Regular charged black hole construction in $2 + 1$ −dimensions

S. Habib Mazharimousavi,[*] M. Halilsoy,[†] and T. Tahamtan[‡]

Department of Physics, Eastern Mediterranean University, G. Magusa, north Cyprus,
Mersin 10 - Turkey, Tel.: +90 392 6301067; fax: +90 3692 365 1604.

Abstract: It is well-known that unlike its chargeless version the charged Banados-Teitelboim-Zanelli (BTZ) black hole solution in $2 + 1-$ dimensional spacetime is singular. We construct a charged, regular extension of the BTZ black hole solution by employing nonlinear Born-Infeld electrodynamics, supplemented with the Hoffmann term and gluing different spacetimes. The role of the latter term is to divide spacetime in a natural way into two regions by a circle and eliminate the inner singularity. Thermodynamics of such a black hole is investigated by Kaluza-Klein reduction to the 1+1-dimensional dilaton gravity.

I. INTRODUCTION

The principal aim in introducing nonlinear electrodynamics was to eliminate the Coulomb divergences that arose in Maxwell electromagnetism. One such prominent member of this class of theories was introduced by Born and Infeld (BI) in 1934 [1]. The characteristic action of the BI theory consisted of a square root term under which the electromagnetic field strengths admitted automatic upper (lower) bounds. This is reminiscent of the relativistic particle Lagrangian where the upper bound turns out to be the speed of light. In the proper limiting procedure one naturally recovers from the BI formalism the linear electromagnetism of Maxwell as expected.

One problem that was invited with the BI theory was the double-valued behavior of the displacement vector $\overrightarrow{D}(\overrightarrow{E})$ as a function of the electric field $\overrightarrow{E}$ [2]. That means, with the same $\overrightarrow{E}$ value at a particular point one could obtain two different values for $\overrightarrow{D}$; a totally unacceptable situation from the physics standpoint. To resolve this problem, shortly after the introduction of BI theory, Hoffmann and Infeld introduced a supplementary term to the BI Lagrangian [2, 3] which came to be known as the 'Hoffmann term'. This amounted to a pair of Lagrangians matching at a natural boundary, removing the double-values in $\overrightarrow{D}(\overrightarrow{E})$, but all at the cost of two Lagrangians in the same theory.

These novel ideas of 1930s may find appropriate arena now in the geometric theory of Einstein, namely the general relativity since boundaries / intersections of different spacetimes are encountered therein naturally. From this token we developed a geometric model of a particle where the outside and inside of the particle correspond to different spacetimes matching consistently at the radius of the particle [4, 5]. This took place in the $4-$ dimensional $(d = 4)$ spacetime, but naturally we can extend the investigation also to nowadays fashionable $d > 4$ spacetimes [6]. In this paper instead of $d > 4$ we shall go the opposite route, namely to $d = 3$ spacetimes and investigate in the presence of the Hoffmann term the impacts of the BI electrodynamics [7]. This suggests that beside its historical importance the BI and Hoffmann terms together constitute viable a physical model. Our Lagrangian is shown (in the Appendix) to satisfy the weak (WEC) and strong (SEC) energy conditions. Gravity coupled BI electrodynamics in $2 + 1$-dimensions alone is known to be singular [8]. Some other forms of the non-linear electrodynamics coupled with gravity in $2 + 1-$dimensions, also reveal that the possible black hole solutions are singular [8].

Our model consists of the Einstein-Hilbert, cosmological (Λ) and BI terms which are supplemented by the logarithmic Hoffmann term. As we had shown in our particle model [4] the Hoffmann term serves to introduce a natural boundary term so that the spacetime can be considered divided in two different regions. The inner part which is singularity-free turns out to contain a uniform electric field in addition to the cosmological constant overall which is nothing but the anti-de Sitter spacetime. The outer part is an entirely different spacetime which matches according to the Israel's junction conditions [9] on the circle, identified as the $2-$ dimensional Friedmann-Robertson-Walker (FRW) universe.

