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Constant curvature $f(R)$ gravity minimally coupled with Yang–Mills field

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Abstract We consider the particular class of $f(R)$ gravities minimally coupled with Yang–Mills (YM) field in which the Ricci scalar $= R_0 =$ constant in all dimensions $d \geq 4$. Even in this restricted class the spacetime has unlimited scopes determined by an equation of state of the form $P_{\text{eff}} = \omega\rho$. Depending on the distance from the origin (or horizon of a black hole) the state function $\omega(r)$ takes different values. It is observed that $\omega \to \frac{1}{3}$ (the ultra relativistic case in 4 dimensions) and $\omega \to -1$ (the cosmological constant) are the limiting values of our state function $\omega(r)$ in a spacetime centered by a black hole. This suggests that having a constant ω throughout spacetime around a charged black hole in $f(R)$ gravity with constant scalar curvature is a myth.

1 Introduction

For a number of reasons, ranging from dark energy and accelerated expansion of the universe to astronomical tests, modified version of general relativity gained considerable interest in recent times. $f(R)$ gravity, in particular, attracted much attention in this context (see [1, 2] for comprehensive reviews of the subject). The reason for this trend may be the dependence of its Lagrangian on the Ricci scalar alone, so that it can be handled relatively simpler in comparison with the higher order curvature invariants. Depending on the structure of the function $f(R)$ the nonlinearity creates curvature sources which may be interpreted as 'sources without sources', manifesting themselves in the Einstein equations. Beside these curvature (or geometrical) sources there may be true physical sources that contribute together with the former to determine the total source in the problem. It should be added that owing to the highly nonlinear structure of the underlying field equations attaining exact solutions is not an easy task at all. In spite of all odds many exact solutions have been obtained from ab initio assumed $f(R)$ functions. To recall an example we refer to the choice $f(R) = R^N$ ($N =$ an arbitrary number), which attains an electromagnetic-like curvature source, so that $N \neq 1$ can be interpreted as an 'electric charge without charge' [3]. That is, the resulting geometry becomes equivalent to the Reissner–Nordstrom (RN) geometry in a spherically symmetry metric ansatz of Einstein's gravity. This particular example reveals that the failure of certain tests related to Solar System/Cosmology in $f(R)$ gravity is accountable by the curvature sources in the Einstein Hilbert action. Equivalence with $f(R) = R +$(scalar fields) provides another such example beside the electromagnetic one. More recently we obtained a large class of non-analytical $f(R)$ gravity solutions minimally coupled with Yang–Mills (YM) field [4]. Even more to this the YM field was allowed to be a nonlinear theory in which the power-YM constitutes a particular example in all higher dimensions. In particular, in $d = 6$, $f(R) = \sqrt{R}$ solves the Einstein–Yang–Mills (EYM) system exactly. For $d = 4$ our solution for nonabelian gauge reduces to an abelian one which may be considered as an Einstein–Maxwell (EM) solution [5].

Previously $f(R)$ gravity coupled non-minimally with Yang–Mills and Maxwell matter sources have been considered [6, 7]. In this paper we consider a particular class within minimally coupled YM field in $f(R)$ gravity with the conditions that the scalar curvature $R = R_0 =$ constant and the trace of the YM energy-momentum tensor is zero. (To see other black hole solutions with matter in $f(R)$ gravity we refer to Ref. [8–13].) Contrary to our expectations this turns out to be a non-trivial class with far-reaching consequences. Our spacetime is chosen spherically symmetric to be in accord with the spherically symmetric Wu–Yang ansatz for the YM field. The field equations admit exact solutions in all dimensions $d \geq 4$ with the physical parameters; mass (m)

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of the black hole, YM charge (Q) and the scalar curvature (R_0) of the space time. In this picture we note that the cosmological constant arises automatically as proportional to R_0. From a physics stand point, considering the equation of state in the form $P_{\mathrm{eff}} = \omega\rho$, with effective pressure (P_{eff}) and density (ρ), important results are obtained as follows. For a critical value of $r = r_c$ we have $-1 < \omega(r) < 0$ for $r > r_c$ and $0 < \omega(r) < \frac{1}{3}$ for $r < r_c$. Remarkably this amounts to a sign shift in the effective pressure to account for the accelerated expansion of a universe centered by a charged black hole. In general the critical distance is thermodynamically unstable so that the universe undergoes the phase of accelerated expansion beyond that particular distance. Absence of the phantom era (i.e. $\omega < -1$) is also manifest. Alternatively, in the limit $r \to 0$ it yields $\omega \to \frac{1}{3}$ (i.e. 4-dimensional ultra relativistic case), while for $r \to \infty$ we have $\omega \to -1$, the case of a pure cosmological constant. Let us note that the latter case corresponds to vanishing of the YM field. Stated otherwise, in the overall space time we do not have a fixed value for ω. Depending on the distance from the center (or horizon) of a black hole we have a varying state parameter $\omega(r)$. The same argument in the Friedmann–Robertson–Walker (FRW) version of the theory implies that beyond a critical time $t = t_c$, $\omega(t)$ changes its role and a different type of matter becomes active. It is known that for the normal and dark matters which provide clustering both the weak energy condition (WEC) and the strong energy condition (SEC) must be satisfied. In the case of dark energy on the other hand WEC is satisfied while SEC is violated. In Appendix we analyze the energy conditions thoroughly covering all dimensions. Although our metric ansatz is chosen to be spherically symmetric so that the constant scalar curvature $R_0 > 0$, in order to prepare ground for the topological black holes we consider the case of $R_0 < 0$ as well.

Organization of the paper is as follows. In Sect. 2 we introduce our formalism and present exact solutions. The analysis of our solution with thermodynamical functions is considered in Sect. 3. We complete the paper with our conclusion, which appears in Sect. 4.

