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Solution of the Dirac equation in the rotating Bertotti–Robinson spacetime

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The Dirac equation is solved in the rotating Bertotti–Robinson spacetime. The set of equations representing the Dirac equation in the Newman–Penrose formalism is decoupled into an axial and an angular part. The axial equation, which is independent of mass, is exactly solved in terms of hypergeometric functions. The angular equation is considered both for massless (neutrino) and massive spin-$\frac{1}{2}$ particles. For the neutrinos, it is shown that the angular equation admits an exact solution in terms of the confluent Heun equation. In the existence of mass, the angular equation does not allow an analytical solution, however, it is expressible as a set of first order differential equations apt for a numerical study. © *2008 American Institute of Physics.* [DOI: 10.1063/1.2912725]

I. INTRODUCTION

The rotating Bertotti–Robinson (RBR) spacetime, which was discovered a long time ago by Carter,[1] is an Einstein–Maxwell solution representing a rotating electromagnetic field. This solution remained unnoticed until the study of Al-Badawi and Halilsoy,[2] who rediscovered it by applying a coordinate transformation to the cross-polarized Bell–Szekeres solution of colliding electromagnetic waves.[3] We can consider the RBR solution as an extended version of the Bertotti–Robinson (BR) solution due to the fact that the RBR solution contains one more degree of freedom to be interpreted as rotation. Adding rotation to the BR creates gravitational curvature, distorts isotropy, and modifies geodesics significantly. For this reason, the RBR solution assumes a more complicated topology compared to the BR solution. The RBR spacetime has the topology of $AdS^2 \times S^2$ and underlying group structure of $SL(2,R) \times U(1)$. Nowadays, spacetimes with the AdS structure are quite popular because of their connection with string theory, higher dimensions, and brane worlds. The RBR solution can also be interpreted as the "throat" connecting two rotating black holes with charges. This is due to the fact that the BR solution is considered as the throat connecting two asymptotically flat Reissner–Nordström regions.[4]

The first study on the Dirac equation in the BR spacetime without charge coupling was considered a long time ago.[5] During the past decade, studies on spin-$\frac{1}{2}$ particles in the BR spacetime gained momentum. For example, Silva-Ortigoza[6] showed how the Dirac equation could be separated when the background is the BR spacetime with cosmological constant. Later on, a more detailed study on the problem of the Dirac equation in the BR spacetime was worked out as well.[7] Here, we extend this recent work by looking for the answer to the following question: "How does a test Dirac particle behave in a rotating spacetime filled with electromagnetic field, i.e., in the RBR spacetime?" We shall ignore the backreaction effect of the spin-$\frac{1}{2}$ particle on the spacetime by the same token done in Ref. 7. The RBR solution represents one of the type-D spacetimes and as it could be followed from literature, studies on spin-$\frac{1}{2}$ particles in type-D spacetimes have

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always attracted attention[8] (and references therein). More recently, the problem of the Dirac equation in the near horizon geometry of an extreme Kerr black hole (Kerr throat) has been studied in Ref. 9. In many aspects, that spacetime of the Kerr throat shares common features with the RBR spacetime. The main difference between them is that the Kerr throat is a vacuum solution, while the RBR is not. On the other hand, they are both regular solutions.

In this paper, our aim is to solve the Dirac equation in the RBR spacetime. To this end, in order to separate the Dirac equation, we employ the well-known method suggested by Chandrasekhar.[8] We separate the Dirac equation into the axial (function of z only) and the angular parts (function of θ only) in such a way that the resulting axial equation remains independent of mass. This advantage leads us to obtain an exact solution for the axial equations in terms of hypergeometric functions. The angular part turns out to be more complicated than the axial part. This is due to the fact that the metric functions are dependent on the variable θ, and also, the angular equations contain the mass term. For the angular part, two separate cases, which are massive and massless (neutrinos) cases, are discussed. In the angular equations of the massive case, we are able to reduce the equations to a set of linear first order differential equations, which can be numerically utilized. However, the massless particle (neutrino) equations are exactly solved by reducing the equations to the confluent Heun equations.