An important result that we obtain within the context of Einstein-Hoffmann- Born-Infeld (EHBI) theory in $d = 3$ is that we can construct regular, charged black hole solutions. This particular point constitutes the main motivation for this letter. Let us note that Cataldo and García [10] found a regular, charged BTZ black hole without identifying an initial Lagrangian for the nonlinear electromagnetism. Unlike the BI case in the absence of such a Lagrangian we

can't investigate its implication in a flat spacetime limit as well as the underlying thermodynamic properties. For the latter analysis we appeal to the $1 + 1$-dimensional dilaton gravity model within the Kaluza-Klein formalism. In this sense our work can be considered as an alternative, charged singularity-free construction of the BTZ black hole [11]. As the charged BTZ black hole has singularity at $r = 0$, with the BI and Hoffmann terms in the action, it eliminates the singularity to yield a regular black hole solution. Expectedly, in proper limits the boundary circle is removed and our spacetime reduces to the uncharged, non-singular BTZ black hole. Although, it does not allow us to interpret such a solution as a particle model it may serve as a toy model in $d = 3$, which can be extended to higher dimensions as a tool to eliminate naked singularities and restore the cosmic conjecture hypothesis, for instance. Further, inclusion of the Hoffmann term may constitute an attractive field theory model of elementary particles.

Technically, by the choice of parameters of the theory (i.e. BI parameter b, electric charge q, mass M and the cosmological constant $\Lambda = -\frac{3}{l^2}$) we can adjust the event horizon to be located inside / outside the circle whose radius is at $r_0 = \left|\frac{q}{b}\right|$ (see Eq. (7) below). For $r < r_0$ the spacetime is regular with negative scalar curvature $R = -\frac{2}{b^2}$. By numerical analysis we obtain solutions with single / double (or no) horizons which are displayed in Fig. 1a.

II. ACTION, FIELD EQUATIONS AND SOLUTIONS

Our $3-$dimensional action in Einstein-Hoffmann-Born-Infeld (EHBI) theory, is given by ($c = \hbar = k_B = G = \frac{1}{4\pi\epsilon_\circ} = 1$)

$$S = \int dx^3 \sqrt{-g} \left(\frac{R - 2\Lambda}{16\pi} + \mathcal{L}(F) \right),$$ (1)

in which $F = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ and

$$\mathcal{L}(F) = \frac{2b^2}{4\pi} \left(1 - \sqrt{1 + \frac{2F}{b^2}} + \ln\left(\frac{1 + \sqrt{1 + \frac{2F}{b^2}}}{2} \right) \right),$$ (2)

in which b is a real parameter (HBI-parameter). The additional $\ln\left(\frac{1 + \sqrt{1 + \frac{2F}{b^2}}}{2} \right)$ term is known as the Hoffmann term which supplements the more familiar BI Lagrangian. Also the limit of the Lagrangian once $b^2 \to \infty$ yields the Einstein Maxwell linear electrodynamics as

$$\lim_{b \to \infty} \mathcal{L}(F) = -\frac{1}{4\pi} F$$ (3)

and in the other limit once $F = 0$ it goes to GR limit. The metric ansaetze are axially symmetric which are given by

$$ds^2 = \begin{cases} -\tilde{f}(r) dt^2 + \tilde{f}(r)^{-1} dr^2 + r_0^2 d\varphi^2, & r \leq r_0 \\ -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 d\varphi^2, & r \geq r_0 \end{cases},$$ (4)

in which r_0 is the constant boundary to be determined. We comment here that these two spacetimes are glued at $r = r_0$. The electric field two-form is chosen as

$$\mathbf{F} = E(r) dt \wedge dr.$$ (5)

and the nonlinear Maxwell equation implies

$$d\left(\mathcal{L}_F \, {}^\star \mathbf{F} \right) = 0$$ (6)
$$\left(\mathcal{L}_F = \frac{\partial \mathcal{L}}{\partial F} \right)$$

