q-Matrix Summability Methods

Şerife Bekar

Submitted to the
Institute of Graduate Studies and Research
in partial fulfilment of the requirements for the Degree of

Doctor of Philosophy

in

Applied Mathematics and Computer Science

Eastern Mediterranean University
December 2010, Famagusta

Approval of the Institute of Graduate Studies and Research

Prof. Dr. Elvan Yılmaz
Director(a)

I certify that this thesis satisfies the requirements as a thesis for the degree of Doctor of Philosophy in Applied Mathematics and Computer Science

Prof. Dr. Agamizra Bashirov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion, it is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy in Applied Mathematics and Computer Science

Asst. Prof. Dr.Hüseyin Aktuğlu

Supervisor

Examining Committee

1. Prof. Dr.Agamizra Bashirov
2. Prof. Dr. Nazim Mahmudov
3. Prof. Dr. Uluğ Çapar
4. Assoc. Prof. Dr.Mehmet Ali Özarslan
5. Asst. Prof. Dr. Hüseyin Aktuğlu

TABLE OF CONTENTS

ABSTRACT **iv**

ÖZ **v**

ACKNOWLEDGEMENTS **vii**

1 INTRODUCTION **1**

2 NOTATION AND BACKGROUND MATERIAL **6**

2.1 Matrix Methods 6

2.2 Cesáro Methods 17

2.3 Hölder Methods 18

2.4 Riesz Methods (Weighted Means) 20

2.5 Hausdorff Methods 21

2.6 Density Functions 25

2.7 Statistical Convergence. 28

2.8 $q-$Integers 31

3 q-CESÁRO METHODS **34**

3.1 Construction and Some Properties of $q-$Cesáro Matrices 34

3.2 $q-$Density function and $q-$Statistical Convergence 45

4 q-HAUSDORFF METHODS **51**

4.1 Construction of q-Hausdorff Matrices 52

4.2 Some Summability Properties of $q-$Hausdorff Matrices 58

REFERENCES **61**

ABSTRACT

In this thesis, we mainly focus on q- analogs of matrix methods such as Cesáro, Hölder, Euler and Hausdorff methods. A summability method which is generated by an infinite matrix is called a matrix method. As it is well known the first order Cesáro summability method (C, 1), which is generated by the Cesáro matrix of order one, plays an important role in the theory of matrix summability methods. For this reason we first introduce a method to find q-analog of the Cesáro matrix of order one. By using the same method we also obtain q-analogs of Cesáro matrices of order α . Summability properties of $C_1(q^k)$, a natural q-analog of the first order Cesáro method are studied. Using $C_1(q^k)$, we define a q-density function and evaluate q-density of some subsets of $\mathbb{N}$. As an application of q-density function, q-statistical convergence which is stronger than statistical convergence is defined. In the last part, we use the relation between Cesáro and Hausdorff matrices to obtain the general form of q- Hausdorff methods. Also, we show that q- Cesáro and q-Hölder matrices can be obtained from the general form of q-Hausdorff matrices. Moreover, by using a q-analog of the generating sequence of Euler method, we can obtain a q-Euler method. Finally, we prove the general summability properties of q-Hausdorff methods.

Keywords: Matrix Summability Methods, Statistical Convergence, q-Integers, Cesáro Matrix, Hausdorff Methods, Density Functions.

ÖZ

Bu tezde esas olarak Cesáro, Hölder, Euler ve Hausdorff gibi matris metodlarının q-genelleştirmeleri üzerine yoğunlaşılmıştır. Bir sonsuz matris tarafından tanımlanan toplanabilirlik metoduna matris toplanabilirlik metodu denir. Birinci dereceden Cesáro matrisi tarafından üretilen, matris toplanabilirlik metodu (C, 1), matris metodlar teorisinde önemli bir rol oynamaktadır. Bu sebepten dolayı öncelikle birinci dereceden q- Cesáro matrislerini bulmak için bir metod verilmiştir. Bu methodu kullanarak $\alpha \in \mathbb{N}$ olmak üzere α. dereceden q- Cesáro matrislerinin genel formu elde edilmiştir. Birinci dereceden Cesáro matrisinin en doğal q-analoğu olarak görülen $C_1(q^k)$ 'nın bazı toplanabilirlik özellikleri verilmiştir. $C_1(q^k)$'yı kullanarak q-yoğunluk fonksiyonu tanımlandı ve bu yoğunluk fonksiyonu yardımı ile $\mathbb{N}$'nin bazı alt kümelerinin q-yoğunlukları hesaplandı. Ayrıca bu q-yoğunluk fonksiyonunun bir uygulaması olarak q-istatistiksel yakınsaklık kavramı verilmiştir. Burada tanımlanan q-istatistiksel yakınsaklığın istatistiksel yakınsaklıktan daha güçlü olduğu ispatlanmıştır. Son kısımda Cesáro ve Hausdorff matris metodları arasındaki ilişki kullanılarak q-Hausdorff matris metodlarının genel formu verilmiştir. Ayrıca bu genel formu kullanarak q- Cesáro ve q-Hölder metodlarının elde edilebildiği gösterilmiştir. Buna ek olarak Euler methodu üreten dizinin bir q-analoğu kullanılarak, bir q-Euler matrisi elde edilmiştir. Son olarak q- Hausdorff metodlarının bazı toplanabilirlik özellikleri ispatlanmıştır.

Anathar Kelimeler: Matris Toplanabilirlik Metodu, İstatistiksel Yakınsaklık, q-tamsayıları,

Cesáro Matrisi, Hausdorff Metodu, Yoğunluk Fonksiyonu.

ACKNOWLEDGEMENTS

First and foremost, I would like to give my deepest, sincerest thanks to my supervisor Asst. Prof. Dr. Hüseyin Aktuğlu for his guadiance, support and encouragement throughout my Phd. period. I will be forever grateful.

I wish to extend my thanks to all staff, especially to Prof. Dr. Agamirza Bashirov, Chairman of the Department of Mathematics.

My family has been absolutely wonderful throughout my entire life and I wish to thank them for everything they have given me. Finally, I would like to give my special thanks to my husband Mehmet Ünlüer whose infinite love, supported and helped me to overcome all problems and difficulties that I had faced during this period.

Chapter 1

INTRODUCTION

If we try to make the definition easier, summability theory is the theory of assignment of limits, which is fundamental in analysis, function theory, topology and functional analysis. The essential evolution of summability started in the end of the nineteenth century. Then, in the first half of the twentieth century summability methods were heavily researched. G. H. Hardy's classical book "Divergent Series [19]" is an important reference for the summability theory. Also historical overviews of the development of summability can be found in Kangro's survey paper [22], which covers the period 1969-1976. Another important reference of summability theory is the book of Johann Boss [6], which includes both Classical and Modern Methods of summability.

In the last thirty years, the study involving $q-$integers and their applications (for example, $q-$analogs of positive linear operators and their approximation properties) have become an active research area. During the same period a large number of research papers on $q-$analogs of existing theories, involving interesting results are published (see for example [4], [5], [20], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [39] and [44]).

The main motivation of the present study was the following question " What kind of results can be achieved by using the idea of $q-$ integers in summability theory?"

In this thesis we deal with summability in a focused manner, that is with assignment of limit to a real or complex sequences which is generated by an infinite matrix. We mainly focus on $q-$ analogs of some well known matrix methods. Since the Cesáro matrix plays and important role in the theory of matrix summability methods we start with $q-$analogs of Cesáro matrices. We obtain a method to find $q-$analogs of Cesáro matrices of order $\alpha \in \mathbb{N}$. We also discuss some summability properties of these $q-$matrix methods. Using the idea, parallel to the ordinary case, the $q-$analog of statistical convergence which is stronger than statistical convergence is defined. Finally using the relation between Cesáro matrices and Hausdorff matrices , we introduce the concept of $q-$ Hausdorff methods.

This thesis is organized as follows. In Chapter 1, which is introduction, we give the brief description of the whole work. In Chapter 2, we deal mainly with matrix summability methods such as Cesáro, Hölder, more generally Riesz and Hausdorff methods. We also introduce some of their basic summability properties.

One of the famous mathematicians Ernesto Cesáro introduced the Cesáro mean (t_n) of real or complex sequence $x = (x_j)$ as $t_n = \frac{x_0+x_1+\cdots+x_n}{n+1}$, $n = 0, 1, ...$. In the case of $\lim_{n\to\infty} t_n = t$, x is said to be Cesáro summable (or $(C, 1)$ summable) to t. The Cesáro methods have played a central role in connection with applications of summability theory to different branches of mathematics, especially in analysis. Most famous application of Cesáro summability is the following classic result due to Fejer which states that:

Let $\{S_n(x; f)\}$ be the sequence of partial sums of the Fourier series for the continuous function f and let $\{t_n(x; f)\}$ be the sequence of Cesáro means that is

$$t_n(x; f) = \frac{1}{n+1}\sum_{k=0}^{n} S_k(x; f).$$

Then $t_n(x; f)$ converges uniformly to f.

Briefly in this chapter,

- the general definition of summability and some basic definitions, related to summability method are given,
- the basic inclusion, comparison and consistency theorems of matrix summability methods are presented,
- some of basic properties of special matrix methods such as Cesáro (C_α), Hölder (H^α), more generally Riesz and Hausdorff are given.

Our contribution starts from Chapter 3. In this Chapter, we introduce a method to find $q-$analogs of Cesáro matrix of order $\alpha \in \mathbb{N},$ for all $q \in \mathbb{R}^+$. It is obvious that $q-$analogs of matrix methods is not unique. Let A be a matrix method then any matrix method $A(q),$ involving a real parameter q, is called a $q-$analog of A if $A(1) = A$. In [7], Bustoz and Gordillo suggested a method to define $q-$analog of Cesáro matrix of order one and they obtained the following $q-$analog $C_1(q) = (a_{nk}(q))$ where

$$a_{nk}(q) = \begin{cases} \frac{1-q}{1-q^{n+1}} q^{n-k} & if\ k \leq n \\ 0 & if\ k > n. \end{cases} \tag{1.0.1}$$

It should be mentioned that, the $q-$analog $C_1(q)$, obtained by Bustoz and Gordillo is valid only for $0 < q < 1$. In their approach they obtain a unique $q-$analog of Cesáro matrix of order one which contradicts with the idea of $q-$analogs given above. In this chapter we introduce a method which can be used to find different $q-$ analogs of the Cesáro matrices of order $\alpha \in \mathbb{N}$. In our approach all $q-$ analogs are valid for all $q \in \mathbb{R}^+$. Also, the $q-$ Cesáro method given in (1.0.1) can be obtained by using our method with an appropriate choice.

On the other hand statistical convergence which is a regular summability method is based on C_1, the Cesáro matrix of order one (see [13]). In the last decades statistical convergence played an important role in the literature and was investigated by several authors (see for example [8], [9], [10], [15], [16], [17], [23] and [38]). At the end of this chapter in a way parallel to [13], we define $q-$ statistical convergence.

The concept of statistical convergence can be extended to $A-$statistical convergence by using nonnegative regular summability matrix A. The concept of A-statistical convergence is examined in [3], [14], [16], [17], [36] and [37]. The $q-$ statistical convergence defined here is a type of $A-$ statistical convergence.

The content of this chapter can be summarize as follows,

- a method to find $q-$analogs of Cesáro matrices of order $\alpha \in \mathbb{N}$ is introduced,
- a $q-$analog of summation matrix is defined,
- some summability properties of our suggested $q-$analog of Cesáro matrix of order one are investigated,
- the density function δ_q, corresponding to the $q-$analog of Cesáro matrix of order one is defined,
- $q-$density of some sets are calculated,
- $q-$statistical convergence is defined.

In Chapter 4, we introduce $q-$ analogs of Hausdorff matrices. As it is well known the class of matrices permutable with C_1 are called Hausdorff matrices [18] and they play an essential role in application of summability methods.

In this chapter, we discuss the following items,

- definition of the $q-$analog of difference matrix Δ_q,

- general form of $q-$Hausdorff matrices,

- q-Hölder and $q-$Euler matrices,

- some properties of $q-$Hausdorff matrices.

Chapter 2

NOTATION AND BACKGROUND MATERIAL

In this chapter, we will summarize some basic definitions and primary properties of matrix summability methods which we need in this thesis. Detailed information about this topics can be found in [6]. Throughout this thesis, we will use the following common notations:

K := The set of all real numbers ($\mathbb{R}$) or complex numbers ($\mathbb{C}$).

$\mathbb{N}$:= The set of all natural numbers.

$\mathbb{N}^0 := \mathbb{N} \cup \{0\}$,

w := The set of all sequences.

m := The set of all bounded sequences.

c := The set of all convergent sequences.

c_0 := The set of all sequences converges to 0.

ϕ := The set of all finitely nonzero sequences.

l := The set of all absolutely summable sequences.

T := The set of all thin sequences. (A sequence $x = (x_k)$ is called thin, if there exist an index sequence (k_v) with $k_{v+1} - k_v \to \infty$ $(v \to \infty)$ and $x_k = 1$ if $k = k_v$ and $x_k = 0$ otherwise).

2.1 Matrix Methods

The definition of a general summability method can be given in the following way.

Definition 2.1.1. *A triple $V = (V, N_V, V - \lim)$ is called summability method which consisting of*

- *a map $V : D_V \to M$, where $D_V \subset w$ and M is a set such that at least on a suitable subset N, $\varnothing \neq N \subset M$, there exist (standard) limit functional $f : N \to K$,*
- *the domain $N_V := V^{-1}(N)$ of V and*
- *the summability functional $V - \lim := f \circ V|_{N_V} : N_V \to K$.*

Briefly, we can say that a summability method is a function whose domain is a subset of w and whose range is a subset of K. It is evident that basic and fundamental parts of summability methods are the domain and the summability map.

Example 2.1.1. *Let $D_V := w$ and consider the map $Z_{\frac{1}{2}}$ defined by*

$$\begin{aligned} Z_{\frac{1}{2}} &: w \to w, \\ x &= (x_k) \to (\frac{x_{n-1} + x_n}{2}). \end{aligned}$$

The domain of $Z_{\frac{1}{2}}$ is

$$c_{Z_{\frac{1}{2}}} = \Big\{ x \in w \mid Z_{\frac{1}{2}} x \in c \Big\} = Z_{\frac{1}{2}}^{-1}(c).$$

Then for every $x \in c_{Z_{\frac{1}{2}}}$, summability functional defined by

$$Z_{\frac{1}{2}} - \lim := \lim_{Z_{\frac{1}{2}}} := \lim \circ Z_{\frac{1}{2}} : c_{Z_{\frac{1}{2}}} \to K,$$

maps x to $\lim Z_{\frac{1}{2}} x$. Hence $(Z_{\frac{1}{2}}, c_{Z_{\frac{1}{2}}}, \lim_{Z_{\frac{1}{2}}})$ is a summability method.

Example 2.1.2. *Take $D_V := w$ and consider the map C_1, defined by*

$$\begin{aligned} C_1 &: w \to w, \\ x &= (x_k) \to (\frac{1}{n+1} \sum_{k=0}^{n} x_k)_n. \end{aligned}$$

The domain of C_1 is

$$c_{C_1} = \{x \in w \mid C_1 x \in c\} = C_1^{-1}(c).$$

Then for each $x \in c_{C_1}$ summability functional defined by

$$C_1 - \lim := \lim_{C_1} := \lim \circ C_1 : c_{C_1} \to K,$$

maps x to $\lim C_1 x$. Therefore $(C_1, c_{C_1}, \lim_{C_1})$ is also a summability method.

