Existence of Reissner-Nordström type black holes in f(R) gravity

S. Habib Mazharimousavi,* M. Kerachian,† and M. Halilsoy‡

Physics Department, Eastern Mediterranean University, G. Magusa north Cyprus, Mersin 10 Turkey.

We investigate the existence of Reissner-Nordström (RN) type black holes in f(R) gravity. Our emphasis is to derive, in the presence of electrostatic source, the necessary conditions which provide such static, spherically symmetric (SSS) black holes available in f(R) gravity. We also study the thermodynamics of the black hole solution.

I. INTRODUCTION

Due to a number of valid reasons $f(R)$ gravity attracted much interest during the recent decade as an extension / modification of Einstein's general relativity [1–6] (for some review works see [7–10]). Here R stands for the Ricci scalar, the simplest among much complicated ones and $f(R)$ is an analytic function of R. Herein we wish to look at $f(R)$ gravity from a different angle which was introduced by Bergliaffa and Nunes in their novel paper [11, 12]. Since the black hole solutions in Einstein's $f(R) = R$, theory has already built enough prominence and play the leading role it should be wise to seek for similar solutions in the more general $f(R)$ theories. This approach concerns directly the existence problem of black holes and it's associated necessary conditions for analog objects in the latter. The existence conditions may simply be dubbed as the "*near-horizon test*" in order to highlight the event horizon of a black hole as a physical reality. It is well-known that physically when the observer approaches the event horizon he / she feels nothing unusual except strong gravity, so this mathematically must reflect analytically on the event horizon. The analytic expansion of a metric function, say $f(r)$, is developed in series of the form $f(r) = f(r_0) + f'(r_0)(r - r_0) + \mathcal{O}\left((r - r_0)^2\right)$ where r_0 is the event horizon and $(r - r_0)$ stands naturally small. When these developed series are substituted back into the Einstein equations they will give conditions of zeroth, first and higher orders. These are precisely what we call the necessary conditions for the existence of certain / analog black hole types. To our amazement these necessary conditions emerge rather restrictive so that we can't propose arbitrarily any polynomial forms of $f(R)$ as the representative black holes. For instance, what are the necessary conditions in order that it will admit Schwarzschild-like black hole solutions? This particular problem without external sources has already been considered and it's found that the possible $f(R)$ must be of the form $f(R) = \alpha\sqrt{R + \beta}$, in which α and β are constants [11, 12]. An analytic expansion reveals that the first term retains the Einstein-Hilbert term with the addition of higher orders in R which seems to be the payoff in the enterprise of $f(R)$ theory. Any $f(R)$ theory is known to create it's own source from the inherent non-linearity of the theory. Beside these, however, additional external sources may be considered (some examples of $f(R)$ black hole with charge are given in [13–19]), which makes the principal aim of the present paper. We consider an external static electric field as source and adopt the Reissner-Nordström (RN)-type black hole within $f(R)$ gravity. Expectedly, the results for necessary conditions for the existence of a RN black hole are more complicated than the case of a Schwarzschild black hole. In this process we obtain an infinite series representation for the near-horizon behavior of our metric functions. The exact determination of the constant coefficients in the series is theoretically possible, at least in the leading orders. The addition of further external sources beside electromagnetism will naturally make the problem more complicated. An equally simple case is the extremal RN black hole which is also considered in our study.

The paper is organized as follows. Section II investigates the necessary conditions for the existence of a RN-type black hole in $f(R)$ gravity. Thermodynamics, and in particular the first law for such black holes are presented in Section III. Section IV is devoted to an extremal RN-type black hole. The paper is completed with Concluding Remarks, which appears in Section V.

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*Electronic address: habib.mazhari@emu.edu.tr
†Electronic address: kerachian.morteza@gmail.com
‡Electronic address: mustafa.halilsoy@emu.edu.tr

