

Quasinormal modes of charged dilaton black holes and their entropy spectra

I. Sakalli¹

Department of Physics, Eastern Mediterranean University, Gazimagosa, North Cyprus, Mersin 10, Turkey. e-mail: izzet.sakalli@emu.edu.tr

The date of receipt and acceptance will be inserted by the editor

Abstract In this study, we employ the scalar perturbations of the charged dilaton black hole (CDBH) found by Chan, Horne and Mann (CHM), and described with an action which emerges in the low-energy limit of the string theory. A CDBH is neither asymptotically flat (AF) nor non-asymptotically flat (NAF) spacetime. Depending on the value of its dilaton parameter a , it has both Schwarzschild and linear dilaton black hole (LDBH) limits. We compute the complex frequencies of the quasinormal modes (QNMs) of the CDBH by considering small perturbations around its horizon. By using the highly damped QNMs in the process prescribed by Maggiore, we obtain the quantum entropy and area spectra of these BHs. Although the QNM frequencies are tuned by a , we show that the quantum spectra do not depend on a , and they are equally spaced. On the other hand, the obtained value of undetermined dimensionless constant ϵ is the double of Bekenstein's result. The possible reason of this discrepancy is also discussed.

Send offprint requests to:

Key words Quasinormal Modes, Entropy Spectrum, Charged Dilaton Black Holes, Zerilli Equation, Hurwitz-Lerch Zeta Function, Confluent Hypergeometric Function.

1 Introduction

There has already been benefits in studying thermodynamics of black holes (BHs). This subject is believed to be threshold of the unification of quantum physics with general relativity, which is the so-called quantum gravity theory (QGT). The reader may see Ref. [1] and references therein for a general review of QGT. However, this theory is still under construction. Recent decades proved that our intricate universe is far from being easily understandable. In this regard, QGT is perceived as a master key which resolves many unanswered questions about the universe. For this reason, the uncompleted form of QGT always stimulates the theoretical physicists for studying on it more and more.

The starting point of QGT dates back to seventies in which Bekenstein proposed that BH entropy is proportional to area of BH horizon and the area is quantized [2,3]. Then Bekenstein [4,5,6] also proved that the BH horizon area is an adiabatic invariant, and according to Ehrenfest's principle it has a discrete and evenly spaced spectrum

$$\mathcal{A}_n = \epsilon n \hbar = \epsilon n l_p^2, \quad (n = 0, 1, 2, \dots), \quad (1)$$

where \mathcal{A}_n denotes the area spectrum of the BH horizon and n is the quantum number. Therefore, the minimum increase of the horizon area is $\Delta \mathcal{A}_{\min} = \epsilon \hbar$ which can be obtained by absorbing a test particle into the BH. In units with $c = G = 1$, the undetermined dimensionless constant ϵ is considered as the order of unity. Bekenstein proposed that the BH

horizon is formed by patches of equal area $\epsilon\hbar$, and moreover professed that $\epsilon = 8\pi$. Motivated by this proposal, many works have been made in this subject in order to compute the entropy spectrum of various BHs. Different spectra with different ϵ have also been presented (see for instance Ref. [7] and references therein). One of the significant contributions in quantizing the entropy of a BH was done by Hod [8,9] who suggested that ϵ can be determined by using the QNM of a BH. As it is well-known, this mode is the characteristic sound of a BH. Based on Bohr's correspondence principle (a reader may refer to Ref. [10]), Hod conjectured that the real part of the asymptotic QNM frequency (ω_R) of a highly damped BH is related to the quantum transition energy between two quantum levels of the BH. Thus, this transition frequency gives rise to a change in the BH mass as $\Delta M = \hbar\omega_R$. Particularly for the Schwarzschild BH, Hod computed the value of the dimensionless constant as $\epsilon = 4 \ln 3$. Later on, Kunstatter [11] used the natural adiabatic invariant I_{adb} for system with energy E and vibrational frequency $\Delta\omega$ (for a BH, E is identified with the mass M) which is given by