Next, we are going to present the basic definitions about inclusion, comparison and the consistency of general summability methods.

Definition 2.1.2. *The summability method $V = (V, N_V, V - \lim)$ is called conservative if $c \subset N_V$.*

Definition 2.1.3. *The summability method $V = (V, N_V, V - \lim)$ is called regular if $c \subset N_V$ and $V - \lim x = \lim x$ for all $x \in c$.*

Assume that $S := (S, N_S, S - \lim)$ and $R := (R, N_R, R - \lim)$ are two summability methods. Then we have the following definitions;

Definition 2.1.4. *S is stronger than R (R is weaker than S) if $N_R \subset N_S$ holds.*

Definition 2.1.5. *S and R are equivalent if $N_S = N_R$.*

Definition 2.1.6. *S and R are called consistent if $S - \lim x = R - \lim x$ for each $x \in N_S \cap N_R$.*

After the above general definition of summability methods, we are ready to define matrix summability method which will be the main interest of this thesis.

Definition 2.1.7. *Given an infinite matrix*

$$A = \begin{bmatrix} a_{00} & a_{01} & a_{02} & \dots & a_{0k} & \dots \\ a_{10} & a_{11} & a_{12} & \dots & a_{1k} & \dots \\ \vdots & \vdots & \vdots & \vdots & & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nk} & \dots \\ \vdots & \vdots & \vdots & \vdots & & \\ \end{bmatrix}$$

then

$$w_A = \left\{ x = (x_k) \in w |\ Ax := \sum_k a_{nk} x_k \textit{ converges for every } n \geq 0 \right\}$$

and

$$c_A := \{x \in w |\ Ax \in c\}$$

are called the application domain and convergence domain of A respectively. The summability method $A = (A, c_A, \lim_A)$ is called a matrix method where

$$\lim_A(x) = \lim Ax.$$

The above definition says that, each infinite matrix A determines a summability method, by using sequence to sequence transformation in which the sequence $x = (x_k)$ is transformed into the sequence $A(x) = (Ax)_n$ where

$$(Ax)_n := \sum_{k=1}^{\infty} a_{nk} x_k$$

provided that the series converges for each $n \in \mathbb{N}^0$.

Example 2.1.3. *The summability method $Z_{\frac{1}{2}}$, given in Example 2.1.1 can be considered* *as a matrix method generated by the infinite matrix*

$$Z_{\frac{1}{2}} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

It is easy to see that $Z_{\frac{1}{2}}$ is stronger than and consistent with the summability method corresponding to the identity matrix I.

Example 2.1.4. *The matrix method $Z_{\frac{1}{2}}$ and*

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & \dots & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+1} & \dots & \dots & \frac{1}{n+1} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

are not equivalent. Obviously

$$y = (y_k) = (1, 0, -1, 1, 0, -1, _{\dots}) \in c_{C_1}$$

with

$$\lim_{C_1} x = 0$$

but $x \notin c_{Z_{\frac{1}{2}}}$.

Example 2.1.5. *Consider the matrix method $Z_{\frac{1}{2}}$ and*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & \dots \\ -1 & 1 & 0 & 0 & 0 & \dots & \dots \\ 0 & -1 & 1 & 0 & 0 & \vdots & \vdots \\ 0 & 0 & -1 & 1 & 0 & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

One can easily see that, $x := (-1)^k \in c_{Z_{\frac{1}{2}}} \setminus c_A$ and $y := (k) \in c_A \setminus c_{Z_{\frac{1}{2}}}$ therefore this two matrix methods are not comparable.

Definition 2.1.8. *We say that a summability matrix A sums x to L (or x is $A-$summable to L) provided $\lim_{n\to\infty}(Ax)_n = L$.*

Example 2.1.6. *Let A be the infinite matrix given by*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

and let $x = (0, 1, 0, 1, \dots)$. Then $Ax = (1, 1, 1, \dots)$ and

$$\lim_{n\to\infty}(Ax)_n = 1.$$

Thus x is $A-$ summable to 1.

Before starting to discuss summability properties of matrix methods, we would like to mention some features of multiplication of infinite matrices. As it is well known, in general matrix multiplication operation is not associative for infinite matrices. Next theorems give us sufficient conditions to ensure that associativity of $A(Bx)$ and $A(BC)$ hold.

Theorem 2.1.1. *([6])Let A and B be two infinite matrices and $x = (x_k) \in w$. If*

(i) *$x \in w_B$ and $A = (a_{nk})$ is row finite (that is, $(a_{nk})_k \in \varphi$, $n \in \mathbb{N}^0$) or*

(ii) *$x \in m$, $\|B\| := \sup_\mu \sum_v |b_{\mu v}| < \infty$ and $(a_{nk})_k \in l$, for each $n \in \mathbb{N}^0$*

holds, then $A(Bx)$ and $(AB)x$ exist and $A(Bx) = (AB)x$.

Theorem 2.1.2. *([6])Let A, B and C be an infinite matrices. If*

(i) BC defined and A is row finite or

(ii) $\|B\| < \infty$, $(c_{vk})_v \in m$ and $(a_{nv})_v \in l$

holds, then $A(BC)$ and $(AB)C$ exist and $A(BC) = A(BC)$.

Definition 2.1.9. *Let A and B be two infinite matrices. If AB exist and $AB = I$, then A is called a left inverse of B, and B is called a right inverse of A. If in addition BA exist and $AB = BA = I$ holds, then the matrix B is called inverse of A. The inverse of A, if it exists, is denoted by A^{-1}.*

Definition 2.1.10. *A matrix $A = (a_{nk})$ is called (lower) triangular if $a_{nk} = 0$ ($k, n \in \mathbb{N}^0$ with $k > n$). A triangular matrix $A = a_{nk}$ with $a_{nn} \neq 0$ $(n \in \mathbb{N}^0)$ is called triangle.*

Theorem 2.1.3. *([6]) If A is triangle, then the following statements hold:*

(i) *For each $y \in w$, there exist a unique solution of system of equations $Ax = y$.*

(ii) *There exist unique right inverse B of A. Moreover, B is also triangle and left inverse. So A^{-1}exists.*

(iii) *The matrix A may have more than one left inverse, but there is exactly one that is also triangle, namely A^{-1}.*

Now, we turn our attention back to the matrix methods and present some inclusion, comparison and consistency theorems.

Definition 2.1.11. *The matrix A is said to be conservative if the convergence of the sequence implies the convergence of $A(x)$, (or equivalently $c \subset c_A$). In addition, if $A(x)$ converges to the limit of x, for each convergent sequence x, then it is called regular.*

The following theorem states the well known characterization of conservative matrices and can be found in any standard summability books (see for example [6], [42]).

Theorem 2.1.4. *(Kojima-Schur)An infinite matrix $A = (a_{nk})$ $n, k = 0, 1, 2, \ldots$ is conservative if and only if*

(i) (Column condition) $\lim_{n\to\infty} a_{nk} = \lambda_k$, for each $k = 0, 1, \ldots$

(ii) (Row sum condition) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \lambda$, and

(iii) (Row norm condition) $\sup_n \sum_{k=0}^{\infty} |a_{nk}| \leq M < \infty$, for some $M > 0$.

Here, of course the limits λ_k and λ are finite. If $\lambda_k = 0$, for all k and $\lambda = 1$ then the above theorem reduces to the well known theorem of Silverman and Toeplitz which provides necessary and sufficient conditions for regularity of the infinite matrix $A = (a_{nk})$ $n, k = 0, 1, 2, \ldots$.

Theorem 2.1.5. *(Silverman-Toeplitz, [42]) The summability matrix $A = (a_{nk})$ $n, k = 0, 1, 2, \ldots$ is regular if and only if*

(i) $\lim_{n\to\infty} a_{nk} = 0$, *for each* $k = 0, 1, \ldots$,

(ii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1$,

(iii) $\sup_n \sum_{k=0}^{\infty} |a_{nk}| \leq M < \infty$, for some $M > 0$

Example 2.1.7. *Matrix methods*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ 1-\frac{1}{n} & \frac{1}{n} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

and

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+1} & \cdots & \cdots & \frac{1}{n+1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

are conservative and regular matrix methods respectively.

Remark 2.1.6. *For any infinite matrix with nonnegative entries, the row sum condition implies the row norm condition.*

If A is conservative then for any sequence $x = (x_k) \in c$, we can use the following limit formula

$$\lim_A x = \chi(A) \lim x + \sum_k \lambda_k x_k$$

where

$$\chi(A) := \lim_n \sum_k a_{nk} - \sum_k \lim_n a_{nk} = \lambda - \sum_k \lambda_k.$$

The number $\chi(A)$ defined above is called the characteristic of A. Furthermore, if A is regular $\chi(A) = 1$.

Conservative matrix methods can be classified as coregular or conull as follows:

Definition 2.1.12. *Let A be a conservative matrix then A and corresponding matrix method obtained from A are called coregular if $\chi(A) \neq 0$ and it is called conull if $\chi(A) = 0$.*

Example 2.1.8. *If we consider matrices A and C_1, given in Example 2.1.7. The conservative matrix A is conull and, the regular matrix C_1 is coregular.*

Summability of various kinds of sequences investigated by famous names Agnew, Mazur, Orlicz, Zeller and Willansky. In 1933, Mazur and Orlicz proved the following celebrated theorem.

Theorem 2.1.7. *([11],) If a conservative matrix A sums a bounded divergent sequence, then it also sums an unbounded sequence. That is, $c_A \subset m$ implies $c_A = c$.*

By Theorem 2.1.7 and Definition 2.1.12, we can state the following corollary.

Corollary 2.1.1. *A conull matrix A must sum both bounded divergent sequences and unbounded sequences. That is $\chi(A) = 0$ implies $c \subsetneq m \cap c_A \subsetneq c_A$*

Following theorems are stated useful result on the comparison and consistency of matrix methods.

Theorem 2.1.8. *([6])Let A, B and C be infinite matrices with $B = CA$ such that $(CA)x$ and $C(Ax)$ exist and $(CA)x = C(Ax)$ for each $x \in c_A$. Then the following statements hold:*

(i) *If C is conservative, then B is stronger than A (that is, $c_A \subset c_B$).*

(ii) *If C is regular, then B is stronger than and consistent with A (that is, $c_A \subset c_B$ and $\lim_B |_{C_A} = \lim_A$).*

If C is row finite then associativity assumptions in Theorem 2.1.8 are satisfied. Thus, if we assume A is a triangle and B is row finite, automatically this implies that C is rowfinite. Therefore, Theorem 2.1.8 can be extended to the following stronger theorem.

Theorem 2.1.9. *([6])Let A be a triangle, B row finite and $C := BA^{-1}$. Then the following statements hold:*

(i) *B is stronger than A if and only if C is conservative.*

(ii) *B is stronger than and consistent with A if and only if C is regular.*

Definition 2.1.13. *Let A be a matrix with bounded columns. Then A is defined to be of type M if $tA = 0$ implies $t = 0$ for every $t \in l$.*

Theorem 2.1.10. *([6])A regular triangle $A = (a_{nk})$ is of type M if A^{-1} has bounded columns.*

Remark 2.1.11. *In particular, any triangle A for which A^{-1} is column finite (that is for all $k \in \mathbb{N}^0$, there exist $n_k \in \mathbb{N}^0$ such that $a_{nk} = 0$ for all $n \geq n_k$) is of type M.*

Corollary 2.1.2. *([6])Let A be a regular triangle of type M and let B be a regular triangular matrix. If $c_A \subset c_B$, then A and B are consistent.*

Definition 2.1.14. *Let $A = (a_{nk})$ be an infinite matrix then we say that A satisfies (or A enjoys) the mean value property with a constant $K > 0$ (or $M_K(A)$ for short) if*

$$\left|\sum_{k=0}^{r} a_{nk}x_k\right| \leq K \sup_{0\leq v\leq r}\left|\sum_{k=0}^{\infty} a_{vk}x_k\right| \tag{2.1.1}$$

where $0 \leq r \leq n \in \mathbb{N}^0$ and $(x_k) \in w_A$.

Condition 2.1.1 is called the mean value condition. The following theorem shows that the mean value property of matrices is sufficient for regular triangle to be of type M.

Theorem 2.1.12. *([6])A regular triangle is of type M if it enjoys the mean value property.*

After becoming familiar with inclusion, comparison and consistency results, we can start to discuss the theory of matrix methods by considering some specific matrix summability methods. In the rest of this chapter we shall discuss consevativity, regularity, mean value and type M properties for some well known matrix methods.

2.2 Cesáro Methods

Definition 2.2.1. *Let α be a real number with $-\alpha \notin \mathbb{N}$ then the regular matrices $C_\alpha := (c_{nk}^{\alpha})$ defined by*

$$c_{nk}^{\alpha} = \begin{cases} \dfrac{\begin{pmatrix} n-k+\alpha-1 \\ n-k \end{pmatrix}}{\begin{pmatrix} n+\alpha \\ n \end{pmatrix}} & if \;\; k \leq n, \;\; n,k = 0,1.. \\ 0 & otherwise \end{cases}$$

and the associated matrix summability methods, are called the Cesáro matrix and Cesáro summability method of order α respectively.

In particular if we choose $\alpha = 1$, we get the first order Cesáro matrix C_1 with the following explicit form,

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \frac{1}{n+1} & \frac{1}{n+1} & \frac{1}{n+1} & \cdots & \frac{1}{n+1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}. \tag{2.2.1}$$

Corresponding summability method is called the first order Cesáro summability method and denoted by $(C, 1)$. The following theorem is the direct result of the theorem of Silverman and Toeplitz which provides necessary and sufficient conditions for regular matrices.

Theorem 2.2.1.

(i) *If $\alpha \geq 0$, then C_α is regular*

(ii) *If $\alpha < 0$, C_α is not conservative or regular.*

The following result shows us monotonicity of the C_α methods.(see [18]).

Theorem 2.2.2. *For all $\alpha, \beta \in \mathbb{R}$ satisfying $-1 < \alpha \leq \beta$, the method C_β is stronger than and consistent with C_α, that is $c_{C_\alpha} \subset c_{C_\beta}$ and $\lim_{C_\alpha} x = \lim_{C_\beta} x$ for each $x \in c_{C_\alpha}$.*

Theorem 2.2.3. *For any $\alpha \geq 0$, the matrix C_α is of type M.*

Theorem 2.2.4. *The matrix C_1 satisfies the mean value property with $K = 1$, whereas for each $\alpha > 1$, the matrix C_α does not enjoy the mean value property. Moreover, for any $\alpha \in (0, 1]$, the Cesáro matrix C_α satisfies $M_1(C_\alpha)$.*

2.3 Hölder Methods

The Hölder matrix , $H^\alpha(\alpha \in \mathbb{N}^0)$ can be obtained from the Cesáro matrix of order one by iteration. A useful feature of the H^α is that, most of the properties can be derived from the corresponding properties of C_1. But we have to use direct handling to get its matrix coefficients, because of there is no simple formula for H^α matrix coefficients.

Definition 2.3.1. *Let C_1 be the Cesáro matrix of order one and $\alpha \in \mathbb{N}^0$. Then*

$$H^\alpha := (C_1)^\alpha \text{ that is } H^0 = I \text{ and } H^\alpha := C_1 H^{\alpha-1} \; (\alpha \geq 1)$$

and the associated matrix summability methods are called Hölder matrix and Hölder method of order α respectively.

Remark 2.3.1. *In general, the Hölder and Cesáro methods are different from each other. Moreover the multiplication of two Cesáro matrix is not a Cesáro Matrix.*

Using above definition and Theorem 2.1.2, we can state the following properties of H^α.