2 The formalism and solution for $R = \text{constant}$

We choose the action as (our unit convention is chosen such that $c = G = 1$ so that $\kappa = 8\pi$)

$$S = \int d^d x \sqrt{-g}\left[\frac{f(R)}{2\kappa} + \mathcal{L}(F)\right] \tag{1}$$

in which $f(R)$ is a real function of Ricci scalar R and $L(F)$ is the nonlinear YM Lagrangian with $F = \frac{1}{4}\operatorname{tr}(F^{(a)}_{\mu\nu} F^{(a)\mu\nu})$. Obviously the particular choice $\mathcal{L}(F) = -\frac{1}{4\pi} F$ will reduce to the case of standard YM theory. The YM field 2-form components are given by

$$\mathbf{F}^{(a)} = \frac{1}{2} F^{(a)}_{\mu\nu}\, dx^\mu \wedge dx^\nu \tag{2}$$

with the internal index (a) running over the degrees of freedom of the nonabelian YM gauge field. Variation of the action with respect to the metric $g_{\mu\nu}$ gives the EYM field equations as

$$f_R R^\nu_\mu + \left(\Box f_R - \frac{1}{2} f\right)\delta^\nu_\mu - \nabla^\nu \nabla_\mu f_R = \kappa T^\nu_\mu \tag{3}$$

in which

$$T^\nu_\mu = \mathcal{L}(F)\delta^\nu_\mu - \operatorname{tr}\big(F^{(a)}_{\mu\alpha} F^{(a)\nu\alpha}\big)\mathcal{L}_F(F), \qquad \mathcal{L}_F(F) = \frac{d\mathcal{L}(F)}{dF}. \tag{4}$$

Our notation here is as follows: $f_R = \frac{df(R)}{dR}$, $\Box f_R = \nabla_\mu \nabla^\mu f_R = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu) f_R$, R^ν_μ is the Ricci tensor and

$$\nabla^\nu \nabla_\mu f_R = g^{\alpha\nu}(f_R)_{,\mu;\alpha} = g^{\alpha\nu}\big[(f_R)_{,\mu,\alpha} - \Gamma^m_{\mu\alpha}(f_R)_{,m}\big]. \tag{5}$$

The trace of the EYM equation (3) yields

$$f_R R + (d-1)\Box f_R - \frac{d}{2} f = \kappa T \tag{6}$$

in which $T = T^\mu_\mu$. The $SO(d-1)$ gauge group YM potentials are given by

$$\mathbf{A}^{(a)} = \frac{Q}{r^2} C^{(a)}_{(i)(j)}\, x^i\, dx^j, \tag{7}$$

$$Q = \text{YM magnetic charge},$$

$$r^2 = \sum_{i=1}^{d-1} x_i^2,$$

$$2 \le j+1 \le i \le d-1, \quad \text{and} \quad 1 \le a \le (d-2)(d-1)/2,$$

$$x_1 = r\cos\theta_{d-3}\sin\theta_{d-4}\ldots\sin\theta_1,$$
$$x_2 = r\sin\theta_{d-3}\sin\theta_{d-4}\ldots\sin\theta_1,$$
$$x_3 = r\cos\theta_{d-4}\sin\theta_{d-5}\ldots\sin\theta_1,$$
$$x_4 = r\sin\theta_{d-4}\sin\theta_{d-5}\ldots\sin\theta_1,$$
$$\ldots$$
$$x_{d-2} = r\cos\theta_1,$$

in which $C^{(a)}_{(b)(c)}$ are the non-zero structure constants of $\frac{(d-1)(d-2)}{2}$-parameter Lie group $\mathcal{G}$ [14–16]. The metric ansatz is spherically symmetric; it reads

$$ds^2 = -A(r)\, dt^2 + \frac{dr^2}{A(r)} + r^2 d\Omega^2_{d-2}, \tag{8}$$

with the only unknown function $A(r)$ and the solid angle element

$$d\Omega^2_{d-2} = d\theta_1^2 + \sum_{i=2}^{d-2}\prod_{j=1}^{i-1}\sin^2\theta_j\, d\theta_i^2, \tag{9}$$

with

$$0 \le \theta_{d-2} \le 2\pi,$$
$$0 \le \theta_i \le \pi,$$
$$1 \le i \le d-3.$$

Variation of the action with respect to $\mathbf{A}^{(a)}$ implies the YM equations

$$\mathbf{d}\left[{}^{\star}\mathbf{F}^{(a)} L_F(F)\right] + \frac{1}{\sigma} C^{(a)}_{(b)(c)} L_F(F) \mathbf{A}^{(b)} \wedge {}^{\star}\mathbf{F}^{(c)} = 0, \tag{10}$$

in which σ is a coupling constant and $\star$ means duality. One may show that the YM invariant satisfies

$$F = \frac{1}{4}\operatorname{tr}\left(F^{(a)}_{\mu\nu} F^{(a)\mu\nu}\right) = \frac{(d-2)(d-3)Q^2}{4r^4} \tag{11}$$

and

$$\operatorname{tr}\left(F^{(a)}_{t\alpha} F^{(a)t\alpha}\right) = \operatorname{tr}\left(F^{(a)}_{r\alpha} F^{(a)r\alpha}\right) = 0, \tag{12}$$

while

$$\operatorname{tr}\left(F^{(a)}_{\theta_i\alpha} F^{(a)\theta_i\alpha}\right) = \frac{(d-3)Q^2}{r^4}, \tag{13}$$

which leads us to the exact form of the energy-momentum tensor:

$$T^{\nu}_{\mu} = \operatorname{diag}\left[\mathcal{L}, \mathcal{L}, \mathcal{L} - \frac{(d-3)Q^2}{r^4}\mathcal{L}_F, \mathcal{L} - \frac{(d-3)Q^2}{r^4}\mathcal{L}_F, \ldots, \mathcal{L} - \frac{(d-3)Q^2}{r^4}\mathcal{L}_F\right]. \tag{14}$$