$$I_{adb} = \int \frac{dE}{\Delta\omega}. \quad (2)$$

At large quantum numbers, the adiabatic invariant is quantized via the Bohr-Sommerfeld quantization; $I_{adb} \simeq n\hbar$. By using the Schwarzschild BH, Kunstatter showed that when ω_R is used as the vibrational frequency, the Hod' result $\epsilon = 4 \ln 3$ is reproduced. In 2008, Maggiore [12] proposed another method that the QNM of a perturbed BH should be considered as a damped harmonic oscillator since the QNM has an imaginary part. Namely, Maggiore considered the proper physical frequency of the harmonic oscillator with a damping term in the form of $\omega = (\omega_R^2 + \omega_I^2)^{\frac{1}{2}}$, where ω_R and ω_I

are the real and imaginary parts of the frequency of the QNM, respectively. In the large n limit or for the highly excited mode, $\omega_I \gg \omega_R$. Consequently one has to use ω_I rather than ω_R in the adiabatic quantity. With this new identification, for the Schwarzschild BH it was found that $\epsilon = 8\pi$, which corresponds to the same area spectrum of Bekenstein's original result of the Schwarzschild BH [13,14]. To date, there are numerous studies in the literature in which Maggiore's method (MM) was employed (some of them can be seen in Refs. [15,16,17,18,19,20,21]).

In this paper, using the MM with the adiabatic invariant expression (2) we investigate the entropy and area spectra for the CDBH [22]. CDBHs are such spacetimes that by tuning the dilaton field one can convert the NAF structure of the spacetime (including LDBH [23,24]) to the AF one, which corresponds to the Schwarzschild BH. Our main motivation is to examine how the influence of dilaton field effects the BH spectroscopy. For this purpose, we first calculate the QNMs of the CDBH and subsequently use them in the MM. The obtained entropy spectrum is equally spaced and independent of the dilaton field. On the other hand, here we get $\epsilon = 16\pi$ which means that the equi-spacing does not coincide with the Bekenstein's result.

The paper is structured as follows. In Sec. 2, we briefly present the CDBH metric and its basic thermodynamical features. Also, we show that how the massless Klein Gordon equation reduces to the Schrödinger-type equation which is the so-called the Zerilli equation [25] in the CDBH geometry. Sec. 3 is devoted to the derivation of QNM of the CDBH by considering the small perturbations around the horizon. In addition to that, in a particular case of highly damped scalar modes, we perform the MM for the CDBH in order

to compute the entropy and area spectra of it. Finally, the summary and concluding remarks are given in Sec. 4.

2 CDBH and the separation of the massless Klein Gordon equation on it

In this section we will first present the geometry and some thermodynamical properties of the CDBH. Then, we will get the radial equation for a massless scalar field in the background of the CDBH. Finally, we represent how the radial equation can be converted to the Zerilli equation [25] which is none other than one-dimensional Schrödinger wave equation.

The 4D Einstein-Maxwell-dilaton (EMD) low-energy action obtained from string theory is given by

$$S = \int d^4x \sqrt{-g} (\mathfrak{R} - 2(\nabla\phi)^2 - e^{-2a\phi} F^2), \quad (3)$$

where ϕ describes the dilaton field which is a scalar field that couples to Maxwell field, a denotes the dilaton parameter and \mathfrak{R} is the curvature scalar. $F^2 = F_{\mu\nu}F^{\mu\nu}$ in which $F_{\mu\nu}$ is the Maxwell field associated with a $U(1)$ subgroup of $E_8 \times E_8$ or $\text{Spin}(32)/Z_2$ [26]. Without loss of generality, throughout the paper we shall use $a > 0$.

