Chaos in Kundt type III Spacetimes

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Abstract

We consider geodesics motion in a particular Kundt type III spacetime in which Einstein-Yang-Mills equations admit solutions. On a particular surface as constraint we project the geodesics into the (x, y) plane and treat the problem as a 2-dimensional one. Our numerical study shows that chaotic behavior emerges under reasonable conditions.

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Kundt's class of solutions present spacetimes with non-expanding, shear-free and twist-free null geodesic congruences [1]. Interest in this class of solutions comes from the fact that these spacetimes admit plane waves, which exhibit geometrically different properties than the *pp*-waves. Due to its potential application in string theory Kundt class still maintains its popularity [2, 3, 4, 5, 6]. Various Petrov types of Kundt solutions were identified, among them especially Kundt type N has been studied in detail for long time by many authors,(*see for instance* [7, 8, 9]). Geodesics in a specific Kundt type N was analyzed in detail by [9] , and it was shown that particular solutions obey a power-law. Recently, interest in the Kundt type III spaces has gained momentum. Firstly, Griffiths *et al* [10] derived and classified a complete family of Kundt type III, which admit cosmological constant Λ_c and/or pure radiation $\Phi 22 \neq 0$. In the absence of the cosmological constant Λ_c, the Kundt type III solutions are further generalized to be the solutions of the Einstein-Yang-Mills (EYM) system by Fuster and Holten [11].

The pioneering study of proving chaos in the spacetimes of plane waves was done for pure impulsive gravitational *pp*-waves [12] . Recently, it has been further demonstrated that under certain conditions the emergence of chaotic motion is possible both in the spacetimes of the superposed electrovac *pp*-waves and in the non-Abelian plane waves of Kundt type N [13], which are the solutions of the $D = 4$ EYM equations. In [13], it was also pointed out that for the non-Abelian plane waves the chaotic effect of gravity dominates over the gauge field. This is due to the fact that such local fields vanish asymptotically and the chaos inherited from gravity renders the whole system chaotic.

Analysis of the geodesic motion for the $D = 4$ Kundt type III is also discussed by [11] in which a possible chaotic motion is highlighted in particular cases of this class.

In this Brief Report, our aim is to investigate the possibility of chaotic geodesics in Kundt class III spacetimes. For this purpose, we consider the following algebraically special line-element [3, 11],

$$ds^2 = 2du[dv + Hdu + Wdz + \overline{W}d\overline{z}] - 2dzd\overline{z}, \qquad (1)$$

where $H = H(u, v, z, \overline{z})$ is a real function while $W = W(u, v, z, \overline{z})$ is a complex function, in general. Here our motivation is to study the geodesic motion for the case $W_{,v} = \frac{-2}{z+\overline{z}}$, referring to the solution of the EYM equations for the metric (1) given by [11]

$$W = W^0(u,z) - \frac{2v}{z+\overline{z}},$$

$$H = H^0(u,z,\overline{z}) + \frac{W^0(u,z)+\overline{W}^0(u,\overline{z})}{z+\overline{z}}v - \frac{v^2}{(z+\overline{z})^2}. \tag{2}$$

where $W^0(u,z)$ is an arbitrary complex function and $H^0(u,z,\overline{z})$ is a real function. The simplest choice of $W^0(u,z)$ for the type III is

$$W^0(u,z) = g(u)z, \tag{3}$$

such that the Weyl scalar $\Psi_3 \neq 0$. On the other hand, imposing the only solution ($\chi^a = \lambda^a(u)z$) on the Yang-Mills (YM) equation in which the energy density is bounded throughout the spacetime, the solution for $H^0(u,z,\overline{z})$ becomes

$$H^0(u,z,\overline{z}) = \left[f(u,z)+\overline{f}(u,\overline{z})\right](z+\overline{z}) - g\overline{g}z\overline{z} + \sigma(u)(z+\overline{z})^2\left[\ln(z+\overline{z})-1\right]. \tag{4}$$

where $f(u,z)$ is an arbitrary complex function and $\sigma(u) = 2\gamma_{ab}\lambda^a(u)\overline{\lambda}^b(u) + g(u)\overline{g}(u)$ is a real function. Here, $\lambda^a(u)$ are bounded complex functions and γ_{ab} is the invariant metric of the Lie group. We note that the condition for being Kundt type III spacetimes is $g(u) \neq 0$. It is trivially seen that $\lambda^a(u) = 0$ corresponds to the vacuum solution.