Here the trace of T^{ν}_{μ} becomes

$$T = T^{\mu}_{\mu} = d\mathcal{L} - 4F\mathcal{L}_F, \tag{15}$$

and therefore with Eq. (3) we find

$$f = \frac{2}{d}\left[f_R R + (d-1)\Box f_R - \kappa(d\mathcal{L} - 4F\mathcal{L}_F)\right]. \tag{16}$$

To proceed further we set the trace of energy-momentum tensor to be zero i.e.,

$$d\mathcal{L} - 4F\mathcal{L}_F = 0 \tag{17}$$

which leads to a power Maxwell Lagrangian [17–21]

$$\mathcal{L} = -\frac{1}{4\pi} F^{\frac{d}{4}}. \tag{18}$$

Here for our convenience the integration constant is set to be $-\frac{1}{4\pi}$. On the other hand, the constant curvature $R = R_0$, and the zero trace condition together imply

$$f'(R_0)R_0 - \frac{d}{2} f(R_0) = 0. \tag{19}$$

This equation admits

$$f(R_0) = R_0^{\frac{d}{2}}, \tag{20}$$

where the integration constant is set to be one. One can easily write the Einstein equations as

$$G^{\nu}_{\mu} = \kappa \tilde{T}^{\nu}_{\mu} \tag{21}$$

where

$$\tilde{T}^{\nu}_{\mu} = \frac{2R_0}{f(R_0)d} T^{\nu}_{\mu} - \frac{\Lambda_{\text{eff}}}{\kappa}\delta^{\nu}_{\mu}, \tag{22}$$

$$\Lambda_{\text{eff}} = \frac{(d-2)R_0}{2d}, \tag{23}$$

and in which T^{ν}_{μ} is given by (4). The constancy of the Ricci scalar amounts to

$$-\frac{r^2 A'' + 2(d-2)rA' + (d-2)(d-3)(A-1)}{r^2} = R_0 \tag{24}$$

which yields

$$A = 1 - \frac{R_0}{d(d-1)}r^2 - \frac{m}{r^{d-3}} + \frac{\sigma}{r^{d-2}}, \tag{25}$$

where σ and m are two integration constants. From the Einstein equations one identifies the constant σ as

$$\sigma = \frac{8}{d(d-2)R_0^{\frac{d-2}{2}}}\left(\frac{(d-3)(d-2)Q^2}{4}\right)^{\frac{d}{4}}. \tag{26}$$

In the next section we investigate physical properties of our solution in all dimensions.

3 Analysis of the solution

3.1 Four dimensions

3.1.1 Thermodynamics

In 4 dimensions, we know that the nonabelian $SO(3)$ gauge field coincides with the abelian $U(1)$ Maxwell field [5]. Due to its importance we shall study the 4-dimensional case separately and give the results explicitly. First of all, in 4 dimensions the metric function becomes

$$A = 1 - \frac{R_0}{12}r^2 - \frac{m}{r} + \frac{Q^2}{2R_0 r^2}, \quad 0 < |R_0| < \infty \tag{27}$$

and the form of action reads

$$S = \int d^4x\sqrt{-g}\left[\frac{f(R)}{2\kappa} + \mathcal{L}(F)\right] \tag{28}$$

in which

$$f(R) = R^2, \tag{29}$$

with $R = R_0$ and

$$\mathcal{L}(F) = -\frac{1}{4\pi}F. \tag{30}$$

By assumption, R_0 gets positive/negative values and the resulting spacetime becomes de Sitter/anti de Sitter type in $f(R) = R^2$ theory, respectively, with effective cosmological constant $\Lambda_{\text{eff}} = \frac{R_0}{4}$. Let us add that in order to preserve the sign of the charge term in (27) we must abide by the choice $R_0 > 0$. However, simultaneous limits $Q^2 \to 0$ and $R_0 \to 0$, so that $\frac{Q^2}{R_0} = \lambda_0 =$ constant, lead also to an acceptable solution within $f(R)$ gravity [3]. It is not difficult to see here that m is the ADM mass of the resulting black hole. Viability of the pure $f(R) = R^2$ model which has recently been considered critically [22] is known to avoid the Dolgov–Kawasaki instability [23]. Further, in the late time behavior of the expanding universe (i.e. for $r \to \infty$) it asymptotes to the de Sitter/anti de Sitter form. With reference to [22] we admit that sourceless $f(R) = R^2$ model does not possess a good record as far as the Solar System tests are concerned. Herein we have sources and wish to address the universe at large. Now, we follow [24–28] to give the form of the entropy akin to the possible black hole solution. From the area relation the entropy of the modified gravity with constant curvature is given by

$$S = \frac{\mathcal{A}_h}{4G} f'(R_0) \tag{31}$$

which upon insertion from (19) becomes

$$S = \frac{\mathcal{A}_h}{2GR_0} f(R_0) = 2\pi R_0 r_h^2 \tag{32}$$

where r_h indicates the event horizon. The Hawking temperature and heat capacity are given, respectively, by

$$T_H = \frac{A'(r_h)}{4\pi} = \frac{4R_0 r_h^2 - 2Q^2 - R_0^2 r_h^4}{16\pi R_0 r_h^3}, \tag{33}$$

and

$$C_Q = T_H \frac{\partial S}{\partial T_H} = \frac{4\pi R_0 r_h^2 (R_0^2 r_h^4 - 4R_0 r_h^2 + 2Q^2)}{(R_0^2 r_h^4 + 4R_0 r_h^2 - 6Q^2)}. \tag{34}$$