In 1995, CHM obtained the CDBH solution to the above action in their landmark paper [22]. By this end, they used a non-constant dilaton field. Their solution is described by the following static and spherically symmetric metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + R(r)^2 d\Omega^2, \quad (4)$$

where $d\Omega^2$ is the standard metric on 2-sphere and the metric functions $f(r)$ and $R(r)$ are given by

$$f(r) = \frac{1}{\gamma^2} r^{\frac{2}{1+a^2}} \left(1 - \frac{r_h}{r}\right), \quad (5)$$

and

$$R(r) = \gamma r^{\dot{N}}, \quad (6)$$

Here r_h denotes the event horizon of the CDBH. γ and \dot{N} are arbitrary real constants. \dot{N} is related to a by

$$\dot{N} = \frac{a^2}{1+a^2}, \quad (7)$$

Furthermore, the dilaton field satisfies

$$\phi = \phi_0 + \phi_1 \ln r, \quad (8)$$

where

$$\phi_0 = -\frac{1}{2a} \ln \left[\frac{Q^2 (1+a^2)}{\gamma^2} \right] \text{ and } \phi_1 = \frac{\dot{N}}{a}, \quad (9)$$

where Q refers to the electric charge. In this case, the solution for the electromagnetic (em) field is found as

$$F_{tr} = \frac{Q e^{2a\phi}}{R(r)^2}, \quad (10)$$

We should emphasize that the magnetically charged version of the CDBH can also be derived. This is possible with simply replacing $a \rightarrow -a$ in the field equations obtained from the action (3) and to consider the em field as $F_{\theta\varphi} = Q \sin \theta$ (it goes without saying that Q would be referred as magnetic charge) [22].

CDBH is not vacuum solution since the action (3) includes a static dilaton fluid which possesses a non-zero energy-momentum. In fact, considering

such a particular fluid model makes the CDBHs so interesting that they are neither AF nor NAF. As shown in Ref. [22], the mass of the BH can be computed by following the quasilocal mass definition of Brown and York [27] as

$$r_h = \frac{2M}{\dot{N}}, \quad (11)$$

We remark also that the horizon at $r = r_h$ hides the singularity located at $r = 0$. In the extreme case $r_h = 0$, metric (4) still exhibits the features of the BH. Because the singularity at $r = 0$ is null and marginally trapped such that it prevents the signals to reach the external observers. Unlike to the other charged BHs, a CDBH has no extremal limits. In other words, it has no zero charge limit. First of all, the eponyms of the LDBH are Clément and Gal'tsov [23]. Metric functions (5) and (6) correspond to the 4D LDBH which is the solution to the EMD theory [23] in the case of $a = 1$ ($\dot{N} = \frac{1}{2}$). Later on, it is shown that in addition to the EMD theory, LDBHs are available in Einstein-Yang-Mills-dilaton and Einstein-Yang-Mills-Born-Infeld-dilaton theories [24]. The most intriguing feature of these BHs is that while radiating, they undergo an isothermal process. Namely, their temperature does not alter with shrinking of the BH horizon or with the mass loss. Furthermore, LDBHs can perform a fading Hawking radiation in which the temperature goes zero with its ending mass when the quantum corrected entropy is taken into account [28]. On the other hand, while $a \rightarrow \infty$ ($\dot{N} = 1$) with $\gamma = 1$, metric (4) reduces to the Schwarzschild BH, which is AF as it is well-known.

Surface gravity of CDBH is calculated through the following expression

$$\kappa = \frac{f'(r)}{2} \Big|_{r=r_h} = \frac{r_h^{(\frac{2N}{a^2}-1)}}{2\gamma^2}, \quad (12)$$

where a prime "′" denotes differentiation with respect to r . Subsequently, one can readily obtain the Hawking temperature T_H of the CDBH (in gravitational units of $c = G = 1$ and $\hbar = l_p^2$) as

$$\begin{aligned} T_H &= \frac{\hbar\kappa}{2\pi}, \\ &= \frac{\hbar r_h^{(\frac{2\dot{N}}{a^2}-1)}}{4\pi\gamma^2} = \frac{\hbar r_h^{(1-2\dot{N})}}{4\pi\gamma^2}, \end{aligned} \quad (13)$$