Our primary interest here is to write the geodesics equations for the metric (1). Similar to the study [11], for the beginning, we eliminate the u-dependence from W^0 and g by the following particular choice

$$W^0 = z \qquad \text{and} \qquad g = 1, \tag{5}$$

Next, introducing real spatial coordinates x and y by $\sqrt{2}z = (x+iy)$, we get the geodesic equations as

$$\ddot{u} - \dot{u}^2\left(1-\frac{v}{x^2}\right) + 2\dot{u}\frac{\dot{x}}{x} = 0, \tag{6}$$

$$\ddot{x} + \dot{u}^2\left[H_{,x} - \left(1-\frac{v}{x^2}\right)\left(x-2\frac{v}{x}\right)\right] + \frac{2}{x}\dot{u}\dot{v} + 2\dot{u}\dot{x}\left(1-2\frac{v}{x^2}\right) = 0, \tag{7}$$

$$\ddot{y} + \dot{u}^2\left[H_{,y} + y\left(1-\frac{v}{x^2}\right)\right] - 2\dot{u}\dot{x}\frac{y}{x} = 0, \tag{8}$$

where the dot denotes $\frac{d}{d\tau}$ with τ being the proper time. In addition the metric condition implies

$$\dot{x}^2 + \dot{y}^2 - 2\dot{u}\dot{v} - 2H\dot{u}^2 - 2\dot{u}\dot{x}(x - 2\frac{v}{x}) + 2y\dot{u}\dot{y} = \epsilon. \tag{9}$$

where $\epsilon = 1, 0, -1$ stands for timelike, null and spacelike geodesics, respectively. The present form of the equation set does not allow us to obtain a 2D (x, y) Hamiltonian system analogous to the previous studies [12, 13]. However, with appropriate choices of u and v surfaces it is possible to project the geodesics into the (x, y) plane in which writing a 2D Hamiltonian becomes possible. Our first intention is to shift the independent variable from τ to u as an affine parameter. If we consider a family of geodesics, which follow the light-cone coordinate u with constant rate of change in the same proper time intervals, the following assumption can be made

$$\dot{u} = \text{constant} \equiv 1, \tag{10}$$

Such an assumption gives rise to a condition on the v surfaces given by

$$v = x^2(1 - 2\frac{x'}{x}), \tag{11}$$

Here " $'$ " denotes $\frac{d}{du}$. By this substitution into Eqs. (7) and (8), we get a 2D dynamical system in the (x, y) plane

$$3x'' - H^0_{,x} - x = 0, \tag{12}$$

$$y'' + H^0_{,y} = 0, \tag{13}$$

which is described by a Super-Hamiltonian [14]

$$\mathcal{H} = \frac{1}{2}\left(P_y^2 - \frac{P_x^2}{3}\right) + V(x, y), \tag{14}$$

with the corresponding potential

$$V(x, y) = H^0 + \frac{x^2}{2}. \tag{15}$$

Let us note that the Super-Hamiltonian defined by the momenta $P_x = -3x'$ and $P_y = y'$ is not positive definite.

Eq. (9) stands for an energy condition in which it should be automatically satisfied by the solutions of Eqs. (12) and (13). Without loss of generality, we can assume that f and λ^a are independent of u. This assumption implies that σ is a positive constant. As we mentioned before that the chaotic effect of gravity dominates over the gauge (YM) field [13], it would be sufficient to investigate chaos in vacuum, i.e. $\sigma = 1$. In other words, once the chaotic motion appears in the vacuum spacetime, the local fields could not be strong enough to avert it into a regular motion.

In general, any $f = kz^n$ $(n = 0, 1, 2...)$, with the multiplicative factor k being an arbitrary real parameter implies a potential

$$V = \sqrt{2}kx\,\mathrm{Re}(z^n) - \frac{y^2}{2} + 2x^2\left[\ln(\sqrt{2}x) - 1\right]. \tag{16}$$

which admits an integrable system for $k = 0$. The logarithmic term in the potential imposes a condition on the x coordinate, namely $x > 0$. Beside this, the case $f = kz$, which describes a flat space for vacuum *pp*-wave spacetimes [1, 12], confesses a regular motion for the geodesics particles. Contrary to the vacuum homogeneous *pp*-waves [12] , here the case $n = 2$ admits a nonintegrable dynamical system. However, the nonintegrable systems in the vacuum *pp*-waves were obtained for the cases with $n \geq 3$, [12] .

