Domain walls in Einstein-Gauss-Bonnet bulk

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(Received 16 August 2010; revised manuscript received 3 September 2010; published 13 October 2010)

We investigate the dynamics of a n-dimensional domain wall in a $n+1$-dimensional Einstein-Gauss-Bonnet bulk. Exact effective potential induced by the Gauss-Bonnet (GB) term on the wall is derived. In the absence of the GB term we recover the familiar gravitational and antiharmonic oscillator potentials. Inclusion of the GB correction gives rise to a minimum radius of bounce for the Friedmann-Robertson-Walker universe expanding with a negative pressure on the domain wall.

DOI: 10.1103/PhysRevD.82.087502 PACS numbers: 04.50.Gh, 04.50.Kd, 04.70.Bw

We consider a n-dimensional domain wall (DW) Σ in a $n+1$-dimensional bulk $\mathcal{M}$. This DW splits the background bulk into two $n+1$-dimensional spacetimes which will be referred to as $\mathcal{M}_\pm$. Here $\pm$ is assumed with respect to the DW. Our action of Gauss-Bonnet (GB) extended gravity is chosen as

$$S = \frac{1}{2\kappa^2}\int_{\mathcal{M}} d^{n+1}x\sqrt{-g}(R+\alpha\mathcal{L}_{\rm GB}) + \frac{1}{\kappa^2}\times\int_\Sigma d^n x\sqrt{-h}\{K\} + \int_\Sigma d^n x\sqrt{-h}\mathcal{L}_{\rm DW}, \quad (1)$$

in which $\mathcal{L}_{\rm DW} = -\sigma =$ constant is the Nambu-Goto form of the DW Lagrangian, and K is the extrinsic curvature of DW with $h = |g_{ij}|$. (Latin indices run over the DW coordinates while Greek indices refer to the bulk's coordinates). The GB Lagrangian $\mathcal{L}_{\rm GB}$ is given by

$$\mathcal{L}_{\rm GB} = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \quad (2)$$

with the GB parameter α. A variation of the action with respect to the space-time metric $g_{\mu\nu}$ yields the field equations

$$G^E_{\mu\nu} + \alpha G^{\rm GB}_{\mu\nu} = 0, \quad (3)$$

where

$$G^{\rm GB}_{\mu\nu} = 2(-R_{\mu\sigma\kappa\tau}R^{\kappa\tau\sigma}{}_\nu - 2R_{\mu\rho\nu\sigma}R^{\rho\sigma} - 2R_{\mu\sigma}R^\sigma_\nu + RR_{\mu\nu}) - \tfrac{1}{2}\mathcal{L}_{\rm GB}g_{\mu\nu}. \quad (4)$$

Our bulk metric is a $n+1$-dimensional static, spherically symmetric space-time,

$$ds_b^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2_{n-1}, \quad (5)$$

in which $f(r)$ is the only metric function to be determined and $d\Omega^2_{n-1}$ is the line element of S^{n-1}. Upon imposing the constraint

$$-f(a)\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{f(a)}\left(\frac{da}{d\tau}\right)^2 = -1, \quad (6)$$

with the DW position at $r = a(\tau)$, the DW's line element takes the form

$$ds^2_{\rm dw} = -d\tau^2 + a(\tau)^2 d\Omega^2_{n-1}. \quad (7)$$

This is the standard Friedmann-Robertson-Walker metric and its only degree of freedom is $a(\tau)$ in which τ is the proper time measured by the observer on the DW. Now, we wish to consider the rules satisfied by the DW as the boundary of $\mathcal{M}_\pm$. These boundary conditions are the generalized Israel conditions which correspond to the Einstein equations on the wall. [1]

The generalized Darmois-Israel junction conditions on Σ apt for the GB extension is [2]

$$-\frac{1}{\kappa^2}(\langle K^j_i\rangle - K\delta^j_i) - \frac{\alpha}{2\kappa^2}\langle 3J^j_i - J\delta^j_i + 2P^j_{imn}K^{mn}\rangle = S^j_i,, \quad (8)$$

where the surface energy-momentum tensor S_{ij} is given by [3]

$$S_{ij} = \frac{1}{\sqrt{-h}}\frac{2\delta}{\delta g^{ij}}\int d^n x\sqrt{-h}(-\sigma). \quad (9)$$