Consistent solution in both regions for the electric field is given by

$$E(r) = \begin{cases} \frac{q}{r_0} = b, & r \leq r_0 \\ \frac{2qr}{r_0^2 + r^2}, & r \geq r_0 \end{cases},$$ (7)

in which q is the electric charge and $r_0 = \left|\frac{q}{b}\right|$ is the radius of a circle dividing the regions. Let us also add that for $r \leq r_0$, the square-root term vanishes and $\mathcal{L}$ reduces to a constant which may be added to the cosmological constant Λ. That is, for $r \leq r_0$ we have the anti-de Sitter spacetime. From the expression (6) it can be seen that the electric potential is

$$V(r) = \begin{cases} -q \ln 2 + b(r_0 - r), & r \leq r_0 \\ -q \ln\left(1 + \frac{r^2}{r_0^2}\right), & r \geq r_0 \end{cases}. \tag{8}$$

It is seen that $E_r = 0$ at $r = 0$. D_r is also zero (and similar result holds when $r \to \infty$) which means that D_r in terms of E_r is single valued.

The energy density $u(r) = \frac{1}{2}\epsilon \vec{E}.\vec{D}$ $\left(\epsilon = \frac{\partial \mathcal{L}}{\partial F}, \ \vec{D} = \epsilon \vec{E}\right)$ can be integrated to find the total energy, which diverges for $r \to \infty$.

Variation of the action with respect to $g_{\mu\nu}$ yields the Einstein equations

$$G^\nu_\mu + \Lambda \delta^\nu_\mu = 8\pi T^\nu_\mu, \tag{9}$$

where

$$T^\nu_\mu = \delta^\nu_\mu \mathcal{L} - \left(F_{\mu\lambda} F^{\ \nu\lambda}\right)\mathcal{L}_F, \tag{10}$$

and in a closed form it reads

$$T^\nu_\mu = \text{diag}\left[\mathcal{L} - 2F\mathcal{L}_F, \mathcal{L} - 2F\mathcal{L}_F, \mathcal{L}\right]. \tag{11}$$

For $r < r_0$ the only nonzero component of the Einstein tensor G^ν_μ is $G^\varphi_\varphi = \frac{1}{2}f''(r)$ in which the tt and rr parts of the Einstein equations simply read

$$\Lambda = 8\pi T^0_0 = 8\pi T^r_r = \mathcal{L} - 2F\mathcal{L}_F = -4b^2 \ln 2, \tag{12}$$

which clearly imposes $\Lambda = -4b^2 \ln 2$, or equivalently

$$\frac{1}{\ell^2} = 4b^2 \ln 2, \tag{13}$$

since $\frac{1}{\ell^2} = -\Lambda$, and this will be used wherever necessary. Let us note that the relation between the two constants Λ and b is not a choice but is a direct consequence of the Einstein equations. The other $\varphi\varphi$ component of the Einstein equation for $r < r_0$ admits a solution which takes the form

$$\tilde{f}(r) = \left(\frac{4q^2}{r_0^2}\right)r^2 + C_1 r + C_2, \quad r \leq r_0. \tag{14}$$

On the other hand the Einstein equations for $r > r_0$ give the metric function

$$f(r) = -M + \frac{r^2}{\ell^2} - 4q^2\left[\frac{r^2}{r_0^2}\ln\left(1 + \frac{r_0^2}{r^2}\right) + \ln\left(\frac{r^2 + r_0^2}{\alpha^2 + r_0^2}\right) - \frac{\alpha^2}{r_0^2}\ln\left(1 + \frac{r_0^2}{\alpha^2}\right)\right], \quad r \geq r_0 \tag{15}$$

in which M is the ADM mass of the black hole and $\alpha > 0$ is a constant introduced for dimensional reasons and, C_1 and C_2 are integration constants to be found below. Once more we note that

$$\lim_{b \to \infty} f(r) = f_{CBTZ}(r) = -M + \frac{r^2}{\ell^2} - 8q^2 \ln\frac{r}{\alpha}. \tag{16}$$

