Quantum Singularities in (2+1) Dimensional Matter Coupled Black Hole Spacetimes

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ABSTRACT

Quantum singularities are considered in matter coupled $2+1$ dimensional spacetimes in Einstein's theory. The occurrence of naked singularities in the spacetimes both in linear and non-linear electrodynamics in Einstein-Maxwell as well as in the Einstein-Maxwell-Dilaton gravity and pure magnetic Einstein-Power-Maxwell theory are considered. It is shown that the inclusion of the matter fields changes the geometry. The classical central singularity at $r = 0$ turns out to be quantum mechanically singular for quantum particles obeying Klein-Gordon equation but nonsingular for fermions obeying Dirac equation in all space times except the class of static pure magnetic spacetime.

The physical properties of the $2+1$ dimensional magnetically charged solutions in Einstein-Power-Maxwell theory with particular power k of the Maxwell field are investigated. The true timelike naked curvature singularity develops when $k > 1$ which constitutes one of the striking effects of the power Maxwell field. For specific power parameter k, the occurrence of timelike naked singularity is analysed in quantum mechanical point of view. It is shown that the class of static pure magnetic spacetime in the power Maxwell theory is quantum mechanically singular when it is probed with fields obeying Klein-Gordon and Dirac equations in the generic case.

Keywords: Quantum singularity, naked singularity, $2+1$ dimensional spacetimes

ÖZ

Einstein teorisi içinde, kuvantum tekillikleri madde eklenmiş 2+1 boyutlu uzay-zamanlarda çalışılmıştır. Çıplak tekilliklerin oluşumu doğrusal ve doğrusal olmayan elektrodinamik Einstein-Maxwell, hem de Einstein-Maxwell-Dilaton ve manyetik Einstein-Üslü-Maxwell teorilerinde incelenmiştir. Madde alanlarının eklenmesiyle geometrinin değiştiği gösterilmiştir. Statik manyetik uzay-zaman haricindeki tüm çalışılan uzay-zamanlarda $r = 0$ noktasındaki klasik merkezi tekilliğin Klein-Gordon denklemine uyan parçacıklar için kuvantum tekil kaldığı fakat Dirac denklemine uyan fermionlar için bu tekilliğin ortadan kalktığı görülmüştür.

Einstein-Üslü-Maxwell teorisinde $2+1$ boyutlu manyetik yüklü çözümlerin fiziksel özellikleri özel k kuvvetiyle incelenmiştir. $k > 1$ değerleri için zamansal, çıplak, eğrilik tekilliğinin oluştuğu görülmüştür ki bu durum üslü Maxwell alanının en büyük etkisidir. Belli bir k değeri için kuvantum mekaniksel açıdan zamansal çıplak tekilliğin oluşumu incelenmiştir. Üslü Maxwell teorisindeki statik manyetik uzay-zamanın Klein-Gordon ve Dirac alanları içerisinde incelendiğinde, kuvatum mekaniksel olarak tekil kaldığı gösterilmiştir.

Anahtar Kelimeler: Kuvantum tekillik, çıplak tekillik, $2+1$ boyutlu uzay-zamanlar

To My Parents

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INTRODUCTION

1.1 BTZ Black Hole

One of the most interesting structures found in the general theory of relativity are the black holes. In recent years, 2+1 dimensional black holes attract more attention because they carry all the characteristic features of a 3+1 dimensional black hole such as the event horizon and Hawking radiation. They also have a simple mathematical structure which provides a better understanding of the general aspects of black hole physics [1].

The standard Einstein equations in 2+1 spacetime dimensions, with a negative cosmological constant, give a black hole solution. This is known as Bañados-Teitelboim-Zanelli (BTZ) black hole which is locally anti-de Sitter space [2,3]. An unwanted outcome of the theory of general relativity is the occurrence of curvature singularities. Even in 2+1 dimension, the formation of curvature singularity is inevitable. A curvature singularity is found in the causal structure of the BTZ black hole. But, when a matter field is coupled true curvature singularity develops. In this thesis our main task is to analyse especially the naked singularities in the view of quantum mechanics.

Consider a spinless BTZ black hole. The three different kind of spacetimes arises depending on mass parameter, m [2,3]. The BTZ metric is described by the following metric,

$$ds^2 = -\left(-m + \frac{r^2}{l^2}\right) dt^2 + \left(-m + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\theta^2. \qquad (1)$$

Case 1: Vacuum state. This represents the case when $m = 0$ and the horizon size goes to zero.

Case 2: For $m > 0$, a black hole solution is admitted with a singularity in the causal structure at $r = 0$ with an additional pathology, a Taub-NUT type singularity (conical). An event horizon given by $r_+ = \sqrt{m}l$ hides the singularity where $l^2 = -\Lambda^{-1}$ and Λ is the cosmological constant. Cosmic censorship hypothesis (CCH) is preserved for this spacetime.

Case 3: As m grows negative with the constraint condition, $-1 < m < 0$, the conical singularity possessed at $r = 0$ becomes a naked singularity which violates the CCH.

When $m = -1$, there is no horizon and no singularity. An anti de-Sitter space is allowed and it is the ground state of the theory of quantum gravity [4]. If $m < -1$, the spacetime represents point sources with negative mass which have no physical meaning at all [5].

1.2 Singularities and Cosmic Censorship Hypothesis

In general, singularities are one of the most important issues in the Einstein's theory of relativity. A singular spacetime is described by geodesic incompleteness in the theory. If the time evolution of timelike or null geodesics is not defined after a proper time, the spacetime is a singular spacetime. The singularities occurring during the gravitational collapse of massive stars, black holes or in big-bang cosmologies are at $r = 0$, which is a typical central singularity.

There is/are horizon(s) around the singularity in black hole spacetimes. When there is/are no horizon(s) around the singularity, the singularity becomes a naked singularity. Naked singularities may be able to communicate with outside observers far away to affect the dynamics of the outside observers [6]. In 1969, Penrose [7] proposed the Cosmic Censorship Hypothesis (CCH) which states that the singularities forming in a general gravitational collapse should always be covered by the event horizons of gravity and remain invisible to any external observer. This hypothesis is not proven yet and it remains as one of the most significant unsolved problems in general relativity and gravitation physics [6].

Naked singularities violate CCH. Joshi states that occurrence of a naked singularity would imply a catastrophic breakdown of predictability (causality) in physics because arbitrary bursts of radiation or matter could be radiated in external universe by a naked singularity [6].

1.3 New Solutions and Studies

The stars which undergo a continual gravitational collapse to a spacetime singularity of infinite curvature and density, quantum gravity effects will become important in the very advance stages of the collapse at the scales of Planck length ($\sim 10^{-33}$cm) [6]. Horowitz and Myers [4] states that the true physics of curvature singularities will not be revealed until one has fully quantized gravity and these singularities will be "smoothed out" or "resolved" in the correct theory of quantum gravity. Therefore, the resolution of these singularities stands as an extremely important problem to be solved. Since naked singularity occurs at very small scale it is expected that quantum theory of gravity replaces classical general relativity. Therefore, it is worth to investigate the nature of this singularity with quantum test fields.

Horowitz and Marolf [8] have developed the idea of Wald [9], to probe the singularities using quantum test fields instead of classical point particles. These wave probes obey the Klein-Gordon equation for static spacetimes having timelike singularities. The propagation of the wave through the singularity may be in a definite and unique way. When you consider the hydrogen atom as an example, the wave function is finite at its origin, which is a classical singularity [10].

In this thesis, the criterion of Horowitz and Marolf [8] is used to probe the naked singularities that form in 2+1 dimensional matter coupled spacetimes. Our motivation here is to investigate the effects of matter fields on the quantum singularity structure of the BTZ spacetime. The surface at $r = 0$ for the BTZ black hole is not a curvature singularity. The curvature singularity is found in the causal structure of the BTZ black hole. But, when a matter field is coupled true curvature singularity develops. The effect of the matter fields, both in nonlinear and linear electrodynamics, as well as in the presence of dilaton field and magnetic charge, changes this picture completely and creates the true curvature singularity at $r = 0$. Furthermore, the spacetime geometry near the origin is not conic anymore. In view of these important physical effects of matter fields the singularity structure will be analysed in quantum mechanical point of view.

Chapter 2

DEFINITION AND CLASSIFICATION OF SINGULARITIES

2.1 Classical Singularities

In classical general relativity, a spacetime is assumed to be smooth without irregular points. A singularity can be thought to be as the boundary or the edge of the spacetime. A classical singularity in a maximal spacetime (i.e. given C^k, Hausdorff manifold M together with Lorentzian metric $g_{\mu\upsilon}$; (M, g)) is indicated by incomplete geodesics and/or incomplete curves of bounded acceleration [11, 12]. Ellis and Schmidt [13] classified the classical singularities depending on the differentiability assumed. In their assumption, the Hausdorff manifold M is C^∞ (which means that infinite times continuously differentiable) and that:

(i) the metric components $g_{\mu\upsilon}$ are continuous with locally bounded weak derivatives

(ii) the curvature tensor components $R^{\lambda}_{\mu\nu\sigma}$ are C^k (or C^{k-}) functions which means curvature tensor is k times continuously differentiable.

Then they call (M, g) a C^k (or C^{k-}) spacetime $(k \geq 0)$. Here, C^{k-} $(k \geq 1)$ means that the $(k-1)^{\text{th}}$ derivatives obeying Lipschitz conditions. A C^0 or C^{0-} function corresponds to continuous or locally bounded Riemann tensors. The classification uses a bundle (b)-boundary construction [14] to show the boundary-singularity relation. The brief review is as follows.

A set of boundary points, ∂M, is attached to M. ∂M is at a finite distance from points $r \in M$. The b-boundary of (M, g) is constructed by using the bundle $O(M)$ of orthonormal frames over M. Each curve $\gamma(\upsilon)$ in M that is incomplete when υ is a generalized affine parameter. Each curve, at least one, ends at a point $q \in \partial M$. In the work of Ellis and Schmidt [13], they let F_q to denote the family of incomplete curves in M ending at q. Then for each $q \in \partial M$, F_q is a nonempty set. In this case, they suggest two possibilities:

1-The point $q \in \partial M$ is a C^r *regular boundary point* $(r \geq 0^-)$ if there is an extension of the spacetime (M, g) into a larger spacetime (M', g') such that the Riemann tensor of (M, g) is $C^r$and q is an interior point of M'. Therefore, the spacetime is extendible and the singularity is removable.

2-The point $q \in \partial M$ is a C^r *singular boundary point* $(r \geq 0^-)$ if it is not a C^r regular boundary point. In this case, it is impossible to extend (M, g) through q in a C^r way.

Then, the singular boundary point q can be classified as:

i. Quasiregular Singularity

ii. Curvature Singularity: Scalar or Nonscalar

In their study, they give detailed information about the singular points and singularities which are summarized in the next sub-sections below.

2.1.1 Quasiregular Singularity

A singular boundary point $q \in \partial M$ is a $C^k (or\ C^{k-})$ *quasiregular singularity* $(k \geq 0)$ if it is not a $C^k (or\ C^{k-})$ curvature singularity. The curvature tensor components $R_{abcd;e_1 \ldots e_k}(\upsilon)$ measured in an orthonormal parallelly propagated frame behave in a $C^0 (or\ C^{0-})$ (bounded) way on all curves $\gamma(\upsilon)$ terminating at q. Near q the space geometry is locally well behaved. Although the singularity, q, cannot be removed the spacetime can be extended locally [13]. Quasiregular singularities are the weakest classical singularities. An observer never sees physical quantities to diverge. There is no curvature or tidal infinity at all. The ending of a classical particle path is associated with a topological obstruction to spacetime construction. These singularities can be observed along an idealized cosmic string and in TAUB-NUT spacetime [15, 16].

