]] = k;
   For[ j = 1, j ≤ m, j ++,
    Y[[j+1]] = Y[[j]] * f[X[[j]], Y[[j]]]^(h/2) * f[X[[j]] + h, Y[[j]]]^(h/2);
    X[[j+1]] = a + h j;];
   Return[Transpose[{X, Y}]] ]

f[x_, y_] = ⅇ^x;
Print["Find numerical solutions to the D.E."];
Print["y* = ", f[x, y] ];
(n = 20;) (pts1 = Runge[0.0, 2.0, 1.0, n];)
(Y1 = Y;)
(Print["The Runge-Kutta solution for  y* = ", f[x, y]];)
 (Print["Using  n = ", n + 1, "  points."];) (Print[pts1];) (Print[""];)
 (Print["The final value is  y(20) = ", yn+1, " = ", Y[[n + 1]]];)
```

Figure 7.4: 2. order Multiplicative Runge Kutta Method Program

7.5 Multiplicative Runge Kutta Order 3

```
Runge[a0_, b0_, α_, m0_] :=
 Module[{a = a0, b = b0, k = α, m = m0},
   h = (b - a)/m;
   Y = X = Table[0, {m + 1}];
   X[[1]] = a;
   Y[[1]] = k;
   For[ j = 1, j ≤ m, j++,
    k1 = N[f[ X[[j]], Y[[j]]]];
    k2 = N[ f[X[[j]] + h/2, Y[[j]]]];
    k3 = N[f[X[[j]] + h, Y[[j]] × k1^((-Log[k2]/Log[k1]) h) × k2^h]];
    Y[[j + 1]] = N[Y[[j]] * k1^(h/3) × k2^(h/3) × k3^(h/3)];
    X[[j + 1]] = N[a + h j];];
  Return[Transpose[{X, Y}]] ]

f[x_, y_] = Exp[x];
Print["Find numerical solutions to the D.E."];
 Print["y* = ", f[x, y] ];
(n = 20;) (pts1 = Runge[0.0, 2.0, 1.0, n];)
(Y1 = Y;)
(Print["The Runge-Kutta solution for  y* = ", f[x, y]];)
 (Print["Using  n = ", n + 1, "  points."];) (Print[pts1];) (Print[""];)
 (Print["The final value is  y(20)  = ", y_{n+1}, " = ", Y[[n + 1]]];)
```

Figure 7.5: 3. order Multiplicative Runge Kutta Method Program

7.6 Runge Kutta Order 4

```
Runge[a0_, b0_, α_, m0_] :=
 Module[{a = a0, b = b0, k = α, m = m0},
  h = (b - a)/m;
  Y = X = Table[0, {m + 1}];
  X[[1]] = a;
  Y[[1]] = k;
  For[ j = 1, j ≤ m, j ++,
   k1 = N[f[ X[[j]], Y[[j]]]];
   k2 = N[f[X[[j]] + h/4, Y[[j]]]];
   k3 = N[f[X[[j]] + h/2, Y[[j]]]];
   k4 = N[f[X[[j]] + h, Y[[j]]]];
   Y[[j + 1]] = N[Y[[j]] × k1^(h/12) × k2^(h/3) × k3^(h/3) × k4^(h/4)];
   X[[j + 1]] = N[a + h j];];
  Return[Transpose[{X, Y}]] ]

f[x_, y_] = e^x;
Print["Find numerical solutions to the D.E."];
 Print["y* = ", f[x, y] ];
(n = 20;) (pts1 = Runge[0, 2.0, 1, n];)
(Y1 = Y;)
(Print["The Runge-Kutta solution for  y* = ", f[x, y]];)
 (Print["Using  n = ", n + 1, "  points."];) (Print[pts1];) (Print[""];)
 (Print["The final value is  y(20) = ", y_(n+1), " = ", Y[[n + 1]]];)
```

Figure 7.6: 4. order Multiplicative Runge Kutta Method Program

CONCLUSION

In this thesis, we have discussed a new kind of calculus which is called the Multiplicative Calculus. The definition of the derivative is given in terms of the multiplicative calculus and by using the new definition of the derivative multiplicative Taylor series and the multiplicative chain rule are defined. After that by combining these definitions Multiplicative Runge-Kutta methods of order 2,3, and 4 are developed. Also Ordinary Runge-Kutta methods are discussed. Thus by using both the ordinary Runge-Kutta methods and the multiplicative Runge-Kutta methods some differential equations are solved. Thus we had the chance to compare these two methods. At the end we see that the multiplicative Runge-Kutta methods gives better results than the ordinary Runge-Kutta methods if the solutions of the differential equations are of exponential nature.

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