From the above expression, we see that while the CDBH losing its M by virtue of the Hawking radiation, T_H increases for $a^2 > 1$, decreases for $a^2 < 1$ and is constant (independent of M) for $a^2 = 1$ (LDBH). Therefore, as mentioned before the LDBH's radiation is such a particular process that the energy (mass, M) transferring out of the BH typically occurs at a slow rate that thermal equilibrium is maintained. The Bekenstein-Hawking entropy is given by

$$\begin{aligned} S_{BH} &= \frac{A_h}{4\hbar}, \\ &= \frac{\pi}{\hbar} R(r)^2 = \frac{\pi}{\hbar} \gamma^2 r_h^{2\dot{N}}, \end{aligned} \quad (14)$$

which leads to

$$dS_{BH} = 4\frac{\pi}{\hbar}\gamma^2 r_h^{(2\dot{N}-1)} dM, \quad (15)$$

With these definitions, the validity of the first law of thermodynamics for the CDBH can be proven via

$$T_H dS_{BH} = dM. \quad (16)$$

In order to find the entropy spectrum by using the MM, here we shall firstly consider the massless scalar wave equation on the geometry of the

CDBH. The general equation of massless scalar field in a curved spacetime is written as

$$\square F = 0, \quad (17)$$

where \square denotes the Laplace-Beltrami operator. Thus, the above equation is equal to

$$\frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} \partial^i F), \quad i = 0 \dots 3, \quad (18)$$

Using the following ansatz for the scalar field F in Eq. (17)

$$F = \frac{\rho(r)}{r^{\dot{N}}} e^{i\omega t} Y_L^m(\theta, \varphi), \quad \text{Re}(\omega) > 0, \quad (19)$$

in which $Y_L^m(\theta, \varphi)$ is the well-known spheroidal harmonics which admits the eigenvalue $-L(L+1)$ [29], one obtains the following Zerilli equation [25] as

$$\left[-\frac{d^2}{dr^{*2}} + V(r) \right] \rho(r) = \omega^2 \rho(r), \quad (20)$$

where the effective potential is computed as

$$V(r) = f(r) \left[\frac{\dot{N}(\dot{N}-1)}{r^2} f(r) + \frac{L(L+1)}{\gamma^2 r_h^{2\dot{N}}} + \frac{\dot{N}}{r} f'(r) \right], \quad (21)$$

The tortoise coordinate r^* is defined as,

$$r^* = \int \frac{dr}{f(r)}, \quad (22)$$

which yields

$$r^* = -\gamma^2 \frac{r^{2\dot{N}}}{r_h} \Phi\left(\frac{r}{r_h}, 1, 2\dot{N}\right), \quad (23)$$

where Φ denotes the Hurwitz-Lerch Zeta function (see Ref. [30]). This function is defined by

$$\Phi(z, s, b) = \sum_{k=0}^{\infty} \frac{z^k}{(k+b)^s}, \quad (24)$$

and $\Phi(\frac{r}{rh}, 1, 2\dot{N})$ can be transformed into the hypergeometric function as

$$\Phi(\frac{r}{rh}, 1, 2\dot{N}) = \frac{1}{2\dot{N}} {}_2F_1(1, 2\dot{N}; 1 + 2\dot{N}; \frac{r}{rh}), \quad (25)$$

where ${}_2F_1$ represents the Gaussian hypergeometric function. Finally, it follows from Eq. (23) that

$$\lim_{r \rightarrow rh} r^* = -\infty \quad \text{and} \quad \lim_{r \rightarrow \infty} r^* = \infty. \quad (26)$$