Recall that the confluent Heun differential equations are less known than the hypergeometric family in literature. The modern mathematical development shows that many physical problems are exactly solved by Heun functions,[10–12] for example, problems involving atomic physics with certain potentials[13] which combine different inverse powers or combine the quadratic potential with inverse powers of 2, 4, 6, etc. Problems in solid state physics, such as dislocation movement in crystalline materials and quantum diffusion of kinks along dislocations, are also solved in terms of the Heun function.[14] For problems in general relativity, Fiziev[15] gave an exact solution of the Regge–Wheeler equation in terms of the Heun functions and applied them in the study of different boundary problems. More recently, Birkandan and Hortacsu[16] gave examples in which the Heun functions admit the solution of the wave equation encountered in general relativity. They have related the solutions of the Dirac equation when the background is Nutku's helicoid spacetime in five dimensions to the double confluent Heun function. Nowadays, modern computer packages have started to involve the Heun functions in their algorithms, as, for instance, it can be seen in the tenth and higher versions of the famous computer package MAPLE.

The paper is organized as follows: In Sec. II, a brief review of the RBR solution is given. Next, we present the basic Dirac equations and separate them in the spacetime of RBR. In Sec. III, we present the exact solution of the axial equation. The angular equation with both massless and massive cases are discussed in Secs. IV and V, respectively. Finally, in Sec. VI, we draw our conclusions.

II. ROTATING BERTOTTI–ROBINSON SPACETIME AND SEPARATION OF THE DIRAC EQUATION ON THIS SPACETIME

The metric describing a rotating electromagnetic field, RBR solution, written in spherical coordinates, is given by[2]

$$ds^2 = \frac{F(\theta)}{r^2}\left[d\tilde{t}^2 - dr^2 - r^2 d\theta^2 - \frac{r^2 \sin^2\theta}{F^2}\left(d\tilde{\phi} - \frac{q}{r}d\tilde{t}\right)^2\right], \tag{1}$$

where the function $F(\theta)$ and the constant q are

$$F(\theta) = 1 + a^2(1+\cos^2\theta),$$

$$q = 2a\sqrt{1+a^2}. \tag{2}$$

in which a is the rotation parameter. It is readily seen that for $a=0$, metric (1) reduces to the BR metric[7] (and references cited therein).

We make the choice of the following null tetrad basis 1-forms $(l,n,m,\bar{m})$ of the Newman–Penrose (NP) formalism[17] in terms of the RBR geometry that satisfies the orthogonality conditions, $l\cdot n=-m\cdot\bar{m}=1$. We note that throughout the paper, a bar over a quantity denotes complex conjugation. We can write the covariant 1-forms as

$$\sqrt{2}l=\frac{1}{2r}\sqrt{F}(d\tilde{t}-dr),$$

$$\sqrt{2}n=\tfrac{2}{r}\sqrt{F}(d\tilde{t}+dr), \tag{3}$$

$$\sqrt{2}m=i\sqrt{F}d\theta+\frac{\sin\theta}{\sqrt{F}}\left(\frac{2a}{r}\sqrt{1+a^2}d\tilde{t}-d\tilde{\phi}\right).$$

We obtain the nonzero Ψ_2 and Φ_{11}, which are known as Weyl and Maxwell scalars, respectively, as

$$\Psi_2=\frac{a^2}{F^3}\left[(1+a^2)\cos 2\theta+a^2\cos^2\theta-\frac{i}{a}\sqrt{1+a^2}(1+a^2+a^2\sin^2\theta)\cos\theta\right],$$
$$\Phi_{11}=\frac{1}{2F^2}. \tag{4}$$

The singularity-free and the type-D characters of the metric can be easily deduced from Ψ_2. It is obvious that for $a=0$, this type-D metric (1) yields a conformally flat spacetime (i.e., the BR spacetime) in which a uniform electromagnetic field, with $\Phi_{11}=\frac{1}{2}$, fills the entire space. As it can be seen from Eq. (4), rotation $(a\neq 0)$ gives rise to anisotropy of the prevailing electromagnetic field.