Here we wish to study the nonintegrable system with the simplest case ($n = 2$), and explore whether the motion depends on the initial conditions or not. If chaos emerges in such a simplest case, intuitively it should also appear for the f functions with higher powers of z. Now, for $n = 2$, it can be seen that the potential (16) has various unstable points according to the range of the multiplicative constant k.

TABLE I: Unstable points depending on the value of k.

Points	Saddle	Repellor
$\left\{x = \frac{-1}{2\sqrt{2}k},\ y = \pm\frac{1}{2\sqrt{2}k}\sqrt{7 + 8\ln(-2k)}\right\}$	$k < -\frac{1}{2}e^{-\frac{7}{8}}$	—
$\left\{x = \frac{2\sqrt{2}}{3k}LambertW\left(\frac{3}{4}k\sqrt{e}\right),\ y = 0\right\}$	$k \geqslant -\frac{1}{2}e^{-\frac{7}{8}}$	$-\frac{4}{3}e^{-\frac{3}{2}} < k < -\frac{1}{2}e^{-\frac{7}{8}}$

Those results in the table show us that the highest possibility of the emergence of chaos corresponds to the case $k < -\frac{1}{2}e^{-\frac{7}{8}}$, in which admits two saddle points. Particularly, the case $-\frac{4}{3}e^{-\frac{3}{2}} < k < -\frac{1}{2}e^{-\frac{7}{8}}$ has an additive repellor point, and whence it may follow a stronger chaos. Conversely, the case $k \geqslant -\frac{1}{2}e^{-\frac{7}{8}}$ upon possessing one saddle point causes a questionable chaotic motion. In order to judge the existence of the chaotic motion, we study the numerical analysis of the evolution of the test particle in the gravitational field.

We integrate numerically the equations of motion given by Eqs. (12) and (13). The initial conditions depend on 3 parameters, $(x_0,\ y_0)$ (at $u = 0$) and k. For a given k value, we may choose $(x_0,\ y_0)$ such that keeping x_0 unchanged, and checking the effect of the y_0 on the geodesic motions, while it varies. To do this, we may set $x_0 = c_1$, a real constant $c_1 > 0$, and $y_0 = -3 + \sum_{j=0}^{18} \frac{j}{3}$. For example, if we take $k = 0$, the solutions are trivially analytic. This is also graphically verified in Fig. 1. Next, by considering the cases $k \neq 0$ the motion can lead to a chaotic motion. It is observed that the chaos has a *movable character* depending on the choices of x_0 and y_0 while $k \geqslant -\frac{1}{2}e^{-\frac{7}{8}}$, see Fig. 2. However, when the multiplicative constant $k < -\frac{1}{2}e^{-\frac{7}{8}}$ chaos is obvious. It is seen that the multiplicative constant k of the function f becomes decisive for the chaotic motion. In other words, k plays the role of critical parameter for the onset of chaos. The chaotic behavior of the geodesics in the case $k < -\frac{1}{2}e^{-\frac{7}{8}}$ is illustrated in Fig. 3. Alternatively, the dynamical system can be investigated by using the Poincaré section method. We use the package POINCARÉ [14] to perform the numerical experiments. Fig. (4) is a demonstration of the Poincaré section, which verifies the chaotic behavior in our dynamical system.

In conclusion, it is shown that the Kundt type III spacetimes may reveal chaotic motion under certain conditions. To our knowledge such a study did not exist in the literature before. Chaos in the spacetimes of electrovac and the specific Kundt type N with YM field was studied before [13]. This report constitutes an extension of that study. It is needless to state that the existence of the chaos in the Kundt type III spacetimes may have further implications for the particle motions in string theory and in higher dimensions.