- $H^1 = C_1$,
- H^α is well defined and a triangle as a product of triangles,
- $H^\alpha = H^{\alpha-1}C_1$ for each $\alpha \in \mathbb{N}$,
- $H^{\alpha+\beta} = H^\alpha H^\beta$ for all $\alpha, \beta \in \mathbb{N}^0$.

Theorem 2.3.2. *([6])For each $\alpha \in \mathbb{N}^0$, the method $H^{\alpha+1}$ is strictly stronger than and consistent with H^α, in other words*

$$c_{H^\alpha} \subsetneq c_{H^{\alpha+1}} \textit{ and } \lim_{H^\alpha} x = \lim_{H^{\alpha+1}} x$$

for each $x \in c_{H^\alpha}$. In particular H^α is a regular summability method.

Theorem 2.3.3. *([6])For any $\alpha \in \mathbb{N}^0$, the matrix H^α is of type M.*

Knopp and Andersen proved the following results which shows us the relation between Hölder methods and Cesáro methods.

Theorem 2.3.4. *For each $\alpha \in \mathbb{N}$, the methods C_α and $C_{\alpha-1}C_1$ are equivalent and consistent.*

Theorem 2.3.5. *For each $\alpha \in \mathbb{N}^0$, the methods H^α and C_α are equivalent and consistent.*

2.4 Riesz Methods (Weighted Means)

One can easily observe that, the sum of each row of C_1 is exactly 1. By this point of view, we can consider the following class of matrix methods which is called Riesz methods. Obviously this class is a generalization of the first order Cesáro method and gives us a simple way to define regular matrices and their inverses.

Definition 2.4.1. *Let $p = (p_k)$, be a sequence of real numbers with $p_0 > 0, p_k \geq 0$, $k \in \mathbb{N}$ and $P_n = \sum\limits_{k=0}^{n} p_k$, then the matrix method $R_p = (r_{nk})$ defined by*

$$r_{nk} = \begin{cases} \frac{p_k}{P_n} & if \;\; k \leq n, \\ 0 & otherwise \end{cases} \quad n, k = 0, 1, ...$$

is called a Riesz matrix (or weighted mean) associated with the sequence p .The corresponding matrix method is called Riesz Method associated with the sequence p.

Example 2.4.1. *C_1 is a weighted mean associated with $e = (1, 1, \ldots)$.*

Theorem 2.4.1. *([6])If $p_n > 0$ for each $n \in \mathbb{N}^0$, then the inverse $R_p^{-1} = (\widehat{r}_{nk})$ of R_p is given by*

$$\widehat{r}_{nk} := \begin{cases} \frac{P_n}{p_n} & if \; k = n \\ -\frac{P_{n-1}}{p_n} & if \; k = n-1 \\ 0 & otherwise \end{cases} \left(n, k \in \mathbb{N}^0\right).$$

Theorem 2.4.2. *([6]) Riesz matrix (or method) R_p, enjoys the mean value property with $K = 1$.*

Theorem 2.4.3. *([6])If $p_n > 0$ for each $n \in \mathbb{N}^0$, then R_p is of type M.*

Applying the Silverman-Toeplitz Theorem to Riesz matrices, we get that each Riesz method is conservative. The following theorem gives us a simple characterization for Riesz methods to be regular.

Theorem 2.4.4. *Let R_p be a Riesz matrix (or Riesz Method) then*

(i) *R_p is conservative.*

(ii) *R_p is regular if and only if $P_n \to \infty$ when $n \to \infty$.*

The following theorem is an important tool to compare, conservative and Riesz matrices.

Theorem 2.4.5. *([6]) Let R_p be a regular Riesz method with $p_k > 0$ for $k = 0, 1, \ldots$ and let $A = (a_{nk})$ be a conservative matrix method. Then A is stronger than R_p if and only if the following conditions hold:*

$i)$ $\lim_{k\to\infty}\left(\frac{a_{nk}}{p_k}\right) = 0$, $n = 0, 1, \ldots$

$ii)$ $\sup_n \sum\limits_k P_k \left|\frac{a_{nk}}{p_k} - \frac{a_{n,k+1}}{p_{k+1}}\right| < \infty$.

2.5 Hausdorff Methods

The class of Hausdorff methods, includes Hölder, Cesáro, Euler and some other matrix methods which plays an essential role in summability theory. Before giving the details of this method, we would like to give the definition of some terms, related with matrices.

Definition 2.5.1. *A matrix D is called a diagonal matrix, provided that each of its elements is zero except those on the diagonal; that is*

$$D = (p_n \delta_{mn})$$

where $\delta_{mn} = \begin{cases} 1; & k = m \\ 0, & k \neq m \end{cases}$ *is Kronecker delta.*

We say, further that a matrix A is reduced to diagonal form by the triangular matrix P provided that

$$PAP^{-1} = (p_n \delta_{mn}) = D.$$

Definition 2.5.2. *Let A and B be two matrices, if $AB = BA$ then A and B are said to be permutable .*

Definition 2.5.3. *The self inverse matrix* $\Delta = (\Delta_{nv})$ *where*

$$\Delta_{nv} := \begin{cases} (-1)^v \binom{n}{v} & \text{if } 0 \leq v \leq n \\ 0 & \text{otherwise} \end{cases}$$

is called the difference matrix.

Remark 2.5.1. Δ *can be obtained by solving the matrix equations*

$$\Delta C_1 \Delta^{-1} = (p_n \delta_{mn})$$

where $p_n = \frac{1}{n+1}$*(see [18]), this says us the Cesáro matrix* C_1 *is reduced to diagonal form by the matrix* Δ.

Theorem 2.5.2. *(see [18]) A necessary and sufficient condition for a triangular matrix* A *to be permutable with* C_1 *is that, it can be reduced to diagonal form by the matrix* Δ*, that is,*

$$D = \Delta A \Delta \text{ or } A = \Delta D \Delta.$$

The following definition shows us the class of matrices which are permutable with C_1. This method can also be viewed as generalization of the first order Cesàro method.

Definition 2.5.4. *Let* $p = (p_n) \in w$ *be any sequence then the matrix defined by;*

$$H_p = (h_{nk}) := (H, p_n) := \Delta^{-1} D \Delta$$

with coefficients

$$h_{nk} = \begin{cases} \binom{n}{k} \sum\limits_{v=0}^{n-k} (-1)^v \binom{n-k}{v} p_{v+k} & if \;\; 0 \leq k \leq n \\ 0 & otherwise \end{cases} \tag{2.5.1}$$

is called a Hausdorff matrix where D *is the diagonal matrix with diagonal elements* $p_n \in w$ *and associated matrix method is called the Hausdorff method generated by the sequence* p.

Example 2.5.1. *Taking* $p_n = \frac{1}{\binom{n+\alpha}{n}}$ *in (2.5.1), gives exactly the Cesáro matrix of order* α*. Therefore the Cesáro matrix of order* α *is a Hausdorff matrix.*

Example 2.5.2. *By Choosing* $p_n = \left(\frac{1}{n+1}\right)^\alpha$ *in (2.5.1), gives exactly the Hölder matrix of order* α *which means the Hölder matrix of order* α *is a Hausdorff matrix as well.*

Example 2.5.3. *Let us use*

$$p_n = \alpha^n$$

in (2.5.1), we obtain the Euler-Knopp matrix of order α *which is given by:*

$$E_\alpha := (e_{nk}^{(\alpha)}) := (H, \alpha^n)$$

with

$$e_{nk}^{(\alpha)} := \binom{n}{k}\alpha^k(1-\alpha)^{n-k} \quad (k \leq n). \tag{2.5.2}$$

The explicit form of E_α *is:*

$$E_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ (1-\alpha) & \alpha & 0 & 0 & 0 & 0 & \cdots \\ (1-\alpha)^2 & 2\alpha(1-\alpha) & \alpha^2 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ (1-\alpha)^n & n\alpha(1-\alpha)^{n-1} & \frac{n(n-1)}{2}\alpha^2(1-\alpha)^{n-2} & \cdots & \alpha^n & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Some basic properties of Euler matrix are given below (see [1] and [6])

Proposition 2.5.1. $i)$ E_α *is conservative for* $0 \leq \alpha \leq 1$.

$ii)$ E_α *is regular for* $0 < \alpha \leq 1$

Proof. $i)$ Since

$$\sum_{k=0}^{n}\binom{n}{k}\alpha^k(1-\alpha)^{n-k} = (\alpha + 1 - \alpha)^n = 1.$$

E_α satisfies the row sum condition with $\lambda = 1$. Moreover using the equation

$$\sum_{k=0}^{n}\left|\binom{n}{k}\alpha^k(1-\alpha)^{n-k}\right| = (|\alpha| + |1-\alpha|)^n \text{ for each } n \in \mathbb{N}^0,$$

one can see that, E_α satisfies the row norm condition if and only if $0 \leq \alpha \leq 1$. On the other hand, if λ_k denotes the limit of the k^{th} column when $n \to \infty$, then a simple calculation shows that for each $0 \leq \alpha \leq 1$ and $k \geq 0,\ \lambda_k$ is exist and finite which means E_α is conservative for $0 \leq \alpha \leq 1$.

$ii)$ It follows from the fact that $\lambda_k = 0$ for all $k \geq 0$ if and only if $0 < \alpha \leq 1$. $\square$

Proposition 2.5.2. $i)$ $E_\alpha.E_\beta := E_{\alpha\beta}$.

$ii)$ *The inverse of* E_α *is* $E_{\frac{1}{\alpha}}$.

Proof. $i)$ Let $s = (s_n)$ be the E_α transformation of $E_\beta x$ for any sequence $x = (x_k)$ then

$$\begin{aligned}
s_n &= \sum_{k=0}^{n} \binom{n}{k} \alpha^k (1-\alpha)^{n-k} \sum_{m=0}^{k} \binom{k}{m} \beta^m (1-\beta)^{k-m} x_k \\
&= \sum_{m=0}^{n} \binom{n}{m} (\alpha\beta)^m \sum_{k=m}^{n} \binom{n-m}{k-m} (\alpha-\alpha\beta)^{k-m} (1-\alpha)^{n-k} x_k \\
&= \sum_{m=0}^{n} \binom{n}{m} (\alpha\beta)^m \sum_{k=0}^{n-m} \binom{n-m}{k} (\alpha-\alpha\beta)^{k} (1-\alpha)^{n-m-k} x_k \\
&= \sum_{m=0}^{n} \binom{n}{m} (\alpha\beta)^m (1-\alpha\beta)^{n-m} x_k.
\end{aligned}$$

Thus the transformation $E_\alpha E_\beta$ is identical with the transformation $E_{\alpha\beta;}$ that is

$$E_\alpha.E_\beta := E_{\alpha\beta.}$$

$ii)$ From part $i)$ $E_\alpha E_{\frac{1}{\alpha}} = E_{\frac{1}{\alpha}} E_\alpha = E_1 = I$, the identity matrix. $\square$

Proposition 2.5.3. *For* $0 < \beta \leq \alpha$, *the method* E_β *is stronger than and consistent with* E_α.

Corollary 2.5.1. *Euler methods are monotone for* $0 < \alpha < \infty$.

The next proposition gives the general properties of all Hausdorff matrices.

Proposition 2.5.4. *Let H_p and H_q be Hausdorff matrices. Then*

(a) *p_n is the coefficient of H_p in the n^{th} position of its diagonal (that is $h_{nn} = p_n$ $(n \in \mathbb{N}^0)$),*

(b) $H_p + H_q = H_{p+q}$,

(c) $H_p H_q = (H, p_n q_n) = H_q H_p$,

(d) $(H_p)^{-1}$ *exist if and only if $p_n \neq 0$. If it exist, $(H_p)^{-1} = (H, p_n^{-1})$.*

(e) *Let $p = (p_n)$ with $p_n \neq p_k$ $(n \neq k)$ be given and let $A = (a_{nk})$ be a lower triangular matrix. Then A is Hausdorff matrix if and only if $AH_p = H_p A$.*

(f) *Regular Hausdorff methods are pairwise consistent,*

(g) *If H_p and H_q be Hausdorff matrices and H_p is triangle, then the following statements hold:*

(i) H_q is stronger than H_p if and only $(H, \frac{q_n}{p_n})$ is conservative.

(ii) H_q is stronger than and consistent with H_p if and only if $(H, \frac{q_n}{p_n})$ is regular.

2.6 Density Functions

A density is a set function satisfying some specific conditions.

Definition 2.6.1. *Let A, B be two subsets of $\mathbb{N}$, the symmetric difference of A and B is denoted by $A \bigtriangleup B$ and defined as be*

$$A \bigtriangleup B = (A \backslash B) \cup (B \backslash A).$$

If the symmetric difference of two sets A and B is finite then we say A and B has "$\sim$" relation. i.e. $A \sim B$ if and only if $A \bigtriangleup B$ is finite.

Definition 2.6.2. *Function δ, defined for all sets of natural numbers and taking values in the closed interval $[0, 1]$, will be called a lower asymptotic density (or just a density) if the following four axioms hold:*

$(d.1)$ $if\ \ A \sim B\ \ then\, \delta(A) = \delta(B)\,;$
$(d.2)$ $if\ A \cap B = \emptyset,\ then\, \delta(A) + \delta(B) \leq \delta(A \cup B)\,;$
$(d.3)$ $for\ all\ \ A, B;\ \delta(A) + \delta(B) \leq 1 + \delta(A \cap B)\,;$
$(d.4)$ $\delta(\mathbb{N}) = 1.$

Definition 2.6.3. *For a density δ we define $\overline{\delta}$, the upper density associated with δ, by*

$$\overline{\delta}(A) = 1 - \delta(\mathbb{N} \backslash A)$$

for any set of natural numbers A.

Proposition 2.6.1. *Let δ be an asymtotic density and $\overline{\delta}$ its associated upper density. For sets A and B of natutal numbers we have*

(i) $A \subseteq B \implies \delta(A) \leq \delta(B)\,;$

(ii) $A \subseteq B \implies \bar{\delta}(A) \leq \bar{\delta}(B)\,;$

(iii) *for all* $A, B,\ \bar{\delta}(A) + \bar{\delta}(B) \geq \bar{\delta}(A \cup B)\,;$

(iv) $\delta(\varnothing) = \overline{\delta}(\varnothing)\,;$

(v) $\bar{\delta}(\mathbb{N}) = 1;$

(vi) $A \sim B \implies \bar{\delta}(A) = \bar{\delta}(B)\,;$

(vii) $\delta(A) \leq \bar{\delta}(A)\,.$

Definition 2.6.4. *A subset $A \subseteq \mathbb{N}$ is said to have natural density with respect to δ, if $\delta(A) = \overline{\delta}(A)\,.$*

Now consider

$$\eta_\delta = \left\{A : \delta(A) = \overline{\delta}(A)\right\} \text{ and } \eta_\delta^0 = \left\{A : \overline{\delta}(A) = 0\right\}.$$

Then for $A \in \eta_\delta$, define $\upsilon_\delta(A) = \delta(A)$ (the natural density of A). Note that $A \in \eta_\delta$ and $\upsilon_\delta(A) = 0$ if and only if $A \in \eta_\delta^0$.

