Ground State H-Atom in Born-Infeld Theory

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Within the context of Born-Infeld (BI) nonlinear electrodynamics (NED) we revisit the non-relativistic, spinless H-atom. The pair potential computed from the Born-Infeld equations is approximated by the Morse type potential with remarkable fit over the critical region where the convergence of both the short and long distance expansions slows down dramatically. The Morse potential is employed to determine both the ground state energy of the electron and the BI parameter.

I. INTRODUCTION

An intriguing example of nonlinear electrodynamics (NED) was introduced in 1934 by Born and Infeld (BI) in order to eliminate divergences in the Coulomb problem [1]. With the advent of quantum electrodynamics (QED), however, divergences were resolved by the well-established scheme of renormalization. Being popular enough, QED suppressed the efforts of BI to the extend of being forgotten until very recent decade. We observe now that BI theory gained recognition anew within string theory; a theory that aims to unify all forces of nature, including quantum gravity, under a common title. Once BI paved the way toward NED, various modifications emerged as alternative theories to the well-known linear electrodynamics of Maxwell. The common feature in all these NED theories is that in the linear limit it recovers the Maxwell's electrodynamics, as it should. The NED Lagrangian is commonly constructed from the invariants $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}{}^{\star}F^{\mu\nu}$($\star$ means dual) as nonpolynomial functions. The field tensor $F_{\mu\nu}$ is defined as in Maxwell's linear electrodynamics, namely, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, in terms of the vector potential A_μ. Since these invariants play role in the vacuum polarization, BI theory, or any version of NED can polarize the vacuum. We note that beside string theory, NED and naturally BI, find applications in different areas, ranging from black holes and cosmology to the model of elementary particles[2]. In this paper we revisit the problem to compute Born-Infeld effects on atomic spectra and employ a Morse type potential simulation to determine the ground level energy of the H-atom. We restrict ourselves entirely to the pure electrical potential (i.e. without magnetic fields) applied to the ground state of the non-relativistic Schrödinger equation and for pedagogical reasons we shall try to avoid the intricacy of NED as much as we can. Let us note that throughout our calculations, following Ref. [5] we shall employ dimensionless variables. We choose c (the speed of light), e (charge of the electron), m_e (mass of the electron) and $\hbar$ all equal to unity. Accordingly, to recover the dimensionful quantities each variable should be multiplied with the appropriate factor. To convert the dimensionless unit of length, for instance, it should be multiplied by the Compton wavelength $\lambda_C = \frac{\hbar}{m_e c}$. Similarly, the unit magnitude of electric field is converted by the factor $\frac{e}{\lambda_C^2}$. Conversion of energy is carried out by the factor $m_e c^2$ and so on. The electrostatic BI Lagrangian can be expressed by

$$L(X) = \frac{4}{\beta^2}\left[1 - \sqrt{1 + \frac{\beta^2}{2}X}\right] \tag{1}$$

where $X = F^2 = F_{\mu\nu}F^{\mu\nu}$ and β is the dimensionless BI parameter. In the limit $\beta \to 0$ by applying the L'Hôpital's rule we recover the Maxwell Lagrangian $L = -X$. The electrostatic potential is given by

$$A_\mu = (V(r), 0, 0, 0) = \delta_\mu^0 V(r) \tag{2}$$

for an r dependent function $V(r)$. The sourceless BI equation reduces to the single equation

$$\partial_\mu\left(\sqrt{|g|}\frac{\partial L}{\partial X}F^{\mu\nu}\right) = 0 \tag{3}$$

in which $\sqrt{|g|}$ refers to the square root of determinant for the spherically symmetric flat metric

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right). \tag{4}$$

Note that the difference of (3) from the standard linear Maxwell equation is the appearance of $\frac{\partial L}{\partial X}$ term inside the parenthesis. As we stated above since the magnetic field is chosen to vanish our electrodynamics equations consist of

(3) alone. Born's solution follows from (3) which is expressed for convenience in the following form [3–5]

$$V(|\mathbf{s}|) = \frac{-\alpha}{\beta} \int_{|\mathbf{s}-\mathbf{s}_e|}^{\infty} \frac{dr}{\sqrt{1+r^4}}. \tag{5}$$

This expression represents the electrostatic potential at $\mathbf{s}$ due to a positively charged particle located at $\mathbf{s}_e$, and the constant $\alpha \left(= \frac{e^2}{\hbar c}\right)$ stands for the fine structure constant. Obviously the latter is introduced to prepare the ground for quantum mechanical treatment. compared with the Coulomb potential $V(|\mathbf{s}|) \sim \frac{1}{|\mathbf{s}|}$, the integral expression of the potential (3) is complicated enough for an analytic treatment. In the next section we shall describe the strategy of handling the even more complicated two-body potential in Born-Infeld theory.