which removes the inner region automatically since $r_0 \to 0$ and leaves us with the charged BTZ (CBTZ) black hole. The other limit is given by $q = 0$, which is the non-rotating BTZ (NBTZ) black hole, namely

$$\lim_{q \to 0} f(r) = f_{NBTZ}(r) = -M + \frac{r^2}{\ell^2}. \tag{17}$$

Now, the boundary surface $F(r) = r - r_0 = 0$ can be considered as a $2-$dimensional FRW circular spacetime i.e.,

$$ds^2_{circle} = -d\tau^2 + a(\tau)^2 d\phi^2 \tag{18}$$

in which τ is the proper time on the circle. Furthermore, the two metric functions in (14) and (15) must fulfill the Israel junction conditions [9, 12, 13] reading

$$S_i^j = -\frac{1}{8\pi} \left(\left\langle K_i^j \right\rangle - \langle K \rangle \delta_i^j \right), \tag{19}$$

where S_i^j is the stress energy tensor on the boundary, K_i^j is the extrinsic curvature, K is the trace of it and $\left\langle K_i^j \right\rangle = \left(K_i^j \right)_{r_\circ^+} - \left(K_i^j \right)_{r_\circ^-}$. One can show that the Israel equations read

$$S_\tau^\tau = \frac{1}{8\pi} \left(\frac{\sqrt{f(a)}}{a} - \frac{\sqrt{\tilde{f}(a)}}{a} \right)_{a=r_0} \tag{20}$$

$$S_\varphi^\varphi = \frac{1}{16\pi} \left(\frac{f'(a)}{\sqrt{f(a)}} - \frac{\tilde{f}'(a)}{\sqrt{\tilde{f}(a)}} \right)_{a=r_0}, \tag{21}$$

where a prime is a derivative with respect to a. A smooth transition from $r \geq r_0$ into $r \leq r_0$ requires that $S_\tau^\tau = S_\phi^\phi = 0$ which are equivalent to both, continuous metric and its first derivatives at $r = r_0$. These conditions fix the constants as

$$C_1 = -\frac{8q^2}{r_0}, \tag{22}$$

$$C_2 = -M + 4q^2 \left[1 - \frac{\alpha^2}{r_0^2} \ln\left(\frac{\alpha^2}{\alpha^2 + r_0^2} \right) - \ln\left(\frac{2r_0^2}{\alpha^2 + r_0^2} \right) \right]. \tag{23}$$

The Ricci and Kretschmann scalars for $r \leq r_0$ are

$$R = -8b^2, \quad K = 64b^4 \tag{24}$$

so that the inner space time is regular as long as $b < \infty$.

From (14) and (15) it can be shown easily that at the boundary $r = r_0$ we have the conditions $\tilde{f}(r_0) = f(r_0)$ and $\tilde{f}'(r_0) = f'(r_0) = 0$, satisfied. For chosen parameters q, α, M and b our metric gives rise, through numerical analysis to different black holes (see Fig. 1a). The interesting case is the one with event horizon from $\tilde{f}(r) = 0$ and an inner horizon from $f(r) = 0$ (Fig. 1a C). The fact that our darkness function for the common parameters satisfies black hole condition, both outside and inside, can be seen from the numerical plots.

A. Tidal force in the inner region $r \leq r_0$

As we found at the end of the previous section, the origin is nonsingular. It should be noted that the tidal force may reveal a singularity even when the curvature invariants are finite.

For checking this we use the formalism introduced in Ref. [14], by considering the line element in the form

$$ds^2 = -\frac{F(r)}{G(r)} dt^2 + \frac{dr^2}{F(r)} + R(r)^2 d\varphi^2, \tag{25}$$

in which $F(r) = f(r)$, $G(r) = 1$ and $R(r) = r_0$ for our case. The only nonzero component of the curvature tensor in the static observer's orthonormal basis is given by

$$R_{trtr} = \frac{1}{2} f''(r) \tag{26}$$

and those related by the symmetry. An observer who is radially falling freely toward the center with conserved energy E is related to the static orthonormal frame by a local Lorantz boost [15]. The instantaneous velocity of the observer is given by

$$v = \sqrt{1 - \frac{F}{GE^2}} = \sqrt{1 - \frac{f(r)}{E^2}} \tag{27}$$

and the only nonzero curvature component in the Lorantz boosted frame happens to be

$$R_{\hat{t}\hat{r}\hat{t}\hat{r}} = R_{trtr} = \frac{1}{2}f''(r) = 4b^2. \tag{28}$$

Having the other Riemann components zero means that the tidal force in the transverse direction vanishes. This could be also seen directly from the Eq. (2.9) in Ref. [14].