Here we note that for the case of zero YM charge ($Q = 0$) one finds

$$C_Q = T_H \frac{\partial S}{\partial T_H} = 4\pi R_0 r_h^2 \frac{(R_0 r_h^2 - 4)}{(4 + R_0 r_h^2)} \tag{35}$$

which clearly shows from (34) that for $R_0 > 0$, the YM source brings in the possibility of having a phase change. This is depicted in Fig. 1. In Figs. 1A and 1C we plot the horizon radius versus mass for $Q = 0$ and $Q = 1$. Similarly in Figs. 1B and 1D we plot the heat capacity C for $Q = 0$ and C_Q for $Q = 1$ to see the drastic difference. It is observed that for $Q = 0$ (Fig. 1B) the heat capacity is regular whereas for $Q = 1$ (Fig. 1D), C_Q is a discontinuous function signaling a phase change.

3.1.2 *Energy conditions*

From the energy conditions (see Appendix A.2) the density and principal pressures are given as

$$\rho = -\tilde{T}_0^0 = \frac{1}{8\pi R_0}\left(F + \frac{1}{4}R_0^2\right),$$
$$p_1 = \tilde{T}_1^1 = -\frac{1}{8\pi R_0}\left(F + \frac{1}{4}R_0^2\right),$$
$$p_i = \tilde{T}_i^i = \frac{1}{8\pi R_0}\left(F - \frac{1}{4}R_0^2\right), \quad i = 2, 3.$$

These conditions imply that for $R_0 \geq 0$, both the WEC and SEC are satisfied. DEC implies, on the other hand, from (A.7) that

$$P_{\text{eff}} = \frac{1}{3}\sum_{i=1}^{3} \tilde{T}_i^i = \frac{1}{24\pi R_0}\left(F - \frac{3}{4}R_0^2\right) \geq 0, \tag{36}$$

which yields

$$R_0 \geq 0 \quad \text{and} \quad F \geq \frac{3}{4}R_0^2 \to r \leq \sqrt[4]{\frac{2Q^2}{3R_0^2}}. \tag{37}$$

In addition to the energy conditions one can impose the causality condition (CC) from (A.9):

$$0 \leq \frac{P_{\text{eff}}}{\rho} = \frac{(F - \frac{3}{4}R_0^2)}{3(F + \frac{1}{4}R_0^2)} < 1, \tag{38}$$

which is satisfied if $F \geq \frac{3}{4}R_0^2$ or $r \leq \sqrt[4]{\frac{2Q^2}{3R_0^2}}$.

Finally, if one introduces a new parameter (as the equation of state function ω) by $\omega = \frac{P_{\text{eff}}}{\rho}$, one observes that in the range for $0 < r < \infty$ we have

$$-1 \leq \omega < \frac{1}{3}. \tag{39}$$

In terms of the physical parameters, if

$$\sqrt[4]{\frac{2Q^2}{3R_0^2}} \leq r \tag{40}$$

then $-1 \leq \omega \leq 0$, and if

$$\sqrt[4]{\frac{2Q^2}{3R_0^2}} > r \tag{41}$$

we have $0 < \omega < \frac{1}{3}$. It is clearly seen that the foregoing bounds serve to define possible critical distances where the sign of the effective pressure changes sign. This may be interpreted as changing phase for example, from contraction to expansion or vice versa in a universe centered by a black hole. We note that scaling the mass and distance by R_0 the results will not be affected. For this reason we set $R_0 = 1$. From Eq. (34) we plot in Fig. 2, C_Q (with $R_0 = 1$) versus r_h and Q. The shaded region for $r_h < r_c$ and $C_Q > 0$, which