In order to study the Dirac equation in the RBR spacetime, we prefer to work in a more convenient coordinate system; therefore, by using the following transformation:

$$z=\frac{1}{2r}(\tilde{t}^2-r^2+1),$$

$$t=\tan^{-1}\left[\frac{1}{2\tilde{t}}(\tilde{t}^2-r^2-1)\right], \tag{5}$$

$$\phi=\frac{1}{2}q\ln\left[\frac{(r-\tilde{t})^2+1}{(r+\tilde{t})^2+1}\right]+\tilde{\phi},$$

metric (1) is transformed into

$$ds^2=F(\theta)\left[(1+z^2)dt^2-\frac{dz^2}{(1+z^2)}-d\theta^2-\frac{\sin^2\theta}{F^2}(d\phi-qzdt)^2\right]. \tag{6}$$

We notice that the metric functions in Eq. (6) explicitly depend on the variable θ, as a result, the angular part of the Dirac equation becomes important. The coordinates $-\infty<t<\infty,-\infty<z<\infty$ covers the entire, singularity-free spacetime.

The covariant 1-forms of the metric (6) can be taken as

$$\sqrt{2}l = \sqrt{F}\left(\sqrt{1+z^2}dt - \frac{dz}{\sqrt{1+z^2}}\right),$$

$$\sqrt{2}n = \sqrt{F}\left(\sqrt{1+z^2}dt + \frac{dz}{\sqrt{1+z^2}}\right), \tag{7}$$

$$\sqrt{2}m = \sqrt{F}d\theta + \frac{i\sin\theta}{\sqrt{2}\sqrt{F}}(d\phi - qzdt),$$

while their corresponding directional derivatives become

$$\sqrt{2}D = \frac{\partial_t}{\sqrt{F}\sqrt{1+z^2}} + \frac{\sqrt{1+z^2}\partial_z}{\sqrt{F}} + \frac{qz\partial_\phi}{\sqrt{F}\sqrt{1+z^2}},$$

$$\sqrt{2}\Delta = \frac{\partial_t}{\sqrt{F}\sqrt{1+z^2}} - \frac{\sqrt{1+z^2}\partial_z}{\sqrt{F}} + \frac{qz\partial_\phi}{\sqrt{F}\sqrt{1+z^2}},$$

$$\sqrt{2}\delta = -\left[\frac{\partial_\theta}{\sqrt{F}} + i\frac{\sqrt{F}}{\sin\theta}\partial_\phi\right], \tag{8}$$

$$\sqrt{2}\bar{\delta} = -\left[\frac{\partial_\theta}{\sqrt{F}} - i\frac{\sqrt{F}}{\sin\theta}\partial_\phi\right],$$

By using the above tetrad, we determine the nonzero NP complex spin coefficients[17] as

$$\tau = -\pi = \frac{-1}{2\sqrt{2}F^{3/2}}[a^2\sin(2\theta) + iq\sin\theta)],$$

$$\epsilon = \gamma = \frac{z}{2\sqrt{2}\sqrt{F}\sqrt{1+z^2}}, \tag{9}$$

$$\alpha = -\beta = \frac{1}{4\sqrt{2}F^{3/2}}[2\cot\theta(1+2a^2) + iq\sin\theta].$$

The Dirac equations in the NP formalism are given by[8]

$$(D + \epsilon - \rho)F_1 + (\bar{\delta} + \pi - \alpha)F_2 = i\mu_p G_1,$$

$$(\delta + \beta - \tau)F_1 + (\Delta + \mu - \gamma)F_2 = i\mu_p G_2,$$

$$(D + \bar{\epsilon} - \bar{\rho})G_2 - (\delta + \bar{\pi} - \bar{\alpha})G_1 = i\mu_p F_{2,} \tag{10}$$

$$(\Delta + \bar{\mu} - \bar{\gamma})G_1 - (\bar{\delta} + \bar{\beta} - \bar{\tau})G_2 = i\mu_p F_1,$$

where $\mu^* = \sqrt{2}\mu_p$ is the mass of the Dirac particle.