3 QNMs and entropy spectrum of CDBH

In this section, we intend to derive the entropy and area spectra of the CDBH by using the MM. Gaining inspiration from the studies [31,32,33], here we use an approximation method in order to define the QNMs. Since the effective potential (8) diverges at the spatial infinity ($r^* \rightarrow \infty$) and vanishes at the horizon ($r^* \rightarrow -\infty$), therefore the QNMs are defined to be those for which we have only ingoing plane wave at the horizon, namely,

$$\rho(r)|_{QNM} \sim e^{i\omega r^*} \text{ at } r^* \rightarrow -\infty, \quad (27)$$

Now we can proceed to solve Eq. (20) in the near horizon limit and then impose the above boundary condition to find the frequency of QNM i.e., ω . Expansion of the metric function $f(r)$ around the event horizon is given by

$$\begin{aligned}
f(r) &= f'(r_h)(r - r_h) + \mathcal{O}(r - r_h)^2, \\
&\simeq 2\kappa(r - r_h),
\end{aligned} \tag{28}$$

where κ is the surface gravity, which is nothing but $\frac{1}{2}f'(r_h)$. From Eq. (22) we now obtain

$$r^* \simeq \frac{1}{2\kappa} \ln(r - r_h), \tag{29}$$

Furthermore, after letting $x = r - r_+$ and inserting Eq. (28) into Eq. (21) together with performing Taylor expansion around $x = 0$, one gets the near horizon form of the effective potential as,

$$V(x) \simeq 2\kappa x \left[\frac{L(L+1)}{\gamma^2 r_h^{2N}} \left(1 - \frac{2\dot{N}x}{r_h}\right) + \frac{2\dot{N}\kappa}{r_h} \left(1 - \frac{x}{r_h}\right) + \frac{2\dot{N}\kappa x}{r_h^2} (\dot{N} - 1) \right], \tag{30}$$

After substituting Eq. (30) into the Zerilli equation (20), we find

$$-4\kappa^2 x^2 \frac{d^2 \rho(x)}{dx^2} - 4\kappa^2 x \frac{d\rho(x)}{dx} + V(x)\rho(x) = \omega^2 \rho(x), \tag{31}$$

Solution of the above equation admits

$$\rho(x) \sim \varepsilon^{\frac{i\omega}{2\kappa}} U(a, b, c), \tag{32}$$

where $U(a, b, c)$ is the confluent hypergeometric function [34]. The parameters of the confluent hypergeometric functions are found as

$$\begin{aligned}
a &= \frac{1}{2} + i \left(\frac{\omega}{2\kappa} - \frac{\hat{\alpha}}{\hat{\beta}\sqrt{\kappa}\gamma} \right), \\
b &= 1 + i \frac{\omega}{\kappa}, \\
c &= i \frac{\hat{\beta}x}{2\gamma r_h \sqrt{\kappa}},
\end{aligned} \tag{33}$$

where

$$\begin{aligned}\hat{\beta} &= 4r_h^{(\dot{N}-\frac{1}{2})} \sqrt{\dot{N}L(L+1) + \dot{N}\kappa\gamma^2(2-\dot{N})r_h^{(2\dot{N}-1)}}, \\ \hat{\alpha} &= L(L+1) + 2\dot{N}\kappa\gamma^2r_h^{(2\dot{N}-1)},\end{aligned}\quad (34)$$

One can easily check that these results are in consistent with the studies done for the 4D LDBH ($\dot{N} = \frac{1}{2}$) [35].

In the limit of $x \ll 1$, the solution (32) becomes

$$\rho(x) \sim c_1 x^{-\frac{i\omega}{2\kappa}} \frac{\Gamma(i\frac{\omega}{\kappa})}{\Gamma(a)} + c_2 x^{\frac{i\omega}{2\kappa}} \frac{\Gamma(-i\frac{\omega}{\kappa})}{\Gamma(1+a-b)}, \quad (35)$$

where constants c_1 and c_2 denote the amplitudes of the near-horizon outgoing and ingoing waves, respectively. Now, since there is no outgoing wave in the QNM at the horizon, the first term of Eq. (35) should be vanished. This is possible with the poles of the Gamma function of the denominator. Therefore, the poles of the Gamma function are the decision makers of the frequencies of the QNMs. Thus, we can read the frequencies of the QNMs of the CDBHs as,