II.   ANALOG RN BLACK HOLES IN $f(R)$ GRAVITY

The proper action in $f(R)$ gravity coupled minimally with Maxwell source in $4-$dimensions is given by

$$S = \int \sqrt{-g} \left(\frac{f(R)}{2\kappa} - \frac{\mathcal{F}}{4\pi} \right) d^4 x \tag{1}$$

in which $f(R)$ is a real function of the Ricci scalar R, $\mathcal{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ is the Maxwell invariant and $\kappa = 8\pi G$ where G is the Newton's constant. Our choice of the spacetime is a RN-type black hole solution whose line element can be written as

$$ds^2 = -e^{-2\Phi} \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)} + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right), \tag{2}$$

where M and Q are two real constants which indicate the mass and the charge of the black hole respectively. Also $\Phi = \Phi(r)$ is an unknown real function which is well behaved everywhere and dies off at large r. The matter source which we shall consider in our consideration is a Maxwell electric field whose two-forms is given by

$$\mathbf{F} = E(r) \, dt \wedge dr \tag{3}$$

where $E(r)$ is the electric field. Following the line element (2) one finds the dual-Maxwell field as

$$^*\mathbf{F} = -E(r) \, e^{\Phi} r^2 \sin\theta d\theta \wedge d\varphi \tag{4}$$

and in turn the Maxwell equation

$$d^*\mathbf{F} = 0 \tag{5}$$

implies

$$E(r) = \frac{q}{r^2} e^{-\Phi} \tag{6}$$

in which the integration constant q is to be identified with Q. Varying the action with respect to $g_{\mu\nu}$ provides the field equations

$$F R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - \nabla_\mu \nabla_\nu F + g_{\mu\nu} \Box F = \kappa T_{\mu\nu} \tag{7}$$

where $F = \frac{df}{dR}$, $\Box F = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \partial^\mu \left(\frac{df}{dR} \right) \right)$ and $\nabla^\nu \nabla_\mu F = g^{\alpha\nu} \left[F_{,\mu,\alpha} - \Gamma^m_{\mu\alpha} F_{,m} \right]$. We obtain

$$\Box F = \Box \frac{df}{dR} = \frac{1}{\sqrt{-g}} \partial_r \left(\sqrt{-g} g^{rr} \partial_r F \right), \tag{8}$$

$$\nabla^t \nabla_t F = \frac{1}{2} g^{tt} g^{rr} g_{tt,r} F', \tag{9}$$

$$\nabla^r \nabla_r F = g^{rr} F'' - g^{rr} \Gamma^r_{rr} F', \tag{10}$$

$$\nabla^\varphi \nabla_\varphi F = \nabla^\theta \nabla_\theta F = \frac{1}{2} g^{\theta\theta} g^{rr} g_{\theta\theta,r} F' \tag{11}$$

in which a prime denotes derivative with respect to r. Also in Eq. (7) the stress-energy tensor T^ν_μ reads as

$$T^\nu_\mu = -\frac{1}{4\pi} \left(\mathcal{F} \delta^\nu_\mu - F_{\mu\lambda} F^{\nu\lambda} \right), \tag{12}$$

which after considering the line element (2) and the Maxwell field (3) together with (6), one finds

$$T_\mu^\nu = \frac{1}{8\pi} \frac{Q^2}{r^4} \text{diag}\left[-1, -1, 1, 1\right]. \tag{13}$$

We note that another dependent equation is the vanishing trace condition

$$FR - 2f + 3\Box F = 0, \tag{14}$$

which is obtained after knowing $T = T_\mu^\mu = 0$. The trace equation may be used to simplify the field equations and therefore Eq. (7) becomes

$$FR_\nu^\mu - \frac{1}{4}\delta_\nu^\mu (FR - \Box F) - \nabla^\mu\nabla_\nu F = \kappa T_\nu^\mu. \tag{15}$$

From the metric given in (2) one finds the event horizon at $r = r_0 = M + \sqrt{M^2 - Q^2}$ or consequently

$$M = \frac{r_0^2 + Q^2}{2r_0}, \tag{16}$$

as the ADM mass. Based on the *near horizon test* introduced in Ref. [11, 12] we expand all the unknown functions about the horizon. This would lead to the expansions

$$R(r) = R_0 + R_0'(r - r_0) + \frac{1}{2}R_0''(r - r_0)^2 + \mathcal{O}\left((r - r_0)^3\right), \tag{17}$$

$$\Phi(r) = \Phi_0 + \Phi_0'(r - r_0) + \frac{1}{2}\Phi_0''(r - r_0)^2 + \mathcal{O}\left((r - r_0)^3\right), \tag{18}$$

$$F = F_0 + F_0'(r - r_0) + \frac{1}{2}F_0''(r - r_0)^2 + \mathcal{O}\left((r - r_0)^3\right), \tag{19}$$

in which the sub zero implies the corresponding quantity evaluated at the horizon. After some manipulation, the field equations would develop as series in different orders of $(r - r_0)$. In the zeroth order one finds two independent equations