Proposition 2.6.2. **(i)** *If $A \sim \mathbb{N}$, then $A \in \eta_\delta$ and $\upsilon_\delta(A) = 1$;*

(ii) *If $A \sim \varnothing$ (i.e, if A is finite), then $A \in \eta_\delta^0$.*

Proposition 2.6.3. (i) *υ_δ is finitely additive, i.e, if $A, B \in \eta_\delta$ and $A \cap B = \varnothing$, then $A \cup B \in \eta_\delta$ and*

$$\upsilon_\delta(A \cup B) = \upsilon_\delta(A) + \upsilon_\delta(B);$$

(ii) *If A_1, A_2,$\ldots$,$A_n \in \eta_\delta^0$, then ${}_{i=1}^{n} A_i \in \eta_\delta^0$;*

(iii) *If $A \in \eta_\delta$, then $(\mathbb{N} \backslash A) \in \eta_\delta$ and $\upsilon_\delta(\mathbb{N} \backslash A) = 1 - \upsilon_\delta(A)$;*

(iv) *If $A \in \eta_\delta$ and $A \sim B$, then $B \in \eta_\delta$ and $\upsilon_\delta(A) = \upsilon_\delta(B)$.*

The following simple example shows that υ_δ is never countably additive;

Example 2.6.1. *Taking $A_i = \{i\}$, $i = 1, 2, ...$, we have*

$$A_i \in \eta_\delta^0 \subset \eta_\delta, \; i = 1, 2, ... \textit{ and } \; A_i \cap A_j = \emptyset \quad (i \neq j),$$

but

$$\cup_{i=1}^{\infty} A_i = \mathbb{N} \text{ and } \upsilon_\delta(\mathbb{N}) = 1 \neq \sum_{i=1}^{\infty} \upsilon_\delta(A_i) = 0.$$

Definition 2.6.5. *(Additive Property (AP)) The density δ is said to have additive property if for each family $\{A_i\} \subset \eta_\delta$, $i = 1, 2, ...$, with $A_i \cap A_j = \emptyset$ $(i \neq j)$, there exists a family $\{B_i\} \subset \eta_\delta$, $i = 1, 2, ...$, such that*

$i)$ $B_i \sim A_i$, $i = 1, 2, ...,$

$ii)$ $\cup_{i=1}^{\infty} B_i \in \eta_\delta$ *and*

$iii)$ $\upsilon_\delta(\cup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \upsilon_\delta(B_i)$.

Definition 2.6.6. *(Additivity Property For Null sets (APO)) The density δ is said to have additive property for null sets if for each family $\{A_i\} \subset \eta_\delta^0$, $i = 1, 2, ...$, with $A_i \cap A_j = \emptyset$ $(i \neq j)$, there exists a family $\{B_i\} \subset \eta_\delta^0$, $i = 1, 2, ...$, such that*

$i)$ $B_i \sim A_i$, $i = 1, 2, ...,$

$ii)$ $\cup_{i=1}^{\infty} B_i \in \eta_\delta^0$ *and*

$iii)\ \upsilon_\delta\left(\cup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \upsilon_\delta\left(B_i\right) = 0.$

If the condition that the sets A_i are disjoint is removed form (APO), we get an apparently stronger property (APO1).

Example 2.6.2. *The term "asymptotic density" is often used for the function*

$$d\left(A\right) = \liminf_{n\to\infty} \frac{A\left(n\right)}{n}$$

where $A\left(n\right)$ is the number of elements in $A \cap \{1, 2, ..., n\}$. If $\varkappa_A$ denotes the characteristic sequence of A (thus $\varkappa_A$ is a sequence of 0's and 1's), and if $C_1 = (c_{nk})$ denotes the Cesáro matrix of order one where

$$c_{nk} = \begin{cases} \frac{1}{n} & \textit{if} \quad 1 \leq k \leq n, \\ 0 & \textit{otherwise} \end{cases}$$

then $A\left(n\right)/n$ is the nth term of the sequence $C_1.\varkappa_A$. Thus

$$d\left(A\right) = \liminf_{n\to\infty} \left(C_1.\varkappa_A\right)_n.$$

This function satisfies axioms $(d.1) - (d.4)$, so it is a density.

The above idea can be extended to a non-negative regular matrix.

Proposition 2.6.4. *Let M be a nonnegative regular matrix and let δ_M be defined by*

$$\delta_M(A) = \lim \inf_{n\to\infty} (M.\chi_A)_n.$$

Then δ_M is density (i.e., satisfies $(d.1) - (d.4)$) and furthermore,

$$\bar{\delta}_M(A) = \lim \inf_{n\to\infty} (M.\chi_A)_n.$$

2.7 Statistical Convergence.

Let $K \subset \mathbb{N}$ be any subset of natural numbers then consider the asymptotic density δ, defined by

$$\delta\left(K\right) = \lim_{n\to\infty} \frac{\left|K\left(n\right)\right|}{n}$$

where $K(n) := \{k \leq n : k \in K\}$ and $|K(n)|$ represents the cardinality of the set $K(n)$. The number $\delta(K)$ is called the asymptotic (or shortly density) of K, provided that limit exists. Densities of some subsets of natural numbers are given in the following examples.

Example 2.7.1. *Let* $K := \{k \in \mathbb{N} : k = m^2\}$, *then we have* $|K(n)| \leq \sqrt{n}$. *Since* $\lim_n \frac{\sqrt{n}}{n} = 0$ *we conclude that* $\delta(K) = 0$.

Example 2.7.2. *It is obvious that both* $K := \{2k : k \in \mathbb{N}\}$ *and* $M := \{2k + 1 : k \in \mathbb{N}\}$ *has density* $\frac{1}{2}$.

Example 2.7.3. *Let* $K := \{ak + b : k \in \mathbb{N}\}$ *then* $\delta(K) = \frac{1}{a}$.

Example 2.7.4. *If* K *is a finite set then obviously* $\delta(K) = 0$.

Consider a subset K of $\mathbb{N}$, one can ask the following question ´´ Is $\delta(K)$ always defined´´. The answer is absolutely ´´No´´.

Example 2.7.5. *Consider the sequence*

$$x_k = (1, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, ...)$$

and define $K = \{k \in \mathbb{N} : x_k = 1\}$, *then for large* m *we have*

$$|K(2^m)| = \begin{cases} \geq 2^{m-1} + 2^{m-3} & \text{if } m \text{ is odd,} \\ \leq 2^{m-2} + 2^{m-3} & \text{if } m \text{ is even.} \end{cases}$$

(If m *is odd there are at least* $2^{m-1} + 2^{m-3}$ *ones, among the first* 2^m *terms of the sequence. Namely from the last block and the block two steps earlier) therefore,*

$$\lim_m \frac{|K(2^m)|}{2^m} = \begin{cases} \geq \frac{5}{8} & \text{if } m \text{ is odd,} \\ \leq \frac{3}{8} & \text{if } m \text{ is even.} \end{cases}$$

which means that limit and therefore $\delta(K)$ *does not exist.*

Definition 2.7.1. *The sequence* $x := (x_k)$ *is said to be statistically convergent to a number* L *if for every* $\varepsilon > 0$,

$$\delta(\{k : |x_k - L| \geq \varepsilon\}) = 0.$$

Statistical convergence of x *to* L *is denoted by* $st - \lim_n x_n = L$.

Theorem 2.7.1. *Ordinary convergence implies statistical convergence.*

Proof. Assume that $\lim_k x_k = L$ (i.e. $x = x_k$ is convergent in the ordinary sense) then for each $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ is finite. Therefore $\delta\left(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}\right) = 0$ or $st - \lim_k x_k = L$. □

Remark 2.7.2. *It is easy to see that if* x *is statistically convergent to a number* L, *then at the outside of each* $\varepsilon-$*neigborhood of* L, *sequence may have infinitely many terms but the density of its indices must be* 0.

Example 2.7.6. *Consider the sequence* $x := (x_k)$ *which is defined by*

$$x_k = \begin{cases} 1 & if \quad k = m^2, \\ 0 & if \quad k \neq m^2. \end{cases}$$

Since $\delta\left(\{k^2 : k \in \mathbb{N}\}\right) = 0$ *we have* $st - \lim_k x_k = 0$, *but* x *is not convergent in the ordinary sense.*

In the ordinary sense, convergence of a sequence implies boundedness, but in the sense of statistical convergence we may have statistically convergent but unbounded sequences.

Example 2.7.7. *Consider the sequence* $x := (x_k)$ *where*

$$x_k = \begin{cases} \sqrt{k} & if \quad k = m^2, \\ 0 & if \quad k \neq m^2. \end{cases}$$

Then $st - \lim_k x_k = 0$, *but* x *is not bounded.*

Theorem 2.7.3. *If $st - \lim x_k = L$ and $st - \lim y_k = \eta$ then*

(i) $st - \lim(x_k + y_k) = L + \eta$.

(ii) $st - \lim(x_k y_k) = L\eta$.

(iii) $st - \lim(\lambda x_k) = \lambda L$ for any $\lambda \in \mathbb{R}$.

As we mention in the previous section, natural density function was generalized, by replacing C_1 with an arbitrary nonegative regular matrix A, that is; $A-$density of $K \subseteq \mathbb{N}$ is defined by

$$\delta_A(K) := \lim_{n\to\infty} \sum_{k\in K} a_{nk} = \lim_{n\to\infty} (A\chi_K)_n$$

provided limit exists. The $A-$density has been used by Kolk [23], to extend statistical convergence as follows.

Definition 2.7.2. *For a nonnegative regular infinite matrix A, a sequence x is said to be A-statistically convergent to the number L if, for every $\varepsilon > 0$,*

$$\delta_A(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0.$$

Remark 2.7.4. *It is obvious that taking $A = C_1$, in the above definition, $A-$convergence reduces to statistical convergence.*

2.8 $q-$Integers

Definition 2.8.1. *The value $[r]$ denotes the $q-$integer of r, which is given by*

$$[r] = [r]_q = \begin{cases} \frac{1-q^r}{1-q}, & q \in \mathbb{R}^+ - \{1\} \\ r, & q = 1 \end{cases} .$$

For a given $q > 0$ let us define

$$\mathbb{N}_q = \{[r]\,, with\, r \in \mathbb{N}\}\,.$$

We see from the Definition 2.8.1 that

$$\mathbb{N}_q = \left\{0, 1, 1+q, 1+q+q^2, 1+q+q^2+q^3, \ldots\right\}. \tag{2.8.1}$$

Obviously, if we put $q = 1$ in (2.8.1), the set of all $q-$integers $\mathbb{N}_q$ reduces to the set of all natural numbers, the set of nonnegative integers $\mathbb{N}$.

Definition 2.8.2. *Given a value $q > 0$, $q-$shifted factorial is defined as*

$$(a; q)_n = (1-a)(1-aq)\ldots(1-aq^{n-1})$$

for all $n \geq 1$ and

$$(a; q)_0 = 1.$$

The infinite version of this product is defined by

$$(a; q)_\infty = \lim_{n\to\infty} (a; q)_n.$$

For a given value $q > 0$, the $q-$factorial, $[r]!$, can also be defined as

$$[r]! = \begin{cases} [r]\,[r-1]\ldots[1], & r \geq 1 \\ 1, & r = 0 \end{cases}.$$

where $r \in \mathbb{N}$.

Definition 2.8.3. *For any integer n and k, $q-$binomial coefficient is defined by*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} \tag{2.8.2}$$

for any $n \geq k \geq 0$.

Another way to write (2.8.2) is

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[n-k]![k]!}$$

which satisfies the following two pascal rules:

$$\begin{bmatrix} n \\ j \end{bmatrix} = \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + q^j \begin{bmatrix} n-1 \\ j \end{bmatrix}$$

and

$$\begin{bmatrix} n \\ j \end{bmatrix} = q^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ j \end{bmatrix}$$

where $1 \leq j \leq n-1$.

Definition 2.8.4. *The $q-$analog $(x-a)^n$ is defined by the polynomial*

$$(x-a)_q^n = \begin{cases} 1, & if\ n=0 \\ (x-a)(x-qa)\ldots(x-q^{n-1}a), & if\ n \geq 1 \end{cases}.$$

Throughout the thesis we will make frequent use of the finite $q-$binomial theorem in the following form ([21])

$$(x-a)_q^n = \sum_{j=0}^{n} (-1)^j\, q^{\frac{j(j-1)}{2}} \begin{bmatrix} n \\ j \end{bmatrix} a^j x^{n-j}.$$

Finally, we have some limit results which are useful in our dissertation.

Example 2.8.1. *If $q<1$,*

$$\lim_{n\to\infty} \frac{1}{[n]} = \lim_{n\to\infty} \frac{1}{\frac{1-q^n}{1-q}} = \lim_{n\to\infty} \frac{1-q}{1-q^n} = 1-q,$$

on the other hand if $q \geq 1$

$$\lim_{n\to\infty} \frac{1}{[n]} = 0.$$

Example 2.8.2. *if $0<q<1$*

$$\begin{aligned} \lim_{n\to\infty} \begin{bmatrix} n \\ j \end{bmatrix} &= \lim_{n\to\infty} \frac{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-j+1})}{(1-q)(1-q^2)\ldots(1-q^j)} \\ &= \frac{1}{(1-q)(1-q^2)\ldots(1-q^j)}. \end{aligned}$$

Chapter 3

q-CESÁRO METHODS

In this chapter, we mainly focus on $q-$analogs of Cesáro matrices of order $\alpha \in \mathbb{N}$ and their properties. Consequently, we determine $q-$density function using a general way to produce a density from nonnegative regular summability matrix.

3.1 Construction and Some Properties of $q-$Cesáro Matrices

In this section, we introduce a method to find $q-$analogs of Cesáro matrices of order $\alpha \in \mathbb{N}$, for all $q \in \mathbb{R}^{+}$. Recall that one can define infinitely many different $q-$analog of an infinite matrix (or matrix methods). Let A be an infinite matrix, then any infinite matrix of the form $A(q)$, where q is a real parameter, and $A(1) = A$, is called a $q-$analog of A. In other words if $A(q)$ is a $q-$analog of A then $A(1) = A$. In [7], Bustoz and Gordillo obtained the following $q-$analog $C_1(q) = (a_{nk}(q))$ of Cesáro matrix of order one where

$$a_{nk}(q) = \begin{cases} \frac{1-q}{1-q^{n+1}} q^{n-k} & if\ k \leq n \\ 0 & if\ k > n. \end{cases} \tag{3.1.1}$$

It should be mentioned that, the $q-$analog $C_1(q)$, obtained by Bustoz and Gordillo, is valid only for $0 < q < 1$. In their approach they obtained unique $q-$analog of Cesáro matrix of order one which is contrary, to the idea that $q-$analog of an infinite matrix is not unique. In this chapter we introduce a method which can be used to find different

$q-$analogs of the Cesáro matrices of order $\alpha \in \mathbb{N}$. In our approach all $q-$ analogs are valid for all $q \in \mathbb{R}^+$. Also, the $q-$ Cesáro method given in (3.1.1) can be obtained by using our method with an appropriate choice.