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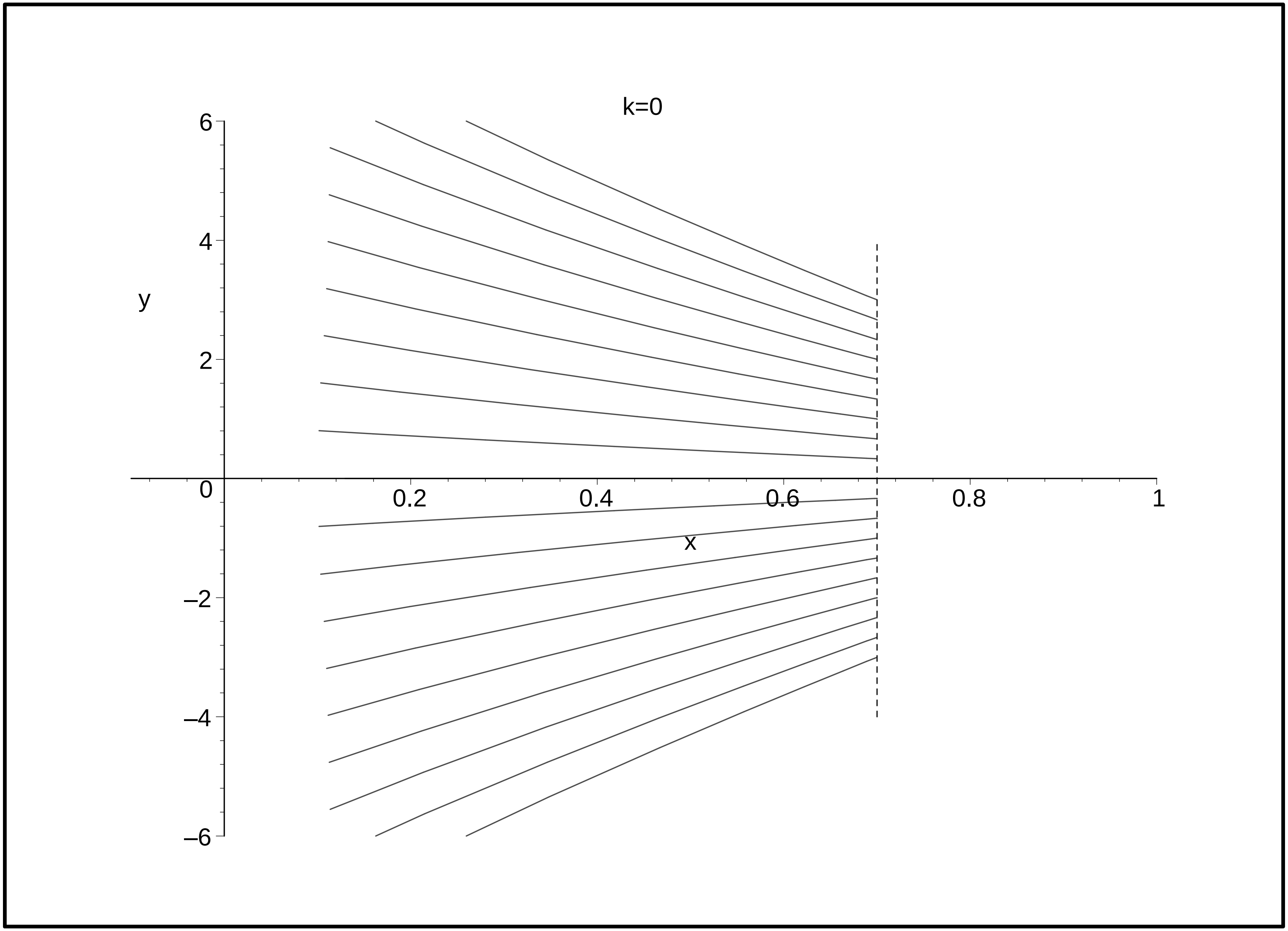
Figures

FIG. 1: For $k = 0$, 2D (x, y) plot of the geodesics. Geodesics start from $\left\{ x_0 = 0.7,\ y_0 = -3 + \sum_{j=0}^{18} \frac{j}{3} \right\}$ (dashed line) and move through non-intersected trajectories. The non-intersected trajectories represent the regular motion.

FIG.2: In case $k > -\frac{1}{2}e^{-\frac{7}{8}}$, $k = 1$ is chosen for the 2D (x, y) plot of the geodesics.The initial positions are $\left\{ x_0 = 0.7,\ y_0 = -3 + \sum_{j=0}^{18} \frac{j}{3} \right\}$ (dashed line). Intersected trajectories signal the existence of chaos. Two trajectories, which have different y_0 initial points initially accelerate in $+x$-direction contrary to the others.