$$f_0 r_0^4 - (E_0 R_0' + 3\Phi_0' F_0) r_0^3 + Q^2 (E_0 R_0' + 3\Phi_0' F_0) r_0 + 2Q^2 (F_0 - 1) = 0, \tag{20}$$

$$f_0 r_0^4 - 2r_0^3 E_0 R_0' + 2Q^2 r_0 E_0 R_0' - 2Q^2 (F_0 - 1) = 0, \tag{21}$$

together with

$$R_0 = \frac{3\Phi_0' \left(r_0^2 - Q^2\right)}{r_0^3}. \tag{22}$$

The first order equations admit another pair of equations

$$F_0 R_0' r_0^4 + \left[\left(2\Phi_0'^2 - 5\Phi_0''\right) F_0 - 3\Phi_0' E_0 R_0' - 3H_0 R_0'^2 + 4f_0 - 3E_0 R_0''\right] r_0^3 - 2\left(3E_0 R_0' + 5\Phi_0' F_0\right) + \tag{23}$$
$$\left[\left(-2\Phi_0'^2 + 5\Phi_0''\right) F_0 + 3\Phi_0' E_0 R_0' + 3H_0 R_0'^2 + 3E_0 R_0''\right] Q^2 r_0 + 6Q^2 E_0 R_0' + 4\Phi_0' F_0 Q^2 = 0,$$

$$F_0 R_0' r_0^4 + 4\left(f_0 - E_0 R_0'' - H_0 R_0'^2 + \frac{1}{2}\Phi_0' E_0 R_0'\right) r_0^3 - 2\left(\Phi_0' F_0 + 3E_0 R_0'\right) r_0^2 + \tag{24}$$
$$\left(4E_0 R_0'' - 2\Phi_0' E_0 R_0' + 4H_0 R_0'^2\right) Q^2 r_0 + 2\Phi_0' F_0 Q^2 = 0$$

together with

$$R_0' = \frac{\left(5\Phi_0'' - 2\Phi_0'^2\right) r_0^3 - 2\Phi_0' r_0^2 + \left(2Q^2\Phi_0'^2 - 5Q^2\Phi_0''\right) r_0 + 8Q^2\Phi_0'}{r_0^4}. \tag{25}$$

In these equations $E = \frac{d^2 f}{dR^2} = \frac{dF}{dR}$ and $H = \frac{d^3 f}{dR^3} = \frac{dE}{dR}$ and a sub "0" implies the value at the horizon. From these equations we find the possible solutions for the unknown coefficients. The following are the results:

$$\Phi = \beta_1 \epsilon + \beta_2 \epsilon^2 + O\left(\epsilon^3\right) \tag{26}$$

$$f = f_0 - \frac{1}{6} \frac{\left(f_0 r_0^4 - 6Q^2\right)\left[2r_0\left(r_0^2 - Q^2\right)\beta_1^2 + 2\left(r_0^2 - 4Q^2\right)\beta_1 - 5r_0\left(r_0^2 - Q^2\right)\beta_2\right]}{r_0^4\left(r_0\left(r_0^2 - Q^2\right)\beta_1 - Q^2\right)} \epsilon + O\left(\epsilon^2\right) \tag{27}$$

$$F = \frac{f_0 r_0^4 - 6Q^2}{6\left(\beta_1 r_0\left(r_0^2 - Q^2\right) - Q^2\right)} + \frac{3\beta_1\left(r_0^2 - Q^2\right)\left(r_0^4 f_0 + 2Q^2\right) - 4f_0 r_0^3 Q^2}{6\left(r_0^2 - Q^2\right)\left(\beta_1 r_0\left(r_0^2 - Q^2\right) - Q^2\right)} \epsilon + O\left(\epsilon^2\right), \tag{28}$$

$$R = \frac{3\beta_1\left(r_0^2 - Q^2\right)}{r_0^3} - \frac{2r_0\left(r_0^2 - Q^2\right)\beta_1^2 + 2\left(r_0^2 - 4Q^2\right)\beta_1 - 5r_0\left(r_0^2 - Q^2\right)\beta_2}{r_0^4} \epsilon + O\left(\epsilon^2\right) \tag{29}$$

in which $\epsilon = r - r_0$, β_1, β_2 are constants and

$$f_0 = -\frac{6Q^2}{r_0^3} \frac{8r_0\left(r_0^2 - Q^2\right)^2 \beta_1^2 - 2\left(r_0^2 - Q^2\right)\left(Q^2 + 5r_0^2\right)\beta_1 - 5r_0\left(r_0^2 - Q^2\right)^2 \beta_2}{16r_0^2\left(r_0^2 - Q^2\right)^2 \beta_1^2 + 2r_0\left(r_0^2 - Q^2\right)\left(5r_0^2 - 23Q^2\right)\beta_1 + 5r_0^2\left(r_0^2 - Q^2\right)^2 \beta_2 + 24Q^4}. \tag{30}$$