The form of the stress-energy tensor can be written as

$$S^j_i = -\sigma\delta^j_i \quad (10)$$

in which $\sigma =$ constant, stands for the wall tension (or energy density of the wall Σ). Considering the energy-momentum tensor in the form $S^j_i = {\rm diag}(-\rho, p, p, \ldots)$, we observe that $\sigma = \rho = -p$, and satisfies the weak energy condition. Here in (8) a bracket implies a jump across Σ. The divergence-free part of the Riemann tensor P_{abcd} and the tensor J_{ab} (with trace $J = J^a_a$) are given by [2]

$$P_{imnj} = R_{imnj} + (R_{mn}g_{ij} - R_{mj}g_{in}) - (R_{in}g_{mj} - R_{ij}g_{mn}) + \tfrac{1}{2}R(g_{in}g_{mj} - g_{ij}g_{mn}), \quad (11)$$

$$J_{ij} = \tfrac{1}{3}[2KK_{im}K^m_j + K_{mn}K^{mn}K_{ij} - 2K_{im}K^{mn}K_{nj} - K^2K_{ij}]. \quad (12)$$

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By employing these expressions through (8) and (10) we find the energy density and surface pressures for a generic metric function $f(r)$, with $r = a(\tau)$. The results are given by [4]

$$-\Delta(n-1)\left[\frac{2}{a} - \frac{4\tilde{\alpha}}{3a3}(\Delta^2 - 3(1+\dot{a}^2))\right] = \kappa^2\sigma, \quad (13)$$

$$\frac{2(n-2)\Delta}{a} + \frac{2\ell}{\Delta} - \frac{4\tilde{\alpha}}{3a^2}\left[3\ell\Delta - \frac{3\ell}{\Delta}(1+\dot{a}^2) + \frac{\Delta^3}{a}(n-4) - \frac{6\Delta}{a}\left(a\ddot{a} + \frac{n-4}{2}(1+\dot{a}^2)\right)\right] = -\kappa^2\sigma, \quad (14)$$

where $\ell = \ddot{a} + f'(a)/2$ and $\Delta = \sqrt{f(a) + \dot{a}^2}$ in which

$$f(a) = f_-(r)|_{r=a}. \quad (15)$$

Note that a dot "·" implies derivative with respect to the proper time.

We differentiate (13) to get (with $\ddot{a} = \ell - f'(a)/2$)

$$\ell = \frac{\Delta^2}{a}\frac{a^2 + \tilde{\alpha}[2(af' - \Delta^2) + 6(1+\dot{a}^2)]}{a^2 + 2\tilde{\alpha}(\Delta^2 + 1 + \dot{a}^2)}, \quad (16)$$

which, after substitution into (14) we recover (13). In other words, Eqs. (13) and (14) are not independent, the solution of one satisfies also the other. Now, we analyze the first equation (13) of the junction conditions. By some manipulation, Δ above can be expressed in the form

$$\Delta = \sqrt[3]{\xi} - \frac{\epsilon}{3\sqrt[3]{\xi}}, \quad (17)$$

where

$$\xi = -\tfrac{1}{2}s \pm \tfrac{1}{18}\sqrt{12\epsilon^3 + 81s^2}, \quad (18)$$

$$s = \frac{3}{8}\frac{\kappa^2\sigma a^3}{\tilde{\alpha}(d-1)}, \qquad \epsilon = \frac{3}{2}\left(1 - f + \frac{a^2}{2\tilde{\alpha}}\right). \quad (19)$$

From $\Delta = \sqrt{f(a) + \dot{a}^2}$ and (17) it follows that

$$\dot{a}^2 + V(a) = 0, \quad (20)$$

where

$$V(a) = f - \left(\sqrt[3]{\xi} - \frac{\epsilon}{3\sqrt[3]{\xi}}\right)^2. \quad (21)$$

In the sequel we consider the wall to be a classical one-dimensional particle which moves with zero total energy under the effective potential $V(a)$. It is clear from (20) that only $V(a) < 0$ has a physical meaning. By plotting $V(a)$ in terms of a we investigate the possible types of motion for the wall.

The metric function $f(r)$ is the solution of the Einstein equations in the $n+1-$ dimensional bulk, i.e., from Eq. (5). In terms of the Arnowitt-Deser-Misner mass and GB parameter $\tilde{\alpha} = (n-2)(n-3)\alpha$, the solution for $f(r)$ is [5]

$$f_\pm(r) = 1 + \frac{r^2}{2\tilde{\alpha}}\left(1 \pm \sqrt{1 + \frac{16\tilde{\alpha}M}{(n-1)r^n}}\right). \quad (22)$$

Here, the negative branch gives the correct limit of general relativity, i.e.,

$$\lim_{\tilde{\alpha}\to 0} f_-(r) = 1 - \frac{4M}{(n-1)r^{n-2}}, \quad (23)$$

$$\lim_{\tilde{\alpha}\to\infty} f_-(r) = 1. \quad (24)$$

For this reason we consider the negative branch solution, which means that $f(a) = f_-(a)$. Upon substitution of $f(a)$ in (21) we observe that

$$\lim_{\tilde{\alpha}\to\infty} V(a) = 1, \quad (25)$$

which corresponds to a nonphysical case [i.e. Eq. (20)] and

$$\lim_{\tilde{\alpha}\to 0} V(a) = V_0 = 1 - \frac{4M}{(n-1)a^{n-2}} - \frac{1}{4}\frac{\kappa^4 a^2\sigma^2}{(n-1)^2}. \quad (26)$$