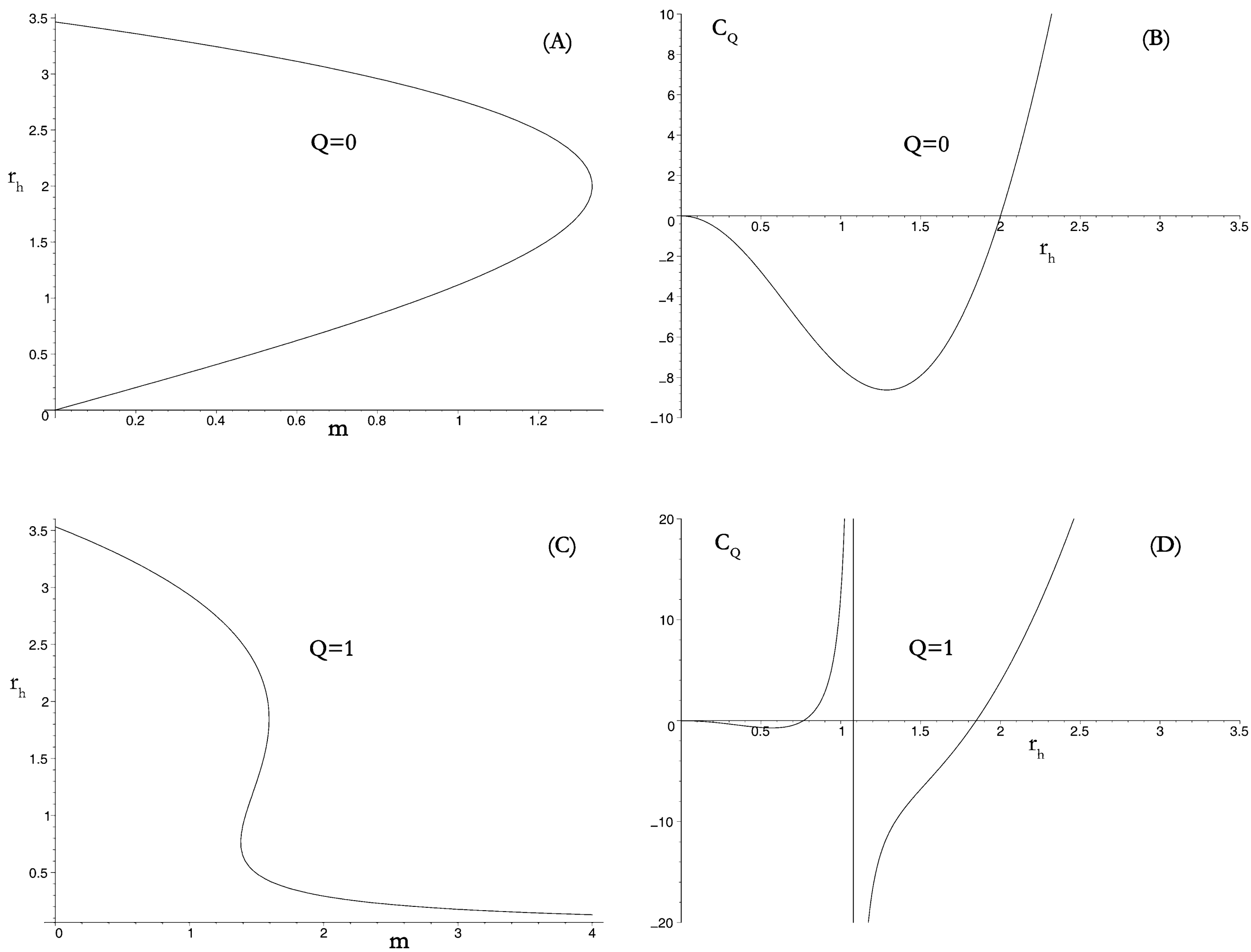

Fig. 1 The plot of horizon radius r_h in 4 dimensions versus mass m for different charges, $Q = 0$ (**A**) and $Q = 1$ (**C**). We also plot the heat capacity C_Q versus the horizon radius for $Q = 0$ (**B**) and $Q = 1$ (**D**). (**D**) Displays in particular the instability caused by the non-zero charge

lies below the curve $r_h = r_c$ is the stable region outside the black hole. All the rest with $C_Q < 0$ is a thermodynamically unstable region. Figure 2 reveals that except for a very narrow band of stability islands there is a vast region of instability for r_c at which the effective pressure turns sign and continues into opposite pressure, i.e. expansion reverses into contraction or vice versa.

3.2 d dimensions

3.2.1 Thermodynamics

In higher dimensions one obtains for the entropy and Hawking temperature the following expressions (for $n \geq 2$):

$$S = \begin{cases} \frac{(2n-1)n\pi^{\frac{2n-1}{2}} r_h^{2(n-1)} R_0^{n-1}}{4\Gamma(n+\frac{1}{2})}, & d = 2n, \\ \frac{(2n+1)n\pi^n r_h^{2n-1} R_0^{(2n-1)/2}}{4\Gamma(n+1)}, & d = 2n+1, \end{cases} \tag{42}$$

$$T_H = \begin{cases} \frac{-1}{8n\pi r_h^{2n-1} R_0^{n-1}} [\frac{4Q^n}{(n-1)} (\frac{(2n-3)(n-1)}{2})^{\frac{n}{2}} \\ \quad + (6n - 4n^2 + R_0 r_h^2) r_h^{2(n-1)} R_0^{n-1}], \\ d = 2n, \\ \frac{-1}{4\pi(2n+1) r_h^{2n} R_0^{\frac{2n-1}{2}}} [\frac{8Q^{\frac{2n+1}{2}}}{2n-1} (\frac{(n-1)(2n-1)}{2})^{\frac{2n+1}{4}} \\ \quad + [R_0 r_h^2 - 2(2n+1)(n-1)] r_h^{2n-1} R_0^{\frac{2n-1}{2}}], \\ d = 2n+1. \end{cases} \tag{43}$$

The specific heat also follows as

$$C_Q = \begin{cases} \frac{\pi^{\frac{2n-1}{2}} r_h^{2(n-1)} n R_0^{n-1} (n-1)(2n-1)\Psi_1}{2\Gamma(n+\frac{1}{2})\Phi_1}, & d = 2n, \\ \frac{\pi^n r_h^{2n-1} R_0^{\frac{2n-1}{2}} n(4n^2-1)\Psi_2}{4\Gamma(n+1)\Phi_2}, & d = 2n+1 \end{cases} \tag{44}$$

in which we have used the following abbreviations:

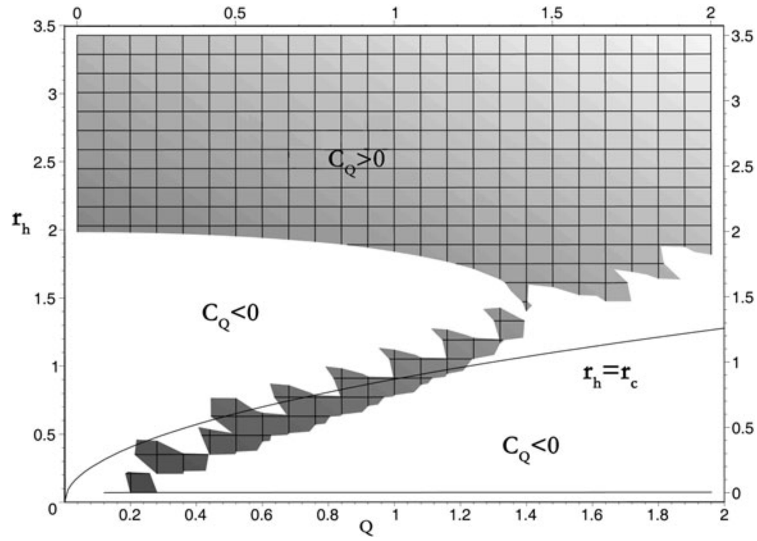

Fig. 2 The 3-dimensional picture of C_Q versus r_h and Q as projected into the (r_h, Q) plane. The *shaded region* with $C_Q > 0$ shows the thermodynamically stable region. From cosmological point of view the region of interest is when the critical r_c is outside the event horizon. As shown, *below the curve* $r_h = r_c$ we obtain stability (*dark*) regions. *Above the curve* $r_h = r_c$, the region is already inside the *black hole* and no stability is expected