III. THERMODYNAMICAL CONSIDERATIONS THROUGH THE 1+1-DIMENSIONAL DILATON GRAVITY

In this section we follow [16] to study the thermodynamics of the EHI black hole found above. To do so we derive the solution (15) for $r > r_0$ from the dilaton gravity. Now we consider

$$ds^2 = g_{ab}dx^a dx^b = \tilde{g}_{\mu\nu}d\tilde{x}^\mu d\tilde{x}^\nu + \phi^2(\tilde{x})d\theta^2 \tag{29}$$

in which ϕ is the radius of the circle S^1 in $M_3 = M_2 \times S^1$. Herein the Greek indices represent the two-dimensional spacetime. After the Kaluza-Klein dimensional reduction, the action (1) becomes

$$S_{2D} = 2\pi \int d\tilde{x}^2 \sqrt{-\tilde{g}}\phi \left(\frac{\tilde{R} - 2\Lambda}{16\pi} + \mathcal{L}(F) \right) \tag{30}$$

in which $\tilde{R}$ is the Ricci scalar of M_2 and for the sake of completeness we shall describe the dilatonic approach briefly. The field equations are given by

$$d(\phi \mathcal{L}_F \,^\star\mathbf{F}) = 0, \tag{31}$$
$$\nabla^2\phi + 2\phi\Lambda = 16\pi\phi(\mathcal{L}(F) - 2F\mathcal{L}_F), \tag{32}$$
$$\tilde{R} - 2\Lambda = -16\pi\mathcal{L}(F) \tag{33}$$

in which $\mathbf{F} = E(\phi)\,dt \wedge d\phi$ and $^\star\mathbf{F} = E(\phi)$ (i.e., $0-$form). Our choice of electric field yields

$$F = -\frac{1}{2}E(\phi)^2 \tag{34}$$

and therefore the Maxwell equation (31) implies

$$\frac{\phi E(\phi)}{1 + \sqrt{1 - \frac{E(\phi)^2}{b^2}}} = q = \text{constant} \tag{35}$$

which admits

$$E(\phi) = \frac{2q\phi}{q^2 b^2 + \phi^2}. \tag{36}$$

Accordingly the Lagrangian $\mathcal{L}(F)$ takes the form

$$\mathcal{L}(F) = \frac{b^2}{2\pi} \left[\frac{2q^2}{q^2 + b^2\phi^2} + \ln\left(\frac{b^2\phi^2}{q^2 + b^2\phi^2} \right) \right] \tag{37}$$

with

$$\mathcal{L}_F = -\frac{q^2 + b^2\phi^2}{4\pi b^2\phi^2}. \tag{38}$$

The other two equations also become

$$\nabla^2\phi = V(\phi) = \frac{2\phi}{\ell^2} + 8\phi b^2 \ln\left(\frac{b^2\phi^2}{q^2 + b^2\phi^2} \right) \tag{39}$$

and

$$\tilde{R} = -V'(\phi) = -\frac{2\phi}{\ell^2} - \frac{16q^2 b^2}{q^2 + b^2 \phi^2} - 8b^2 \ln\left(\frac{b^2 \phi^2}{q^2 + b^2 \phi^2}\right). \tag{40}$$