This shows that vanishing of the GB parameter yields a potential on the DW which contains a gravitational and antiharmonic oscillator potentials. The exact potential (with $\alpha \neq 0$), however, has a rather intricate structure which can be expanded in terms of the α as

$$V(a) = V_0 + V_1\alpha + V_2\alpha^2 + \ldots \quad (27)$$

for V_0 was given in Eq. (26)

$$V_1 = \frac{(n-2)(n-3)}{(n-1)^2}\left(\frac{\sigma^4\kappa^8 a^2}{6(n-1)^2} + \frac{4M\sigma^2\kappa^4}{(n-1)a^{n-2}} + \frac{16(n-3)M^2}{a^{2(n-1)}}\right), \quad (28)$$

and

$$V_2 = \frac{(n-2)^2(n-3)^2}{(n-1)^3}\left(-\frac{7}{36}\frac{\sigma^6\kappa^{12}a^2}{(n-1)^3} - \frac{20}{3} \times \frac{M\sigma^4\kappa^8}{(n-1)^2 a^{n-2}} - \frac{64M^2\sigma^2\kappa^4}{(n-1)a^{2(n-1)}} - \frac{128M^3}{a^{3n-25}}\right). \quad (29)$$

In Figs. 1–3 we display $V(a)$ and $f(a)$ for $\kappa^2 = 1$, $\sigma = 1$, $n = 4$, with changing α and M. For different α and M values we may obtain similar plots, such as for example $2a$ and $3c$. This implies that the effect of α may be compensated with that of M and vice versa. Once inside the event horizon of the black hole the DW has no chance but crush to the central singularity as it should. This is the ultimate fate of our DW universe if it lies inside a large black hole. For favorable condition of the potential [i.e. $V(a) < 0$] and in the vicinity (outside) of the horizon the DW collapses into the black hole much like shells [6]. The overall view, however, whether we have a black hole or not is that the potential provides a minimum bounce for the DW which is determined by the GB parameter α.

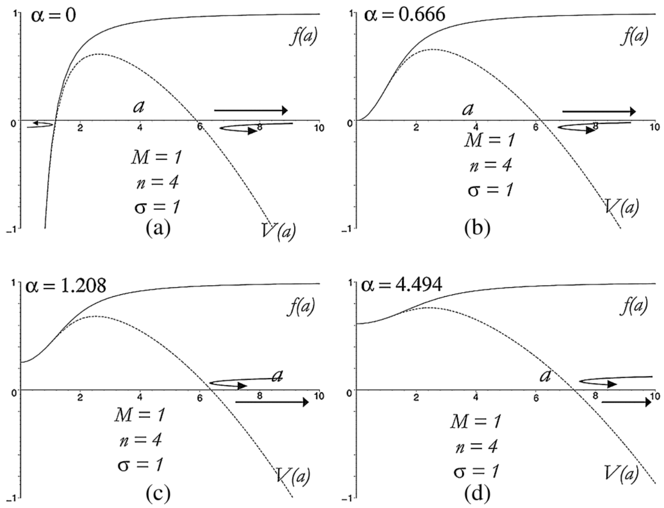

FIG. 1. For 5-dimensional bulk, we have our DW as 4-dimensional FRW universe. With increasing α the bouncing radius of the DW universe increases also. Fig. 1(a) is a black hole, while Fig. 1(b) can be interpreted as a pointlike black hole. Figs 1(c) and 1(d) are non-black hole cases with differing bounce radii. The arrows show the possible motions of DW including bounces.

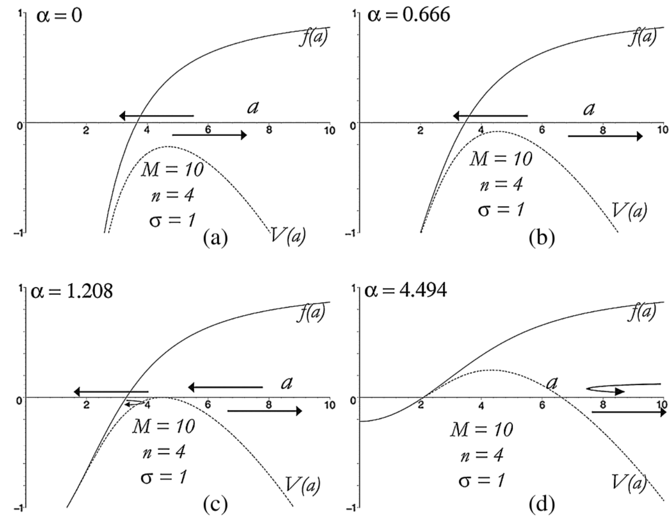

FIG. 3. The minimum bounces of the DW universe in 3(a) and 3(b) occur at the horizon so that the DW collapses into the black hole. In 3(c) we have also $V(a_c) = 0$. For $a > a_c$, the bounce does not occur at a_c. For $a < a_c$ the DW collapses into the black hole while in 3(d), it has no chance to fall into the black hole. Once inside the horizon, its fate ends at the central singularity.