2.1.2 Curvature Singularity

Ellis and Schmidt [13] define a singular boundary point $q \in \partial M$ is a $C^k(or\ C^{k-})$ *curvature singularity* $(k \geq 0)$ where at least one curvature tensor component $R_{abcd;e_1 \ldots e_k}(v)$ does not behave in a $C^0(or\ C^{0-})$ (not bounded) way when an orthonormal tetrad parallel along a curve $\gamma(v)$ is used as a basis. As one come near to q, the space geometry is not locally well behaved. In curvature singularities, making an extension is prevented by the curvature of the spacetime.

2.1.2.1 Scalar Curvature Singularity

A point $q \in \partial M$ is a $C^k(or\ C^{k-})$ *scalar singularity* $(k \geq 0)$ where some scalars from the tensors g_{ab}, $\eta^{abcd}, R_{abcd;e_1 \ldots e_k}$ do not behave in a $C^0(or\ C^{0-})$ way (not bounded). Near q, physical quantities such as energy density and tidal forces diverge for all observers. This singularity is the strongest of the classical singularities. It is associated with infinite curvature scalar such as the centre of a black hole or the beginning of a Big Bang cosmology [13, 17].

2.1.2.2 Nonscalar Curvature Singularity

A curvature singularity $q \in \partial M$ is a $C^k(or\ C^{k-})$ *nonscalar singularity* $(k \geq 0)$ if it is not a $C^k(or\ C^{k-})$ scalar singularity. All scalars from tensors g_{ab}, $\eta^{abcd}, R_{abcd;e_1 \ldots e_k}$ behave in a $C^0(or\ C^{0-})$ way (or are bounded). An observer experiences infinite tidal forces at q. Whimper cosmologies or Bell-Szekeres solution which describes the non-linear interaction of two oppositely moving electromagnetic shock waves are examples of this type of singularities [13, 16].

In general, singularities are one of the most important issues in the Einstein's theory of relativity. There is an extensive effort to remove these singularities. As we mentioned before the scales where the singularities occur in black hole spacetimes or in big-bang cosmologies are smaller than the Planck scales. Therefore, the methods of classical general relativity are not applicable. In order to solve this important problem at these small scales we need to replace classical methods by the quantum theory of gravity which is still under consideration.

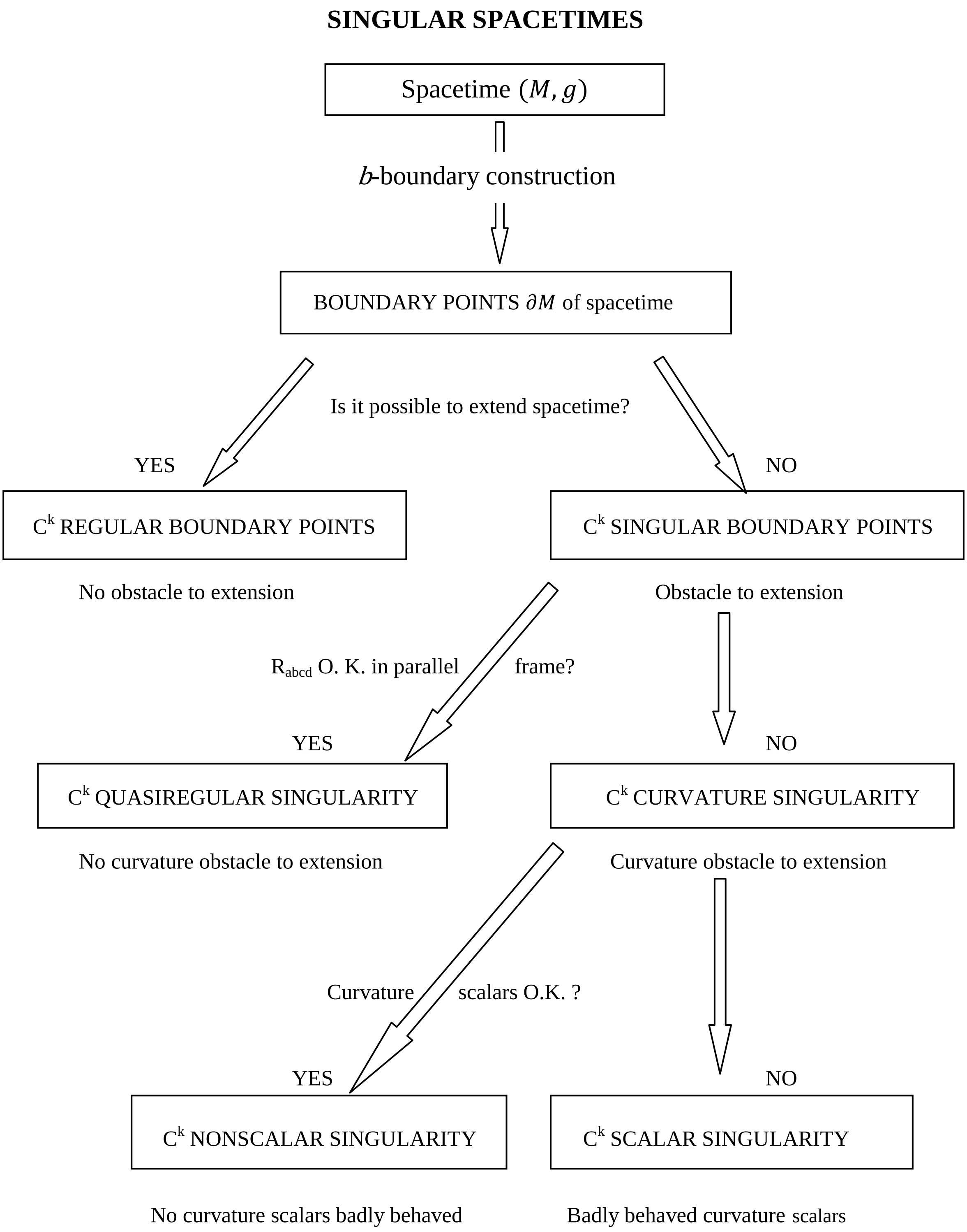

Fig. 1. The different types of finite boundary points and singularities for a spacetime (M, g) for classical point of view [13].

2.2 Quantum Singularities

Horowitz and Marolf [8] proposed that a spacetime is quantum mechanically singular if the evolution of a test wave packet is not uniquely determined by the initial data. They found the criteria to test the classical singularities with quantum test particles that obey the Klein-Gordon equation for static spacetime having timelike singularities. According to this criterion, the singular character of the spacetime is defined as the ambiguity in the evolution of the wave functions. That is the singular character is determined by attempting to find self-adjoint extension of the operator to the entire space. If the extension is unique, then the space is accepted quantum mechanically nonsingular.

An operator, A, is called self-adjoint if

(1) $A = A^{\dagger}$

(2) $Domain(A) = Domain(A^{\dagger})$

where $A^{\dagger}$ is the adjoint of A. An operator is essentially self-adjoint if

(1) is met and,

(2) can be met by expanding the domain of the operator or its adjoint so that it is true [18,19].

Horowitz and Marolf [8] considered a static spacetime $(M, g_{\mu\nu})$ with a timelike Killing vector field ξ^{μ}. Let t denotes the Killing parameter and Σ denote a static slice. The Klein-Gordon equation on this space is

$$\left(\nabla^{\mu}\nabla_{\mu} - m^2\right)\psi = 0 \tag{2}$$

This equation can be written in the form of

$$\frac{\partial^2\psi}{\partial t^2} = \sqrt{f}D^i\left(\sqrt{f}D_i\psi\right) - fm^2\psi = -\mathrm{A}\psi \tag{3}$$

in which $f = -\xi^\mu\xi_\mu$ and D_i is the spatial covariant derivative on Σ. The Hilbert space $\left(L^2(\Sigma)\right)$ is the space of square integrable functions on Σ. The domain of the operator A, $Domain(A)$ is taken in such a way that it does not include the spacetime singularities. An appropriate set is $C_0^\infty(\Sigma)$, the set of smooth functions with compact support on Σ. The self-adjoint extensions of operator A always exist since it is real, positive and symmetric. A is called essentially self-adjoint, if it has a unique extension A_E [18, 19, 20]. The Klein-Gordon equation for a free particle satisfies

$$i\frac{d\psi}{dt} = \sqrt{A_E}\psi \tag{4}$$

with the solution

$$\psi(t) = exp\left[-it\sqrt{A_E}\right]\psi(0) \tag{5}$$

If A is not essentially self-adjoint, the future time evolution of the wave function (equation (5)) is ambiguous. Horowitz and Marolf [8] define such a spacetime as quantum mechanically singular. However, if there is only one self-adjoint extension, the operator A is said to be essentially self-adjoint and the quantum evolution described by equation (5) is uniquely determined by the initial conditions. This spacetime is said to be quantum mechanically regular (nonsingular).

In order to determine the number of self-adjoint extensions, the concept of deficiency indices is used. The deficiency subspaces $N_\pm$ are defined by (see [10] for a detailed mathematical background),

$$N_{+} = \{\psi \in D(A^{*}), \qquad A^{*}\psi = Z_{+}\psi, \qquad ImZ_{+} > 0\} \quad \text{with dimension } n_{+} \tag{6}$$

$$N_{-} = \{\psi \in D(A^{*}), \qquad A^{*}\psi = Z_{-}\psi, \qquad ImZ_{-} < 0\} \quad \text{with dimension } n_{-} \tag{7}$$

The dimensions (n_{+}, n_{-}) are the deficiency indices of the operator A. The indices $n_{+}(n_{-})$ are completely independent of the choice of $Z_{+}(Z_{-})$ depending only on whether Z lies in the upper (lower) half complex plane. Generally one takes $Z_{+} = i\lambda$ and $Z_{-} = -i\lambda$, where λ is an arbitrary positive constant necessary for dimensional reasons. The determination of deficiency indices then reduces to counting the number of solutions of $A^{*}\psi = Z\psi$; (for $\lambda = 1$),

$$A^{*}\psi \pm i\psi = 0 \tag{8}$$

that belong to the Hilbert space $\mathcal{H}$. If there is no square integrable solutions (i.e. $n_{+} = n_{-} = 0$), the operator A is essentially self-adjoint as it possesses a unique self-adjoint extension. As a result, a sufficient condition for the operator A to be essentially self-adjoint is to investigate the solutions satisfying equation (8) that do not belong to the Hilbert space.

In general, for an $(n + 2)$-dimensional static spacetime is defined by the metric

$$ds^{2} = -V^{2}dt^{2} + h_{ij}dx^{i}dx^{j}. \tag{9}$$

A function space is chosen on each t=constant hypersurface Σ as the usual Hilbert space described by

$$\mathcal{H} = \{\Psi \mid \parallel \Psi \parallel < \infty\}, \tag{10}$$

with the following norm

$$\|\Psi\|^2 = \frac{q^2}{2}\int_{\Sigma} d\Sigma\, V^{-1}\Psi^*\Psi < \infty \qquad (11)$$

where q^2 is a positive constant, and $d\Sigma = \mathrm{d}^{\mathrm{n}+1}\mathrm{x}\sqrt{\mathrm{n}}$ is the natural volume element on Σ.

Another approach to remove the quantum singularity is to choose the function space to be the Sobolev space ($\mathcal{H}^1$). This function space is used to study quantum singularities first time by Ishibashi and Hosoya [10]. Here, the norm defined in as,

$$\|\Psi\|^2 = \frac{q^2}{2}\int_{\Sigma} d\Sigma\, V^{-1}\Psi^*\Psi + \frac{1}{2}\int_{\Sigma} d\Sigma\, V h_{ij} D_i \Psi^* D_j \Psi, \qquad (12)$$

in which D_i is the covariant derivative with respect to the induced metric h_{ij} on Σ. The square of the norm (12) involves both the wave function and its derivative to be square integrable. The failure in the square integrability indicates that the operator A is essentially self-adjoint and thus, the spacetime is "wave regular". It should be noted that the Sobolev space is not the natural quantum mechanical Hilbert space.