Here for our future use we find the asymptotical behavior of the potential $V(\phi)$ and its first derivative $V'(\phi)$ which are given by

$$V_{CBTZ}(\phi) = \lim_{b\to\infty} V(\phi) = \frac{2\phi}{\ell^2} - \frac{8q^2}{\phi}, \tag{41}$$

$$V'_{CBTZ}(\phi) = \lim_{b\to\infty} V'(\phi) = \frac{2\phi}{\ell^2} + \frac{8q^2}{\phi}. \tag{42}$$

These two latter equations correspond to the $2-$dimensional field equations of dilaton gravity with an action

$$S_{2D} = \int_{M_2} d\tilde{x} dt \sqrt{-\tilde{g}} \left(\phi\tilde{R} + V(\phi)\right) \tag{43}$$

and the line element

$$ds^2 = -f(\tilde{x}) dt^2 + \frac{d\tilde{x}^2}{f(\tilde{x})}. \tag{44}$$

The field equations simply read

$$\nabla^2 \phi(\tilde{x}) = f(\tilde{x}) \phi''(\tilde{x}) + f'(\tilde{x}) \phi'(\tilde{x}) = V(\phi) \tag{45}$$

(in which a prime means derivative with respect to the argument) and

$$\tilde{R} = -f''(\tilde{x}) = -V'(\phi). \tag{46}$$

For a linear dilaton ansatz which we impose at this stage

$$\phi(\tilde{x}) = \tilde{x} \tag{47}$$

it simply yields from (45) that

$$f'(\tilde{x}) = V(\phi). \tag{48}$$

Now, we consider the results found above with

$$ds^2 = -f(\phi) dt^2 + \frac{d\phi^2}{f(\phi)} \tag{49}$$

where $f(\phi)$ is aptly expressed by

$$f(\phi) = J(\phi) - \mathcal{C} \tag{50}$$

in which $J(\phi)$ is defined as

$$J(\phi) = \int V(\phi) d\phi = \frac{\phi^2}{\ell^2} - 4q^2 \ln\left(\frac{q^2 + b^2 \phi^2}{q^2 + b^2 \phi_0^2}\right) + 4b^2 \phi^2 \ln\left(\frac{b^2 \phi^2}{q^2 + b^2 \phi^2}\right) - 4b^2 \phi_0^2 \ln\left(\frac{b^2 \phi_0^2}{q^2 + b^2 \phi_0^2}\right). \tag{51}$$

Let us comment at this point that any choice other than the linear dilaton (47), may not coincide with the $2 + 1-$dimensional solution. With reference to [16] $\mathcal{C}$ represents the ADM mass of the EHBI black hole (See the Appendix of Ref. [16]). Herein ϕ_0 is just a reference potential which is set to be $\phi_0 = \alpha = \ell$ with $\Lambda = -\frac{1}{\ell^2} = -4b^2 \ln 2$. Following [16], the extremal value for ϕ_+ is found by $V(\phi_+ = \phi_e) = 0$, which is given by

$$\phi_e^2 = \frac{q^2}{b^2} = r_0^2 \tag{52}$$

leading us to an extremal mass as

$$M_e = \left(\frac{1 + 4q^2 \ln 2}{\ln 2} \right) \ln \left(1 + 4q^2 \ln 2 \right) - 4q^2 \ln \left(8q^2 \ln 2 \right). \tag{53}$$

By trading the Maxwell terms for a dilaton potential, we now have the charge parameter q appearing in the action. Efectively, this restricts our analysis to the fixed charge (canonical) ensemble, as that parameter can no longer be varied [17]. To have at least one horizon one should set $M \geq M_e$ so that ϕ_+ indicates the outer horizon at which the Hawking temperature is defined by

$$T_H = \frac{V(\phi_+)}{4\pi} = \frac{2b^2\phi_+}{\pi} \ln \left(\frac{2b^2\phi_+^2}{q^2 + b^2\phi_+^2} \right). \tag{54}$$