$$\omega_{\tilde{n}} = \frac{2\sqrt{\kappa}\hat{\alpha}}{\hat{\beta}\gamma} + i(2\tilde{n}+1)\kappa, \quad (\tilde{n} = 1, 2, 3, \dots) \quad (36)$$

where \tilde{n} is the overtone quantum number of the QNM. Thus, the imaginary part of the frequency of the QNM is

$$\omega_I = (2\tilde{n}+1)\kappa = \frac{2\pi}{\hbar}(2\tilde{n}+1)T_H, \quad (37)$$

where $T_H = \frac{\hbar\kappa}{2\pi}$ which is called the Hawking temperature [1]. Hence the transition frequency between two highly damped neighboring states becomes $\Delta\omega = \omega_{\tilde{n}+1} - \omega_{\tilde{n}} = 4\pi T_H$. So the adiabatic invariant quantity (2) in this case results with

$$I_{adb} = \frac{\hbar}{4\pi} \int \frac{dM}{T_H}, \quad (38)$$

Recalling the first law of thermodynamics (16), we easily see that

$$I_{adb} = \frac{S_{BH}}{4\pi} \hbar, \quad (39)$$

Finally, according to the Bohr-Sommerfeld quantization rule $I_{adb} = \hbar n$, one gets the spacing of the entropy spectrum as

$$S_n = 4\pi n, \quad (40)$$

Since $S = \frac{A}{4\hbar}$, the area spectrum is obtained as

$$\mathcal{A}_n = 16\pi n \hbar, \quad (41)$$

From the above, we can simply measure the area spacing as

$$\Delta \mathcal{A} = 16\pi \hbar. \quad (42)$$

It is easily seen that the spectroscopy of the CDBH is completely independent of the dilaton parameter a . Besides, the spacings between the levels are double of the Bekenstein's original result. This means that $\epsilon = 16\pi$. The discussion on this differentness is made in the conclusion part. Nevertheless, the obtained entropy and area spectra are evenly spaced. The latter result is in agreement with the Wei et al.'s conjecture [17] which proposes that static BHs of Einstein's gravity theory has equidistant quantum spectra of both entropy and area .

4 Conclusion

In this paper, the quantum spectra of the CDBH are investigated through the MM, which is based on adiabatic invariance of BHs. In order to obtain the QNM of the CDBH, we applied an approximation method given in Refs. [31,32,33] to the Zerilli equation (20). After a straightforward calculation, by using the MM which employs the proper frequency as the imaginary part instead of the real part of the QNMs the entropy and area spectra of the CDBH are derived. Both spectra are independent of the dilaton parameter and equally spaced as such as in the case of the LDBH [35]. However, we obtained $\epsilon = 16\pi$ which results that the equi-spacing is different than its usual Schwarzschild value: $\epsilon = 8\pi$. This discrepancy may arise due to the Schwinger mechanism [36]. Because, in the Bekenstein's original work [3], one gets the entropy spectrum by combining both the Schwinger mechanism and the Heisenberg quantum uncertainty principle. However, the QNM method that applied herein considers only the uncertainty principle via the Bohr-Sommerfeld quantization (40). Therefore, as stated in Ref. [9], the spacings between two neighboring levels may become different depending on the which method is applied. Thus, getting $\epsilon = 16\pi$ rather than its usual value $\epsilon = 8\pi$ is not surprising. Finally, we would like to point out that it will be interesting to apply the same analysis to the other dilatonic BHs like the dyonic BHs [37,38]. This is going to be our next problem in the near future.

References

1. C. Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, 2007).
2. J.D. Bekenstein, *Lett. Nuovo Cimento* **4**, 737 (1972).
3. J.D. Bekenstein, *Phys. Rev. D* **7**, 2333 (1973).
4. J.D. Bekenstein, *Lett. Nuovo Cimento* **11**, 467 (1974).