Let $S := (s_{nk})$ be the summation matrix with

$$s_{nk} = \begin{cases} 1; & k \leq n \\ 0; & otherwise \end{cases}$$

and I be the identity matrix. For any sequence $x = (x_k)$, define

$$B_n^0 x = I(x) = x_n, \tag{3.1.2}$$

$$B_n^1 x = S(x) = \sum_{\nu=0}^{n} x_v = \sum_{\nu=0}^{n} B_\nu^0(x) \tag{3.1.3}$$

and

$$B_n^\alpha(x) = S^\alpha(x) = \sum_{\nu=0}^{n} B_\nu^{\alpha-1} x. \tag{3.1.4}$$

$\alpha \in \mathbb{N}$, $\alpha \geq 2$. Recall that the entries s_{nk}^α of the matrix S^α can be determined in the following way. By 3.1.4 we have

$$\sum_{k=0}^{n} s_{nk}^\alpha x_k = B_n^\alpha(x).$$

But,

$$(1-z)\sum_n B_n^\alpha(x) z^n = \sum_n (B_n^\alpha(x) - B_{n-1}^\alpha(x)) z^n = \sum_n B_n^{\alpha-1}(x) z^n$$

with $B_{-1}^\alpha(x) = 0$, therefore,

$$\sum_n B_n^\alpha(x) z^n \quad = \quad \frac{1}{(1-z)^\alpha} \sum_n B_n^0(x) z^n$$

.

and substitute series representation of $\frac{1}{(1-z)^\alpha}$ and using Cauchy product, we get

$$= \sum_n \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} x_k z^n.$$

By comparing coefficients of z^n, we have

$$B_n^\alpha(x) = \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} x_k, \text{ with } n, k = 0, 1, \ldots,\ k \le n,$$

or equivalently,

$$s_{nk}^\alpha = \binom{n-k+\alpha-1}{n-k}.$$

On the other hand, the sum of the n^{th} row is;

$$\sum_{k=0}^{n} s_{nk}^\alpha = \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} = \binom{n+\alpha}{n}$$

and the matrix defined by

$$c_{nk}^\alpha := \frac{1}{\binom{n+\alpha}{n}} s_{nk}^\alpha \tag{3.1.5}$$

gives exactly the Cesáro matrix of order $\alpha \in \mathbb{N}$. Although the above calculations are not new and can be found in standard summability books (see [6]), they can be modified to obtain $q-$ analogs of Cesáro matrices of order $\alpha \in \mathbb{N}$. Before giving more details of this process we need the following definition.

Definition 3.1.1. *Let*

$$S_q(a_{nk}(q)) = \begin{cases} a_{nk}(q), & \text{if } k \le n \\ 0, & \text{otherwise} \end{cases}$$

be the infinite, lower triangular matrix, satisfying

$$a_{nk}(1) = 1,$$

then $S_q(a_{nk}(q))$ (or S_q for short) is called the $q-$analog of the summation matrix S associated with $a_{nk}(q)$.

Example 3.1.1. *By choosing $a_{nk}(q) = q^k$ we have*

$$S_q(a_{nk}(q)) = \begin{cases} q^k, & \text{if } k \leq n \\ 0, & \text{otherwise} \end{cases}$$

which gives

$$S_q(q^k) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & q & 0 & 0 & 0 & 0 & \cdots \\ 1 & q & q^2 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & q & q^2 & \cdots & q^n & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Example 3.1.2. *If we take $a_{nk}(q) = q^{-k}$ we get*

$$S_q(a_{nk}(q)) = \begin{cases} q^{-k}, & \text{if } k \leq n \\ 0, & \text{otherwise} \end{cases}$$

where its implicit form is given by

$$S_q(q^{-k}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & \frac{1}{q} & 0 & 0 & 0 & 0 & \cdots \\ 1 & \frac{1}{q} & \frac{1}{q^2} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & \frac{1}{q} & \frac{1}{q^2} & \cdots & \frac{1}{q^n} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Replacing S by its $q-$analog in the above process, we will obtain a $q-$analog of the Cesáro matrix of order $\alpha \in \mathbb{N}$ (or $q-$ Cesáro matrix generated by $a_{nk}(q)$). In the following theorem, we introduce a general formula for Cesáro matrix of order one associated with $a_{nk}(q)$.

Theorem 3.1.1. *The $q-$analog of the Cesáro matrix of order one associated with $a_{nk}(q)$ is $C_1(a_{nk}(q)) = (c^1_{nk}(a_{nk}(q))$ where*

$$c^1_{nk}(a_{nk}(q)) = \begin{cases} a_{nk}(q)\left(\sum\limits_{k=0}^{n} a_{nk}(q)\right)^{-1}, & if\ \ k \leq n \\ 0, & otherwise \end{cases} \tag{3.1.6}$$

$n, k = 0, 1, \ldots$.

Proof. Let S_q be the $q-$analog of S associated with $a_{nk}(q)$. By applying above process for $\alpha = 1$, equations (3.1.2) and (3.1.3) become

$$B^0_n x = I(x) = x_n,$$

$$B^1_n x = \left(S_q(x)\right)_n = \sum_{\nu=0}^{n} a_{n\nu}(q)x_\nu,$$

respectively. The matrix multiplication yields that $S_q = (s^1_{nk}(a_{nk}(q)))$ where

$$s^1_{nk}(a_{nk}(q)) = \begin{cases} a_{nk}(q) & if\ \ k \leq n \\ 0 & otherwise \end{cases}, \ n, k = 0, 1, \ldots$$

Now, the sum of the n^{th} row is $a_{n0}(q) + a_{n1}(q) + \cdots + a_{nn}(q) = \sum\limits_{k=0}^{n} a_{nk}(q)$, therefore in a way parallel to (3.1.5) one can obtain the $q-$Cesáro matrix of order one which is given in (3.1.6). □

It is obvious that in the case $q = 1$, $C_1(a_{nk}(q))$ reduces to the ordinary Cesáro matrix C_1, given in (2.2.1) for $\alpha = 1$.

Remark 3.1.2. *It should be mentioned that, under the conditions $a_{nk}(q) = a_k(q)$, for all n, with $a_0(q) > 0$, and $a_k(q) \geq 0$, $k \in \mathbb{N}$, $C_1(a_{nk}(q))$ is a Riesz method associated with $a_k(q)$.*

Remark 3.1.3. *The $q-$Cesáro matrix associated with $a_{nk}(q) = q^{-k}$ for $0 < q < 1$ is the $q-$analog of the Cesáro matrix suggested by Bustoz and Gordillo given in* (3.1.1).

Of course, there are many ways to define $q-$ analogs of Cesáro matrices. In the following theorem, we suggest a suitable $q-$analog of the Cesáro matrix of order one, order two and order $\alpha \in \mathbb{N}$.

Theorem 3.1.4. *If* $a_{nk}(q) = q^k$, *then* $C_1(q^k) = (c^1_{nk}(q^k))$ *with*

$$c^1_{nk}(q^k) = \begin{cases} \frac{q^k}{[n+1]_q} & if \ \ k \le n \\ 0 & otherwise \end{cases} , \tag{3.1.7}$$

$n, k = 0, 1, \ldots$. *and* $C_2(q^k) = (c^2_{nk}(q^k))$ *with*

$$c^2_{nk}(q^k) = \begin{cases} [n-k+1]_q \, q^{2k} \left(\sum\limits_{k=0}^{n} q^{2k} \, [n-k+1]_q \right)^{-1} & if \ \ k \le n \\ 0 & otherwise \end{cases} , \tag{3.1.8}$$

$n, k = 0, 1, \ldots$., *more generally* $C_\alpha(q^k) = \left(c^\alpha_{nk}(q^k)\right)$ *where*

$$c^\alpha_{nk}(q^k) = \begin{cases} \dfrac{q^{\alpha k} \sum\limits_{m_1=0}^{n-k} q^{m_1} \sum\limits_{m_2=0}^{m_1} q^{m_2} \cdots \sum\limits_{m_{\alpha-2}=0}^{m_{\alpha-1}} q^{m_{\alpha-2}} [m_{\alpha-2}+1]_q}{\sum\limits_{k=0}^{n} \left(q^{\alpha k} \sum\limits_{m_1=0}^{n-k} q^{m_1} \sum\limits_{m_2=0}^{m_1} q^{m_2} \cdots \sum\limits_{m_{\alpha-2}=0}^{m_{\alpha-1}} q^{m_{\alpha-2}} [m_{\alpha-2}+1]_q \right)} & if \ \ k \le n \\ 0 & otherwise \end{cases} , \tag{3.1.9}$$

$n, k = 0, 1, \ldots$., *with* $\alpha > 2, \alpha \in \mathbb{N}$.

Proof. To find $C_1(q^k)$, it is enough to replace $a_{nk}(q)$ by q^k in Theorem 3.1.1. noindent

For $C_2(q^k)$, take $a_{nk}(q) = q^k$, then equations (3.1.2),(3.1.3) and (3.1.4) become

$$B^0_n x = I(x) = x_n,$$

$$B^1_n x = (S^1_q(x))_n = \sum_{\nu=0}^{n} q^v x_v$$

and

$$B^2_n(x) = \left(S^2(x)\right)_n = \sum_{\nu=0}^{n} B^1_\nu x$$

respectively. Matrix multiplication yields that, second order $q-$summation matrix is $S^2_q = (s^2_{nk}(q^k))$ where

$$s^2_{nk}(q^k) = \begin{cases} [n-k+1]_q \, q^{2k} & if \ k \le n \\ 0 & otherwise \end{cases}$$

and the row sum of the n^{th} row of S^2_q is $\sum\limits_{k=0}^{n} [n-k+1] \, q^{2k}$. Therefore in a way parallel to (3.1.5), one can obtain the second order $q-$Cesaro matrix as

$$c^2_{nk}(q^k) = \frac{[n-k+1]_q \, q^{2k}}{\sum\limits_{k=0}^{n} q^{2k} \, [n-k+1]_q} \text{ for } k \le n$$

Similarly, for $C_\alpha(q^k)$ take $a_{nk}(q) = q^k$ and apply the process described above. □

Recall that in the ordinary case the sum of the n^{th} row of the summation matrix S was $n+1$, and the most natural $q-$analog of $n+1$ is $[n+1]_q$. To have the sum $[n+1]_q$ on the n^{th} row of S_q, the generating sequence can be selected as $a_{nk}(q) = q^k$. Therefore, $C_\alpha(q^k)$ is a suitable $q-$analog of the Cesáro matrix C_α.

The matrix method $C_1(q^k)$ and the corresponding summability method are called $q-$ Cesáro matrix and $q-$Cesáro summability method of order one respectively.

In the rest of this thesis we shall focus on the matrix $C_1(q^k)$ which has the following explicit form;

$$C_1(q^k) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{[2]_q} & \frac{q}{[2]_q} & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{[3]_q} & \frac{q}{[3]_q} & \frac{q^2}{[3]_q} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \frac{1}{[n+1]_q} & \frac{q}{[n+1]_q} & \frac{q^2}{[n+1]_q} & \cdots & \frac{q^n}{[n+1]_q} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

Definition 3.1.2. *A sequence $x = (x_k)$ is called $q-$Cesáro summable to L if*

$$\lim_{n\to\infty} \sum_{k=0}^{n} c^1_{nk}(q^k) x_k = L.$$

Example 3.1.3. *For any fixed $q \leq 1$, the divergent sequence $x = (x_k)$ with*

$$x_k = \begin{cases} \frac{1}{q} & k = 0, 2, \dots \\ -\frac{1}{q^2} & k = 1, 3, \dots \end{cases}$$

is $C_1(q^k)-$summable to 0.

Example 3.1.4. *For any fixed $q < 1$, the divergent sequence $x = x_k = \left(q^{-k}\right)$ is not $C_1(q^k)-$summable. Indeed*

$$\left(C_1(q^k)x\right)_n = \sum_{k=0}^{n} q^{-k} \frac{q^k}{[n+1]_q} = \frac{n+1}{[n+1]_q}$$

and

$$\lim_{n\to\infty} \frac{n+1}{[n+1]_q} = \infty.$$

Theorem 2.1.4 and the Theorem of Silverman -Toeplitz give us the following characterization for $C_1(q^k)$:

Lemma 3.1.1. (i) *$C_1(q^k)$ is conservative for each $q \in \mathbb{R}^+$,*

(ii) *$C_1(q^k)$ is regular for each $q \geq 1$.*

Proof. Since the sum of each row is 1 and $C_1(q^k)$ satisfies row norm condition, it is enough to prove column limit condition of Theorem 2.1.4 and the Theorem 2.1.5.

(i) a) For $q = 1$ we have nothing to do because $C_1(q^k)$ reduces to the ordinary Cesáro matrix which is regular.

b) Assume that $0 < q < 1$, then

$$\begin{aligned} \lim_{n\to\infty} \frac{q^k}{[n+1]_q} &= \lim_{n\to\infty} \frac{q^k(1-q)}{1-q^n} \\ &= q^k(1-q) \end{aligned}$$

therefore $C_1(q^k)$ satisfies column limit condition with $\lambda_k = q^k(1-q)$ for $k = 0, 1, \ldots$.

c) Assume that $q > 1$ then

$$\begin{aligned} \lim_{n\to\infty} \frac{q^k}{[n+1]_q} &= \lim_{n\to\infty} \frac{q^k(1-q)}{1-q^n} \\ &= \lim_{n\to\infty} \frac{1}{q^n} \frac{q^k(1-q)}{\left(\frac{1}{q^n} - 1\right)} \\ &= 0 \quad \text{for } k = 0, 1, \ldots \end{aligned}$$

Therefore, $C_1(q^k)$ satisfies column limit condition with $\lambda_k = 0$ for $k = 0, 1, \ldots$.and $q > 1$.

From a) , b) and c) $C_1(q^k)$ is conservative for each $q \in \mathbb{R}^+$.

(ii) For $q = 1, C_1(q^k) = C_1$ and it is regular. Assume that $q > 1$, then by the discussion given in section c) $C_1(q^k)$ is regular.

$\square$

Remark 3.1.5. *If* $q_1 \neq q_2$ *then* $C_1(q_1^k) \neq C_1(q_2^k)$, *moreover if* $q_1 > 1$ *then* $C_1(q_1^k)$ *is regular but* $C_1(q_2^k)$ *is not regular for* $q_2 = q_1^{-1}$.

Now it is natural to ask how the strength of $C_1(q^k)$ changes with q. The answer is given in the following Theorem.

Theorem 3.1.6. $C_1(q_1^k)$ *is equivalent to* $C_1(q_2^k)$, *for* $1 < q_1 < q_2$.