FIG. 3: In case $k < -\frac{1}{2}e^{-\frac{7}{8}}$, $k = -0.21$ $(-\frac{4}{3}e^{-\frac{3}{2}} < -0.21 < -\frac{1}{2}e^{-\frac{7}{8}})$ is chosen 2D (x, y) plot of the geodesics.The initial positions are $\left\{ x_0 = 0.7,\ y_0 = -3 + \sum_{j=0}^{18} \frac{j}{3} \right\}$ (dashed line). The chaotic behavior is evident from the trajectories. The symmetry in Eq. (13) shows itself along the y-axis.

FIG. 4: Poincaré sections of (x', x) for $k = -0.21$ and $H = 0.2$ across $y = 0$ KAM surface. Some points are distributed randomly in a finite region to form a chaotic sea, however the large island surrounded by the chaotic sea indicates the existence of quasi-periodic orbits. (Here, $x \rightarrow q1$, and $x' \rightarrow p1$)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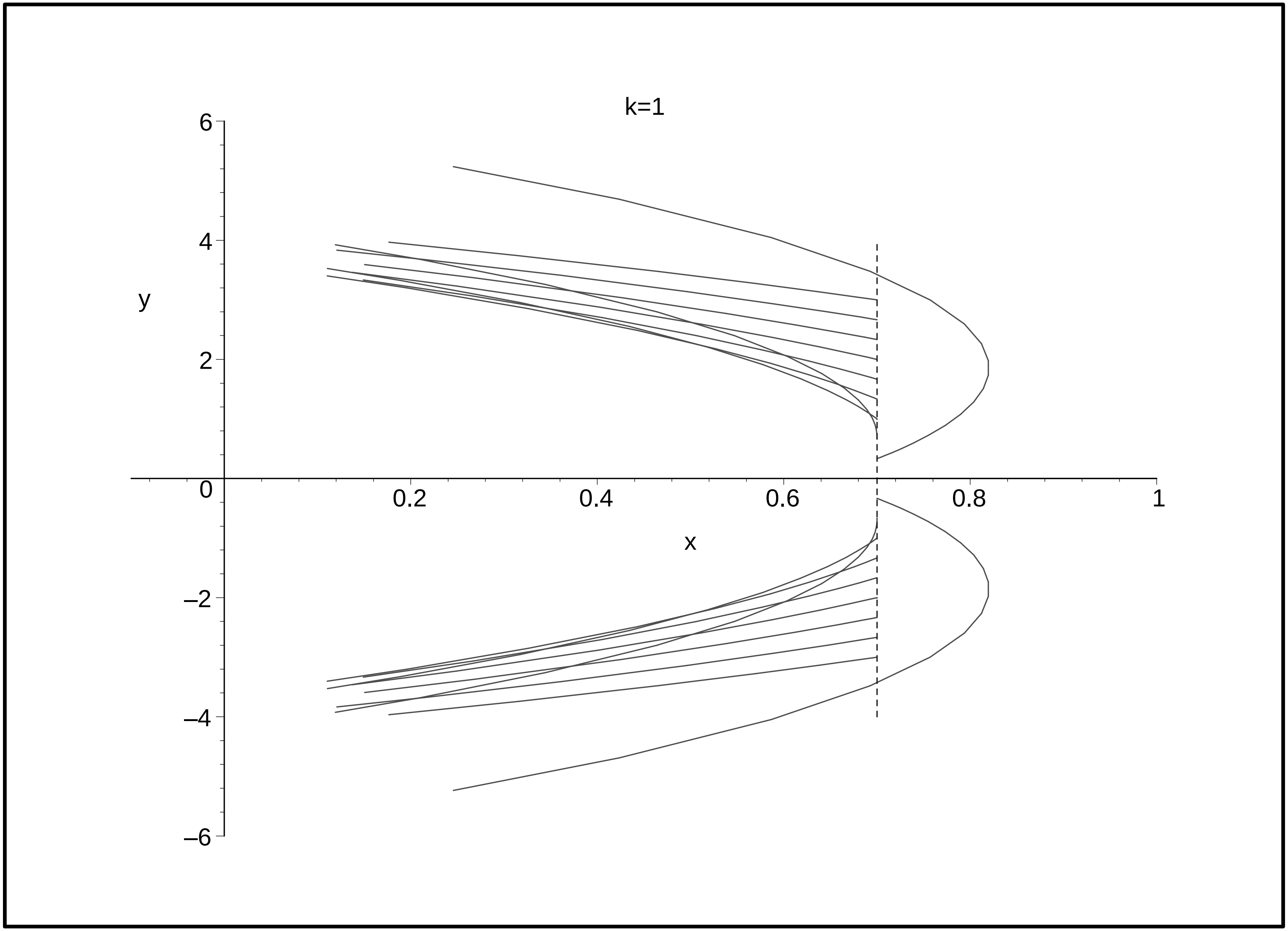
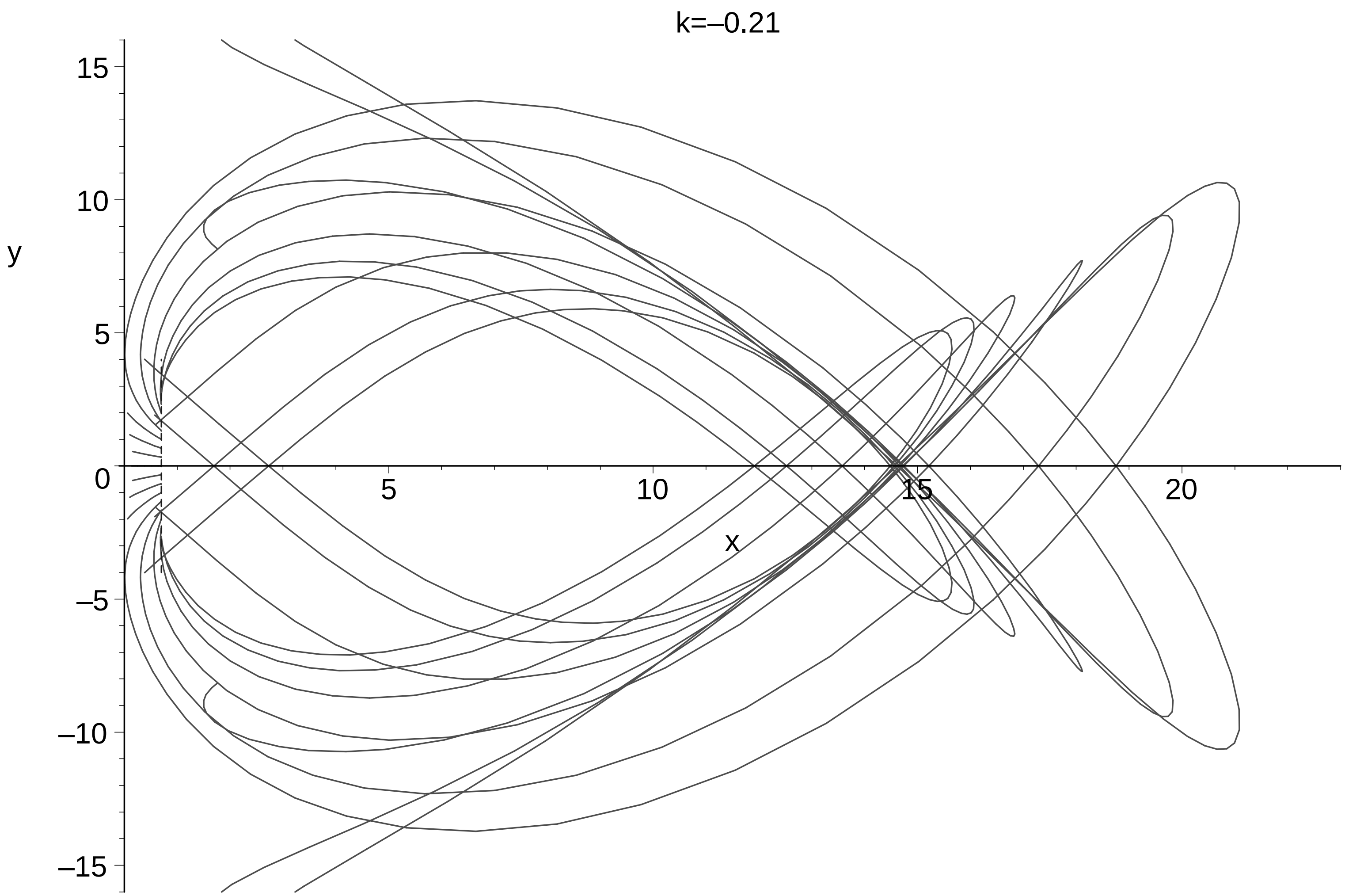
4
2
0
–2
–4
–6
0.2
0.4
0.6
0.8
1