II. MORSE POTENTIAL VERSUS BORN POTENTIAL IN THE HYDROGEN-ATOM

In [5], Carley and Kiessling addressed (see also [3, 4]) the old problem of computing Born-Infeld effects on the Schrödinger spectrum of the hydrogen atom. They use as Schrödinger potential $V(r,\beta)$, given by

$$V(r,\beta) = -\frac{\alpha}{\beta}\left[W(r/\beta) + \frac{1}{4}B\left(\frac{1}{4},\frac{1}{4}\right)\right], \tag{6}$$

in which

$$\begin{aligned} W(r/\beta) &= \int_{2\sqrt{2}\beta/r}^{\infty} \frac{f'(y)}{\sqrt{1+x^4}} dx, \\ f(y) &= \sqrt{\frac{1}{4} + y^2 - y\sqrt{1+y^2}}, \\ (f'(y) &= \frac{df}{dy}),\ r = |\mathbf{s}_{proton} - \mathbf{s}_{electron}|, \\ B(.,.) &= \text{Euler's Beta function} \end{aligned} \tag{7}$$

and $xy = \frac{\beta}{r}$. Following this expression they showed that in two different regions, namely for $r > 2\sqrt{2}\beta$ and $r < 2\sqrt{2}\beta$, one finds convergent expansions for $W(r/\beta)$ as

$$W(r/\beta) = \begin{cases} \sum_{k=0}^{\infty} a_k (r/\beta)^{4k+1} & , \ r < 2\sqrt{2}\beta \\ \sum_{k=0}^{\infty} b_k (\beta/r)^k & , \ r > 2\sqrt{2}\beta \end{cases} \tag{8}$$

in which the first few constants were given explicitly[4]. As a result of their rigorous analysis it turned out that $r = 2\sqrt{2}\beta$ is "the breakdown point" for both expansions. That is, for small $\frac{r}{\beta}$ the expansion fails to converge for $r/\beta > 2\sqrt{2}$, and for large r/β it fails similarly for $r/\beta < 2\sqrt{2}$. Getting closer to $r/\beta = 2\sqrt{2}$ requires more terms in both expansions, which make them unpractical in this region. For practical purposes one of course can always evaluate the integral (7) by straightforward numerical quadratures. Yet, whenever two expansions from opposite ends of an interval fail to overlap in their respective domains of convergence, it is desirable to have some interpolating formula which bridges the two domains of convergence. In this note we report on such a simple interpolation, involving only 4 parameters and the well-known Morse potential [6], given by (note that the subscript 'a' refers to approximate expression)

$$W_a(r/\beta) = \left[G\left(1 - e^{-\kappa(r/\beta - b)}\right)^2 + V_\circ\right] \tag{9}$$

where the dimensionless constants are chosen as

$$\begin{aligned} G &= -1.8300, \\ V_\circ &= 0.09805 \\ \kappa &= 0.58520 \\ b &= -0.45720. \end{aligned} \tag{10}$$

After standard separation of variables, the radial part of the Schrödinger equation reads, ($s-$state)

$$-\frac{1}{2r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)R\left(r\right)+V_a\left(r,\beta\right)R\left(r\right)=\varepsilon R\left(r\right) \tag{11}$$

where the approximate potential $V_a\left(r,\beta\right)$ is obtained by plugging (9) into (6). We impose now the following change of variables

$$r=\beta\rho \text{ and } u\left(\rho\right)=\rho R\left(\rho\right) \tag{12}$$

to get

$$-\frac{1}{2}\frac{d^2}{d\rho^2}u\left(\rho\right)-\alpha\beta\left[G\left(1-e^{-\kappa(\rho-b)}\right)^2+V_\circ+\frac{1}{4}B\left(\frac{1}{4},\frac{1}{4}\right)\right]u\left(\rho\right)=\frac{\varepsilon}{\alpha^2}\left(\alpha\beta\right)^2u\left(\rho\right). \tag{13}$$

The latter equation, by introducing $x=\kappa\left(\rho-b\right)$ yields

$$-\frac{1}{2}\frac{d^2}{dx^2}u\left(x\right)-\left|A\right|\left(2e^{-x}-e^{-2x}\right)u\left(x\right)=Eu\left(x\right) \tag{14}$$

where

$$\begin{aligned}\left|A\right| &= \frac{\alpha\beta}{\kappa^2}\left|G\right|>0, \\ E &= \frac{1}{\kappa^2}\left[\frac{\varepsilon}{\alpha^2}\left(\alpha\beta\right)^2+\alpha\beta\left(V_\circ+\frac{1}{4}B\left(\frac{1}{4},\frac{1}{4}\right)-\left|G\right|\right)\right]<0.\end{aligned} \tag{15}$$