In the study of Horowitz and Marolf [8] they analysed the four dimensional negative mass Schwarzschild solution whose singularity is at $r = 0$. They found that the solution remains singular when probed with quantum test particles that obey the Klein-Gordon equation. In a similar study of Ishibashi and Hosoya [10], they found the solution to be regular if the function space is chosen to be the Sobolev space.

There are many studies that analyse singularities in quantum mechanical point of view by using the criterion of Horowitz and Marolf [8]. For example, there are studies about the singularities in quasi-regular spacetimes [15, 16]. In one of these studies four dimensional Gal'tsov-Letelier-Todd spacetime is analysed [15]. The quasi-regular singularity of this spacetime is probed with the Klein-Gordon-Maxwell and Dirac fields. The spacetime is found to be quantum mechanically singular independent of the type of field used to probe.

Pitelli and Letelier [5] analysed the singularity in the BTZ black hole without matter fields coupled. The BTZ black hole possesses a naked singularity when the mass parameter is bounded to $-1 < m < 0$. The Klein-Gordon and Dirac fields are used to probe the naked singularity. It is shown that the singularity remains quantum singular when tested by Klein-Gordon field and the singularity is healed when tested by fermions.

Pitelli and Letelier [21, 22] also studied the singularity of the global monopole. There is a scalar curvature singularity in the spacetime around a global monopole. This spacetime represents a symmetric cloud of cosmic strings, where strings intersect at a single point $r = 0$. The singularity is probed with Klein-Gordon field. It is found that the singularity remains singular quantum mechanically.

There are other similar studies that apply the criterion of Horowitz and Marolf [8]. For example, there are singularity analyses of bubbles and cylindrical shells [23] and of four dimensional static spherically symmetric Einstein-Maxwell-Dilaton black holes [24a]. Quantum singularities are also considered in Lovelock gravity [24b].

Chapter 3

REVIEW OF MATTER COUPLED 2+1 DIMENSIONAL SOLUTIONS AND SPACETIME STRUCTURES IN EINSTEIN'S THEORY

In this section 2+1 dimensional matter coupled solutions in Einstein-Maxwell, Einstein-Maxwell-Dilaton and Einstein-Power-Maxwell theories will be reviewed.

3.1 BTZ Black hole Coupled with Nonlinear Electrodynamics

The action describing (2+1) - dimensional Einstein theory coupled with non-linear electrodynamics is given by Cataldo [25] as,

$$S = \int \sqrt{g} \left(\frac{1}{16\pi} (R - 2\Lambda) + L(F) \right) d^3x \tag{13}$$

The Einstein-Maxwell field equations via variational principle read as,

$$\mathrm{G_{ab}} + \Lambda \mathrm{g_{ab}} = 8\pi \mathrm{T_{ab}} \tag{14}$$

$$\mathrm{T_{ab}} = \mathrm{g_{ab}}\, L(F) - F_{ac} F_b^c L_{,F} \tag{15}$$

$$\nabla_a \left(F^{ab} L(F) \right) = 0 \tag{16}$$

in which $L_{,F}$ stands for the derivative of $L(F)$ with respect to $\mathrm{F} = \frac{1}{4} F_{ab} F^{ab}$.

The non-linear field is chosen to make the energy momentum tensor (15) having a vanishing trace. The trace of the tensor gives,

$$\mathrm{T} = \mathrm{T_{ab}} \mathrm{g}^{ab} = 3\, L(F) - 4F L_{,F} \tag{17}$$

Hence, to have a vanishing trace, the electromagnetic Lagrangian is obtained as

$$L = c|F|^{\frac{3}{4}} \tag{18}$$

where c is an integration constant. With reference to the paper [25], the complete solution to the above action is given by the metric,

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\theta^2 \tag{19}$$

where the metric function $f(r)$ is given by,

$$f(r) = -m + \frac{r^2}{l^2} + \frac{4q^2}{3r} \tag{20}$$

Here $m > 0$ is the mass, q is the electric charge and $l^2 = -\Lambda^{-1}$ the case $\Lambda > 0$ $(\Lambda < 0)$, that corresponds with an asymptotically de-Sitter (anti de-Sitter) spacetime. This metric represents the BTZ spacetime in non-linear electrodynamics. If $\Lambda = 0$, we have an asymptotically flat solution coupled with Coulomb-like field

$$ds^2 = -\left(-m + \frac{4q^2}{3r}\right)dt^2 + \left(-m + \frac{4q^2}{3r}\right)^{-1} dr^2 + r^2 d\theta^2 \tag{21}$$

The Kretschmann scalar which indicates the occurrence of curvature singularity is given by,

$$\mathcal{K} = \frac{12}{l^4} + \frac{6\beta^2}{r^6} \tag{22}$$

in which $\beta = \frac{4q^2}{3}$. It is clear that $\mathrm{r} = 0$ is a typical central curvature singularity. According to the values of Λ, m and q, this singularity may be clothed by a single or double horizons (see the paper [25] for details).

To find the condition for naked singularities the metric function (20) is written in the following form,

$$f(r) = -\frac{m}{r}\left(r + \widetilde{\Lambda} r^3 - \frac{4\tilde{q}^2}{3}\right) \tag{23}$$

where $\widetilde{\Lambda} = \frac{\Lambda}{m}$ and $\tilde{q}^2 = \frac{q^2}{m}$. Since the range of coordinate r varies from 0 to infinity, the negative root will indicate the condition for a naked singularity. In order to find the roots, we set $f(r) = 0$ which yields

$$r^3 + \frac{r}{\widetilde{\Lambda}} - \frac{4\tilde{q}^2}{3\widetilde{\Lambda}} = 0. \tag{24}$$

To solve the equation, we introduce a new variable defined by

$$r = z - \frac{1}{3\Lambda z} \tag{25}$$

that transforms the equation to

$$27\tilde{\Lambda}^3 z^6 - 36\tilde{\Lambda}^2 q^2 z^3 - 1 = 0 \tag{26}$$

The solution of the equation is

$$r = u^{1/3} - \frac{1}{3\tilde{\Lambda} u^{1/3}} \tag{27}$$

in which $u = \frac{12\tilde{q}^2\tilde{\Lambda} \pm 2\sqrt{3\tilde{\Lambda}(12\tilde{q}^4\tilde{\Lambda}+1)}}{18\tilde{\Lambda}^2}$, with a constraint condition $3\tilde{\Lambda}(12\tilde{q}^4\tilde{\Lambda}+1) > 0$.

The equation (27) can be easily written as

$$r = a^{1/3}\left\{\left(1 \pm \frac{b}{a}\right)^{1/3} + \left(1 \mp \frac{b}{a}\right)^{1/3}\right\} \tag{28}$$

where $a = \frac{2\tilde{q}^2}{3\tilde{\Lambda}}$ and $b = \frac{\sqrt{3\tilde{\Lambda}(12\tilde{q}^4\tilde{\Lambda}+1)}}{9\tilde{\Lambda}^2}$. It can be verified easily that the expression inside the curly bracket in equation (28) is always positive. Hence, the only possibility for a negative root is $a < 0$. This implies $\tilde{\Lambda} < 0$. Therefore, the condition $12\tilde{q}^4\tilde{\Lambda} + 1 < 0$ is imposed from the constraint condition. As a result, for a naked singularity, $\tilde{\Lambda} < -\frac{1}{12\tilde{q}^4}$ or $\Lambda < -\frac{m^3}{12q^4}$ should be satisfied.

In the next chapter, we investigate the quantum singularity structure of the naked singularity that may arise if the constant coefficients satisfy $\Lambda < -\frac{m^3}{12q^4}$.

3.2 BTZ Black hole Coupled with Linear Electrodynamics

The metric for the charged BTZ spacetime in linear electrodynamics is given by Martinez [26],

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\theta^2 \tag{29}$$

with the metric function

$$f(r) = -m + \frac{r^2}{l^2} - 2q^2 ln\left(\frac{r}{l}\right) \tag{30}$$

where $m > 0$ is the mass, q is the electric charge and $l^2 = -\Lambda^{-1}$. The Kretschmann scalar is given by,

$$\mathcal{K} = \frac{12}{l^4} - \frac{8q^2}{r^2 l^2} + \frac{4q^4}{r^4} \tag{31}$$

which displays a power-law central curvature singularity at $r = 0$. According to the values of m, l and q, this central singularity is clothed by horizons or it remains naked. We investigate the quantum mechanical behaviour of the naked singularity. In order to find the condition for naked singularity, we set $f(r_h) = 0$ and the solution for $l = 1$ is

$$r = exp\left\{-\frac{m}{2q^2} - \frac{1}{2} LambertW\left(-\frac{1}{q^2} e^{-\frac{m}{q^2}}\right)\right\}, \tag{32}$$

in which LambertW represents the Lambert function [27]. Figure 1 displays (unmarked region) the possible values of m and q that result in naked singularity.

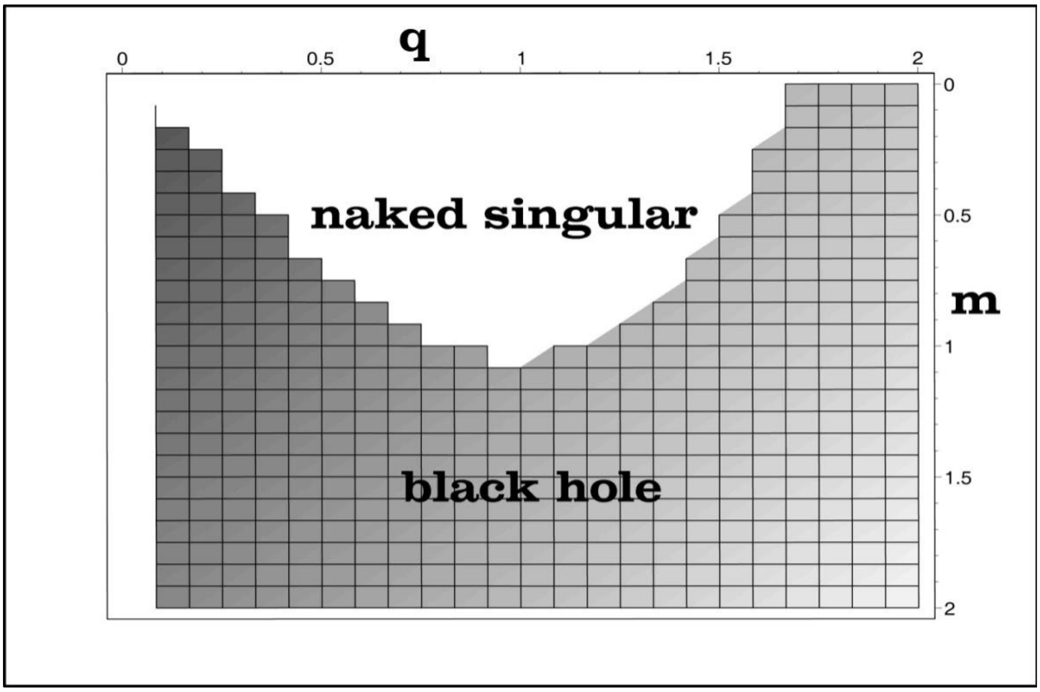

Figure 2. Graph of mass (m) versus electric charge (q) when $l = 1$.