$$\begin{aligned}\Psi_1 &= 4Q^n\left(\frac{(2n-3)(n-1)}{2}\right)^{\frac{n}{2}} \\ &\quad + \left(6n-4n^2+R_0r_h^2\right)r_h^{2(n-1)}R_0^{n-1} \\ \Phi_1 &= -4Q^n\left(\frac{(2n-3)(n-1)}{2}\right)^{\frac{n}{2}}(2n-1) \\ &\quad + \left(-6n+4n^2+R_0r_h^2\right)r_h^{2(n-1)}R_0^{n-1} \\ \Psi_2 &= 4Q^{\frac{2n+1}{2}}\left(\frac{(n-1)(2n-1)}{2}\right)^{\frac{2n+1}{4}} \\ &\quad + \frac{(2n-1)}{2}\left[R_0r_h^2-2(2n+1)(n-1)\right]r_h^{2n-1}R_0^{\frac{2n-1}{2}} \\ \Phi_2 &= -8nQ^{\frac{2n+1}{2}}\left(\frac{(n+1)(2n-1)}{2}\right)^{\frac{2n+1}{4}} \\ &\quad + \frac{(2n-1)}{2}\left[R_0r_h^2+2(2n+1)(n-1)\right]r_h^{2n-1}R_0^{\frac{2n-1}{2}}.\end{aligned} \tag{45}$$

We notice that in odd dimensions from $f(R_0) = R_0^{\frac{d}{2}}$, R_0 cannot get negative values for $d =$ odd integer. The details can be seen in the appendix.

3.2.2 The first law of thermodynamics

As was shown in Ref. [4] the first law of thermodynamics in $f(R)$ gravity can be expressed as

$$T\,dS - dE = P\,dV \tag{46}$$

in which E is the Misner–Sharp [29–35] energy stored inside the horizon such that

$$dE = \frac{1}{2\kappa}\left[\frac{(d-2)(d-3)}{r_h^2}f_R + (f - Rf_R)\right]\mathcal{A}_h\,dr_h, \tag{47}$$

$T = \frac{A'}{4\pi}$ is the Hawking temperature, $S = \frac{2\pi\mathcal{A}_h}{\kappa}f_R$, is the entropy of the black hole $P = T_r^r = T_0^0$ is the radial pressure of matter fields at the horizon and $dV = \mathcal{A}_h\,dr_h$ is the change of volume of the black hole at the horizon. In the case of constant curvature, i.e., $R = R_0$, one gets

$$dE = \frac{1}{2\kappa}\left[\frac{(d-2)(d-3)}{r_h^2}\frac{d}{2R_0} + \left(1-\frac{d}{2}\right)\right]R_0^{\frac{d}{2}}\mathcal{A}_h\,dr_h \tag{48}$$

which implies

$$E = \frac{(d-2)}{4\kappa}\left[\frac{d(d-3)}{r_h(d+1)R_0} - \frac{r_h}{d-1}\right]R_0^{\frac{d}{2}}\mathcal{A}_h. \tag{49}$$

Here we show that the first law of thermodynamics for the metric function (25) is satisfied. Herein $P = -\frac{1}{4\pi}\left(\frac{(d-2)(d-3)Q^2}{4r_h^4}\right)^{\frac{d}{4}}$ and therefore the right hand side reads

$$P\,dV = -\frac{1}{4\pi}\left(\frac{(d-2)(d-3)Q^2}{4r_h^4}\right)^{\frac{d}{4}}\mathcal{A}_h\,dr_h. \tag{50}$$

On the other side we have

$$\begin{aligned}T\,dS - dE &= A'\frac{\mathcal{A}_h}{4\kappa}\frac{d(d-2)}{r_h}R_0^{\frac{d}{2}-1}\,dr_h \\ &\quad - \frac{1}{2\kappa}\left[\frac{(d-2)(d-3)}{r_h^2}\frac{d}{2R_0} + \left(1-\frac{d}{2}\right)\right] \\ &\quad \times R_0^{\frac{d}{2}}\mathcal{A}_h\,dr_h.\end{aligned} \tag{51}$$

We combine the latter with (46) and (50) to rewrite the first law as

$$\begin{aligned}&A'\frac{1}{4\kappa}\frac{d(d-2)}{r_h}R_0^{\frac{d}{2}-1} \\ &\quad - \frac{1}{2\kappa}\left[\frac{(d-2)(d-3)}{r_h^2}\frac{d}{2R_0} + \left(1-\frac{d}{2}\right)\right]R_0^{\frac{d}{2}} \\ &= -\frac{1}{4\pi}\left(\frac{(d-2)(d-3)Q^2}{4r_h^4}\right)^{\frac{d}{4}}\end{aligned} \tag{52}$$

or equivalently

$$A' = \frac{(d-3)}{r_h} - \frac{r_h}{d} R_0 - \frac{8}{d(d-2) r_h^{d-1} R_0^{\frac{d}{2}-1}} \times \left(\frac{(d-2)d - 3Q^2}{4} \right)^{\frac{d}{4}}, \quad (53)$$

which is the derivative of the metric function at $r = r_h$. This shows that the first law of thermodynamics by using the generalized form of the entropy for the Misner–Sharp energy is satisfied. To conclude this section of thermodynamics we must admit that we do not feel the necessity of addressing the second law. This originates from the fact that we are entirely in the static gauge so that the entropy change is assumed trivially satisfied i.e. $\Delta S = 0$.