The form of the Dirac equation suggests that we assume,[8]

$$
\begin{aligned}
F_1 &= f_1(z)A_1(\theta)e^{i(kt+m\phi)},\\
F_2 &= f_2(z)A_3(\theta)e^{i(kt+m\phi)},\\
G_1 &= g_1(z)A_2(\theta)e^{i(kt+m\phi)},\\
G_2 &= g_2(z)A_4(\theta)e^{i(kt+m\phi)}.
\end{aligned}
\tag{11}
$$

Here, we consider the corresponding Compton wave of the Dirac particle as in the form of $f(z)A(\theta)e^{i(kt+m\phi)}$, where k is the frequency of the incoming wave and m is the azimuthal quantum number of the wave. The temporal and azimuthal dependencies are chosen to be the same but the axial and angular dependencies are chosen to be different for different spinors.

Substituting the appropriate spin coefficients (9) and the spinors (11) into the Dirac equation (10), we obtain

$$
\begin{aligned}
\frac{\tilde{Z}f_1}{f_2} - \frac{LA_3}{A_1} &= i\mu^* \frac{g_1}{f_2}\frac{A_2}{A_1}\sqrt{F},\\
\frac{\bar{\tilde{Z}}f_2}{f_1} + \frac{L^\dagger A_1}{A_3} &= -i\mu^* \frac{g_2}{f_1}\frac{A_4}{A_3}\sqrt{F},\\
\frac{\tilde{Z}g_2}{g_1} + \frac{\pounds^\dagger A_2}{A_4} &= i\mu^* \frac{f_2}{g_1}\frac{A_3}{A_4}\sqrt{F},\\
\frac{\bar{\tilde{Z}}g_1}{g_2} - \frac{\pounds A_4}{A_2} &= -i\mu^* \frac{f_1}{g_2}\frac{A_1}{A_2}\sqrt{F},
\end{aligned}
\tag{12}
$$

where the axial and the angular operators, respectively, are

$$
\tilde{Z} = \sqrt{1+z^2}\partial_z + \frac{1}{2\sqrt{1+z^2}}[z + 2i(k+mqz)],
$$

$$
\bar{\tilde{Z}} = \sqrt{1+z^2}\partial_z + \frac{1}{2\sqrt{1+z^2}}[z - 2i(k+mqz)],
$$

and

$$
\begin{aligned}
L &= \partial_\theta + \frac{\cot\theta}{2} - \frac{a^2\sin 2\theta}{4F} + \frac{mF}{\sin\theta} - i\frac{q\sin\theta}{4F},\\
L^\dagger &= \partial_\theta + \frac{\cot\theta}{2} - \frac{a^2\sin 2\theta}{4F} - \frac{mF}{\sin\theta} - i\frac{q\sin\theta}{4F},\\
\pounds &= \partial_\theta + \frac{\cot\theta}{2} - \frac{a^2\sin 2\theta}{4F} + \frac{mF}{\sin\theta} + i\frac{q\sin\theta}{4F},\\
\pounds^\dagger &= \partial_\theta + \frac{\cot\theta}{2} - \frac{a^2\sin 2\theta}{4F} - \frac{mF}{\sin\theta} + i\frac{q\sin\theta}{4F}.
\end{aligned}
\tag{13}
$$

It is obvious that L and $L^{\dagger}$ are purely angular operators and $L=\bar{\pounds}, L^{\dagger}=\bar{\pounds}^{\dagger}$.