3.3 (2+1) Dimensional Einstein-Maxwell-Dilaton Theory

We consider 3D black holes described by the Einstein-Maxwell-Dilaton action,

$$S = \int d^3x\sqrt{(-g)}\left(R - \frac{B}{2}(\nabla\ \phi)^2 - e^{-4a\phi}F_{\mu\nu}F^{\mu\nu} + 2e^{b\phi}\Lambda\right), \tag{33}$$

where ϕ is the dilaton field, R is the Ricci scalar, $F_{\mu\nu}$ is the Maxwell field and Λ, a, b, and B are arbitrary couplings. The general solution to this action is given by Chann and Mann [28],

$$ds^2 = -f(r)dt^2 + \frac{4r^{\left(\frac{4}{N}\right)-2}dr^2}{N^2\gamma^{\frac{4}{N}}f(r)} + r^2d\theta^2, \tag{34}$$

where

$$f(r) = Ar^{\left(\frac{2}{N}\right)-1} + \left(\frac{8\Lambda r^2}{(3N-2)N}\right) + \left(\frac{8Q^2}{(2-N)N}\right). \tag{35}$$

Here, A is an integration constant which is proportional to the quasilocal mass $\left(A = \frac{-2m}{N}\right)$, γ is a constant of integration and Q is the electric charge. The dilaton field is given by

$$\phi = \frac{2k}{N}ln\left(\frac{r}{\beta(\gamma)}\right) \tag{36}$$

in which $\beta(\gamma)$ is a γ related constant parameter. Note that, the above solution for $N = 2$ contains both the vacuum BTZ metric if one takes $Q = A = 0$ (where $f(r) = \frac{r^2}{l^2}$ with $\Lambda = \frac{-1}{l^2}$) and the BTZ black hole [2] below if $A < 0, Q = 0$.

$$ds^2 = -\left(-m + \frac{r^2}{l^2}\right)dt^2 + \left(-m + \frac{r^2}{l^2}\right)^{-1}dr^2 + r^2d\theta^2 \tag{37}$$

where $m > 0$.

However, if the constant parameters are chosen appropriately, the resulting metric represents black hole solutions with prescribed properties. For example, when $N = \frac{6}{5}$, $A = -\frac{5m}{3}$, the metric function given in equation (35) becomes

$$f(r) = -\frac{5m}{3} r^{\frac{2}{3}} + \frac{25\Lambda}{6} r^2 + \frac{25Q^2}{3}, \tag{38}$$

and therefore the corresponding metric is

$$ds^2 = -f(r)dt^2 + \frac{\alpha r^{\frac{4}{3}} dr^2}{f(r)} + r^2 d\theta^2, \tag{39}$$

where $\alpha = \frac{25}{9\gamma^{\frac{10}{3}}}$ is a constant parameter.

The Kretschmann scalar for this solution is given by

$$\mathcal{K} = \frac{25\left\{12m^2 r^{\frac{5}{3}} + 5\Lambda r^3\left[55\Lambda r^{\frac{4}{3}} - 4m\right] + 40 r^{\frac{1}{3}} Q^2\left[2\left(5Q^2 - m r^{\frac{2}{3}}\right) - 5\Lambda r^2\right]\right\}}{81\alpha^2 r^7}, \tag{40}$$

which indicates a central curvature singularity at $r = 0$ that is clothed by the event horizon. To find the location of horizons, g_{tt} is set to zero and we have

$$r^2 - \frac{2m}{5\Lambda} r^{\frac{2}{3}} + \frac{2Q^2}{\Lambda} = 0. \tag{41}$$

There are three possible cases to be considered.

Case 1: If $\frac{Q^2}{\Lambda} < \left(\frac{2m}{15\Lambda}\right)^{\frac{3}{2}}$, the equation admits two positive roots indicating inner and outer horizons of the black hole.

Case 2: If $\frac{Q^2}{\Lambda} = \left(\frac{2m}{15\Lambda}\right)^{\frac{3}{2}}$, this is an extreme case and the equation (41) has one real positive root. This means that there is only one horizon.

Case 3: If $\frac{Q^2}{\Lambda} > \left(\frac{2m}{15\Lambda}\right)^{\frac{3}{2}}$, there is no real positive root and the solution does not admit black hole so that the singularity at $r = 0$ is naked. With reference to the detailed analysis given in the paper [28], the Penrose diagram of the solution illustrates the timelike character of the singularity at $r = 0$. Our aim in the next chapter is to investigate the behaviour of this naked singularity when probed with Klein-Gordon and Dirac fields in the framework of quantum mechanics.

3.4 (2+1) Dimensional Magnetically Charged Solutions in Einstein-Power-Maxwell Theory

The 3-dimensional action in Einstein-power-Maxwell theory of gravity with a cosmological constant Λ $(8\pi G = 1)$ is given in our work [29] as

$$I = \frac{1}{2}\int dx^3\sqrt{-g}\left(R - \left(\frac{2}{3}\right)\Lambda - F^k\right), \tag{42}$$

in which $\mathcal{F}$ is the magnetic Maxwell invariant

$$\mathcal{F} = F_{\mu\nu}F^{\mu\nu}, \tag{43}$$

and the field 2-form

$$\boldsymbol{F} = B(r)dr \wedge d\theta. \tag{44}$$

where $B(r)$ stands for the magnetic field to be determined.

The metric anzats for 3-dimensions, is chosen as

$$ds^2 = -f_1(r)dt^2 + \frac{dr^2}{f_2(r)} + f_3(r)d\theta^2. \tag{45}$$

in which $f_i(r)$ are some unknown functions to be found. The parameter k in the action is a real constant which is restricted by the energy conditions (see the Appendix A). Note that $k = 1$ is a linear Maxwell limit and in our treatments we consider the case $k \neq 1$, so that our treatment do not cover the linear Maxwell limit. By varying with respect to the gauge potentials the Maxwell equation is obtained as

$$\mathbf{d}(\,{}^*\boldsymbol{F}\mathcal{F}^{k-1}) = 0, \tag{46}$$

where ***** means duality and **d**(.) stands for the exterior derivative. Remaining field equations are

$$G_\mu^\nu + \frac{1}{3}\Lambda\delta_\mu^\nu = T_\mu^\nu, \tag{47}$$

in which

$$T_\mu^\nu = -\frac{1}{2}(\delta_\mu^\nu F^k - 4k(F_{\mu\lambda}F^{\nu\lambda})\mathcal{F}^{k-1}), \tag{48}$$

is the energy-momentum tensor due to the non-linear electrodynamics (NED). Nonlinear Maxwell equation (46) determines the unknown magnetic field in the form

$$B^2 = \frac{f_3(r)}{f_2(r)} \frac{P^2}{f_1(r)^{\frac{1}{2k-1}}}, \tag{49}$$

in which P is interpreted as the magnetic charge. Imposing this into the energy-momentum tensor (48) results in

$$T_\nu^\mu = \frac{1}{2}\mathcal{F}^k diag(-1,2k-1,2k-1), \tag{50}$$

and the explicit form of $\mathcal{F}$ is given by

$$\mathcal{F} = 2\frac{P^2}{f_1(r)^{\frac{1}{2k-1}}}. \tag{51}$$

The exact solution comes after solving the Einstein equations (47), which is expressed by the metric functions

$$f_1(r) \equiv A(r) = -M + \frac{|\Lambda|}{3} r^2 = \frac{|\Lambda|}{3}(r^2 - r_+^2), \tag{52}$$

$$f_2(r) = \frac{1}{r^2}\left(r^2 + \frac{9\tilde{P}^2(2k-1)^2}{(k-1)\Lambda^2}\right)A(r), \tag{53}$$

$$f_3(r) = \frac{r^2}{A(r)} f_2(r), \quad k \neq 1, \tag{54}$$

where M may be interpreted as the mass, $\tilde{P}^2 = 2^{k-1}P^{2k}$. Note that $r_+^2 = \left|\frac{3M}{\Lambda}\right|$ which shouldn't be taken as a horizon radius since our solution doesn't represent a black hole. Ricci and Kretschmann scalars are given as

$$R = -2|\Lambda| - 8\tilde{P}^2\left(k - \frac{3}{4}\right)A^{-\frac{k}{2k-1}}, \tag{55}$$

$$\mathcal{K} = \frac{4}{3}\Lambda^2 + \frac{32}{3}\tilde{P}^2\left(k - \frac{3}{4}\right)|\Lambda| A^{-\frac{k}{2k-1}} + 4(8k(k-1)+3)\tilde{P}^4 A^{-\frac{2k}{2k-1}}. \tag{56}$$

As one observes, depending on k, one can put the solution into three general categories. In the first category, $\frac{1}{4} \leq k < \frac{1}{2}$ and therefore R and $\mathcal{K}$ are regular as the Weak Energy Condition (WEC) and Strong Energy Condition (SEC) (see Appendix) are both satisfied. Since, there may be $f_3(r_o) = 0$ for some r_o it suggests that the coordinate patch is not complete and needs to be revised. In such case we set

$$x^2 = r^2 - r_o^2 \tag{57}$$

which leads to the line element

$$ds^2 = -g_1(x)dt^2 + \frac{dx^2}{g_2(x)} + g_3(x)d\theta^2 \tag{58}$$

with the metric functions

$$g_1(x) = \frac{|\Lambda|}{3}(x^2 + {r_o}^2 - r_+^2), \tag{59}$$

$$g_2(x) = \left(x^2 + {r_o}^2 - \frac{9\tilde{P}^2(2k-1)^2}{|k-1|\Lambda^2} g_1(x)^{\frac{k-1}{2k-1}}\right)\frac{g_1(x)}{x^2}, \tag{60}$$

$$g_3(x) = \left(x^2 + {r_o}^2 - \frac{9\tilde{P}^2(2k-1)^2}{|k-1|\Lambda^2} g_1(x)^{\frac{k-1}{2k-1}}\right), \quad k \neq 1, \tag{61}$$

Here, one can show that for $x \in [0, \infty)$ then $g_3(x) < 0$, which implies a non-physical solution and hence the power in this interval $\frac{1}{4} \leq k < \frac{1}{2}$ should be excluded. The second category of solutions can be found by setting $\frac{1}{2} < k < 1$ in which $g_3(x) > 0$ possessing a non-singular solution. It should be noted that the case for $k = 1$ is already considered in [30, 31, 32, 33] where the resulting spacetime has no curvature singularity.

The third category of solutions is when $k > 1$ which results in a curvature singularity. Therefore, by shifting the coordinate in accordance with $y^2 = r^2 - {r_+}^2$ we relocate the singularity to the point $y = 0$ which will be a naked singularity. Our interest in this thesis will be confined entirely to this third category of solutions. In this new coordinate the line element reads as

$$ds^2 = -h_1(y)dt^2 + \frac{dy^2}{h_2(y)} + h_3(y)d\theta^2 \tag{62}$$

$$h_1(y) = \left(\frac{1}{3}\right)|\Lambda|y^2, \tag{63}$$

$$h_2(y) = \left(y^2 + r_+^2 + \frac{9\tilde{P}^2(2k-1)^2}{(k-1)\Lambda^2}\left(\frac{1}{3}|\Lambda|y^2\right)^{\frac{k-1}{2k-1}}\right)\left(\frac{|\Lambda|}{3}\right) \tag{64}$$

$$h_3(y) = \frac{3}{|\Lambda|}h_2(y), \quad k \neq 1, \tag{65}$$

Ricci and Kretschmann scalars are given as

$$R \;=\; -2|\Lambda| - 8\tilde{P}^2\left(k - \frac{3}{4}\right)\left(\frac{1}{3}|\Lambda|y^2\right)^{-\frac{k}{2k-1}}, \tag{66}$$

$$\mathcal{K} \;=$$

$$\frac{4}{3}\Lambda^2 + \frac{32}{3}\tilde{P}^2\left(k - \frac{3}{4}\right)|\Lambda|\left(\frac{1}{3}|\Lambda|y^2\right)^{-\frac{k}{2k-1}} + 4(8k(k-1)+3)\tilde{P}^4\left(\frac{1}{3}|\Lambda|y^2\right)^{-\frac{2k}{2k-1}}. \tag{67}$$

It can be seen that for $k > 1$, both R and K are singular at $y = 0$, and this singularity can easily be shown that it is naked timelike.