4 Conclusion

A relatively simpler class of solutions within $f(R)$ gravity is the one in which the scalar curvature R is a constant R_0 (both $R_0 > 0$ and $R_0 < 0$). We have concentrated on this particular class with the supplementary condition of zero energy-momentum trace. The general spherically symmetric spacetime minimally coupled with nonlinear Yang–Mills (YM) field is presented in all dimensions ($d \geq 4$). The YM field can even be considered in the power-law form in which the YM Lagrangian is expressed by $L(F) \sim (F^a.F^a)^{\frac{d}{4}}$. Since exact solutions in $f(R)$ gravity with external matter sources, are rare, such solutions must be interesting. The equation of state for effective matter is considered in the form $P_{\text{eff}} = \omega\rho$, which is analyzed in Appendix. The general forms of $\omega(r)$ given in (A.21) determine ω within the ranges of $-1 < \omega < \frac{1}{d-1}$ and $0 < \omega < \frac{1}{d-1}$, respectively. The fact that $\omega < -1$ does not occur eliminates the possibility of ghost matter, leaving us with the YM source and the scalar curvature R_0. In case that the YM field vanishes ($Q \to 0$) the only source to remain is the effective cosmological constant $\Lambda_{\text{eff}} = \frac{(d-2)R_0}{2d}$, which arises naturally in $f(R_0)$ gravity. Another interesting result to be drawn from this study is that the effective pressure P_{eff} changes sign before/after a critical distance. Thus, it is not possible to introduce a simple $\omega =$ constant, so that the pressure preserves its sign in the presence of a physical field (here YM) in the entire spacetime. From cosmological considerations the interesting case is when the critical distance lies outside the event horizon. This is depicted in the projective plot (Fig. 2) of the heat capacity versus horizon and the charge. Finally it should be added that although $f(R) = R^{d/2}$ gravities face viability problems in experimental tests the occurrence of sources may render them acceptable in this regard.

Appendix: Energy conditions

When a matter field couples to any system, energy conditions must be satisfied for physically acceptable solutions. We follow the steps as given in [36–38].

A.1 $R_0 > 0$

Weak Energy Condition (WEC) The WEC states that

$$\rho \geq 0, \qquad \rho + p_i \geq 0. \quad (A.1)$$

In which ρ is the energy density and p_i are the principal pressure components given by

$$\rho = -\tilde{T}_0^0 = \frac{R_0}{2\pi d}\left(\frac{F^{\frac{d}{4}}}{R_0^{\frac{d}{2}}} + \frac{(d-2)}{8} \right),$$
$$p_i = \tilde{T}_i^i = \frac{R_0}{2\pi d}\left(\frac{2}{(d-2)} \frac{F^{\frac{d}{4}}}{R_0^{\frac{d}{2}}} - \frac{(d-2)}{8} \right), \quad i = 2, \ldots, (d-1),$$
$$p_1 = \tilde{T}_1^1 = -\frac{R_0}{2\pi d}\left(\frac{F^{\frac{d}{4}}}{R_0^{\frac{d}{2}}} + \frac{(d-2)}{8} \right). \quad (A.2)$$

Both conditions are satisfied. So WEC is held.

Strong Energy Condition (SEC) This condition states that

$$\rho + \sum_{i=1}^{d-1} p_i \geq 0, \qquad \rho + p_i \geq 0. \quad (A.3)$$

The second condition is satisfied but the first condition implies that

$$\rho + \sum_{i=1}^{d-1} p_i = \frac{R_0}{2\pi d}\left(2\frac{F^{\frac{d}{4}}}{R_0^{\frac{d}{2}}} - \frac{(d-2)^2}{8} \right) \geq 0 \quad (A.4)$$

or consequently

$$\left(2\left(\frac{F}{R_0^2}\right)^{\frac{d}{4}} - \frac{(d-2)^2}{8} \right) \geq 0. \quad (A.5)$$

By a substitution from (11) for F one finds that for $r < r_c$ the condition is satisfied in which

$$r_c = \sqrt[d]{\frac{16}{(d-2)^2}} \sqrt[4]{\frac{(d-2)(d-3)Q^2}{4R_0^2}}. \quad (A.6)$$

Dominant Energy Condition (DEC) In accordance with DEC, the effective pressure must not be negative. This amounts to

$$P_{\text{eff}} = \frac{1}{d-1}\sum_{i=1}^{d-1} T_i^i = \frac{1}{(d-1)}\frac{R_0}{2\pi d} \times \left(\frac{F^{\frac{d}{4}}}{R_0^{\frac{d}{2}}} - \frac{(d-2)(d-1)}{8}\right) \geq 0, \tag{A.7}$$

which for $r < \tilde{r}_c$ it is fulfilled in which

$$\tilde{r}_c = \sqrt[d]{\frac{8}{(d-2)(d-1)}}\sqrt[4]{\frac{(d-2)(d-3)Q^2}{4R_0^2}}. \tag{A.8}$$

Causality Condition (CC) In addition to the energy conditions one can impose the causality condition

$$0 \leq \frac{P_{\text{eff}}}{\rho} = \frac{(F^{\frac{d}{4}} R_0^{\frac{-d}{2}} - \frac{(d-2)(d-1)}{8})}{(d-1)(F^{\frac{d}{4}} R_0^{\frac{-d}{2}} + \frac{(d-2)}{8})} < 1. \tag{A.9}$$

This is equivalent to

$$F^{\frac{d}{4}} R_0^{\frac{-d}{2}} - \frac{(d-2)(d-1)}{8} > 0 \tag{A.10}$$

which for $r < \tilde{r}_c$ is satisfied.