Proof. Assume that $1 < q_1 < q_2$ then,

$$\begin{aligned}
\sup_n \sum_{k=0}^{n} [k+1]_{q_2} \left| \frac{q_1^k}{[n+1]_{q_1} q_2^k} - \frac{q_1^{k+1}}{[n+1]_{q_1} q_2^{k+1}} \right| &\leq \sup_n \sum_{k=0}^{n} \frac{[k+1]_{q_2}}{q_2^k} \left| \frac{q_1^k}{[n+1]_{q_1}} - \left(\frac{q_1}{q_2}\right) \frac{q_1^k}{[n+1]_{q_1}} \right| \\
&\leq \sup_n \sum_{k=0}^{n} \frac{[k+1]_{q_2}}{q_2^k} \left(\frac{q_1^k}{[n+1]_{q_1}} \right) \\
&\leq \sup_n \sum_{k=0}^{n} \frac{[k+1]_{q_2}}{q_2^k} \leq \sup_n \sum_{k=0}^{n} \left(\frac{1}{q_2}\right)^k \\
&\leq \frac{q_2}{q_2 - 1}.
\end{aligned}$$

Conversely

$$\begin{aligned}
\sup_n \sum_{k=0}^{n} [k+1]_{q_1} \left| \frac{q_2^k}{[n+1]_{q_2} q_1^k} - \frac{q_2^{k+1}}{[n+1]_{q_2} q_1^{k+1}} \right| &\leq \sup_n \sum_{k=0}^{n} \frac{[k+1]_{q_1}}{q_1^k} \left| \frac{q_2^k}{[n+1]_{q_2}} - \frac{q_2^{k+1}}{[n+1]_{q_2} q_1} \right| \\
&\leq \sup_n \sum_{k=0}^{n} \frac{[k+1]_{q_1}}{q_1^k} \left| \frac{q_2^k}{[n+1]_{q_2}} - \frac{q_2}{q_1} \frac{q_2^k}{[n+1]_{q_2}} \right|
\end{aligned}$$

$$\leq \sup_n \sum_{k=0}^{n} \frac{[k+1]_{q_1}}{q_1^k} \left(\frac{q_2}{q_1} \frac{q_2^k}{[n+1]_{q_2}} - \frac{q_2^k}{[n+1]_{q_2}} \right)$$

$$\leq \sup_n \frac{q_2}{q_1} \sum_{k=0}^{n} \frac{[k+1]_{q_1}}{q_1^k} \left(\frac{q_2^k}{[n+1]_{q_2}} \right)$$

$$\leq \sup_n \frac{q_2}{q_1} \sum_{k=0}^{n} \frac{[k+1]_{q_1}}{q_1^k}$$

$$\leq \sup_n \frac{q_2}{q_1} \frac{q_1}{q_1 - 1} = \frac{q_2}{q_1 - 1}.$$

The proof is completed using Theorem 2.4.5 and the fact that $C_1(q_1^k)$ and $C_1(q_2^k)$ are both regular, row finite matrices. □

Theorem 3.1.7. *The summability method C_1 is stronger than $C_1(q^k)$ for $q \geq 1$.*

Proof. For $q = 1$, $C_1(q^k)$ reduces to C_1, therefore without loss of generality, we may assume that $q > 1$. By using Theorem 2.4.5 and the fact that C_1 is a row finite regular method, it is enough to show that

$$\sup_n \frac{1}{n+1} \left(\frac{q-1}{q} \right) \sum_{k=0}^{n} \frac{[k+1]_q}{q^k} < \infty. \tag{3.1.10}$$

Using

$$\frac{[k+1]_q}{q^k} = \sum_{i=0}^{k} \frac{1}{q^i} \leq \frac{q}{q-1}$$

in (3.1.10), completes the proof. □

Theorem 3.1.8. *For $q \leq 1$, $c \subsetneq c_{C_1(q^k)}$.*

Proof. For any fixed $q \leq 1$, the divergent sequence $x = (x_k)$ with

$$x_k = \begin{cases} \frac{1}{q} & k = 0, 2, \dots \\ -\frac{1}{q^2} & k = 1, 3, \dots \end{cases}$$

is $C_1(q^k)-$summable to 0. □

As a direct consequence of previous theorem and Theorem 2.1.7, we can state the following lemma:

Lemma 3.1.2. *$C_1(q^k)$ sums at least one unbounded sequence for $q \leq 1$.*

Proof. For any fixed $q \leq 1$, choosing an index sequence as $r_j = j^{j+1}(j \in \mathbb{N})$ and $r_0 = 0$, unbounded sequence $x = (x_k)$ with

$$x_k = \begin{cases} \sum_{i=0}^{j} \frac{1}{q(i+1)} & k = 0, 2, \ldots \text{ and } r_j \leq k < r_{j+1} \\ \sum_{i=0}^{j} \frac{-1}{q^2(i+1)} & k = 1, 3, \ldots \text{ and } r_j \leq k < r_{j+1} \end{cases},$$

is $C_1(q^k)-$summable to 0. □

Using the fact that $C_1(q^k)$ is a Riesz method, the inverse of $C_1(q^k)$ is

$$C_1^{-1}(q^k) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{-1}{q} & \frac{[2]}{q} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{-[2]}{q^2} & \frac{[3]}{q^2} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & \cdots & \frac{-[n]}{q^n} & \frac{[n+1]}{q^n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Theorem 3.1.9. *$C_1(q^k)$ is of type M for $q \in \mathbb{R}^+$.*

Proof. Let's choose $t \in l$ with $tC_1(q^k) = 0$. Since $C_1^{-1}(q^k)$ is column finite, according to Theorem 2.1.2 $(tC_1(q^k))C_1^{-1}(q^k)$ and $t(C_1(q^k)C_1^{-1}(q^k))$ exist and $t(C_1(q^k)C_1^{-1}(q^k)) = (tC_1(q^k))C_1^{-1}(q^k)$. Thus

$$t = t(C_1(q^k)C_1^{-1}(q^k)) = (tC_1(q^k))C_1^{-1}(q^k) = 0.$$

which proves the theorem. □

As an immediate consequence of Corollary 2.1.2 and Theorem 3.1.7, we can state the following corollary.

Corollary 3.1.1. *$C_1(q^k)$ and C_1 are consistent for $q \geq 1$.*

Remark 3.1.10. *$C_1(q^k)$ and C_1 are not equivalent for $q \geq 1$.*

Theorem 3.1.11. *$C_1(q^k)$ satisfies the mean value property with $K = 1$*

Proof. By a direct calculation we have the following;

$$\begin{aligned} \left|\sum_{k=0}^{r} \frac{q^k}{[n+1]_q} x_k\right| &= \frac{1}{[n+1]_q}\left|\sum_{k=0}^{r} q^k x_k\right| \\ &= \frac{[r+1]_q}{[n+1]_q}\left|\sum_{k=0}^{r} \frac{q^k}{[r+1]_q} x_k\right| \\ &\leq \left|\sum_{k=0}^{r} \frac{q^k}{[r+1]_q} x_k\right| \end{aligned}$$

since $\frac{[r+1]}{[n+1]} \leq 1$ for $r \leq n$. This means that $C_1(q^k)$ satisfies the mean value property with $K = 1$. $\square$

3.2 $q-$Density function and $q-$Statistical Convergence

As we mentioned in Section 2.5, Freedman and Sember [14] showed that each non-negative regular matrix A can be associated by a density function

$$\delta_A(K) = \lim_{n\to\infty} \inf (A\chi_K)_n, \tag{3.2.1}$$

where χ_K denotes the characteristic function of $K \subset \mathbb{N}$. Replacing A by C_1 and $\lim\inf$ by ordinary limit in 3.2.1, we obtain the well-known natural density function

$$\delta(K) = \delta_{C_1}(K) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} \chi_K(k)$$

provided that limit exists. Using regularity of $C_1(q^k)$ (for short C_1^q) for $q \geq 1$, and replacing A by C_1^q in (3.2.1) we can define the following density functions $\delta_{C_1^q}$, between the subsets of natural numbers and the interval $[0,1]$;

$$\delta_q(K) = \delta_{C_1^q}(K) = \lim_{n\to\infty}\inf\,(C_1^q\chi_K)_n\,, \tag{3.2.2}$$

$$= \lim_{n\to\infty}\inf\sum_{k\in K}\frac{q^{k-1}}{[n]}\,,\, q\geq 1. \tag{3.2.3}$$

Remark 3.2.1. *If K is finite subset of $\mathbb{N}$, then obviously $\delta_q(K)=0$.*

Before giving the $q-$density of some infinite sets we need the following lemma.

Lemma 3.2.1. *For $q>1$, there exist M such that*

$$1+q+\ldots+q^n \leq Mq^{n+1}$$

Proof. For $q>1$,

$$\begin{aligned}
\frac{1+q+\ldots+q^n}{q^{n+1}} &= \frac{1}{q^{n+1}}+\frac{1}{q^n}+\ldots+\frac{1}{q}\\
&= \sum_{k=1}^{n+1}\left(\frac{1}{q}\right)^k\\
&\leq \sum_{k=1}^{\infty}\left(\frac{1}{q}\right)^k = \frac{\frac{1}{q}}{1-\frac{1}{q}} = \frac{1}{q-1} = M
\end{aligned}$$

□

Recall that in the ordinary case, $\delta(\mathbb{N}^2)=0$, $\delta(2\mathbb{N})=\delta(2\mathbb{N}+1)=\frac{1}{2}$ and more generally $\delta(a\mathbb{N}+b)=\frac{1}{a}$ where a and b are positive integers. In the following lemma we obtain parallel results for δ_q.

Lemma 3.2.2. (i) $\delta_q(2\mathbb{N})=\delta_q(2\mathbb{N}+1)=\frac{1}{[2]}$

(ii) $\delta_q(a\mathbb{N}+b)=\frac{1}{[a]}$ *where a and b are positive integers.*

(iii) $\delta_q(\mathbb{N}^2)=0$

Proof. $i)$ By the definition

$$\delta_q(2\mathbb{N}) = \lim_{n\to\infty} \inf \sum_{k\in 2\mathbb{N}} \frac{q^{k-1}}{[n]}$$

where

$$\sum_{k\in 2\mathbb{N}} \frac{q^{k-1}}{[n]} = \begin{cases} \sum_{k=1}^{\frac{n}{2}} \frac{q^{2k-1}}{[n]}. & \text{if } n \text{ is even} \\ \sum_{k=1}^{\frac{n-1}{2}} \frac{q^{2k-1}}{[n]} & \text{if } n \text{ is odd} \end{cases}$$

If n is even then n^{th} partial sum is

$$s_n = \frac{q}{[n]} + \frac{q^3}{[n]} + \cdots + \frac{q^{n-1}}{[n]} \tag{3.2.4}$$

and

$$q^2 s_n = \frac{q^3}{[n]} + \frac{q^5}{[n]} + \cdots + \frac{q^{n+1}}{[n]} \tag{3.2.5}$$

combining (3.2.4) and (3.2.5) we have

$$s_n = \frac{q(1-q^n)}{(1-q^2)\,[n]}$$

and

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{q(1-q^n)}{(1-q^2)\,[n]} = \frac{q}{1+q}.$$

If n is odd then n^{th} partial sum is

$$s_n = \frac{q}{[n]} + \frac{q^3}{[n]} + \cdots + \frac{q^{n-2}}{[n]} \tag{3.2.6}$$

and

$$q^2 s_n = \frac{q^3}{[n]} + \frac{q^5}{[n]} + \cdots + \frac{q^{n}}{[n]} \tag{3.2.7}$$

similarly combining (3.2.6) and (3.2.7) yields

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{q-q^n}{(1-q^2)\,[n]} = \frac{1}{1+q} = \frac{1}{[2]_q}$$

.Since $q \geq 1$, we have $\frac{1}{1+q} \leq \frac{q}{1+q}$ or equivalently,

$$\delta_q(2\mathbb{N}) = \lim_{n\to\infty} \inf \sum_{\substack{k\in 2\mathbb{N}\\ k\leq n}} \frac{q^{k-1}}{[n]} = \frac{1}{[2]_q}.$$

By using the above technique one can prove that $\delta_q(2\mathbb{N}+1) = \frac{1}{[2]}$

$ii)$ Since $\{a\mathbb{N}+ j : j = 0, 1, \dots, a-1\}$ is a partition for $\mathbb{N}$ and using the method of (i), we have

$$\lim_{n\to\infty}\left(\sum_{k\in a\mathbb{N}+\mathbb{J}} \frac{q^{k-1}}{[n]_q}\right) = \frac{q^{a-1-j}}{[a]_q}$$

for fixed $j \in \{0, 1, \dots, a-1\}$, and

$$\delta_q(a\mathbb{N}+b) = \inf\left\{\frac{q^{a-1-j}}{[a]} : j = 0, 1, \dots, a-1.\right\} = \frac{1}{[a]_q}.$$

$iii)$ By the definition

$$\delta_q(\mathbb{N}^2) = \lim_{n\to\infty} \inf \sum_{k\in\mathbb{N}^2} \frac{q^{k-1}}{[n]}.$$

Consider the subsequence

$$\mathrm{t}_{(m^2-1)} = \sum_{k=1}^{m-1} \frac{q^{k^2-1}}{[m^2-1]} \text{ of } t_m = \sum_{\substack{k\in\mathbb{N}^2 \\ k\leq m}} \frac{q^{k-1}}{[m]}$$

then

$$\begin{aligned}\lim_{m\to\infty} t_{(m^2-1)} &= \lim_{m\to\infty} \frac{q^0+q^3+\ldots+q^{(m-1)^2-1}}{[m^2-1]} \\ &\leq \lim_{m\to\infty} \frac{Mq^{(m-1)^2}}{[m^2-1]} = 0.\end{aligned}$$

thus

$$\delta_q(\mathbb{N}^2) = \lim_{n\to\infty} \inf \sum_{k\in\mathbb{N}^2} \frac{q^{k-1}}{[n]} = 0.$$

$\square$

Finally, we shall define a new type convergence, $q-$statistical convergence which is different from statistical convergence.

Definition 3.2.1. *A number sequence $x = (x_k)$ is called $q-$statistical convergent to L, written st^q-lim $x = L$, if for every $\varepsilon > 0$, $\delta_q(K_\varepsilon) = 0$, where $K_\varepsilon = \{k : |x_k - L| \geq \varepsilon\}$.*

Example 3.2.1. *Consider the sequence $x_k = \left(\underbrace{1}_{2^0}, \underbrace{0,0}_{2^1}, \underbrace{1,1,1,1}_{2^2}, \underbrace{0,0,0,...0}_{2^3}, 1, \ldots,\right)$ and define the set $K = \{k \in \mathbb{N} \, : \, x_k = 1\}$ then $\delta(K)$ does not exists (see [15]) therefore x_k is not statistically convergent. On the other hand since $[C_1^q \chi_K]_{2^{2n}-1} \to 0$, $st^q - \lim x_k = 0$*

Theorem 3.2.2. *If $\delta(K) = 0$ for an infinite set K then $\delta_q(K) = 0$.*

Proof. Assume that $K := \{k_1 < k_2 < \cdots < k_n < \cdots\}$. Since $\delta(K) = 0$, we have

$$\sup_{n\in\mathbb{N}} \{k_n - k_{n-1} : n = 2, 3, \cdots\} = +\infty. \tag{3.2.8}$$

Using (3.2.8), we can find a monoton increasing sequence $(k_{\nu(n)} - k_{\nu(n-1)})_{n\in\mathbb{N}}$ with $k_{\nu(n)} - k_{\nu(n-1)} \to \infty$, when $n \to \infty$. Define

$$s_n = \left(\sum_{\substack{k\in K \\ k\leq n}} \frac{q^{k-1}}{[n]}\right)$$

then by the definition of δ_q we have,

$$\delta_q(K) = \liminf_n s_n.$$

Now consider the subsequence

$$s_{k_{\nu(n)}-1} := \left(\sum_{\substack{k\in K \\ k\leq k_{\nu(n)}-1}} \frac{q^{k-1}}{[k_{\nu(n)} - 1]}\right)$$

of s_n, we have

$$\left(\sum_{\substack{k\in K \\ k\leq k_{\nu(n)}-1}} \frac{q^{k-1}}{[k_{\nu(n)} - 1]}\right) \leq \left(\sum_{\substack{k=1 \\ k\leq k_{\nu(n)}-1}}^{k_{\nu(n-1)}} \frac{q^{k-1}}{[k_{\nu(n)} - 1]}\right)$$

$$\leq \left(\frac{1+q+q^2+\cdots+q^{k_{\nu(n-1)}-1}}{\left[k_{\nu(n)}-1\right]}\right)$$

$$\leq \left(\frac{Mq^{k_{\nu(n-1)}}}{\left[k_{\nu(n)}-1\right]}\right)$$

now take limit from both sides as $n \to \infty$, we have

$$\begin{aligned}
\lim_{n\to\infty} s_{(k_{\nu(n)}-1)} &\leq \lim_{n\to\infty}\left(\sum_{\substack{k\in K\\ k\leq k_{\nu(n)}-1}} \frac{q^{k-1}}{\left[k_{\nu(n)}-1\right]}\right)\\
&\leq \lim_{n\to\infty}\left(\frac{Mq^{k_{\nu(n-1)}}}{\left[k_{\nu(n)}-1\right]}\right)\\
&\leq \lim_{n\to\infty}\frac{M(1-q)q^{k_{\nu(n-1)}}}{1-q^{k_{\nu(n)}-1}}\\
&\leq \lim_{n\to\infty}\frac{q^{k_{\nu(n-1)}}M(1-q)}{q^{k_{\nu(n-1)}}\left(1-q^{k_{\nu(n)}-k_{\nu(n-1)}-1}\right)}\\
&\leq \lim_{n\to\infty}\frac{M(1-q)}{\left(1-q^{k_{\nu(n)}-k_{\nu(n-1)}-1}\right)}=0
\end{aligned}$$

since $k_{\nu(n)} - k_{\nu(n-1)} \to \infty$ when $n \to \infty$. Therefore $\delta_q(K) = 0$. $\square$

Remark 3.2.3. *If a sequence x is statistically convergent to L then by Theorem 3.2.2 x is $q-$statistically convergent to L.*

Chapter 4

q-HAUSDORFF METHODS

In the ordinary case it is well known that C_α belongs to an important class of summability method called Hausdorff Methods. The main idea of the present chapter is to introduce and discuss the class of $q-$ Hausdorff matrices. But before starting to discuss Hausdorff matrices in q generalized sense, we would like to repeat a very brief outline of Hausdorff methods and the relation between Hausdorff matrices and C_1. Assume that A and B are two regular matrices with $AB = BA$, since permutable regular matrices define consistent methods, summability methods corresponding to A and B are consistent to each other. In other words if x is any sequence in $c_A \cap c_B$ then A and B assing the same limit value to x.