Let us note that Φ_0 remains unknown, but since it can be absorbed into the redefinition of time it can be set as $\Phi_0 = 0$. What we have here are some complicated relations between the forms of f, F and R in terms of β_1 and β_2 which are arbitrary. In the zeroth order the conditions on any $f(R)$ can be written as

$$f|_{r_0} = -\frac{6Q^2}{r_0^3} \frac{8r_0\left(r_0^2 - Q^2\right)^2 \beta_1^2 - 2\left(r_0^2 - Q^2\right)\left(Q^2 + 5r_0^2\right)\beta_1 - 5r_0\left(r_0^2 - Q^2\right)^2 \beta_2}{16r_0^2\left(r_0^2 - Q^2\right)^2 \beta_1^2 + 2r_0\left(r_0^2 - Q^2\right)\left(5r_0^2 - 23Q^2\right)\beta_1 + 5r_0^2\left(r_0^2 - Q^2\right)^2 \beta_2 + 24Q^4} \tag{31}$$

and

$$F|_{r_0} = \frac{f_0 r_0^4 - 6Q^2}{6\left(\beta_1 r_0\left(r_0^2 - Q^2\right) - Q^2\right)}. \tag{32}$$

A. Examples of $f(R) = R$ and $f(R) = R^2$

For instance, in the case of R gravity we have $f_0 = R_0$ and $F = 1$. The latter yields

$$\frac{\left(\beta_1\left(r_0^2 - Q^2\right)\right)r_0 - 2Q^2}{2\left(\beta_1 r_0\left(r_0^2 - Q^2\right) - Q^2\right)} = 1 \tag{33}$$

or consequently $\beta_1 = 0$. Having $\beta_1 = 0$ implies

$$f_0 = R_0 \rightarrow -\frac{6Q^2}{r_0^3} \frac{-5r_0\left(r_0^2 - Q^2\right)^2 \beta_2}{5r_0^2\left(r_0^2 - Q^2\right)^2 \beta_2 + 24Q^4} = 0 \tag{34}$$

which clearly leads to $\beta_2 = 0$. Therefore $f(R) = R$ satisfies our general conditions with $\beta_1 = 0 = \beta_2$. Next we test the case of $f(R) = R^2$ for which the above conditions become

$$R_0^2 = -\frac{6Q^2}{r_0^3} \frac{8r_0\left(r_0^2 - Q^2\right)^2 \beta_1^2 - 2\left(r_0^2 - Q^2\right)\left(Q^2 + 5r_0^2\right)\beta_1 - 5r_0\left(r_0^2 - Q^2\right)^2 \beta_2}{16r_0^2\left(r_0^2 - Q^2\right)^2 \beta_1^2 + 2r_0\left(r_0^2 - Q^2\right)\left(5r_0^2 - 23Q^2\right)\beta_1 + 5r_0^2\left(r_0^2 - Q^2\right)^2 \beta_2 + 24Q^4} \tag{35}$$

and

$$2R_0 = \frac{R_0^2 r_0^4 - 6Q^2}{6\left(\beta_1 r_0\left(r_0^2 - Q^2\right) - Q^2\right)}. \tag{36}$$