Finally we introduce $\omega = \frac{P_{\text{eff}}}{\rho}$, given by

$$\omega = \frac{((\frac{F}{R_0^2})^{\frac{d}{4}} - \frac{(d-2)(d-1)}{8})}{(d-1)((\frac{F}{R_0^2})^{\frac{d}{4}} + \frac{(d-2)}{8})}, \tag{A.11}$$

which is bounded as

$$-1 \leq \omega < \frac{1}{d-1}. \tag{A.12}$$

It is observed that

$$\begin{cases} 0 \leq \omega < \dfrac{1}{d-1} & \text{if } r < \tilde{r}_c \\ -1 \leq \omega < 0 & \text{if } \tilde{r}_c < r. \end{cases} \tag{A.13}$$

A.2 $R_0 < 0$

As one may see, presence of $R_0^{\frac{d}{2}}$ in the definition of ρ and p_i imposes that $d \neq 2n+1$ where for $n = 2, 3, 4, \ldots$. For $d = 4n$ we get

$$\rho = -\tilde{T}_0^0 = \frac{-|R_0|}{8\pi n}\left(\frac{F^n}{R_0^{2n}} + \frac{2n-1}{4}\right),$$
$$p_i = \tilde{T}_i^i = \frac{-|R_0|}{8\pi n}\left(\frac{1}{2n-1}\frac{F^n}{R_0^{2n}} - \frac{2n-1}{4}\right), \tag{A.14}$$
$$p_1 = \tilde{T}_1^1 = \frac{|R_0|}{8\pi n}\left(\frac{F^n}{R_0^{2n}} + \frac{2n-1}{4}\right).$$

WEC: These expressions reveal that the condition $\rho \geq 0$ and $\rho + p_i \geq 0$ are not satisfied. Similarly the SEC is also violated and since the source is exotic we shall not consider it any further here. A case of interest for $R_0 < 0$ is the choice $d = 4n + 2$ for $n = 1, 2, 3, \ldots$ in which

$$\rho = -\tilde{T}_0^0 = \frac{|R_0|}{4\pi(2n+1)}\left(\frac{F^{\frac{2n+1}{2}}}{|R_0|^{2n+1}} - \frac{n}{2}\right),$$
$$p_i = \tilde{T}_i^i = \frac{|R_0|}{4\pi(2n+1)}\left(\frac{1}{2n}\frac{F^{\frac{2n+1}{2}}}{|R_0|^{2n+1}} + \frac{n}{2}\right), \quad i = 2, \ldots, (d-1), \tag{A.15}$$
$$p_1 = \tilde{T}_1^1 = -\frac{|R_0|}{4\pi(2n+1)}\left(\frac{F^{\frac{2n+1}{2}}}{|R_0|^{2n+1}} - \frac{n}{2}\right).$$

WEC: $\rho \geq 0$ yields

$$\frac{F^{\frac{2n+1}{2}}}{|R_0|^{2n+1}} - \frac{n}{2} \geq 0 \tag{A.16}$$

or

$$r < \bar{r}_c \tag{A.17}$$

where

$$\bar{r}_c = \sqrt[4n+2]{\frac{2}{n}}\sqrt[4]{\frac{n(4n-1)Q^2}{|R_0|^2}}. \tag{A.18}$$

SEC: The conditions are simply satisfied.

DEC: This amounts to

$$P_{\text{eff}} = \frac{1}{4n+1}\frac{|R_0|}{4\pi(2n+1)}\left(\frac{F^{\frac{2n+1}{2}}}{|R_0|^{2n+1}} + \frac{n}{2} + 2n^2\right) \geq 0, \tag{A.19}$$

which is also satisfied.

CC: The causality condition implies

$$0 \leq \frac{P_{\text{eff}}}{\rho} = \frac{(\frac{F^{\frac{2n+1}{2}}}{|R_0|^{2n+1}} + \frac{n}{2} + 2n^2)}{(4n+1)(\frac{F^{\frac{2n+1}{2}}}{|R_0|^{2n+1}} - \frac{n}{2})} < 1, \tag{A.20}$$

or equivalently

$$|R_0|^{2n+1}\frac{1+4n}{4} < F^{\frac{2n+1}{2}} \tag{A.21}$$

which is satisfied for

$$r < \check{r}_c \tag{A.22}$$

where

$$\check{r}_c = \sqrt[4n+2]{\frac{4}{1+4n}}\sqrt[4]{\frac{n(4n-1)Q^2}{|R_0|^2}}. \tag{A.23}$$

Here the state function $\omega = \frac{P_{\text{eff}}}{\rho}$ becomes

$$\omega = \frac{(\frac{F^{\frac{2n+1}{2}}}{|R_0|^{2n+1}} + \frac{n}{2} + 2n^2)}{(4n+1)(\frac{F^{\frac{2n+1}{2}}}{|R_0|^{2n+1}} - \frac{n}{2})}, \tag{A.24}$$

which is bounded as

$$-1 \leq \omega < \frac{1}{4n+1}. \tag{A.25}$$

One can show that

$$\begin{cases} 0 \leq \omega < \dfrac{1}{4n+1} & \text{if } r < \bar{r}_c, \\ -1 \leq \omega < 0 & \text{if } \bar{r}_c < r. \end{cases} \tag{A.26}$$

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