Chapter 4

QUANTUM SINGULARITIES IN (2+1) DIMENSIONAL MATTER COUPLED SPACETIMES

4.1 Analysis for Nonlinear Electrodynamics

4.1.1 Klein-Gordon Fields

The BTZ spacetime has the metric [2]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\theta^2. \tag{68}$$

By using separation of variables, $\psi = R(r)e^{in\theta}$, the radial portion of equation (8) is obtained as

$$R_n'' + \frac{(fr)'}{fr}R_n' - \frac{n^2}{fr^2}R_n - \frac{m^2}{f}R_n \pm \frac{i}{f^2}R_n = 0, \tag{69}$$

where a prime denotes derivative with respect to r.

i. When $r \to \infty$:

The Coulomb-like field in metric function (20) becomes negligibly small and hence, the metric function and the metric take the form

$$f(r) = \frac{r^2}{l^2}, \tag{70}$$

$$ds^2 \simeq -\left(\frac{r^2}{l^2}\right)dt^2 + \left(\frac{r^2}{l^2}\right)^{-1}dr^2 + r^2 d\theta^2, \tag{71}$$

respectively.

The metric (71) shows an asymptotically anti-de Sitter spacetime. When we insert the metric function in the general radial equation (69), it becomes

$$R_n'' + \frac{3}{r} R_n' - \frac{n^2 l^2}{r^4} R_n - \frac{m^2 l^2}{r^2} R_n \pm \frac{i l^4}{r^4} R_n = 0, \tag{72}$$

As $r \to \infty$, the last three terms become negligible and we get the final equation

$$R_n'' + \frac{3}{r} R_n' = 0. \tag{73}$$

Its solution is

$$R_n(r) = C_{1n} + C_{2n} r^{-2} \tag{74}$$

where C_{1n} and C_{2n} are arbitrary constants. When we check the square integrability of the solution

$$\text{first part} \qquad \sim \int \frac{1}{r} dr = \ln r \to \infty \tag{75}$$

$$\text{second part} \qquad \sim \int \frac{1}{r^5} dr = \frac{-1}{r^4} < \infty \tag{76}$$

So, $R(r)$ is square integrable only if $C_{1n} = 0$. So the asymptotic behaviour of $R(r)$ is given by $\sim \frac{1}{r^2}$. This particular case overlaps with the results already reported in [5]. Hence, no new result arises for this particular case. This is expected because the effect of source term vanishes for large values of r.

ii. When $r \to 0$:

The case near origin is topologically different compared to the analysis reported in [5]. Here, the spacetime is not conic. The metric function (20) becomes

$$f(r) = \frac{\beta}{r}, \tag{77}$$

where β is $\frac{4q^2}{3}$.

The approximate metric is given by,

$$ds^2 \simeq -\left(\frac{\beta}{r}\right) dt^2 + \left(\frac{\beta}{r}\right)^{-1} dr^2 + r^2 d\theta^2. \tag{78}$$

This metric can also be interpreted as the 2+1 dimensional topological Schwarzschild-like black hole geometry.

For the solution of the radial equation (69), a massless case (i.e. $m = 0$) is assumed since it is known that the initial value problem is well posed for $m \neq 0$.

$$R_n'' \pm i\frac{r^2}{f^2}R_n - \frac{n^2}{\beta r}R_n = 0. \tag{79}$$

We ignore the term $i\frac{r^2}{f^2}R_n$ term since it can be neglected near the origin. Then the final form is ,

$$R_n'' - \frac{n^2}{\beta r}R_n = 0. \tag{80a}$$

The solution is given as

$$R_n(r) = A_{1n}\sqrt{r}J_1(ik) + A_{2n}\sqrt{r}N_1(ik). \tag{80b}$$

In order to make analysis in a simpler way we prefer to write the above solution in terms of modified Bessel functions,

$$R_n(r) = C_{1n}\sqrt{r}I_1(k) + C_{2n}\sqrt{r}K_1(k). \tag{81}$$

where $I_1(k)$ and $K_1(k)$ are the first and second kind modified Bessel functions and $k = \sqrt{\frac{4n^2 r}{\beta}}$. The modified Bessel functions for real $\nu \geq 0$ as $r \to 0$ are given by;

$$I_\nu(x) \simeq \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu, \tag{82}$$

$$K_\nu(x) \simeq \begin{Bmatrix} -\left[ln\left(\frac{x}{2}\right) + 0.5772\,...\right], & \nu = 0 \\ \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu, & \nu \neq 0 \end{Bmatrix}, \tag{83}$$

thus $I_1(k) \sim \frac{1}{\Gamma(2)}\left(\frac{k}{2}\right)$ and $K_1(k) \sim \frac{\Gamma(1)}{2}\left(\frac{2}{k}\right)$. Checking for the square integrability of the solution (81) requires the behaviour of the integral for

$$\int r\frac{r}{\beta}\left|\sqrt{r}I_1(k)\right|^2 \approx \int r^4 dr \tag{84}$$

$$\int r\frac{r}{\beta}\left|\sqrt{r}K_1(k)\right|^2 \approx \int r^2 dr \tag{85}$$

which are both convergent as $r \to 0$. Any linear combination is also square integrable. The operator A described in equation (8) is not essentially self-adjoint because the solution (81) belong to the Hilbert space, $\mathcal{H}$. Therefore, the naked singularity at $r = 0$ is quantum mechanically singular if it is probed with quantum particles.

According to Sobolev norm, the first integral is square integrable while the second integral behaves for the functions $I_1(k)\ as\ \approx \int_0 dr$ and $K_1(k)$ integral vanishes. As a result, the wave functions are square integrable and thus the spacetime is quantum mechanically wave singular.

4.1.2 Dirac Fields

We apply the same methodology as in [5] for finding a solution to Dirac equation. Since the fermions have only one spin polarization in 2+1 dimensions [34] Dirac matrices are reduced to Pauli matrices [35] so that,

$$\gamma^{(j)} = (\sigma^{(3)}, i\sigma^{(1)}, i\sigma^{(2)}), \tag{86}$$

where latin indices represent internal (local) indices. Pauli matrices are given as

$$\sigma^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{87}$$

The anticommutator relation is given as

$$\{\gamma^i, \gamma^j\} = 2\eta^{(ij)} I_{2\times 2}, \tag{88}$$

where $\eta^{(ij)}$ is the Minkowski metric in 2+1 dimensions and $I_{2\times 2}$ is the identity matrix. The coordinate dependent metric tensor $g_{\mu\upsilon}(x)$ and matrices $\sigma^{\mu}(x)$ are related to the triads $e_{\ \mu}^{(i)}(x)$ by

$$g_{\mu\upsilon}(x) = e_{\ \mu}^{(i)}(x) e_{\ \mu}^{(i)}(x) \eta_{(ij)}, \tag{89}$$

$$\sigma^{\mu}(x) = e_{\ (i)}^{\mu}(x) \gamma^{(i)}, \tag{90}$$

where μ and ν are the external (global) indices.

The Dirac equation in 2+1 dimensional curved spacetime for a free particle with mass M becomes

$$i\sigma^{\mu}(x)\left[\partial_{\mu} - \Gamma_{\mu}(x)\right]\Psi(x) = M\Psi(x), \tag{91}$$

where $\Gamma_{\mu}(x)$ is the spinorial affine connection and is given by

$$\Gamma_{\mu}(x) = \left(\frac{1}{4}\right) g_{\lambda\alpha}\left[e_{\ \nu,\mu}^{(i)}(x)e_{\ (i)}^{\alpha}(x) - \Gamma_{\nu\mu}^{\alpha}(x)\right]s^{\lambda\nu}(x), \tag{92}$$

$$s^{\lambda\nu}(x) = \left(\frac{1}{2}\right)\left[\sigma^{\lambda}(x), \sigma^{\nu}(x)\right] \tag{93}$$

The causal structure of the spacetime indicates that there are two singular cases to be investigated. For the asymptotic case, $r \to \infty$, the triad for the metric (71) is chosen as

$$e_{\ \mu}^{(i)}(t, r, \theta) = diag\left(\frac{r}{l}, \frac{l}{r}, r\right) \tag{94}$$

The spinorial affine connection and the coordinate dependent gamma matrix are found to be

$$\Gamma_{\mu}(x) \ = \left(\frac{r\sigma^{(2)}}{2l^2}, 0, \frac{ir}{2l}\sigma^{(3)}\right). \tag{95}$$

$$\sigma^{\mu}(x) \ = \left(\frac{l}{r}\sigma^{(3)}, i\frac{r}{l}\sigma^{(1)}, \frac{i\sigma^{(2)}}{r}\right), \tag{96}$$

For the spinor

$$\Psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}, \tag{97}$$

the Dirac equation is written as

$$\frac{il}{r}\frac{\partial\psi_1}{\partial t} - \frac{r}{l}\frac{\partial\psi_2}{\partial r} + \frac{i}{r}\frac{\partial\psi_2}{\partial\theta} - \frac{1}{l}\psi_2 - M\psi_1 = 0, \tag{98}$$

$$-\frac{il}{r}\frac{\partial\psi_2}{\partial t} - \frac{r}{l}\frac{\partial\psi_1}{\partial r} - \frac{i}{r}\frac{\partial\psi_1}{\partial\theta} - \frac{1}{l}\psi_1 - M\psi_2 = 0. \tag{99}$$

The following anzats will be employed for the positive frequency solutions:

$$\Psi_{n,E}(t, x) = \begin{pmatrix}R_{1n}(r) \\ R_{2n}(r)e^{i\theta}\end{pmatrix} e^{in\theta}e^{-iEt}. \tag{100}$$

As $r \to \infty$, two coupled equations are obtained:

$$R'_{2n} + \frac{1}{r}R_{2n} + \frac{lM}{r}R_{1n}e^{-i\theta} = 0 \tag{101}$$

$$R'_{1n} + \frac{1}{r}R_{1n} + \frac{lM}{r}R_{2n}e^{i\theta} = 0. \tag{102}$$

Therefore, for both components same equation is obtained as

$$R''_j + \frac{3}{r}R'_j + \frac{1}{r^2}(1 - M^2l^2)R_j = 0 \quad (j = 1,2). \tag{103}$$

Neglecting the higher order terms give the equation

$$R''_j + \frac{3}{r}R'_j = 0, \tag{104}$$

with the solution

$$R(r) = Ar^{-2} + B. \tag{105}$$

A and B are constant spinors. The condition for the Dirac operator to be quantum-mechanically regular requires that both solutions should belong to the Hilbert space $\mathcal{H}$ .The case above has already been analysed by [5]. The solution (105) is square-integrable only if $B = 0$. Then the solution is finite near infinity and there is no need for extra boundary conditions.

