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Comment on "Static and spherically symmetric black holes in f(R) theories"

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We consider the interesting "near-horizon test" reported in S. E. P. Bergliaffa and Y. E. C. de O. Nunes, Phys. Rev. D **84**, 084006 (2011) for any static, spherically symmetric black hole solution admitted in f(R) gravity. Before adopting the necessary conditions for the test, however, revisions are needed as we point out in this Comment.

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In order to derive the necessary conditions for the existence of static, spherically symmetric (SSS) black holes, we consider the series expansions of all expressions in the vicinity of the event horizon [1].

Our four-dimensional action that represents the Einstein-$f(R)$ gravity is given by

$$S = \frac{1}{2\kappa}\int \sqrt{-g} f(R) d^4x, \tag{1}$$

in which $\kappa = 8\pi G$, and $f(R)$ is a real arbitrary function of the Ricci scalar R. The four-dimensional SSS black hole's line element is chosen to be as [1]

$$ds^2 = -e^{-2\Phi}\left(1-\frac{b}{r}\right)dt^2 + \frac{1}{(1-\frac{b}{r})}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{2}$$

where Φ and b are two unknown real functions of r and at the horizon, $r = r_0$, we have $b(r_0) = b_0 = r_0$.

Variation of the action with respect to the metric gives the following field equations:

$$f_R R^\nu_\mu - \frac{f}{2}\delta^\nu_\mu - \nabla^\nu\nabla_\mu f_R + \delta^\nu_\mu \Box f_R = 0, \tag{3}$$

in which $f_R = \frac{df}{dR}$, $\Box = \nabla^\mu\nabla_\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu)$, and $\nabla^\nu\nabla_\mu h = g^{\lambda\nu}\nabla_\lambda h_{,\mu} = g^{\lambda\nu}(\partial_\lambda h_{,\mu} - \Gamma^\beta_{\lambda\mu}h_{,\beta})$ [2,3]. This leads to the field equations

$$f_R R^t_t - \frac{f}{2} + \Box f_R = \nabla^t\nabla_t f_R, \tag{4}$$

$$f_R R^r_r - \frac{f}{2} + \Box f_R = \nabla^r\nabla_r f_R, \tag{5}$$

$$f_R R^\theta_\theta - \frac{f}{2} + \Box f_R = \nabla^\theta\nabla_\theta f_R, \tag{6}$$

which are independent. Note that the $\varphi\varphi$ equation is identical with the $\theta\theta$ equation. By adding the four equations (i.e., tt, rr, $\theta\theta$, and $\varphi\varphi$) we find

$$f_R R - 2f + 3\Box f_R = 0, \tag{7}$$

which is the trace of Eq. (3). We note that this is not an independent equation from the other three equations. One may consider this equation with only two of the others. In other words, if one considers the latter equation with the other three equations, two of them become identical. The Eqs. (3–6) of Ref. [1] involve unfortunate errors so that we evaluate each Ricci tensor component in some detail. In the following we shall expand the unknown functions about the horizon that will determine the near-horizon behavior. To do so we introduce

$$\epsilon x = r - r_0, \qquad |\epsilon| \ll 1, \tag{8}$$

so that

$$\Phi = \Phi_0 + \Phi_0'\epsilon x + \frac{1}{2}\Phi_0''\epsilon^2x^2 + \ldots, \tag{9}$$

$$b = b_0 + b_0'\epsilon x + \frac{1}{2}b_0''\epsilon^2x^2 + \ldots, \tag{10}$$

$$f = f_0 + f_0'\epsilon x + \frac{1}{2}f_0''\epsilon^2x^2 + \ldots, \tag{11}$$

$$R = R_0 + R_0'\epsilon x + \frac{1}{2}R_0''\epsilon^2x^2 + \ldots, \tag{12}$$

$$F = \frac{df}{dR} = F_0 + F_0'\epsilon x + \frac{1}{2}F_0''\epsilon^2x^2 + \ldots, \tag{13}$$

$$E = \frac{d^2f}{dR^2} = E_0 + E_0'\epsilon x + \frac{1}{2}E_0''\epsilon^2x^2 + \ldots, \tag{14}$$

$$H = \frac{d^3f}{dR^3} = H_0 + H_0'\epsilon x + \frac{1}{2}H_0''\epsilon^2x^2 + \ldots, \tag{15}$$

in which a prime denotes derivative with respect to r. Another notation is such that $Y_0 = Y(r_0)$, $Y_0' = \frac{dY}{dr}|_{r=r_0}$, $Y_0'' = \frac{d^2Y}{dr^2}|_{r=r_0}$ and so on, in which Y represents any function used here. We evaluate also the near-horizon form of $\Box f_R$ and the other similar terms:

$$\Box f_R = \frac{d^3f}{dR^3}g^{11}(R')^2 + \frac{d^2f}{dR^2}\Box R \tag{16}$$

where

$$\Box R = \left(R'' + \frac{R'}{r} - \Phi'R'\right)\left(1-\frac{b}{r}\right) + \frac{R'}{r}(1-b'), \tag{17}$$

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which, up to the first order in ϵ it would read

$$\Box f_R \simeq -\frac{E_0 R_0'(b_0'-1)}{r_0} - \frac{(H_0 R_0'^2 - (E_0\Phi_0' - E_0')R_0' + 2E_0 R_0'')(b_0'-1) + R_0' E_0 b_0''}{r_0}\epsilon x.$$