We should also add that in our analysis we were unable to see a maximum bounce. This implies that the GB extension of general relativity does not suffice to provide a closed universe on DW.

Figure 4 plots the same quantities in $n = 5$, for comparison with the previous ones in $n = 4$. What we observe is that going into higher dimensions does not change the general features except that some nonblack hole cases will turn into black holes. We should remark also that although the coupling constant σ between the bulk and DW has been fixed as $\sigma = 1$, its effect can be investigated by taking different values for σ. In general, larger σ results smaller bouncing radii and vice versa.

In conclusion, if our 4-dimensional universe, assumed as a Friedmann-Robertson-Walker universe on a DW laying in a 5-dimensional Einstein-Gauss-Bonnet (EGB) bulk, the

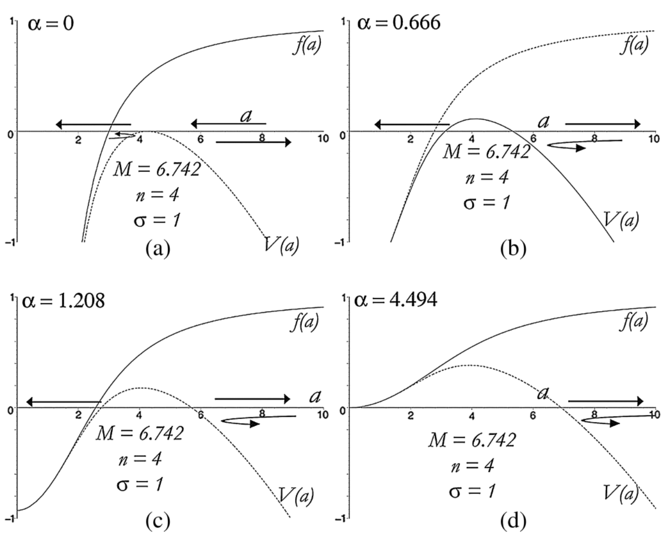

FIG. 2. Beside the minimum bouncing radii in the DW the smaller region between the horizon and allowable potential may have a DW which has no chance other than collapsing into the black hole. Fig. 2(a) has a critical radius a_c for which $V(a_c) = 0$. The nature of a DW inside the black hole of course changes, since it turns into a dynamic and collapsing object toward the central singularity. This occurs in 2(a) and 2(b) more clearly. Fig. 2(d) is similar to 1(b), which means that the mass difference compensates with the difference in α.

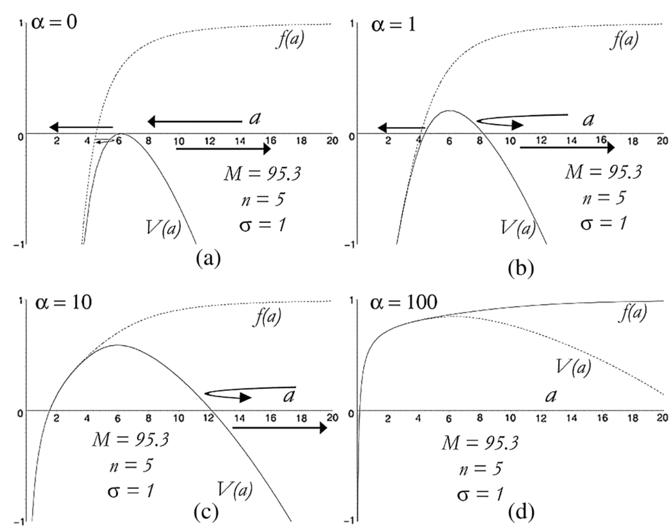

FIG. 4. For 6-dimensional bulk, we have DW as a 5-dimensional universe. Fig 4(a)/4(b) has a similar feature with 3(c)/3(d). Fig 4(c) is also similar to 2(b). Fig. 4(d) represents a black hole with a very small horizon but with a very large bouncing radius.

GB term protects us against the big crunch. Inclusion of physical fields such as Maxwell and Yang-Mills will definitely enrich our world on such a DW. Abiding by a bulk consisting of pure geometrical terms alone, however, the hierarchy of GB, known as the Lovelock gravity must be taken into account.

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