These two conditions admit

$$\beta_1 = \frac{1}{6} \frac{4Q + 2\sqrt{4Q^2 - 2r_0^4}}{r_0\left(r_0^2 - Q^2\right)} \tag{37}$$

and

$$\beta_2 = \tag{38}$$

$$\frac{-4Q\left\{\left[Q^2\left(20Q^2 - 11r_0^4\right) + 15r_0^2\left(r_0^4 - 2Q^2\right)\right]\sqrt{4Q^2 - 2r_0^4} + Q\left[20Q^2\left(2Q^2 - 3r_0^2\right) + r_0^4\left(r_0^2\left(2r_0^2 + 45\right) - 32Q^2\right)\right]\right\}}{45r_0^2\left(r_0^2 - Q^2\right)^2\left(2\left(r_0^4 - Q^2\right) - Q\sqrt{4Q^2 - 2r_0^4}\right)}$$

which are only acceptable if $4Q^2 - 2r_0^4 \geq 0$. Specifically, once the equality holds one has

$$r_0^4 = 2Q^2 \tag{39}$$

while from (16) we have

$$r_0^2 - 2Mr_0 + Q^2 = 0. \tag{40}$$

These together become

$$r_0^2 - 2Mr_0 + \frac{r_0^4}{2} = 0 \tag{41}$$

or

$$M = r_0 + \frac{1}{2}r_0^3. \tag{42}$$

Also in this case we find

$$\beta_1 = \frac{2}{3} \frac{Q}{r_0\left(r_0^2 - Q^2\right)}, \tag{43}$$

$$\beta_2 = \frac{8}{45} \frac{4r_0^2 - 15}{\left(r_0^2 - 2\right)^2} \tag{44}$$

and $f_0 = R_0 = 1$ while $F_0 = 2$.

These examples can further be extended to cover more general polynomial forms of $f(R)$ to justify the validity of our existence conditions, however, we shall be satisfied with an extremal-RN example in the following section.

B. Extremal RN-type black hole

An interesting case which can be considered here is the case for $M = Q$ in (2). This will make the extremal RN-type black hole with the line element

$$ds^2 = -e^{-2\Phi}\left(1 - \frac{b_0}{r}\right)^2 dt^2 + \frac{dr^2}{\left(1 - \frac{b_0}{r}\right)^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{45}$$

in which $b_0 = Q$. Taking this into account would lead from the general equations

$$R = 6\frac{\beta}{r_0^2}\epsilon - \frac{6\beta}{r_0^2}\left(2\beta + \frac{5}{r_0}\right)\epsilon^2 + \frac{\beta}{r_0^2}\left(\frac{93\beta^2}{4} + \frac{71\beta}{r_0} + \frac{90}{r_0^2}\right)\epsilon^3 + O\left(\epsilon^4\right), \tag{46}$$

$$f = 6\frac{\beta}{r_0^2}\epsilon - \frac{3\beta}{r_0^2}\left(3\beta + \frac{10}{r_0}\right)\epsilon^2 + \frac{\beta}{r_0^2}\left(\frac{57\beta^2}{4} + \frac{49\beta}{r_0} + \frac{90}{r_0^2}\right)\epsilon^3 + O\left(\epsilon^4\right), \tag{47}$$

$$F = \left(\frac{df}{dR}\right) = 1 + \beta\epsilon - \frac{\beta}{2}\left(\beta + \frac{2}{r_0}\right)\epsilon^2 + \beta\left(\frac{3\beta^2}{8} + \frac{3\beta}{4r_0} + \frac{1}{r_0^2}\right)\epsilon^3 + O\left(\epsilon^4\right) \tag{48}$$

and

$$\Phi = \Phi_0 + \beta\epsilon - \frac{\beta}{8}\left(5\beta + \frac{8}{r_0}\right)\epsilon^2 + \beta\left(\frac{73\beta^2}{120} + \frac{73\beta}{60r_0} + \frac{1}{r_0^2}\right)\epsilon^3 + O\left(\epsilon^4\right) \tag{49}$$

in which β is an arbitrary, non-zero constant and $\epsilon = (r - r_0)$. As before, we absorb Φ_0 into time. It is remarkable observe that R is zero at the horizon and so is f, but $\left(\frac{df}{dR}\right) = 1$. This is an indication that a proper candidate for such an $f(R)$ is of the form

$$f(R) = R + a_2 R^2 + a_3 R^3 + a_4 R^4 + ... \tag{50}$$

in which the constant coefficients a_i can be determined, using above conditions. For instance, if we restrict ourselves up to the third order we get $f(R) \sim R + \frac{r_0^2}{12}R^2 + r_0^3\left(\frac{5}{72}r_0 + \frac{19}{108\beta}\right)R^3$. One can easily check that this form of $f(R)$ satisfies all the conditions given above up to the second order. Subsequent implication of the results found above is that $f(R) \sim R^\nu$. Here any ν can not satisfy the conditions without choosing $\beta = 0$, which is the case of $\nu = 1$ or GR. Another example which at least satisfies the above conditions up to first order is $f(R) = \frac{R}{1-R}$.