The case of $r \to 0$ is not conical so there is a topological difference in the spacetime near $r = 0$. Hence, the suitable triads for the metric (78) are given by,

$$e_{\ \mu}^{(i)}(t,r,\theta) = diag\left(\left(\frac{\beta}{r}\right)^{\frac{1}{2}}, \left(\frac{r}{\beta}\right)^{\frac{1}{2}}, r\right), \tag{106}$$

The spinorial affine connection and the coordinate dependent gamma matrices are given by

$$\Gamma_\mu(x) = \left(\frac{-\beta\sigma^{(2)}}{4r^2}, 0, \frac{i}{2}\left(\frac{\beta}{r}\right)^{\frac{1}{2}}\sigma^{(3)}\right). \tag{107}$$

$$\sigma^\mu(x) = \left(\left(\frac{r}{\beta}\right)^{\frac{1}{2}}\sigma^{(3)}, i\left(\frac{\beta}{r}\right)^{\frac{1}{2}}\sigma^{(1)}, \frac{i\sigma^{(2)}}{r}\right), \tag{108}$$

Now, for the spinor given in (97) the Dirac equation can be written as

$$i\left(\frac{r}{\beta}\right)^{\frac{1}{2}}\frac{\partial\psi_1}{\partial t}-\left(\frac{\beta}{r}\right)^{\frac{1}{2}}\frac{\partial\psi_2}{\partial r}+\frac{i}{r}\frac{\partial\psi_2}{\partial\theta}-\frac{1}{4}\left(\frac{\beta}{r^3}\right)^{\frac{1}{2}}\psi_2-M\psi_1=0, \tag{109}$$

$$-i\left(\frac{r}{\beta}\right)^{\frac{1}{2}}\frac{\partial\psi_2}{\partial t}-\left(\frac{\beta}{r}\right)^{\frac{1}{2}}\frac{\partial\psi_1}{\partial r}-\frac{i}{r}\frac{\partial\psi_1}{\partial\theta}-\frac{1}{4}\left(\frac{\beta}{r^3}\right)^{\frac{1}{2}}\psi_1-M\psi_2=0. \tag{110}$$

As a result, two coupled equations are obtained:

$$R'_{2n}+\left[\frac{1}{4r}+\frac{n+1}{\sqrt{r\beta}}\right]R_{2n}+\left[M\sqrt{\frac{r}{\beta}}-\frac{Er}{\beta}\right]R_{1n}e^{-i\theta}=0 \tag{111}$$

$$R'_{1n}+\left[\frac{1}{4r}-\frac{n}{\sqrt{r\beta}}\right]R_{1n}+\left[M\sqrt{\frac{r}{\beta}}+\frac{Er}{\beta}\right]R_{2n}e^{i\theta}=0. \tag{112}$$

With further analysis and simplification, the radial parts of the Dirac equation for investigating the behaviour as $r\to 0$, are

$$R''_{1n}+\frac{\alpha_1}{\sqrt{r}}R'_{1n}+\frac{\alpha_2}{r^{3/2}}R_{1n}=0, \tag{113}$$

$$R''_{2n}+\frac{\alpha_3}{\sqrt{r}}R'_{2n}+\frac{\alpha_4}{r^{3/2}}R_{2n}=0. \tag{114}$$

where $\alpha_1=\frac{2M-E}{2M\sqrt{\beta}}$, $\alpha_2=\frac{-7E+4M(4n+1)}{16M\sqrt{\beta}}$, $\alpha_3=\frac{2M+E}{2M\sqrt{\beta}}$ and $\alpha_4=\frac{7E-4M(4n+3)}{16M\sqrt{\beta}}$. Then, for the sake of making the analysis in a simpler way we prefer to express the solutions as,

$$R_1(r)=e^{-\frac{b}{2}\rho}\left\{C_1\sqrt{\rho}\,\mathrm{Whittaker_M}(a,1,b\rho)+C_2\sqrt{\rho}\,\mathrm{Whittaker_W}(a,1,b\rho)\right\}, \tag{115}$$

$$R_2(r)=e^{-\frac{b'}{2}\rho}\left\{C_3\sqrt{\rho}\,\mathrm{Whittaker_M}(a',1,b'\rho)+C_4\sqrt{\rho}\,\mathrm{Whittaker_W}(a',1,b'\rho)\right\}, \tag{116}$$

where $\rho=\sqrt{r}$, $a=\frac{-9E+8M(1+2n)}{4(2M-E)}$, $a'=\frac{9E-8M(1+2n)}{4(2M+E)}$, $b=2\alpha_1$, and $b'=2\alpha_3$.

When we look for the square integrability of the above solutions, we obtained that both functions $\mathrm{Whittaker_M}$ and $\mathrm{Whittaker_W}$ are square integrable near $\rho=0$ (or $r=0$) for both $R_1(r)$ and $R_2(r)$. One has,

$$\int rf^{-1}\mid R\mid^2 dr\approx\int\rho^6 e^{-b\rho}[\mathrm{Whittaker_M}(a,1,b\rho)]^2 d\rho<\infty, \tag{117}$$

and

$$\approx\int\rho^6 e^{-b\rho}[\mathrm{Whittaker_W}(a,1,b\rho)]^2 d\rho<\infty. \tag{118}$$

We note that these results are verified first by expanding the Whittaker functions in series form up to the order of $O(\rho^6)$ and then by integrating term by term in the limit as $r \to 0$.

For the spacetime (79), the set of solutions for the Dirac equation is given by

$$\Psi_{n,E}(t,x)$$
$$= \begin{pmatrix} e^{-\frac{b}{2}\rho}\{C_1\sqrt{\rho}\text{Whittaker}_\text{M}(a,1,b\rho) + C_2\sqrt{\rho}\text{Whittaker}_\text{W}(a,1,b\rho)\} \\ e^{-\frac{b'}{2}\rho}\{C_3\sqrt{\rho}\text{Whittaker}_\text{M}(a',1,b'\rho) + C_4\sqrt{\rho}\text{Whittaker}_\text{W}(a',1,b'\rho)\}e^{i\theta} \end{pmatrix} e^{in\theta}e^{-iEt}. \tag{119}$$

and an arbitrary wave packet can be written as

$$\Psi(t,x) = \sum_{n=-\infty}^{+\infty} C_n \begin{pmatrix} e^{-\frac{b}{2}\rho}\sqrt{\rho}T(a,1,b\rho) \\ e^{-\frac{b'}{2}\rho}\sqrt{\rho}T'(a',1,b'\rho)e^{i\theta} \end{pmatrix} e^{in\theta}e^{-iEt} \tag{120}$$

where C_n is an arbitrary constant, and

$$T(a,1,b\rho) = \text{Whittaker}_\text{M}(a,1,b\rho) + \text{Whittaker}_\text{W}(a,1,b\rho) \tag{121}$$

$$T'(a',1,b'\rho) = \text{Whittaker}_\text{M}(a',1,b'\rho) + \text{Whittaker}_\text{W}(a',1,b'\rho). \tag{122}$$

Hence, initial condition $\Psi(0,x)$ is sufficient to determine the future time evolution of the particle. The spacetime is then quantum regular when tested by fermions.

4.2 Analysis for Linear Electrodynamics

4.2.1 Klein-Gordon Fields

The causal structure is similar to the case considered in the previous section. There are two singular cases to be investigated. The case for $r \to \infty$ is approximately the same case considered in [5] where the approximate metric is given as

$$ds^2 \approx -\left(\frac{r^2}{l^2}\right)dt^2 + \left(\frac{r^2}{l^2}\right)^{-1}dr^2 + r^2d\theta^2 \tag{123}$$

For small r $(r \to 0)$ values, the approximate metric can be written in the following form

$$ds^2 \approx -(2q^2 \mid ln(\tilde{r}) \mid)dt^2 + (2q^2 \mid ln(\tilde{r}) \mid)^{-1}dr^2 + r^2 d\theta^2, \qquad (124)$$

in which $\tilde{r} = \frac{r}{l} << 1$. The radial equation becomes

$$R_n'' + \frac{1}{\tilde{r}}\left(1 + \frac{1}{ln\tilde{r}}\right)R_n' + \frac{n^2}{2q^2 r^2 ln\tilde{r}}R_n = 0, \qquad (125)$$

since $\frac{r}{l} << 1$, we can transform the equation by writing $-x^2 = ln\tilde{r}$. As $\tilde{r} \to 0$, $x \to \infty$. The new equation becomes

$$\frac{d^2}{dx^2}R + \frac{1}{x}\frac{d}{dx}R - \frac{2n^2}{q^2}R = 0, \qquad (126a)$$

where its solution can be written in terms of zeroth order first and second kind Bessel functions,

$$R_n(x) = A_{1n}J_0\left(\frac{\sqrt{2}n}{q}ix\right) + A_{2n}N_0\left(\frac{\sqrt{2}n}{q}ix\right). \qquad (126b)$$

As we have done before, to make analysis in a simpler way we prefer to write the above solution in terms of modified Bessel functions,

$$R_n(x) = C_{1n}I_0\left(\frac{\sqrt{2}n}{q}x\right) + C_{2n}K_0\left(\frac{\sqrt{2}n}{q}x\right). \qquad (127)$$

The modified Bessel functions for $x >> 1$ are given as

$$I_0(x) \simeq \frac{e^x}{\sqrt{2\pi x}} \qquad (128)$$

$$K_0(x) \simeq \sqrt{\frac{\pi}{2x}}e^{-x} \qquad (129)$$

These functions are always square integrable for $x \to \infty$, that is

$$\int rf^{-1} \mid R \mid^2 dr \approx \int xe^{-2x^2}f^{-1} \mid R \mid^2 dx < \infty. \qquad (130)$$

These results indicate that charged BTZ black hole in linear electrodynamics is quantum mechanically singular when probed with quantum test particles obeying Klein-Gordon equation.

If we use the Sobolev norm (12), the second integral which involves the derivative of the wave function $I_0(x) \simeq \frac{e^x}{\sqrt{2\pi x}}$ becomes $\approx \int x^{-2} e^{2x}(2x-1)^2 dx$. Numerical integration has revealed that as

$$x \to \infty, \sim \int x^{-2} e^{2x}(2x-1)^2 dx \to \infty. \qquad (131)$$

On the other hand for the wave function $K_0(x) \simeq \sqrt{\frac{\pi}{2x}} e^{-x}$, the second integral in the Sobolev norm is solved numerically as

$$x \to \infty, \sim \int x^{-2} e^{-2x}(2x+1)^2 dx < \infty \qquad (132)$$

which is square integrable. As a result, charged coupled BTZ black hole in linear electrodynamics is quantum mechanically wave regular if and only if the arbitrary constant parameter is $C_{2n} = 0$ in equation (127).

Consequently, if the naked singularity both in linear and non-linear electrodynamics is probed with quantum test particles, the following results are obtained:

1) In classical point of view, the Kretschmann scalar in non-linear case diverges faster than in the linear case.

2) In quantum mechanical point of view, if the chosen function space is Sobolev space, spacetime remains singular for non-linear case, but the spacetime can be made wave regular for linear case.

From these results we may conclude that the structure of the naked singularity in the non-linear electrodynamics is deeper rooted than the singularity in the linear case.

4.2.2 Dirac Fields

The effect of the charge when $r \to \infty$ does not contribute as much as the term that contains the cosmological constant. Therefore, we ignore the mass and the charged terms in the metric function (30). This particular case has already been analysed in section 4.1.2.(i) and in the paper [5].

The contribution of the charge is dominant when $r \to 0$. The Dirac equation for the metric (124) is solved by using the same method demonstrated in the previous section. The chosen triad is

$$e_{\mu}^{\ (i)}(t,r,\theta) = diag\left(\alpha(\ln\tilde{r})^{\frac{1}{2}}, \frac{1}{\alpha(\ln\tilde{r})^{\frac{1}{2}}}, r\right), \tag{133}$$

where $\alpha^2 = 2q^2$. The spinorial affine connection and the coordinate dependent gamma matrices are given by

$$\Gamma_{\mu}(x) = \left(\frac{\alpha^2\sigma^{(2)}}{4r}, 0, \frac{i\alpha(\ln\tilde{r})^{\frac{1}{2}}}{2}\sigma^{(3)}\right). \tag{134}$$

$$\sigma^{\mu}(x) = \left(\frac{1}{\alpha(\ln\tilde{r})^{\frac{1}{2}}}\sigma^{(3)}, i\alpha(\ln\tilde{r})^{\frac{1}{2}}\sigma^{(1)}, \frac{i\sigma^{(2)}}{r}\right), \tag{135}$$

All these findings are inserted into Dirac equation (91) and for the anzats (100), two coupled equations are obtained:

$$R_{2n}' + \left[\frac{1}{2r} + \frac{1}{4r(\ln\tilde{r})^{\frac{1}{2}}} + \frac{n+1}{r\alpha(\ln\tilde{r})^{\frac{1}{2}}}\right]R_{2n} + \left[\frac{M}{\alpha(\ln\tilde{r})^{\frac{1}{2}}} - \frac{E}{\alpha^2\ln\tilde{r}}\right]R_{1n}e^{-i\theta} = 0 \tag{136}$$

$$R_{1n}' + \left[\frac{1}{2r} + \frac{1}{4r(\ln\tilde{r})^{\frac{1}{2}}} - \frac{n}{r\alpha(\ln\tilde{r})^{\frac{1}{2}}}\right]R_{1n} + \left[\frac{M}{\alpha(\ln\tilde{r})^{\frac{1}{2}}} + \frac{E}{\alpha^2\ln\tilde{r}}\right]R_{2n}e^{i\theta} = 0 \tag{137}$$

The radial equations are simplified to one single equation in the limit $r \to 0$ as

$$R_j'' + \frac{1}{r}R_j' - \frac{R_j}{4r^2} = 0, \qquad j = 1,2 \tag{138}$$

whose solution is given by

$$R_j(r) = C_{1j}\sqrt{r} + \left(\frac{C_{2j}}{\sqrt{r}}\right). \tag{139}$$

where C_{1j} and C_{2j} are arbitrary constants. The solution given in equation (139) is square integrable for both parts

$$\int \frac{r^2}{\ln \tilde{r}} dr < \infty \tag{140}$$

$$\int \frac{1}{\ln \tilde{r}} dr < \infty. \tag{141}$$

The arbitrary wave packet can be written as,

$$\Psi(t,x) = \sum_{n=-\infty}^{+\infty} C_n \begin{pmatrix} \sqrt{r} + \frac{1}{\sqrt{r}} \\ \left(\sqrt{r} + \frac{1}{\sqrt{r}}\right) e^{i\theta} \end{pmatrix} e^{in\theta} e^{-iEt} \tag{142}$$

Thus, the spacetime is quantum mechanically regular when probed with fermions.