Similarly, the other terms to the first order read

$$\nabla^t\nabla_t f_R = -g^{tt}\frac{d^2 f}{dR^2}R'\Gamma^r_{tt} \simeq -\frac{E_0 R_0'(b_0'-1)}{2r_0} - \frac{r_0 E_0 R_0' b_0'' + (b_0'-1)[E_0' R_0' r_0 + E_0(R_0'' r_0 - 2R_0'[1+\Phi_0' r_0])]}{2r_0^2}\epsilon x, \quad (18)$$

$$\nabla^r\nabla_r f_R = g^{rr}\left(R''\frac{d^2 f}{dR^2} + R'^2\frac{d^3 f}{dR^3} - \frac{d^2 f}{dR^2}R'\Gamma^r_{tt}\right)$$
$$\simeq -\frac{E_0 R_0'(b_0'-1)}{2r_0} - \frac{r_0 E_0 R_0' b_0'' + (b_0'-1)[R_0' r_0(2R_0' H_0 + E_0') - E_0(2R_0' - 3r_0 R_0'')]}{2r_0^2}\epsilon x, \quad (19)$$

and

$$\nabla^\theta\nabla_\theta f_R = -g^{\theta\theta}\frac{d^2 f}{dR^2}R'\Gamma^r_{\theta\theta} \simeq -\frac{E_0 R_0'(b_0'-1)}{r_0^2}\epsilon x. \quad (20)$$

Accordingly, the Ricci tensor components become

$$R^t_t = \frac{b_0'' - 3(b_0'-1)\Phi_0'}{2r_0} + \frac{(b_0'-1)[(2\Phi_0'^2 - 5\Phi_0'')r_0 + 2\Phi_0'] - b_0''(3\Phi_0' r_0 + 1) + r_0 b_0'''}{2r_0^2}\epsilon x, \quad (21)$$

$$R^r_r = R^t_t + \frac{2(b_0'-1)\Phi_0'}{r_0^2}\epsilon x, \quad (22)$$

and finally

$$R^\varphi_\varphi = R^\theta_\theta = \frac{b_0'}{r_0} - \frac{(b_0'-1)(\Phi_0' r_0 + 2) - b_0'' r_0 + 2}{r_0^3}\epsilon x. \quad (23)$$

Next, we rewrite the field equations up to the first order in ϵ. After matching the zeroth order terms we find from (4) and (5)

$$-(b_0'-1)(3\Phi_0' F_0 + E_0 R_0') + F_0 b_0'' - f_0 r_0 = 0, \quad (24)$$

while from (6) it yields

$$(b_0'-1)(F_0 - E_0 R_0' r_0) + F_0 - \frac{1}{2}f_0 r_0^2 = 0. \quad (25)$$

From these equations one finds

$$\frac{f_0}{F_0} = -2\frac{(3\Phi_0' r_0 + 1)(b_0'-1) + 1 - b_0'' r_0}{r_0^2}, \quad (26)$$

and

$$\frac{F_0}{E_0} = \frac{r_0(b_0'-1)R_0'}{6b_0' - 3\Phi_0' r_0(b_0'-1) + b_0'' r_0 - 2R_0 r_0^2}, \quad (27)$$

knowing that the explicit form of R_0 and R_0' are given by

$$R_0 = \frac{2 + b_0'' r_0 + (2 - 3\Phi_0' r_0)(b_0'-1)}{r_0^2}, \quad (28)$$

and

$$R_0' = \frac{[(2\Phi_0'^2 - 5\Phi_0'')r_0^2 + 2\Phi_0' r_0 - 4](b_0'-1) - r_0^2(3\Phi_0' b_0'' - b_0''') - 4 + b_0'' r_0}{r_0^3}. \quad (29)$$

Unlike the result of Eq. (14) in Ref. [1], here from Eqs. (26) and (27) we obtain

$$\frac{f_0}{F_0} = 2R_0 - \frac{6b_0'}{r_0^2} \quad \text{and} \quad \frac{F_0}{E_0} = \frac{R_0' r_0(b_0'-1)}{4b_0' - R_0 r_0^2}. \quad (30)$$

On the other hand, from (11)–(15), one finds

$$F_0 = \frac{f_0'}{R_0'}; \qquad E_0 = \frac{F_0'}{R_0'}; \qquad H_0 = \frac{E_0'}{R_0'}. \quad (31)$$

As we mentioned before, Eq. (7) is not independent and is identically satisfied. After the zeroth order terms one may look at the first order equations from which upon combination of Eqs. (4) and (5) we get

$$r_0(H_0 R_0'^2 + E_0 R_0'' + E_0 R_0'\Phi_0') + 2\Phi_0' F_0 = 0, \quad (32)$$

which consequently implies

$$\frac{H_0}{E_0} = \frac{(4R_0'' + 6R_0'\Phi_0')(b_0' - 1) + (R_0'' + R_0'\Phi_0')(4 - R_0 r_0^2)}{(R_0 r_0^2 - 4b_0')R_0'^2}. \tag{33}$$

The other equations are rather complicated so that we will not write them openly.

Our conclusion for a general SSS black hole solution in $f(R)$ gravity is that the necessary conditions for an $f(R)$ gravity to have $b_0 = r_0$ type of solution are given by (28)–(31). To have a Schwarzschild-like black hole with $b(r) = M =$ constant, these conditions reduce to the simpler condition as

$$\frac{f_0}{F_0} = 2R_0, \tag{34}$$

which, for example, in $f(R) = \alpha(R + \beta)^n$, with α, β constants, metric is viable only for $n = \frac{1}{2}$. We add that for the Schwarzschild case (i.e. $n = 1$, $\alpha = 1$, $\beta = 0$) condition (34) is trivially satisfied so that the "near-horizon test" concerns non-Schwarzschild SSS metrics in $f(R)$ gravity. Our result also conforms with Ref. [4].

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