III. THERMODYNAMICS OF THE ANALOG BLACK HOLE

After having the solution one may be curious about the thermodynamical properties of the solution. This is doable in exact form because of the metric function which is known about the horizon. First of all the horizon will remain as $r = r_0$ and the Hawking temperature is found by

$$T_H = \left.\frac{\frac{\partial}{\partial r}g_{tt}}{4\pi}\right|_{r=r_0} = T_H^{(RN)} = \frac{1}{4\pi r_0}\left(1 - \frac{Q^2}{r_0^2}\right) \tag{51}$$

in which $T_H^{(RN)}$ implies RN Hawking temperature. The form of Entropy is given by

$$S = \left.\frac{\mathcal{A}}{4G}F\right|_{r=r_0} = \pi r_0^2 F_0 \tag{52}$$

in which $\mathcal{A}|_{r=r_0} = 4\pi r_0^2$ is the surface area of the black hole at the horizon and $F|_{r=r_0} = F_0$. We note that T_H and S are both exact. Having T_H and S one may find the heat capacity of the black hole

$$C_q = T\left(\frac{\partial S}{\partial T}\right)_Q = C_q^{(RN)}\mathcal{I} \tag{53}$$

in which

$$\mathcal{I} = 12Q^2\left(r_0^2 - Q^2\right)\Pi \tag{54}$$

where

$$\Pi = \frac{\left[5r_0^3\left(\left(r_0^4 - Q^4\right)\beta_1 - 4Q^2 r_0\right)\beta_2 + 16r_0^3\beta_1^3\left(r_0^4 - Q^4\right) + 4Q^2 r_0^2\beta_1^2\left(7r_0^2 - 23Q^2\right) + \frac{2Q^2\left(24Q^4 - r_0\beta_1\left(15r_0^4 + 32r_0^2 Q^2 - 59Q^4\right)\right)}{\left(r_0^2 - Q^2\right)}\right]}{\left[r_0^2\left(r_0^2 - Q^2\right)^2\left(5\beta_2 + 16\beta_1^2\right) + 2r_0\left(r_0^2 - Q^2\right)\left(5r_0^2 - 23Q^2\right)\beta_1 + 24Q^4\right]^2} \tag{55}$$

We comment that $\mathcal{I}$ in the RN limit (i.e., $\beta_i \to 0$) becomes unit as expected. Let us note also that the form of C_q is exact. In order to study the thermodynamics of the extremal solution we use the general results found in the non-extremal RN type solution. One easily finds that $T_H = 0$, and $C_q = 0$.

A. First Law of Thermodynamics

Furthermore, in this section, we would like to show that in general, the above solution also satisfies the first law of thermodynamics. This is somehow a generalization of what was introduced in Ref. [20] to find a higher dimensional form of the Misner-Sharp (MS) energy [21] and was used in SSS black hole in $f(R)$ gravity in Ref. [22]. To this end we rewrite the field equation in the following form

$$G_\mu^\nu = \kappa \left[\frac{1}{F} T_\mu^\nu + \frac{1}{\kappa} \check{T}_\mu^\nu \right], \tag{56}$$

in which G_μ^ν is the Einstein tensor,

$$\check{T}_\mu^\nu = \frac{1}{f_R} \left[\nabla^\nu \nabla_\mu F - \left(\Box F - \frac{1}{2} f + \frac{1}{2} RF \right) \delta_\mu^\nu \right] \tag{57}$$

and for our later convenience we consider

$$ds^2 = -e^{-2\Phi} U d^2 t + \frac{1}{U} d^2 r + r^2 d\Omega^2. \tag{58}$$

In turn, the tt component of the latter field equation would read

$$G_0^0 = \kappa \left[\frac{1}{F} T_0^0 + \frac{1}{\kappa} \frac{1}{F} \left[\nabla^0 \nabla_0 F - \left(\Box F - \frac{1}{2} f + \frac{1}{2} RF \right) \right] \right] \tag{59}$$

in which

$$G_0^0 = \frac{U'r - 1 + U}{r^2}, \tag{60}$$

$$\nabla^0 \nabla_0 F = \frac{1}{2} \left(-2\Phi' U + U' \right) F' \tag{61}$$

and $\Box F = \frac{2}{3} f - \frac{1}{3} RF$. At the horizon (where the MS energy is introduced) $U(r_0) = 0$, which yields $G_0^0 = \frac{U_0' r_0 - 1}{r_0^2}$, $\nabla^0 \nabla_0 F = \frac{1}{2} U_0' F_0'$. A substitution in (40) and calculating everything at the horizon $r = r_0$ yields

$$\frac{F_0 U_0'}{r_0} - \frac{F_0}{r_0^2} = \kappa T_0^0 + \left(\frac{1}{2} U_0' F_0' - \frac{1}{6} (f_0 + R_0 F_0) \right). \tag{62}$$