k=1
y
6
4
2
0
–2
–4
–6
0.2
0.4
0.6
0.8
1
x



k=–0.21
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x
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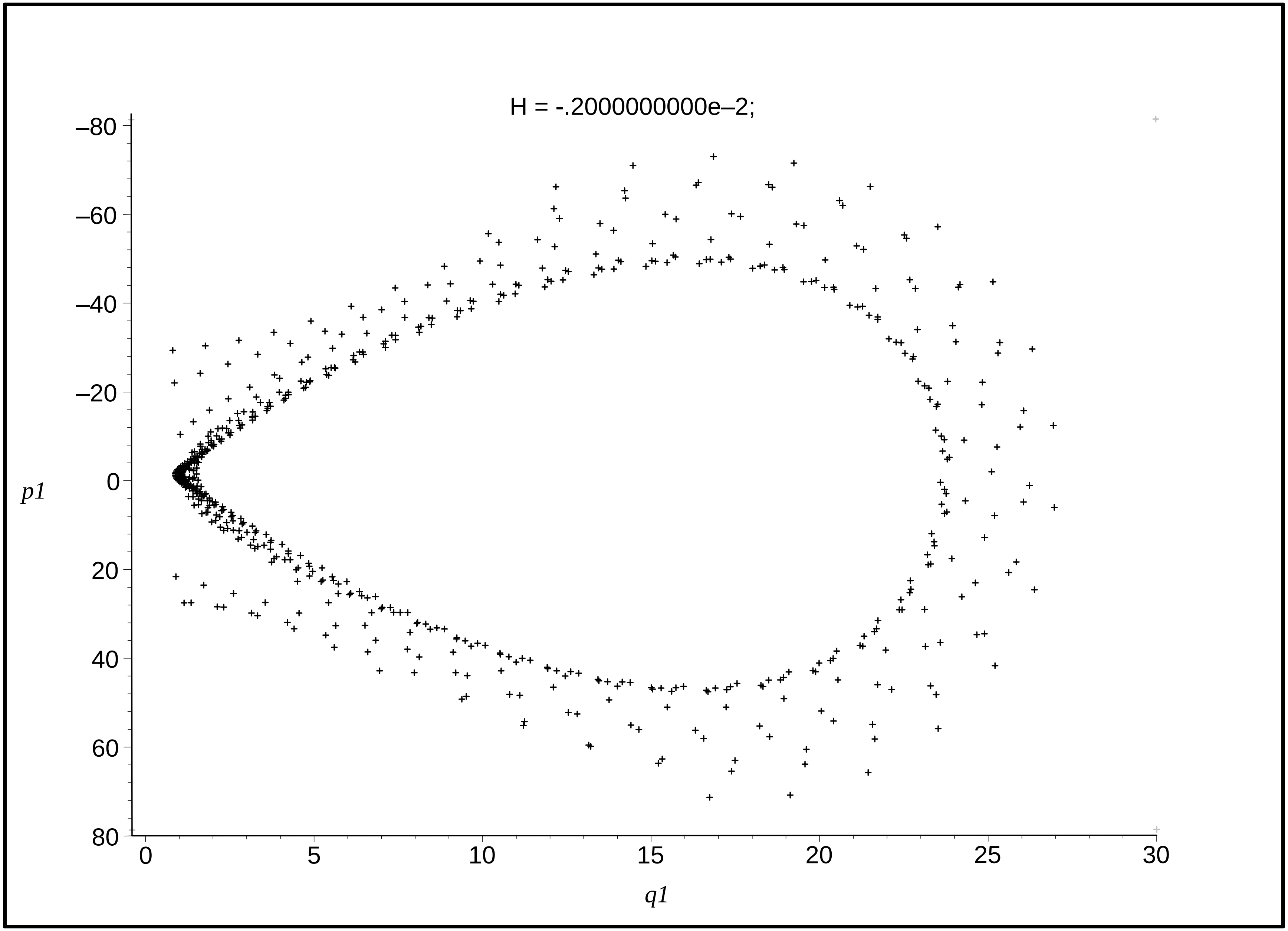
10
5
0
–5
–10
–15
5
10
15
20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–60
–40
–20
0
20
40
60
80
p1
0
5
10
15
20
25
30
q1