In order to separate the Dirac equation (12) into axial and angular parts, we choose $f_1=g_2$, $f_2=g_1, A_2=\overline{A_1}$, and $A_4=\overline{A_3}$ and introduce a real separation constant λ as

$$\tilde{Z}g_2 = -\lambda g_1, \tag{14}$$

$$\bar{\tilde{Z}}g_1 = -\lambda g_2, \tag{15}$$

and

$$LA_3 + i\mu^* \bar{A}_1\sqrt{F} = -\lambda A_1, \tag{16}$$

$$L^{\dagger}A_1 + i\mu^* \bar{A}_3\sqrt{F} = \lambda A_3. \tag{17}$$

III. SOLUTION OF THE AXIAL EQUATION

The structure of the axial equations (14) and (15) admits $g_1=\bar{g}_2$. Thus, it is enough to decouple the axial equations for g_2, namely,

$$\bar{\tilde{Z}}(\tilde{Z}g_2) = \lambda^2 g_2. \tag{18}$$

The explicit form of Eq. (18) can be obtained as

$$(1+z^2)g_2''(z) + 2zg_2'(z) + \frac{1}{2(1+z^2)}\left[1 + \frac{1}{2}z^2 + 2(k+mqz)^2 - 2i(kz-mq) - 2\lambda^2(1+z^2)\right]g_2(z) = 0. \tag{19}$$

(Throughout the paper, a prime denotes the derivative with respect to its argument.)

Let us introduce a new variable y such that $z=i(1-2y)$, therefore Eq. (19), becomes

$$y(1-y)g_2''(y) + (1-2y)g_2'(y) - \frac{1}{4y(1-y)}\left\{\frac{1}{4} - m^2q^2(1-2y)^2 + y(1-y) + k(k+1-2y) - 4\lambda^2 y(1-y) + imq[1+2k(1-2y)]\right\}g_2(y) = 0. \tag{20}$$

The exact solution of the axial part is found in terms of the Gauss hypergeometric functions as

$$g_2(y) = C_1 y^{\alpha}(y-1)^{\beta}{}_2F_1\left(\tfrac{1}{2}+k-\gamma, \tfrac{1}{2}+k+\gamma, \tfrac{3}{2}+k+imq; y\right) + C_2 y^{-\alpha}(y-1)^{\beta}{}_2F_1\left(-(\gamma+imq), \gamma-imq, \tfrac{1}{2}-k-imq; y\right), \tag{21}$$

where the parameters are

$$\alpha = \tfrac{1}{2}\left(k+\tfrac{1}{2}+imq\right),$$

$$\beta = \tfrac{1}{2}\left(k-\tfrac{1}{2}-imq\right), \tag{22}$$

$$\gamma = \sqrt{\lambda^2 - m^2q^2},$$

and C_1, C_2 are complex constants.

IV. REDUCTION OF THE ANGULAR EQUATION TO HEUN EQUATION: THE MASSLESS CASE

The aim of this section is to show that the angular equations (16) and (17) for the massless Dirac particles (such as neutrinos) can be decoupled to a confluent Heun differential equation.

For $\mu^*=0$, Eqs. (16) and (17) can be explicitly written as

$$A_1'(\theta) + (K-M)A_1(\theta) = \lambda A_3(\theta), \tag{23}$$

$$A_3'(\theta) + (K+M)A_3(\theta) = -\lambda A_1(\theta), \tag{24}$$

where

$$K = \frac{\cot\theta}{2} - \frac{a^2 \sin 2\theta}{4F} - i\frac{q\sin\theta}{4F}, \tag{25}$$