Next, we multiply both sides by the spherical volume element at the horizon i.e. $dV_0 = \mathcal{A} dr_0$ to get

$$\frac{F_0 U_0'}{r_0} \mathcal{A} dr_0 = \left(\frac{F_0}{r_0^2} + \frac{1}{2} U_0' F_0' - \frac{1}{6} (f_0 + R_0 F_0) \right) \mathcal{A} dr_0 + \kappa T_0^0 dV_0. \tag{63}$$

Using $\frac{\mathcal{A}}{r_0} = \frac{1}{2} \frac{d}{dr_0} \mathcal{A}$ and some manipulation one finds

$$\frac{U_0'}{4\pi} \frac{d}{dr_0} \left(\frac{2\pi \mathcal{A}}{\kappa} F_0 \right) dr_0 = \frac{1}{\kappa} \left(\frac{F_0}{r_0^2} + U_0' F_0' - \frac{1}{6} (f_0 + R_0 F_0) \right) \mathcal{A} dr_0 + T_0^0 dV_0 \tag{64}$$

which is nothing but the first law of thermodynamics i.e., $TdS = dE + PdV$. This is due to the definition which we have for Hawking temperature $T = \frac{U_0'}{4\pi}$, entropy of the black hole $S = \frac{2\pi \mathcal{A}}{\kappa} F_0$, the radial pressure $P = T_r^r = T_0^0$ and the MS energy as

$$E = \frac{1}{\kappa} \int \left(\frac{F_0}{r_0^2} + U_0' F_0' - \frac{1}{6} (f_0 + R_0 F_0) \right) \mathcal{A} dr_0 \tag{65}$$

in which the integration constant is set to zero [20, 23–26] (also for a BH-like solutions see [27]). Here we comment that all quantities are calculated at the horizon and due to this the Hawking temperature becomes $T = \frac{\left(e^{-2\Phi} U \right)'}{4\pi} \Big|_{r_0} = \frac{U_0'}{4\pi}$.

The above results imply that, using (65) as MS energy, the first law of thermodynamic is satisfied. Once more we wish to add that our results are exact.

IV.   CONCLUDING REMARKS

In this paper we have applied the "*near-horizon test*" to the Reissner-Nordström (RN)-type black holes in $f(R)$ gravity. Necessary conditions, not the sufficient ones that a RN-type black hole exists are derived. These are nothing but the regularity conditions of the metric functions in the vicinity of the event horizon. Our metric ansatz consists of a general static, spherically symmetric (SSS) case adopted from the Einstein's general relativity. We considered also the extremal case as an analog black hole in $f(R)$ gravity and derived the underlying conditions. Due to their intricacy we didn't attempt to solve those equations in general. To the zeroth order, however, they can be obtained exactly while to the first order approximation is also tractable. Our analysis shows that a closed form of $f(R)$ doesn't seem possible: With a given source we can determine $f(R)$ implicitly as an infinite series in $(r - r_0)$, since $R(r)$ also is expressed in similar series. This is against the strategy adapted so far, namely, an explicit form of $f(R)$ is assumed a priori to be tested whether it fits physical requirements. In our opinion, the "*near-horizon test*", introduced in [11, 12] and developed here further constitutes a more fundamental test than any other arguments in connection with black holes. We admit that since our necessary conditions for the existence of RN-type black holes are entirely local they don't involve the requirements for asymptotic flatness. Stability of such black holes must also be considered separately when one considers exact solutions. Our test must naturally be supplemented with $\frac{d^2 f}{dR^2} > 0$ and $\frac{df}{dR} > 0$, for stability and no-ghost requirements [28, 29]. We have shown also that the thermodynamic of these analog black holes can be studied through the Misner-Sharp formalism to verify the validity of the first law. Finally, we remark that solution for $f(R)$ gravity admitting an electromagnetic field with similar thermodynamics was reported before [30].

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