We notice that since $0<\rho<\infty$ then $\kappa b<x<\infty$ in which infinity is in contrast with the dimension of the atom. Also we note that this is (an energy-dependent-potential) Schrödinger equation and therefore in the solution we identify V_0 (consequently $\alpha\beta$) and E simultaneously. One can show that the proper solution which satisfies the boundary conditions

$$\lim_{x\to\infty}u\left(x\right)=0,\qquad \lim_{x\to\kappa|b|}u\left(x\right)=0 \tag{16}$$

is given by

$$\begin{aligned}u\left(x\right) &= C\,WhittakerM\left(a,\nu,2ae^{-x}\right), \\ a &= \sqrt{2\left|A\right|},\quad \nu=\sqrt{2\left|E\right|},\end{aligned} \tag{17}$$

in which $0<\nu\in\mathbb{R}$ and $WhittakerM\left(a,\nu,2ae^{-\kappa|b|}\right)=0$ [7]. In fact here ν is not a quantum number but it is a new parameter to adjust the results, and $2ae^{-\kappa|b|}=X_\nu$ is the first root of $WhittakerM\left(a,\nu,2ae^{-x}\right)=0$. It is remarkable to observe that once we choose ν, we identify both potential and energy of the system at the same time.

Hence one can show that

$$\begin{aligned}\frac{\varepsilon}{\alpha^2} &= \frac{1}{\left(\alpha\beta\right)^2}\left[\frac{\kappa^2\nu^2}{2}-\alpha\beta\left(V_\circ+\frac{1}{4}B\left(\frac{1}{4},\frac{1}{4}\right)-\left|G\right|\right)\right], \\ \alpha\beta &= \frac{\kappa^2a^2}{2\left|G\right|}.\end{aligned} \tag{18}$$

This closed expression helps us to adjust ν in order to set the ground-state energy of the $H-$atom in accordance with the empirical data and consequently to find the corresponding value for Born's parameter β. The following table gives $\alpha\beta$ and three first s-states of a sample $H-$atom by using the above considerations.

(Table 1)

	$\alpha\beta$	$-\frac{\varepsilon_{100}}{\alpha^2}$	$-\frac{\varepsilon_{200}}{\alpha^2}$	$-\frac{\varepsilon_{300}}{\alpha^2}$
Empirical		0.49973	0.12493	0.05553
Our results(Morse-type-potential)	1.82337	0.49973	0.20101	0.08006

We plot the exact potential (7) and its Morse-like approximation $V_a\left(r,\beta\right)=-\frac{\alpha}{\beta}\left[W_a\left(r/\beta\right)+\frac{1}{4}B\left(\frac{1}{4},\frac{1}{4}\right)\right]$, in Fig. 1 with a remarkable fit over the range $\frac{r}{\beta}=0$ to 10, bridging the two expansion regions mentioned above. However, note that for larger $\frac{r}{\beta}$ our interpolation formula does not apply, as the Morse type potential decays exponentially fast instead of inversely.

III. CONCLUSION AND DISCUSSION

In conclusion, we recall that our attempt was to show that one can always find a simulation for the functions whose closed forms are not known. This provides analytical approximations for the final results. Having such analytical solutions for physical systems always have some significant features consisting the application of the results for the similar systems. In particular we simulated the Born-Infeld-Coulomb (BIC) potential into a Morse-type potential to find an estimation for the Born's parameter as well as the accurate ground state energy level. We have shown that in this approach, Born's parameter has a value between the original value proposed by Born and the value found by Carley and Kiessling. We must emphasize also that our Morse simulation is independent of the value of the Born parameter. It is interesting that a non-linear electrodynamics developed in 1930s by Born and Infeld can be simulated remarkably by a potential developed in 1920s by Morse which proved to be useful in $2-$atom molecules. With the difference that the subject now is the H-atom consisted of a fixed proton and a moving electron. From Table 1 it is seen that for $n=2$ and $n=3$ the Morse-fitting is inefficient. By invoking a different Born parameter for each excited state may overcome this discrepancy, however, our concern in this study is confined to the ground state. Further, since our approach is non-relativistic with zero angular momentum, consistency with the spectra obtained in Ref. [8] should not be expected. Our ground state, however, is consistent with the Schrödinger spectrum (with zero angular momentum) of Ref. [8], as it should. Morse-simulation of the Dirac equation in Born-Infeld electrodynamics may be considered as a separate future project. Finally we wish to remark that our interest in the Born-Infeld electrodynamics was aroused while searching for pure electrically charged, regular black holes in the Einstein-Born-Infeld theory [9].

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Figure Caption:

Fig. 1: $\frac{\beta}{\alpha}V(r,\beta)$ (discrete curve) and the simulated Morse-like $\frac{\beta}{\alpha}V_s(r,\beta)$ (continuous curve) versus $\frac{r}{\beta}$. Also in this figure we give 2-terms (DOT), $3-$terms (BOX) and $4-$terms (CIRCLE) approximation for $\frac{\beta}{\alpha}V(r,\beta)$ given in Ref. [5] for both regions $r<2\sqrt{2}$ and $r>2\sqrt{2}$, which reveal that once r approaches to $2\sqrt{2}$ from both sides we need more terms to get a good approximation.

Table Caption:

Comparison of first three s-state energy levels with our Morse-type-simulated results. It is seen that we obtain the Born parameter as $\beta=\frac{1.82337}{\alpha}$.