5. J.D. Bekenstein, gr-qc/9710076 (1997).
6. J.D. Bekenstein, gr-qc/9808028 (1998).
7. L.Y Jia, P.J. Mao, and J.R. Ren, Eur. Phys. J. C **71**, 1518 (2011).
8. S. Hod, Phys. Rev. Lett. **81**, 4293 (1998); arXiv:gr-qc/9812002.
9. S. Hod, Phys. Rev. D. **59**, 024014 (1998); arXiv:gr-qc/9906004.
10. R.L. Liboff, Found. Phys. **5**, 271 (1975).
11. G. Kunstatter, Phys. Rev. Lett. **90**, 161301 (2003); arXiv:gr-qc/0212014.
12. M. Maggiore, Phys. Rev. Lett. **100**, 141301 (2008); arXiv:0711.3145.
13. E.C. Vagenas, J. High Energy Phys. **0811**, 073 (2008); arXiv:0804.3264.
14. A.J.M. Medved, Class. Quantum Gravity **25**, 205014 (2008); arXiv:0804.4346.
15. D. Kothawala, T. Padmanabhan, and S. Sarkar, Phys. Rev. D **78**, 104018 (2008); arXiv:0807.1481
16. W. Li, L. Xu, and J. Lu, Phys. Lett. B **676**, 117 (2009).
17. S. Wei, R. Li, Y. Liu, and J. Ren, J. High Energy Phys. **03**, 076 (2009); arXiv:0901.0587.
18. A.L. Ortega, Phys. Lett. B **682**, 85 (2009).
19. K. Ropentko, Phys. Rev. D **82**, 044037 (2010); arXiv:0911.5635.
20. M.R. Setare and D. Momeni, Mod. Phys. Lett. A **26**, 151 (2011).
21. C.Z. Liu, Eur. Phys. J. C **72**, 2009 (2012).
22. K.C.K. Chan, J.H. Horne, and R.B. Mann, Nucl. Phys. B **447**, 441 (1995); arXiv:gr-qc/9502042.
23. G. Clément, D. Gal'tsov, and C. Leygnac, Phys. Rev. D **67**, 024012 (2003); arXiv:hep-th/0208225.
24. S.H. Mazharimousavi, I. Sakalli, M. Halilsoy, Phys. Lett. B **672**, 177;
25. S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, Oxford, 1983).
26. D. Garfinkle, G. T. Horowitz, and A. Strominger, Phys. Rev. D **43**, 3140 (1991); **45**, 3888(E) (1992).
27. J.D. Brown and J.W. York, Phys. Rev. D **47**, 1407 (1993).
28. I. Sakalli, M. Halilsoy, and H. Pasaoglu, Astrophys. Space Sci. **340**, 155 (2012); arXiv:1202.3259.

29. P. Du and B. Wang, *Phys. Rev. D* **70**, 064024 (2004).
30. H.M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions* (Kluwer Academic Publishers, Dordrecht, 2001).
31. T.R. Choudhury and T. Padmanabhan, *Phys. Rev. D* **69**, 064033 (2004); arXiv:gr-qc/0311064.
32. T. Padmanabhan, *Class. Quantum Gravity* **21**, L1 (2008); arXiv:gr-qc/0310027.
33. B.R. Majhi, *Phys. Lett. B* **686**, 49 (2010).
34. M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).
35. I. Sakalli, *Int. J. Mod. Phys. A* **26**, 2263 (2011); **28**, 1392002 (E) (2013).
36. S.P. Kim, *J. Korean Phys. Soc.* **53**, 1095 (2008); arXiv:0709.4313.
37. S. Yazadjiev, *Class. Quant. Grav.* **22**, 3875 (2005).
38. P.I. Slavov and S. Yazadjiev, *Phys. Rev. D* **86**, 084042 (2012).