The heat capacity and free energy are given respectively by

$$C_Q(\phi_+) = 4\pi \left(\frac{V(\phi_+)}{V'(\phi_+)} \right) = \frac{4\pi\phi_+ \ln \left(\frac{2b^2\phi_+^2}{q^2 + b^2\phi_+^2} \right)}{\frac{2q^2}{q^2 + b^2\phi_+^2} + \ln \left(\frac{2b^2\phi_+^2}{q^2 + b^2\phi_+^2} \right)} \tag{55}$$

and

$$F(\phi_+) = J(\phi_+) - J(\phi_e) - \phi_+ V(\phi_+) = 4q^2 \ln \left(\frac{2q^2}{q^2 + b^2\phi_+^2} \right) - 4b^2\phi_+^2 \ln \left(\frac{2b^2\phi_+^2}{q^2 + b^2\phi_+^2} \right), \tag{56}$$

which complete the thermodynamical analysis of our EHBI black hole. We note that, in this stage we can not take the limit $b \to \infty$ (which is CBTZ limit) because we have set some of our constants in terms of each other to satisfy the Israel boundary conditions. Instead, to use these results we may keep our parameters as they appeared before our conditions are imposed and then our results provide the correct CBTZ limit i.e.,

$$T_H^{(CBTZ)} = \lim_{b \to \infty} \frac{V(\phi_+)}{4\pi} = \frac{1}{4\pi} \left(\frac{2\phi_+}{\ell^2} - \frac{8q^2}{\phi_+} \right), \tag{57}$$

and the same for the other cases. Nevertheless with the limit $b \to \infty$ (i.e. $r_0 \to 0$) it automatically removes the inner space. This leads to the case that we don't need to consider any boundary conditions and therefore the relation between ℓ and b will no longer be needed. Under these circumstances we can use the above limits.

Through the explicit expressions of T_H, $C_Q(\phi_+)$ and $F(\phi_+)$ one simply finds that by a rescaling $b^2\phi_+^2 \to \tilde{\phi}_+^2$ one can eliminate b from the expressions. This means that the different values for b (with fixed q) does not change the general behavior of the thermodynamical properties. In such case we set $b = 1$ and plot Fig.s 1 and 2 without loss of generality of the problem. Fig. 1a displays through the Israel's junction conditions that there are three possible gluing: i) $M < M_e$ corresponds to no horizon, ii) $M = M_e$ corresponds to a single horizon and finally iii) $M > M_e$ corresponds to two distinct horizons in which $r_- < r_0 < r_+$. Note also that $r = r_0$ is the only minimum point of the metric (see Fig. 1a). Fig. 1b is a plot of extremal mass M_e versus charge q. This reveals the regions in which the mass M must be chosen in terms of q to have two horizons.

Fig. 2 displays the thermodynamical quantities versus horizon. Let us add that the only feasible case we are interested is for $\phi_+ > r_0$. Fig.s 2a, 2b and 2c confirm that the spacetime is thermodynamically stable.

IV. CONCLUSION

The objective of this study was to construct an electrically charged regular black hole solution as an extension of the uncharged BTZ black hole. Within the familiar linear Maxwell electromagnetism this is not available. Next attempt naturally is to consider nonlinear electrodynamics as a potential candidate that may host such a black hole. Now, with the additional Hoffmannn term in the BI Lagrangian an alternative method of gluing different spacetimes renders construction of regular black holes possible. As explained, the role of the Hoffmann term, is to introduce a natural circle as boundary with radius $r_0 = \left| \frac{q}{b} \right|$ (with q =electric charge and b =BI-parameter) which divides spacetime into inside ($r < r_0$) and outside ($r > r_0$) regions. In the process of matching of spacetimes at the junction we employ Israel's junction conditions. The inner region is nothing but the 3-dimensional anti-de Sitter spacetime in which the cosmological constant is proportional to the BI parameter. It should also be added that there is no lower bound on r_0,

that is, $r_0 \to 0$ with $b \to \infty$, which removes the inner region, leaving behind the singular, charged BTZ spacetime. We obtained a variety of black hole states as depicted in Fig. 1a. From the thermodynamical requirements (i.e. $T_H > 0$) we single out the class that are constrained by the feasibility conditions. This means automatically that occurrence of horizons outside and on the circle with radius r_0 become the only physical cases. Finally the thermodynamics of constructed EHBI black hole has been analyzed in accordance with the Kaluza-Klein reduction to $1+1$-dimensional dilaton gravity. The relevant thermodynamical quantities are depicted in Fig. 2.