4.3 Analysis for Einstein-Maxwell-Dilaton Theory

4.3.1 Klein-Gordon Fields

We get the same results as in 4.1.1.(i) for very large values of r ($r \to \infty$). So we obtain the radial equation for the metric (39) and consider the massless case ($m = 0$) as,

$$R_n'' + \frac{\left(fr^{\frac{1}{3}}\right)'}{fr^{\frac{1}{3}}} R_n' - \frac{\alpha n^2}{fr^{\frac{2}{3}}} R_n \pm \frac{i\alpha r^{\frac{4}{3}}}{f^2} R_n = 0. \tag{143}$$

where $f(r) = -\frac{5}{3} m r^{\frac{2}{3}} + \frac{25}{6}\Lambda r^2 + \frac{25}{3} Q^2$.

The behaviour of the radial equation as $r \to 0$ is

$$R_n'' + \frac{1}{3r} R_n' - \frac{k^2}{r^{\frac{2}{3}}} R_n = 0, \tag{144}$$

where $k = \frac{3\alpha n^2}{25Q^2}$. The solution is given by

$$R_n(r) = C_{1n} cosh\left(\frac{3k}{2} r^{\frac{2}{3}}\right) + iC_{2n} sinh\left(\frac{3k}{2} r^{\frac{2}{3}}\right) \tag{145}$$

Both solutions are square integrable in Hilbert space, that is, $\int r g_{rr} \mid R \mid^2 dr < \infty$. Therefore, the spacetime is quantum mechanically singular when probed with quantum particles that obey Klein-Gordon equation.

If we use the Sobolev norm,

$$\parallel R \parallel^2 \sim \int r g_{rr} \mid R \mid^2 dr + \int r g_{rr}^{-1} \mid \frac{\partial R}{\partial r} \mid^2 dr, \tag{146}$$

although the first integral of the solution is square integrable, the second integral for $C_{1n} = 0$ fails to be square integrable and the spacetime is quantum mechanically wave regular.

4.3.2 Dirac Fields

To solve the Dirac equation, we set the triad as

$$e_{\mu}^{(i)}(t, r, \theta) = diag\left(\sqrt{f}, \frac{\left(\alpha r^{\frac{4}{3}}\right)^{\frac{1}{2}}}{\sqrt{f}}, r\right). \tag{147}$$

The spinorial affine connection and the coordinate dependent matrix are found to be

$$\Gamma_{\mu}(x) = \left(\frac{5\left(-2m+15\Lambda r^{\frac{4}{3}}\right)\sigma^{(2)}}{36 r\sqrt{\alpha}}, 0, i\left[\frac{5\left(-2m+5\Lambda r^2+10Q^2\right)}{24\alpha r^{\frac{4}{3}}}\right]^{\frac{1}{2}}\sigma^{(3)}\right) \tag{148}$$

$$\sigma^{\mu}(x) = \left(\frac{\sigma^{(3)}}{\sqrt{f}}, \frac{i\sqrt{f}}{\left(\alpha r^{\frac{4}{3}}\right)^{\frac{1}{2}}}\sigma^{(1)}, \frac{i\sigma^{(2)}}{r}\right), \tag{149}$$

Then, the Dirac equation can be written as,

$$\frac{i}{\sqrt{f}}\psi_{1,t} - \frac{\sqrt{f}}{\sqrt{\alpha} r^{\frac{2}{3}}}\psi_{2,r} + \frac{i}{r}\psi_{2,\theta} - \left\{\frac{5\left(-2m+15\Lambda r^{\frac{4}{3}}\right)}{36 r\sqrt{\alpha f}} + \frac{\sqrt{f}}{2\sqrt{\alpha} r^{\frac{5}{3}}}\right\}\psi_2 - M\psi_1 = 0, \tag{150}$$

$$\left(\frac{-i}{\sqrt{f}}\right)\psi_{2,t} - \frac{\sqrt{f}}{\sqrt{\alpha} r^{\frac{2}{3}}}\psi_{1,r} - \frac{i}{r}\psi_{1,\theta} - \left\{\frac{5\left(-2m+15\Lambda r^{\frac{4}{3}}\right)}{36 r\sqrt{\alpha f}} + \frac{\sqrt{f}}{2\sqrt{\alpha} r^{\frac{5}{3}}}\right\}\psi_1 - M\psi_2 = 0. \tag{151}$$

By using the same anzats as in (100), two coupled equations are obtained,

$$R'_{2n} + \left[\frac{1}{2r} + \frac{5\left(-2m+15\Lambda r^{\frac{4}{3}}\right)}{36r^{\frac{1}{3}}f} + \frac{\sqrt{\alpha}(n+1)}{r^{\frac{1}{3}}\sqrt{f}}\right] R_{2n} + \sqrt{\alpha} r^{\frac{2}{3}}\left[\frac{M}{\sqrt{f}} - \frac{E}{f}\right] R_{1n} e^{-i\theta} = 0 \qquad (152)$$

$$R'_{1n} + \left[\frac{1}{2r} + \frac{5\left(-2m+15\Lambda r^{\frac{4}{3}}\right)}{36r^{\frac{1}{3}}f} - \frac{\sqrt{\alpha}n}{r^{\frac{1}{3}}\sqrt{f}}\right] R_{1n} + \sqrt{\alpha} r^{\frac{2}{3}}\left[\frac{M}{\sqrt{f}} + \frac{E}{f}\right] R_{2n} e^{i\theta} = 0. \qquad (153)$$

The radial part of the Dirac equation reduces to one single equation as

$$R''_n + \frac{1}{3r} R'_n - \left(\frac{7}{12r^2}\right) R_n = 0, \qquad n = 1,2 \qquad (154)$$

which has a solution

$$R_n(r) = C_{1n} r^{\frac{7}{6}} + \frac{C_{2n}}{\sqrt{r}}. \qquad (155)$$

Both parts of the solution are square integrable.

$$\approx \int \frac{r^{\frac{14}{3}}}{f} dr < \infty \qquad (156)$$

$$\approx \int \frac{r^{\frac{4}{3}}}{f} dr < \infty \qquad (157)$$

This is verified first by expanding the functions in series and then by integrating term by term in the limit as $r \to 0$. Consequently, the spacetime is quantum mechanically regular when probed with Dirac fields. An arbitrary wave packet can be written as

$$\Psi(t,x) = \sum_{n=-\infty}^{+\infty} C_n \begin{pmatrix} r^{\frac{7}{6}} + \frac{1}{\sqrt{r}} \\ \left(r^{\frac{7}{6}} + \frac{1}{\sqrt{r}}\right) e^{i\theta} \end{pmatrix} e^{in\theta} e^{-iEt}. \qquad (158)$$

4.4 Analysis for Einstein-Power-Maxwell Theory

4.4.1 Klein-Gordon Fields

We simplify the metric (62) by restricting our analysis to a specific parameter $k = 2$ and the new metric is given as

$$ds^2 = -h_1(y)dt^2 + \frac{dy^2}{\tilde{h}_2(y)} + \tilde{h}_3(y)d\theta^2 \qquad (159)$$

$$h_1(y) = \frac{1}{3}|\Lambda|y^2, \tag{160}$$

$$\tilde{h}_2(y) = \left(y^2 + r_+^2 + \alpha y^{\frac{2}{3}}\right)\frac{|\Lambda|}{3}, \tag{161}$$

$$\tilde{h}_3(y) = \frac{3}{|\Lambda|}\tilde{h}_2(y), \tag{162}$$

where $\alpha = \frac{81\tilde{P}^2}{\sqrt[3]{3}|\Lambda|^{\frac{5}{3}}} > 0$ is a constant. The Kretschmann scalar for this particular parameter is given by

$$K = \frac{4}{3}\Lambda^2 - \frac{40\tilde{P}^2|\Lambda|^{\frac{1}{3}}}{\sqrt[3]{3}y^{\frac{4}{3}}} + \frac{(76\tilde{P}^4)3^{\frac{4}{3}}}{|\Lambda|^{\frac{4}{3}}y^{\frac{8}{3}}}. \tag{163}$$

Clearly $y = 0$ is a true curvature singularity. Applying separation of variables, $\psi = F(y)e^{in\theta}$, we obtain the radial portion of equation (8) as

$$\frac{d^2F(y)}{dy^2} + \frac{1}{y}\left\{1 + \frac{y}{\tilde{h}_2(y)}\frac{d\left(\tilde{h}_2(y)\right)}{dy}\right\}\frac{dF(y)}{dy} + \frac{1}{\tilde{h}_2(y)}\left\{\frac{c}{\tilde{h}_3(y)} - m \pm \frac{i}{h_1(y)}\right\}F = 0 \tag{164}$$

where $c \in \mathbb{R}$ is a separation constant. Since the singularity is at $y = 0$, for small values of y each term in the above equation simplifies for massless ($m = 0$) case to

$$\frac{d^2F}{dy^2} + \frac{1}{y}\frac{dF(y)}{dy} \pm \frac{\nu^2}{y^2}iF(y) = 0, \tag{165}$$

where $\nu^2 = \frac{9}{|\Lambda|^2 r_+^2} > 0$. The solution of the above equation is

$$F(y) = C_{1\nu}y^{\sqrt{\pm i}\nu} + C_{2\nu}y^{-\sqrt{\pm i}\nu}, \tag{166}$$

in which $C_{1\nu}$ and $C_{2\nu}$ are arbitrary constants. In order to check the square integrability, the function space is defined on each $t = \text{constant}$ hypersurface Σ as $\mathcal{H} = \{F \mid \|F\| < \infty\}$ with the following norm given for the metric (159) as,

$$\|F\|^2 = \frac{q^2}{2}\int_0^{\text{constant}}\frac{1}{\sqrt{h_1(y)}}\sqrt{\left(\frac{\tilde{h}_3(y)}{\tilde{h}_2(y)}\right)}|F|^2 dy \sim \int_0^{\text{constant}}\frac{|F|^2}{y^2}dy, \tag{167}$$

where q is a constant parameter. The above solution is checked for the square integrability near $y = 0$, for each sign of the found in equation (166). It is found that the solution becomes square integrable if and only if the constant parameter $C_{2\nu} = 0$. For each sign of the equation (166), we have

$$\|F\|^2 \sim \int_0^{\text{constant}} y^{\sqrt{2}\nu - 1} dy = \frac{y^{\sqrt{2}\nu}}{\sqrt{2}\nu} |_0^{\text{constant}} < \infty. \tag{168}$$

Therefore the operator A has deficiency indices $n_+ = n_- = 1$, which shows that A is not essentially self-adjoint and the spacetime is quantum-mechanically singular.