$$M = \frac{mF}{\sin\theta}, \tag{26}$$

By introducing the scalings

$$A_1(\theta) = H_1(\theta)e^{-\int(K-M)d\theta}, \tag{27}$$

$$A_3(\theta) = H_3(\theta)e^{-\int(K+M)d\theta}, \tag{28}$$

we get

$$H_1'(\theta) = \lambda H_3(\theta)e^{-2\int M d\theta}, \tag{29}$$

$$H_3'(\theta) = -\lambda H_1(\theta)e^{2\int M d\theta}, \tag{30}$$

By decoupling Eqs. (29) and (30) in Eq. (29) for $H_1(\theta)$, we obtain

$$H_1''(\theta) + 2MH_1'(\theta) + \lambda^2 H_1(\theta) = 0. \tag{31}$$

Similarly, if we decouple the axial equations for $H_3(\theta)$, the resulting equation turns out to be

$$H_3''(\theta) - 2MH_3'(\theta) + \lambda^2 H_3(\theta) = 0. \tag{32}$$

By introducing a new variable $\theta=\cos^{-1}(1-2z)$, Eqs. (31) and (32) are cast into the general confluent form of the Heun equation, namely,

$$H_1''(z) + \left[-4a^2m + \frac{\frac{1}{2}+m+2a^2m}{z} + \frac{\frac{1}{2}-(m+2a^2m)}{z-1}\right]H_1'(z) - \frac{\lambda^2}{z(z-1)}H_1(z) = 0, \tag{33}$$

$$H_3''(z) + \left[4a^2m + \frac{\frac{1}{2}-(m+2a^2m)}{z} + \frac{\frac{1}{2}+m+2a^2m}{z-1}\right]H_3'(z) - \frac{\lambda^2}{z(z-1)}H_3(z) = 0. \tag{34}$$

Recall that the general confluent form of the Heun equation[12] is given as follows.

$$H''(z) + \left[A + \frac{B}{z} + \frac{C}{z-1}\right]H'(z) + \frac{ADz-h}{z(z-1)}H(z) = 0. \tag{35}$$

After matching Eqs. (33) and (34) with Eq. (35), one can get the following correspondences. (a) For Eq. (33),

$$A = -4a^2m, \quad B = \tfrac{1}{2} + m + 2a^2m, \quad C = \tfrac{1}{2} - (m + 2a^2m), \quad D = 0, \quad \text{and } h = \lambda^2. \tag{36}$$

(b) For Eq. (34),

$$A = 4a^2m, \quad B = \tfrac{1}{2} - (m + 2a^2m), \quad C = \tfrac{1}{2} + m + 2a^2m, \quad D = 0, \quad \text{and } h = \lambda^2. \tag{37}$$

Determining when the solutions of a confluent Heun equation are expressible in terms of more familiar functions would be obviously useful. Expansion of solutions to the confluent Heun equation in terms of hypergeometric and confluent hypergeometric functions are studied in detail by Ref. 12. In Ref. 12, it is also shown that the confluent Heun functions can be normalized to constitute a group of orthogonal complete functions. Here, as an example, we follow the intermediate steps in Ref. 12 (page 102) in order to express the solutions of Eq. (35) with $D=0$ in terms of the hypergeometric functions. The transformation between the confluent Heun function and the hypergeometric function is given with the Floquet expansion,[12,14] namely,

$$H_j(z) = \sum_{n=-\infty}^{\infty} g_{n\,2}F_1(\xi_1, \xi_2; B; z), \tag{38}$$

where

$$\xi_1 = -n - \nu_j \quad \text{and } \xi_2 = n + \nu_j + C + B - 1, \tag{39}$$

and ν_j are known as the Floquet exponents. The coefficients g_n satisfy a three-term recurrence relation:

$$\Lambda_n g_{n-1} + Q_n g_n + \Phi_n g_{n+1} = 0, \tag{40}$$

where

$$\Lambda_n = A b_{n-1,n},$$

$$Q_n = \xi_1 \xi_2 - h + \frac{A}{2} b_{n,n}, \tag{41}$$

$$\Phi_n = A b_{n+1,n},$$

and the coefficients, $b_{n-1,n}$, $b_{n,n}$, and $b_{n+1,n}$ expressed in terms of the parameters ξ_1, ξ_2, are given explicitly by Ref. 12. (The only difference between our notation and Ronveaux's notation is $B \equiv \gamma$.)