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Figure captions

Fig. 1a: Plot of the metric function $f(r)$ versus r for different masses M, expressed in terms of the extremal mass M_e. Although we analyzed the case $r \geq r_0$ the continuity requirements through Israel junction conditions demand also to cover the region $r \leq r_0$. Three different cases have been shown: double horizons $(M = 2M_e)$ single horizon $(M = M_e)$ and no horizons $(M = 0.5M_e)$. We note that from (13) in the text we have chosen $\ell^2 = \frac{1}{4b^2 \ln 2}$ and also $\alpha = \ell$.

Fig. 1b. Extremal mass M_e versus the charge q.

Fig. 2: The thermodynamical quantities for the EHBI black hole. (a) The Hawking temperature $T_H\left(\phi_+\right)$, which is positive for $\phi_+ \geq 1$ (the event horizon). (b) The specific heat $C\left(\phi_+\right)$, behaves also well for $\phi_+ \geq 1 = r_0$. The phase change occurs for $\phi_+ \leq 1 = r_0$ which will clearly lie in the infeasible region. (c) The free energy $F\left(\phi_+\right)$ which is negative for the $\phi_+ \geq 1$.

APPENDIX: Energy Conditions

Our energy momentum tensors in two regions are given by:

$$T_\mu^\nu = \begin{cases} \text{diag}\left[-\frac{q^2}{2\pi r_0^2}\ln\left(1+\frac{r_0^2}{r^2}\right), -\frac{q^2}{2\pi r_0^2}\ln\left(1+\frac{r_0^2}{r^2}\right), -\frac{q^2}{2\pi r_0^2}\ln\left(1+\frac{r_0^2}{r^2}\right)+\frac{q^2}{\pi\left(r_0^2+r^2\right)}\right], & r \geq r_0 \\ \text{diag}\left[-\frac{q^2}{2\pi r_0^2}\ln(2), -\frac{q^2}{2\pi r_0^2}\ln(2), -\frac{q^2}{2\pi r_0^2}\ln(2)+\frac{q^2}{2\pi r_0^2}\right], & r \leq r_0 \end{cases} . \tag{A1}$$

Accordingly one finds the energy density ρ and the principal pressures p_i as

$$\rho = -T_t^t = \begin{cases} \frac{q^2}{2\pi r_0^2}\ln\left(1+\frac{r_0^2}{r^2}\right), & r \geq r_0 \\ \frac{q^2}{2\pi r_0^2}\ln(2), & r \leq r_0 \end{cases}, \qquad p_r = T_r^r = \begin{cases} -\frac{q^2}{2\pi r_0^2}\ln\left(1+\frac{r_0^2}{r^2}\right), & r \geq r_0 \\ -\frac{q^2}{2\pi r_0^2}\ln(2), & r \leq r_0 \end{cases}, \tag{A2}$$

$$p_\varphi = T_\varphi^\varphi == \begin{cases} -\frac{q^2}{2\pi r_0^2}\ln\left(1+\frac{r_0^2}{r^2}\right)+\frac{q^2}{\pi\left(r_0^2+r^2\right)}, & r \geq r_0 \\ -\frac{q^2}{2\pi r_0^2}\ln(2)+\frac{q^2}{2\pi r_0^2}, & r \leq r_0 \end{cases} .$$

A.   Weak Energy Condition (WEC)

The WEC states that,

$$\rho \geq 0 \qquad \text{and} \qquad \rho + p_i \geq 0 \qquad (i = 1, 2) \tag{A3}$$

which are satisfied in both regions.

B.   Strong Energy Condition (SEC)

This condition states that;

$$\rho + \sum_{i=1}^{2} p_i \geq 0 \qquad \text{and} \qquad \rho + p_i \geq 0, \tag{A4}$$

which are also satisfied.

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