Recall that a matrix A is called diagonal if $A = (\delta_{mn} a_m)$ where $a_m \neq 0$ for all m and δ_{mn} is the Kronecker delta. Moreover we say that the matrix A is reduced to the diagonal form by the triangular matrix P if and only if

$$PAP^{-1} = (p_n \delta_{mn}).$$

As we stated in Section 2.5, C_1 is reduced to diagonal form with diagonal elements $\frac{1}{n+1}$ by the triangular matrix, $\Delta = (d_{nk})$ where

$$d_{mk} = (-1)^k \binom{n}{k}.$$

It should also be mentioned that Δ is self inverse that is $\Delta = \Delta^{-1}$ and A triangular matrix A is permutable with C_1 if and only if $\Delta A\Delta = D$ or equivalently A =$\Delta D\Delta$.

4.1 Construction of q-Hausdorff Matrices

First of all we will apply a method parallel to the ordinary case to obtain the invertible matrix Δ_q, the $q-$analog of the difference matrix Δ.

Theorem 4.1.1. *If D is a diagonal matrix then the matrix equation*

$$\Delta_q C_1\left(q^k\right) = D\Delta_q \tag{4.1.1}$$

has the solution $\Delta_q = (\lambda_{nv})$ with

$$\lambda_{nv} = (-1)^v \begin{bmatrix} n \\ v \end{bmatrix} q^{\left(\frac{n-v)(n-v-1)}{2}\right.}, \; v = 0, 1, \ldots n.$$

The diagonal matrix D is given by $D = (p_n\delta_{nv})$ with

$$p_n = \frac{q^n}{[n+1]_q} = \frac{q^n(1-q)}{1-q^{n+1}}.$$

Proof. Consider the matrix equation $\Delta_q C_1(q^k) = D\Delta_q$, or equivalently

$$\sum_{k=v}^{n} \lambda_{nk} c_{kv}(q) = \sum_k \delta_{nk} p_k \lambda_{kv}. \tag{4.1.2}$$

substituting $c_{k\nu}(q)$ in (4.1.2) we have,

$$\sum_{k=v}^{n} \lambda_{nk} \frac{q^v(1-q)}{(1-q^{k+1})} = p_n\lambda_{nv}. \tag{4.1.3}$$

Taking $\nu = n$ then

$$\lambda_{nn} \frac{q^n(1-q)}{(1-q^{n+1})} = p_n\lambda_{nn}$$

and since $\lambda_{nn} \neq 0$, we get that

$$p_n = \frac{q^n(1-q)}{(1-q^{n+1})}. \tag{4.1.4}$$

Now substitute (4.1.4) in (4.1.3), we obtain that

$$\lambda_{nv} = \sum_{k=v}^{n} \frac{q^v(1-q^{n+1})}{q^n(1-q^{k+1)})} \lambda_{nk}$$

and

$$\lambda_{nv} - \lambda_{n(v+1)} = \sum_{k=v}^{n} \frac{q^v(1-q^{n+1})}{q^n(1-q^{k+1})}\lambda_{nk} - \sum_{k=v+1}^{n} \frac{q^{v+1}(1-q^{n+1})}{q^n(1-q^{k+1})}\lambda_{nk}.$$

Rewriting the terms we have

$$\lambda_{nv} - \lambda_{n(v+1)} = \sum_{k=v}^{n} \frac{q^v(1-q^{n+1})}{q^n(1-q^{k+1})}\lambda_{nk} - q\sum_{k=v}^{n} \frac{q^v(1-q^{n+1})}{q^n(1-q^{k+1})}\lambda_{nk} + \frac{q^{v+1}(1-q^{n+1})}{q^n(1-q^{v+1})}\lambda_{nv}$$

or

$$\lambda_{nv} - \lambda_{n(v+1)} = \lambda_{nv} - q\lambda_{nv} + \frac{q^{v+1}(1-q^{n+1})}{q^n(1-q^{v+1})}\lambda_{nv}.$$

Finally we get

$$\lambda_{n(v+1)} = \frac{q^{n+1}-q^{v+1}}{q^n(1-q^{v+1})}\lambda_{nv}$$

or equivalently the recursion formula

$$\lambda_{nv} = \frac{q^n(1-q^{v+1})}{q^{n+1}-q^{v+1}}\lambda_{n(v+1)} = \frac{q^{n-1}(1-q^{v+1})}{q^n-q^v}\lambda_{n(v+1)}. \tag{4.1.5}$$

Consequently, by repeating application of the recursion formula 4.1.5, we have

$$\begin{aligned}
\lambda_{nv} &= \frac{q^{n-1}(1-q^{v+1})}{q^n-q^v}\frac{q^{n-1}(1-q^{v+2})}{q^n-q^{v+1}}\cdots\frac{q^{n-1}(1-q^n)}{q^n-q^{n-1}}\lambda_{nn} \\
&= \frac{q^{n-1}(1-q^{v+1})q^{n-1}(1-q^{v+2})\ldots q^{n-1}(1-q^n)}{q^n(1-q^{v-n})q^n(1-q^{v-n+1})\ldots q^n(1-q^{-1})}\lambda_{nn} \\
&= \frac{q^{n-1}(1-q^{v+1})q^{n-1}(1-q^{v+2})\ldots q^{n-1}(1-q^n)}{q^n(1-q^{v-n})q^n(1-q^{v-n+1})\ldots q^n(1-q^{-1})}\lambda_{nn} \\
&= \frac{q^{(n-1)(n-v)}(1-q^{v+1})(1-q^{v+2})\ldots(1-q^n)}{q^n(\frac{q^{n-v}-1}{q^{n-v}})q^n(\frac{q^{n-v-1}-1}{q^{n-v-1}})\ldots q^n(\frac{q-1}{q})}\lambda_{nn} \\
&= \frac{q^{(n-1)(n-v)}(1-q^{v+1})(1-q^{v+2})\ldots(1-q^n)}{(-1)q^v(1-q^{n-v})(-1)q^{v+1}(1-q^{n-v-1})\ldots(-1)q^{n-1}(1-q)}\lambda_{nn} \\
&= \frac{q^{(n-1)(n-v)}(-1)^{n-v}}{q^{\frac{n(n-1)-v(v-1)}{2}}}.\frac{(1-q^{v+1})(1-q^{v+2})\ldots(1-q^n)}{(1-q^{n-v})(1-q^{n-v-1})\ldots(1-q)}\lambda_{nn}
\end{aligned} \tag{4.1.6}$$

hence we can rewrite (4.1.6) as

$$\frac{q^{\frac{(n-v)(n-v-1)}{2}}(-1)^{n-v}(q;q)_n}{(q;q)_{n-v}(q;q)v}\lambda_{nn}$$

and finally

$$\lambda_{nv}=(-1)^{n-v}q^{\frac{(n-v)(n-v-1)}{2}}\left[\begin{array}{c} n \\ v \end{array}\right]q^{(\frac{n-v)(n-v-1)}{2}}\lambda_{nn}. \tag{4.1.7}$$

Now any nonzero choice of λ_{nn} will give us a matrix in the desired form. Therefore taking $\lambda_{nn}=(-1)^n$ in (4.1.7) we have

$$\lambda_{nv}=(-1)^v q^{\frac{(n-v)(n-v-1)}{2}}\left[\begin{array}{c} n \\ v \end{array}\right]$$

and this completes the proof. $\square$

Explicit form of $q-$difference matrix is;

$$\Delta_q=(\lambda_{nv})=\left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & \cdots & \\ 1 & -1 & 0 & 0 & \cdots & \\ q & -[2] & 1 & 0 & \cdots & \\ q^3 & -q\,[3] & [3] & -1 & \cdots & \\ \vdots & \vdots & \vdots & & \ddots & \\ q^{\frac{n(n-1)}{2}} & -q^{\frac{(n-1)(n-2)}{2}}[n] & q^{\frac{(n-2)(n-3)}{2}}\frac{[n][n-1]}{2} & \cdots & (-1)^n & 0 \\ \vdots & \vdots & \vdots & & & \ddots \end{array}\right].$$

Different from the ordinary case, the invertible matrix Δ_q is not self-inverse. But it is easy to see that the inverse of Δ_q is given by $\Delta_q^{-1}=(\mu_{nv})$ where

$$\mu_{n\nu}=\begin{cases} (-1)^v\left[\begin{array}{c} n \\ v \end{array}\right], & \nu\leq n \\ 0, & otherwise. \end{cases}$$

The explicit form of Δ_q^{-1} is

$$\Delta_q^{-1} = (\mu_{nv}) = \begin{bmatrix} 1 & 0 & 0 & 0 & & \cdots & \\ 1 & -1 & 0 & 0 & & \cdots & \\ 1 & -[2] & 1 & 0 & & \cdots & \\ 1 & -[3] & [3] & -1 & & \cdots & \\ \vdots & \vdots & \vdots & & \ddots & & \\ 1 & -[n] & \frac{[n]\,[n-1]}{[2]} & -\frac{[n]\,[n-1][n-2]}{[3]} & \cdots & (-1)^n & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \ddots \end{bmatrix}.$$

Definition 4.1.1. *A lower triangular matrix of the form $H_{q,p} = \Delta_q^{-1} D \Delta_q$ where D is the diagonal matrix with diagonal elements $p = (p_n) \in w$ and corresponding matrix method are called $q-$Hausdorff matrix and $q-$Hausdorff method respectively associated (or generated) by $p = (p_n)$.*

Theorem 4.1.2. *Given a sequence p_n, let D be the diagonal matrix with diagonal elements p_n. Then $H_{q,p} = (H_q, p_n) = (h^q_{nk})$ where*

$$h^q_{nk} = \begin{cases} \begin{bmatrix} n \\ v \end{bmatrix} \sum_{v=0}^{n-k} (-1)^v \begin{bmatrix} n-k \\ v \end{bmatrix} q^{\frac{v(v-1)}{2}} p_{v+k} & \text{if } 0 \le k \le n \\ 0 & \text{if } k > n \end{cases}. \tag{4.1.8}$$

In particular $h^q_{nn} = p_n$.

Proof. Using $H_{q,p} = \Delta_q^{-1} D \Delta_q$, we immediately get the following equalities for all $k \le n$,

$$\begin{aligned} h^q_{nk} &= \sum_v \mu_{nv} p_v \lambda_{vk} \\ &= \sum_{v=k}^{n} (-1)^v \begin{bmatrix} n \\ v \end{bmatrix} p_v (-1)^k \begin{bmatrix} v \\ k \end{bmatrix} q^{\frac{(v-k)(v-k-1)}{2}} \\ &= \sum_{v=0}^{n-k} (-1)^v \begin{bmatrix} n \\ v+k \end{bmatrix} \begin{bmatrix} v+k \\ k \end{bmatrix} q^{\frac{v(v-1)}{2}} p_{v+k} \end{aligned}$$

$$= \left[\begin{array}{c} n \\ k \end{array} \right] \sum_{v=0}^{n-k} (-1)^v \left[\begin{array}{c} n-k \\ v \end{array} \right] q^{\frac{v(v-1)}{2}} p_{v+k}.$$

□

Remark 4.1.3. *One can easily see that, in the case $q = 1$, $\Delta_q^{-1} D \Delta_q$ reduces to $\Delta D \Delta$. Therefore in the case of $q = 1$, each $q-$Hausdorff matrix reduces to an ordinary Hausdorff matrix.*

Next examples show that $q-$analogs of Cesáro methods of order α can be obtain from (4.1.8).

Example 4.1.1. *Take $p_n = \frac{q^n}{[n+1]_q}$ (the main diagonal of $C_1(q^k)$) in (4.1.8), we have*

$$\begin{aligned} h_{nk}^q &= \left[\begin{array}{c} n \\ k \end{array} \right] \sum_{v=0}^{n-k} (-1)^v \left[\begin{array}{c} n-k \\ v \end{array} \right] q^{\frac{v(v-1)}{2}} \frac{q^{\nu+k}}{[v+k+1]_q} \\ &= \left[\begin{array}{c} n \\ k \end{array} \right] q^k \sum_{v=0}^{n-k} (-1)^v \left[\begin{array}{c} n-k \\ v \end{array} \right] \frac{q^{\frac{v^2+v}{2}}}{[v+k+1]_q} \end{aligned}$$

which gives exactly $C_1 \left(q^k\right)$.

Example 4.1.2. *Similarly take $p_n = q^{\alpha n} / \left[\begin{array}{c} n+\alpha \\ n \end{array} \right]$ in (4.1.8), then corresponding $q-$Hausdorff matrix is the $q-$analog of the Cesáro matrix of order α, given in (3.1.9).*

Example 4.1.3. *Recall that $H_q^\alpha = (C_1(q^k))^\alpha$ for each $\alpha \in \mathbb{N}$. Therefore the diagonal of H^α will be the $\alpha-$th power of the diagonal of $C_1 \left(q^k\right)$. In other words if we take $p_n = \frac{q^{n\alpha}}{[n+1]_q^\alpha}$ then we have*

$$\begin{aligned} h_{nk}^q &= \left[\begin{array}{c} n \\ k \end{array} \right] \sum_{v=0}^{n-k} (-1)^v \left[\begin{array}{c} n-k \\ v \end{array} \right] q^{\frac{v(v-1)}{2}} \frac{q^{(\nu+k)\alpha}}{[v+k+1]^\alpha} \\ &= \left[\begin{array}{c} n \\ k \end{array} \right] q^{\alpha k} \sum_{v=0}^{n-k} (-1)^v \left[\begin{array}{c} n-k \\ v \end{array} \right] \frac{q^{\frac{\nu^2-\nu+2\alpha\nu}{2}}}{[v+k+1]^\alpha} \end{aligned}$$

which gives the $q-$analog of the Hölder Method of order α. More precisely for $\alpha = 2$,

$$H_q^2 = \begin{bmatrix} 1 & 0 & \cdots & & & \\ \frac{1}{[2]_q} + \frac{q}{[2]_q^2} & \frac{q^2}{[2]_q^2} & 0 & \cdots & & \\ \frac{1}{[3]_q} + \frac{q}{[2]_q[3]_q} + \frac{q^2}{[3]_q^2} & \frac{q^2}{[2]_q[3]_q} + \frac{q^3}{[3]_q^2} & \frac{q^4}{[3]_q^2} & 0 & \cdots & \\ \cdots & \cdots & \cdots & \cdots & & \\ \frac{1}{[n+1]_q} + \frac{q}{[2]_q[n+1]_q} + \cdots + \frac{q^n}{[n+1]_q^2} & \frac{q^2}{[2]_q[n+1]_q} + \cdots + \frac{q^{n+1}}{[n+1]_q^2} & \cdots & \frac{q^{2n}}{[n+1]_q^2} & 0 & \cdots \\ \vdots & \vdots & & \vdots & & \end{bmatrix}$$

Finally we will give an example of $q-$analog of the Euler Matrix.