4.4.2 Dirac Fields

The suitable triads for the metric (159) are given by,

$$e_\mu^{(i)}(t, y, \theta) = diag\left(y\sqrt{\frac{3}{|\Lambda|}}, \left(\frac{3}{|\Lambda| y^2 \left(y^2 + r_+^2 + \alpha y^{\frac{2}{3}}\right)}\right)^{\frac{1}{2}}, \left(y^2 + r_+^2 + \alpha y^{\frac{2}{3}}\right)^{\frac{1}{2}} \right), \tag{169}$$

The spinorial affine connection and coordinate dependent gamma matrices are given by

$$\Gamma_\mu(x) = \left(\frac{|\Lambda| \left(y^2 + r_+^2 + \alpha y^{\frac{2}{3}}\right)^{\frac{1}{2}} \sigma^{(2)}}{6}, 0, \frac{i\sqrt{|\Lambda|}}{6 y^{\frac{1}{3}} \sqrt{3}} \left(3y^{\frac{4}{3}} + \alpha\right) \sigma^{(3)} \right). \tag{170}$$

$$\sigma^\mu(x) = \left(\left(\sqrt{\frac{3}{|\Lambda|}}\right) \frac{\sigma^{(3)}}{y}, i\left(\frac{|\Lambda| \left(y^2 + r_+^2 + \alpha y^{\frac{2}{3}}\right)}{3}\right)^{\frac{1}{2}} \sigma^{(1)}, \frac{i\sigma^{(2)}}{\left(y^2 + r_+^2 + \alpha y^{\frac{2}{3}}\right)^{\frac{1}{2}}} \right), \tag{171}$$

Now, for the spinor (97), the Dirac equation can be written as

$$\frac{i}{y}\sqrt{\frac{3}{|\Lambda|}} \frac{\partial \psi_1}{\partial t} - \left(\frac{|\Lambda| \left(y^2 + r_+^2 + \alpha y^{\frac{2}{3}}\right)}{3}\right)^{\frac{1}{2}} \frac{\partial \psi_2}{\partial y} + \frac{i}{\sqrt{\left(y^2 + r_+^2 + \alpha y^{\frac{2}{3}}\right)}} \frac{\partial \psi_2}{\partial \theta} -$$

$$\left(\frac{\sqrt{|\Lambda|}\left(3y^{\frac{4}{3}} + \alpha\right)}{6 y^{\frac{1}{3}} \sqrt{3} \left(y^2 + r_+^2 + \alpha y^{\frac{2}{3}}\right)^{\frac{1}{2}}} + \frac{\sqrt{3|\Lambda| \left(y^2 + r_+^2 + \alpha y^{\frac{2}{3}}\right)}}{6y} \right) \psi_2 - m\psi_1 = 0 \tag{172}$$

$$-\frac{i}{y}\sqrt{\frac{3}{|\Lambda|}}\frac{\partial\psi_2}{\partial t}-\left(\frac{|\Lambda|\left(y^2+r_+^2+\alpha y^{\frac{2}{3}}\right)}{3}\right)^{\frac{1}{2}}\frac{\partial\psi_1}{\partial y}-\frac{i}{\sqrt{\left(y^2+r_+^2+\alpha y^{\frac{2}{3}}\right)}}\frac{\partial\psi_1}{\partial\theta}-$$

$$\left(\frac{\sqrt{|\Lambda|}\left(3y^{\frac{4}{3}}+\alpha\right)}{6y^{\frac{1}{3}}\sqrt{3}\left(y^2+r_+^2+\alpha y^{\frac{2}{3}}\right)^{\frac{1}{2}}}+\frac{\sqrt{3|\Lambda|\left(y^2+r_+^2+\alpha y^{\frac{2}{3}}\right)}}{6y}\right)\psi_1-m\psi_2=0 \tag{173}$$

The following anzats will be employed for the positive frequency solutions:

$$\Psi_{n,E}(t,x)=\begin{pmatrix}Z_{1n}(y)\\Z_{2n}(y)e^{i\theta}\end{pmatrix}e^{in\theta}e^{-iEt}. \tag{174}$$

The radial part of the Dirac equation becomes,

$$Z'_{2n}(y)+\left\{\frac{\sqrt{3}(n+1)}{\sqrt{|\Lambda|}\left(y^2+r_+^2+\alpha y^{\frac{2}{3}}\right)}+\frac{\left(3y^{\frac{4}{3}}+\alpha\right)}{6y^{\frac{1}{3}}\left(y^2+r_+^2+\alpha y^{\frac{2}{3}}\right)}+\frac{1}{2y}\right\}Z_{2n}(y)+$$

$$\frac{1}{\sqrt{\left(y^2+r_+^2+\alpha y^{\frac{2}{3}}\right)}}\left\{m\sqrt{\frac{3}{|\Lambda|}}-\frac{3E}{|\Lambda|y}\right\}Z_{1n}(y)e^{-i\theta}=0 \tag{175}$$

$$Z'_{1n}(y)+\left\{-\frac{\sqrt{3}n}{\sqrt{|\Lambda|}\left(y^2+r_+^2+\alpha y^{\frac{2}{3}}\right)}+\frac{\left(3y^{\frac{4}{3}}+\alpha\right)}{6y^{\frac{1}{3}}\left(y^2+r_+^2+\alpha y^{\frac{2}{3}}\right)}+\frac{1}{2y}\right\}Z_{1n}(y)+$$

$$\frac{1}{\sqrt{\left(y^2+r_+^2+\alpha y^{\frac{2}{3}}\right)}}\left\{m\sqrt{\frac{3}{|\Lambda|}}+\frac{3E}{|\Lambda|y}\right\}Z_{2n}(y)e^{i\theta}=0 \tag{176}$$

The behaviour of the Dirac equation near $y=0$ reduces to,

$$Z''_j(y)+\frac{2}{y}Z'_j(y)+\frac{\beta^2}{y^2}Z_j(y)=0,\qquad \mathrm{j}=1,2 \tag{177}$$

where $\beta^2=\frac{1}{4}+\left(\frac{3E}{|\Lambda|r_+}\right)^2$. The solution is given by

$$Z_j(y)=C_{1j}y^{\gamma_1}+C_{2j}y^{\gamma_2}, \tag{178}$$

where C_{1j} and C_{2j} are arbitrary constants. The exponents are given by

$$\gamma_1=-\frac{1}{2}+i\frac{3|E|}{|\Lambda|r_+}\qquad \gamma_2=-\frac{1}{2}-i\frac{3|E|}{|\Lambda|r_+}. \tag{179}$$

The condition for the Dirac operator to be quantum-mechanically regular requires that both solutions should belong to the Hilbert space $\mathcal{H}$. Squared norm of this solution

$$\left\|Z_j(y)\right\|^2 \sim \int_0^{\text{constant}} \frac{\left|Z_j(y)\right|^2}{y} dy \sim \int_0^{\text{constant}} y^{-2}\, dy \sim \frac{1}{y}\Big|_0^{constant} \to \infty \quad (180)$$

diverges. This implies that the solution do not belong to the Hilbert space. Consequently, if the classical singularity at $y = 0$ is probed with fermions the spacetime behaves quantum mechanically singular.

Chapter 5

CONCLUSION

In this thesis, the formation of naked singularities in the matter coupled 2+1 dimensional spacetimes in Einstein's theory is analysed in quantum mechanical point of view. In the analysis, naked singularity at $r = 0$ is probed with quantum fields that obey the Klein-Gordon and Dirac equations. Einstein-Maxwell extension of the BTZ black hole both in linear and nonlinear electrodynamics is considered. The condition for a naked singularity is explicitly displayed. A similar analysis is also considered in Einstein-Maxwell-Dilaton theory. As a final example the occurrence of naked singularities in Einstein-Power-Maxwell theory with magnetic charge is considered.

The analysis performed in this thesis has revealed that for the matter coupled 2+1 dimensional black hole spacetimes in Einstein-Maxwell theory with linear and nonlinear electrodynamics and Einstein-Maxwell-Dilaton theory are shown to share similar quantum mechanical singularity structure as in the case of pure BTZ black hole. The inclusion of matter fields changes the topology and creates true curvature singularity at $r = 0$. The effect of the matter fields allows only specific frequency modes in the solution of Klein-Gordon and Dirac fields. If the quantum singularity analysis is based on the natural Hilbert space of quantum mechanics which is the linear function space with square integrability L^2, the singularity at $r = 0$ turns out to be quantum mechanically singular for particles obeying the Klein- Gordon equation and regular for fermions obeying the Dirac equation. We have proved that the quantum singularity structure of 2+1 dimensional black hole spacetimes are generic for Dirac particles and the character of the singularity in quantum mechanical point of view is irrespective whether the matter field is coupled or not. This result suggests that the Dirac fields preserve the cosmic censorship hypothesis in the considered spacetimes that exhibit timelike naked singularities. The repulsive barrier is replaced against the propagation of Dirac fields instead of horizons (that covers the singularity in the black hole cases). However, for particles obeying Klein - Gordon fields, the singularity becomes worse when a matter field is coupled.

However, we have also shown that in the charged BTZ (in linear electrodynamics) and dilaton coupled black hole spacetimes specific choice of waves exhibit quantum mechanical wave regularity when probed with waves obeying Klein-Gordon equation, if the function space is Sobolev with the norm defined in (12). The singularity at $r = 0$ is stronger in the non-linear electrodynamics case. It should be reminded that, one may not feel comfortable to use Sobolev norm in place of natural linear function space of quantum mechanics.

However, when we consider the solution in 3D Einstein-Power-Maxwell with magnetic charge it does not admit black hole solution. Similar studies in the linear Maxwell theory ($k = 1$) have revealed a regular solution. The main contribution of the nonlinear Maxwell field in our solutions is to create timelike naked singularities for specific values of parameter $k > 1$ which is non-existent in the linear theory. This singularity has been analysed from the quantum mechanical point of view. The singularity is probed with quantum test particles obeying the Klein-Gordon and the Dirac equation.

The analysis of the naked singularity from quantum mechanical point of view has revealed that the considered spacetime is generically quantum singular when it is probed with fields obeying Klein-Gordon and Dirac equations. It is interesting to note that, in contrast to the considered spacetime, the probe of naked singularity with Dirac fields in other 3D metrics, namely BTZ [5] and matter coupled BTZ [1] spacetimes was shown to be quantum mechanically regular. It is also shown in this study that for general modes of spin zero Klein-Gordon fields, the spacetime is still singular. From these results we conclude that the occurrence of naked singularity in Einstein-Power-Maxwell theory with magnetic charge is stronger than the other considered spacetimes.

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APPENDIX

Appendix A: Energy Conditions

Coupling of a matter field to any system requires energy conditions to be satisfied for physically acceptable solutions. The steps are followed as given in [36] and [24] to find the bounds of the power parameter k of the Maxwell field.

Weak Energy Condition (WEC)

The WEC states that,

$$\rho \geq 0 \qquad \text{and} \qquad \rho + p_i \geq 0 \qquad (i = 1,2)$$

in which p_i are the principal pressures and ρ is the energy density given by

$$\rho = -T_t^t = \frac{1}{2}\mathcal{F}^k, \qquad p_i = T_i^i = \frac{2k-1}{2}\mathcal{F}^k \qquad \text{(no sum)}.$$

This condition imposes that $k > 0$.

Strong Energy Condition (SEC)

This condition states that;

$$\rho + \sum_{i=1}^{2} p_i \geq 0 \qquad \text{and} \qquad \rho + p_i \geq 0,$$

which amounts, together with the WEC to constrain the parameter $k \geq \frac{1}{4}$.

Dominant Energy Condition (DEC)

In accordance with DEC, the effective pressure p_{eff} should not be negative i.e. p_{eff} where

$$p_{eff} = \frac{1}{2}\sum_{i=1}^{2} T_i^i$$

Together with SEC and WEC, DEC impose the following condition on the parameter k as

$$k > \frac{1}{2}.$$

Causality Condition (CC)

In addition to the energy conditions, the causality condition (CC) is imposed

$$0 \leq \frac{p_{eff}}{\rho} < 1$$

which implies that

$$\frac{1}{2} \leq k < 1.$$

The CC is clearly violated in our solutions since we abide by the parameter k>1, throughout the paper.