Finally, it should be noted that one can see a brief analysis of the confluent Heun equation as well as its power series solution and polynomial solution in Ref. 18.

V. REDUCTION OF THE ANGULAR EQUATION INTO A SET OF LINEAR FIRST ORDER DIFFERENTIAL EQUATIONS: THE CASE WITH MASS

In this section, we shall reduce the angular equations (16) and (17) into a set of linear set of first order differential equations for the case of the Dirac particle with mass. To this end, let us make the following substitutions into Eqs. (16) and (17).

$$A_1(\theta) = [A_0(\theta) + iB_0(\theta)]e^{-\int(\cot\theta/2 - a^2\sin(2\theta)/4F)d\theta}, \tag{42}$$

$$A_3(\theta) = [M_0(\theta) + iN_0(\theta)]e^{-\int(\cot\theta/2 - a^2\sin(2\theta)/4F)d\theta}, \tag{43}$$

where $A_0(\theta)$, $B_0(\theta)$, $M_0(\theta)$, and $N_0(\theta)$ are real functions. After separating the real and the imaginary parts, one obtains a set of first order differential equations:

$$A_0'(\theta) - \frac{mF}{\sin\theta}A_0(\theta) + \frac{q\sin\theta}{4F}B_0(\theta) = \lambda M_0(\theta) - \mu^*\sqrt{F}N_0(\theta),$$

$$B_0'(\theta) - \frac{mF}{\sin\theta}B_0(\theta) - \frac{q\sin\theta}{4F}A_0(\theta) = \lambda N_0(\theta) - \mu^*\sqrt{F}M_0(\theta),$$

$$M_0'(\theta) + \frac{mF}{\sin\theta}M_0(\theta) + \frac{q\sin\theta}{4F}N_0(\theta) = -\lambda A_0(\theta) - \mu^*\sqrt{F}B_0(\theta), \tag{44}$$

$$N_0'(\theta) + \frac{mF}{\sin\theta}N_0(\theta) - \frac{q\sin\theta}{4F}M_0(\theta) = -\lambda B_0(\theta) - \mu^*\sqrt{F}A_0(\theta).$$

By introducing a new variable $x=\cos\theta$, one can remove the trigonometric functions in the set of first order differential equations (44). However, analytically solving the entire system does not seem possible. To our knowledge, in literature, such a system does not exist. Nevertheless, one can analyze the system via an appropriate numerical technique, which may need an advanced computational study.

VI. CONCLUSION

In this paper, our target was not only to separate the Dirac equation for a test spin-$\frac{1}{2}$ particle in the rotating electromagnetic spacetime (RBR) but to explore exact solutions as well. By this way, we wanted to make a contribution to the wave mechanical aspects of the Dirac particles in the RBR geometry.

Due to the metric functions, which are only functions of the angular variable θ, the angular part of the Dirac equation in the RBR background is the harder part to be tackled compared to the axial part. Another advantage of the axial part is that the axial equations do not involve the mass term. These simplifications in the axial equations guided us in obtaining the general solution of the axial part in terms of the hypergeometric functions. On the other hand, although we could not obtain the general analytic solution of the angular part, we succeeded in overcoming the difficulties in the angular part in the massless case and obtained its exact solution in terms of the Heun polynomials. Inclusion of mass prevents us from obtaining an analytic solution for the angular part. As an alternative way to the analytic solution, in the last section, we showed that the angular part could be written as a set of first order differential equations, which are suitable for numerical investigations.

Finally, the study of the charged Dirac particles in the RBR spacetime may reveal more information compared to the present case. This is going to be our next problem in the near future.

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