Example 4.1.4. *Choose $p_n = \alpha^n$ in (4.1.8) then we have*

$$\begin{aligned} h_{nk}^q &= \begin{bmatrix} n \\ k \end{bmatrix} \sum_{v=0}^{n-k} (-1)^v \begin{bmatrix} n-k \\ v \end{bmatrix} q^{\frac{v(v-1)}{2}} \alpha^{\nu+k} \\ &= \begin{bmatrix} n \\ k \end{bmatrix} \alpha^k \sum_{v=0}^{n-k} (-1)^v \begin{bmatrix} n-k \\ v \end{bmatrix} q^{\frac{v(v-1)}{2}} \alpha^{\nu} \\ &= \begin{bmatrix} n \\ k \end{bmatrix} \alpha^k \, (1-\alpha)_q^{n-k}. \end{aligned}$$

which gives a $q-$analog of E_α^q the Euler matrix with Explicit form,

$$E_\alpha^q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & \\ (1-\alpha)_q^1 & \alpha & 0 & 0 & 0 & \cdots & \\ (1-\alpha)_q^2 & [2]\,\alpha\,(1-\alpha)_q^1 & [\alpha]^2 & 0 & 0 & \cdots & \\ \cdots & \cdots & \cdots & & \cdots & & \cdots \\ (1-\alpha)_q^n & [n]\,\alpha\,(1-\alpha)_q^{n-1} & \frac{[n][n-1]}{[2]}\alpha^2\,(1-\alpha)_q^{n-2} & \cdots & \alpha^n & 0 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \ddots & \ddots \end{bmatrix}.$$

4.2 Some Summability Properties of $q-$Hausdorff Matrices

Now we are going to show that, some basic properties of Hausdorff matrices can be given for $q-$Hausdorff matrices. Next theorem states that each $q-$Hausdorff matrix satisfies the row sum condition.

Theorem 4.2.1. *If $H_{q,p} = (h^q_{nk})$ is any $q-$Hausdorff matrix, then $\sum_k h^q_{nk} = p_0$ $(n \in \mathbb{N}^0)$. In other words $H_{q,p}$ satisfies row sum condition with limit p_0.*

Proof. For every $n \in \mathbb{N}^0$, we have

$$\left(\sum_k h^q_{nk}\right)_n = (\Delta_q^{-1} D \Delta_q e)_n = (\Delta_q^{-1} D e^0)_n = p_0$$

where $e = (1, 1, ...)$ and $e^0 = (1, 0, ...)$. □

Theorem 4.2.2. *$q-$Hausdorff matrices are commutative and product of two $q-$Hausdorff matrices is also $q-$Hausdorff matrix. That is;*

$$H_{q,p}.H_{q,r} = (H_q, p_n r_n) = H_{q,r}.H_{q,p}.$$

Proof. Let $H_{q,p}$ and $H_{q,r}$ be $q-$Hausdorff matrices. Since $q-$Hausdorff matrices are row finite, we get

$$\begin{aligned} H_{q,p}.H_{q,r} &= \left(\Delta_q^{-1} D_1 \Delta_q\right)\left(\Delta_q^{-1} D_2 \Delta_q\right) = \Delta_q^{-1} D_1 \left(\Delta_q \Delta_q^{-1}\right) D_2 \Delta_q \\ &= \Delta_q^{-1} D_1 D_2 \Delta_q = \Delta_q^{-1} D_2 D_1 \Delta_q = \left(\Delta_q^{-1} D_2 \Delta_q\right)\left(\Delta_q^{-1} D_1 \Delta_q\right) = H_{q,r}.H_{q,p} \end{aligned}$$

where D_1 and D_2 are the diagonal matrices with diagonal elements $p = (p_n)$ and $r = (r_n)$ respectively.

On the other hand, consider the diagonal matrix $D_3 = D_1 D_2$, then

$$\Delta_q^{-1} D_3 \Delta_q = (H_q, p_n q_n)$$

which means that $H_{q,p}.H_{q,r} = (H_q, p_n q_n)$ is Hausdorff. □

Proposition 4.2.1. *$(H_{q,p})^{-1}$ exists if and only if $p_n \neq 0$ for all $n \in \mathbb{N}^0$. If it exists, $(H_{q,p})^{-1} = (H_q, p_n^{-1})$.*

Proof. Assume that $(H_{q,p})^{-1}$ exists. Since $(H_{q,p})$ is an invertible, triangular matrix, $p_n \neq 0$, for each n.

Conversely assume that $p_n \neq 0$, for each n.If $H_{q,p^{-1}} = (H_q, p_n^{-1})$ is the $q-$Hausdorff matrix generated by p_n^{-1}, we get

$$\begin{aligned} H_{q,p}H_{q,p^{-1}} &= (H_q, p_n p_n^{-1}) = (H_q, e) = I \\ H_{q,p^{-1}}H_{q,p} &= (H_q, p_n^{-1} p_n) = (H_q, e) = I \end{aligned}$$

which proves $H_{q,p^{-1}}$ is the inverse of $H_{q,p}$. □

Theorem 4.2.3. *Assume that $H_{q,p}$ is a $q-$Hausdorff matrix generated by $p = (p_n)$ with $p_n \neq p_k$ $(n \neq k)$. Then a lower triangular matrix A is $q-$Hausdorff if and only if $AH_{q,p} = H_{q,p}A$.*

Proof. If A is $q-$Hausdorff matrix then from Theorem 4.2.2 $AH_{q,p} = H_{q,p}A$. To prove sufficiency, let us assume that $AH_{q,p} = H_{q,p}A$, according to the definition of $H_{q,p}$, we get

$$A\Delta_q^{-1}D\Delta_q = \Delta_q^{-1}D\Delta_q A \tag{4.2.1}$$

where D is diagonal matrix with diagonal elements $p = (p_n)$. Multiply both sides of (4.2.1) from left by Δ_q and from right with Δ_q^{-1}, we obtain

$$\Delta_q A\Delta_q^{-1}D\Delta_q\Delta_q^{-1} = \Delta_q\Delta_q^{-1}D\Delta_q A\Delta_q^{-1} \tag{4.2.2}$$

$$\Delta_q A\Delta_q^{-1}D = D\Delta_q A\Delta_q^{-1}. \tag{4.2.3}$$

Therefore if we choose $B = (b_{nk}) = \Delta_q A\Delta_q^{-1}$, (4.2.2) reduced to

$$BD = DB$$

or

$$b_{nk}p_k = p_n b_{nk}.$$

Since $p_n \neq p_k$ for $k \neq n$, we have $b_{nk} = 0$, when $n \neq k$. Hence $B = \Delta_q A \Delta_q^{-1}$ is the diagonal matrix with diagonal elements b_{nn}.Thus $A = \Delta_q^{-1} B \Delta_q$ which completes the proof. □

Theorem 4.2.4. *Regular $q-$Hausdorff methods are pairwise consistent.*

Proof. Let $H_{q,p}$ and $H_{q,r}$ be two regular $q-$Hausdorff matrices and $x \in c_{H_{q,p}} \cap c_{H_{q,r}}$ be given, that is $H_{q,p}x \in c$ and $H_{q,r}x \in c$. Then

$$\begin{aligned} \lim_{H_{q,p}} x &= \lim H_{q,p}x = \lim_{H_{q,r}} H_{q,p}x = \lim(H_{q,r}H_{q,p})x \\ &= \lim(H_{q,p}H_{q,r})x = \lim_{H_{q,p}} H_{q,r}x = \lim H_{q,r}x = \lim_{H_{q,r}} x. \end{aligned}$$

□

Next, we give a theorem to compare two Hausdorff matrices.

Theorem 4.2.5. *If $H_{q,p}$ and $H_{q,r}$ are $q-$Hausdorff matrices and $H_{q,p}$ is triangle then*

(i) $H_{q,r}$ is stronger than $H_{q,p}$ if and only if $(H_q, \frac{r_n}{p_n})$ is conservative.

(ii) $H_{q,r}$ is stronger than and consistent with $H_{q,p}$ if and only if $(H_q, \frac{r_n}{p_n})$ is regular.

Proof. Since $H_{q,p}$ is triangle, $H_{q,r}$ is row finite and $(H_{q,p})^{-1} = (H_q, \frac{1}{p_n})$. Therefore, (i) and (ii) satisfied by the help of Theorem 2.1.9. □

REFERENCES

[1] Agnew, R.P., Euler Transformation, American journal of Mathematics, vol. 66.,No. 2, 313-338, 1944.

[2] Aktuğlu H., Bekar Ş, q-Cesáro matrix and q-Statistical convergence, Journal of Computational and Applied Mathematics 2010, (accepted for publication).

[3] Aktuğlu H., Özarslan M.A., Gezer H., A-equistatistical convergence of positive linear operators. J. Comput. Anal. Appl. 12 (2010), no. 1-A, 24–36. (SCI-Expanded)

[4] Aktuğlu H., Özarslan M.A., Duman O., Matrix summability methods on the approximation of multivariate q-MKZ operators, Bulletin of the Malaysian Math. Soc., In Press.

[5] Altin A., Dogru O. and Özarslan M.A., Rates of Convegence of Meyer-König and Zeller Operators Based on q-Integers, WSEAS Trans. on Math., 4 (4) (2005), 313-318.

[6] Boss, J., Cass, P., Classical and Modern Methods in Summability, Oxford University Press, 2000.

[7] Bustoz, J., Gordillo, L. F., $q-$Hausdorff Summability, Journal of Computational Analysis and Applications 7, 35-48, 2005.

[8] Connor, J. and Kline, J., On statistical limit points and consistency of statistical convergence. J. Math. Anal. Appl. 197(2), 392-399 1996.

[9] Connor, J., On strong matrix summability with respect to a modulus and statistical convergence, Can. Math. Bull. 32(2), 194-198, 1989.

[10] Connor, J., The statistical and strong p-Cesáro convergence of sequences, Analysis 8(1-2), 47-63, 1988.

[11] Cooke, R., Infinite Matrices and Sequence Spaces (Russian translation), Fizmatgiz, Moscow (1960).

[12] DeBranges, L. and Trutt, D., Quantum Cesáro Operators, Topics in Functional Analysis, Academic Press, Newyork, 1978.

[13] Fast, H., Sur la convergence statistique, Collog. Math. 2(1951), 241-244.

[14] Freedman, A. R., Sember, J. J., Densities and Summability, Pacific J. Math. 95, 293-305, 1981.

[15] Fridy, J. A., On statistical convergence, Analysis 5, 301-313, 1985.

[16] Fridy, J. A.and H.I. Miller, A matrix characterization of statistical convergence, Analysis 11(1), 59-66, 1991.

[17] Fridy, J. A. and C. Orhan, Lacunary statistical summability, J. Math. Anal. Appl. 173(2), 497-504, 1993.

[18] Garabedian, H.L., Hausdorff matrices, The American Mathematical Monthly, vol 46,no 7, 390-410, 1939.

[19] Hardy, G. H., Divergent Series, Chelsea Publishing Company, 1949.

[20] Il'inskii, A., Ostrovska, S., Convergence of generalized Bernstein Polynomials, J. Approx. Theory, 116(1), 100-112, 2002.

[21] Kac, V., Pokman, C., Qu antum Calculus, Springer-Verlag, 2002.

[22] Kangro, G. F., Theory of summability of sequences and series, J. Sov Math 5, 1-45 1976.

[23] Kolk, E., Matrix summability of statistical convergent sequences, Analysis 13(1-2), 77-83, 1993.

[24] Kuttner, B. and Parameswaran, M. R., A comparison Between Euler and Cesàro Methods Of Summability, Proceedings Of Mathematical Society, vol. 122, no.3, 787-790,1994.

[25] Knopp, K., Über das Eulersche Summierungsverfahren, Math. Z., 15, 226-253, 1922.

[26] Lupaş, A., A q-analogue of the Bernstein operators, University of Cluj-Napoca, Seminar on Numerical and Statistical Calculus, No. 9, 1987.

[27] Mahmudov, Nazim I., Kaffaoglu, H., On q-Szász-Durrmeyer operators. Cent. Eur. J. Math. 8 (2010), no. 2, 399–409.

[28] Mahmudov, N., Sabancigil, P., Some approximation properties of q-parametric BBH operators. J. Comput. Anal. Appl. 12 (2010), no. 1-A, 111–123.

[29] Mahmudov, N., The moments for q-Bernstein operators in the case 0¡q¡1. Numer. Algorithms 53 (2010), no. 4, 439–450.

[30] Mahmudov, N. I., Sabancigil, P., q-parametric Bleimann Butzer and Hahn operators. J. Inequal. Appl. 2008.

[31] Mahmudov, N. I., Convergence properties and iterations for q-Stancu polynomials in compact disks. Comput. Math. Appl. 59 (2010), no. 12, 3763–3769.

[32] Mahmudov, N. I., On q-parametric Szász-Mirakjan operators. Mediterr. J. Math. 7 (2010), no. 3, 297–311.

[33] Mahmudov, N. I., Statistical approximation of Baskakov and Baskakov-Kantorovich operators based on the q-integers. Cent. Eur. J. Math. 8 (2010), no. 4, 816–826.

[34] Mahmudov, N. I., Sabancigil, Pembe On genuine q-Bernstein-Durrmeyer operators. Publ. Math. Debrecen 76(2010), no. 3-4, 465–479.

[35] Özarslan M.A., q-Laguerre Type Linear Positive Operators, Stud. Sci. Math. Hungarica, Vol. 44 (1) (2007), 65-80.

[36] Özarslan M.A., Duman O. and Doğru O., Rates of A-statistical convergence of approximating operators, Calcolo 42 (2005), 93-104.

[37] Özarslan M.A., Duman O. and Doğru O., A-Statistical convergence for a class of positive linear operators, Rev. Anal. Numer. Theor. Approx., 35 (2) (2006), 161-172.

[38] Özarslan M.A., Duman O. and Srivastava H.M., Statistical approximation results for Kantorovich-type operators involving some special functions, Math. Comput. Modelling, 48 (2008), 388 – 401

[39] Philips, G. M., On generalized Bernstein polynomials, Numerical Analysis, 263-269, World Sci. Publ. , River Edge, NJ, 1996.

[40] Philips, G. M., Interpolation and Approximation by Polynomials, CMS books in Mathematics, Springer-Verlag, 2003.

[41] Pólya, G., Szego,G., Problems in Analysis, Springer-Verlag, 1972.

[42] Powel, R. E., Shah, S.M., Summability Theory and its Applications, Van Nostrand, London, 1972.

[43] Rhoades, B.E., Some Hausdorff Matrices Not of Type M, Proceedings Of Mathematical Society, vol. 15, no. 3, 661-365, 1964.

[44] Zorlu S., Aktuğlu H., Özarslan M.A., An estimation to the solution of an initial value problem via \$q\$-Bernstein polynomials. J. Comput. Anal. Appl. 12 (2010), no. 3, 637–645.