A Class of Transcendent Solutions to Ernst System.

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Summary. – We present a method that generates a class of transcendent solutions to Ernst system in Einstein-Maxwell theory of general relativity.

In an earlier paper (1) we presented the general similarity integral to the Ernst equations in Einstein-Maxwell theory (2), where the similarity variables were arbitrary harmonic functions in cylindrical co-ordinates. It has been shown later (3) that by employing a particular independent variable the basic differential equation transforms into the type of Painleve's fifth transcendent, resulting therefore in a transcendental solution to Einstein-Maxwell equations. In this paper we shall give criteria about how this class of transcendental solutions can be increased in number.

The powerful technique that aided in complete similarity integral was based on the harmonic mappings, $\varphi^A : M \to M'$, where

$$(1) \qquad \begin{cases} \Phi^A = \{\xi, \bar{\xi}, \eta, \bar{\eta}\} = \{\Phi^1, \Phi^2, \Phi^3, \Phi^4\}\,, \\ M : \mathrm{d}s^2 = g_{\nu\mu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}\varrho^2 + \mathrm{d}z^2 + \varrho^2\,\mathrm{d}\varphi^2 \qquad (\mu, \nu = 1, 2)\,, \\ M' : \mathrm{d}s'^2 = g_{\mathrm{AB}}(\Phi)\,\mathrm{d}\Phi^A\,\mathrm{d}\Phi^B \qquad (A, B = 1, 2, 3, 4)\,, \end{cases}$$

$$(2) \qquad \mathrm{d}s'^2 = (\xi\bar{\xi} + \eta\bar{\eta} - 1)^{-2}\{\mathrm{d}\xi\,\mathrm{d}\bar{\xi}(1 - \eta\bar{\eta}) + \mathrm{d}\eta\,\mathrm{d}\bar{\eta}(1 - \xi\bar{\xi}) + \xi\bar{\eta}\,\mathrm{d}\eta\,\mathrm{d}\bar{\xi} + \eta\bar{\xi}\,\mathrm{d}\xi\,\mathrm{d}\bar{\eta}\}\,.$$

Upon variation of the energy functional ($\equiv$ the action)

$$(3) \qquad E[\Phi] = \int_M g_{\mathrm{AB}}(\Phi)\,\frac{\partial\Phi^A}{\partial x^\mu}\,\frac{\partial\Phi^B}{\partial x^\nu}\,g^{\mu\nu}|g|^{\frac{1}{2}}\,\mathrm{d}^2x\,,$$

(1) M. HALILSOY: *Lett. Nuovo Cimento*, **37**, 231 (1983).
(2) F. J. ERNST: *Phys. Rev.*, **168**, 1415 (1968).
(3) B. LEAUTE and G. MARCILHACY: *Lett. Nuovo Cimento*, **40**, 102 (1984).

one obtains the pair of Ernst equations (Ernst system)

$$\left\{\begin{aligned}(\xi\bar{\xi}+\eta\bar{\eta}-1)\,\nabla^2\xi &= 2\,\nabla\xi\cdot(\bar{\xi}\,\nabla\xi+\bar{\eta}\,\nabla\eta)\,,\\ (\xi\bar{\xi}+\eta\bar{\eta}-1)\,\nabla^2\eta &= 2\,\nabla\eta\cdot(\bar{\xi}\,\nabla\xi+\bar{\eta}\,\nabla\eta)\,.\end{aligned}\right. \tag{4}$$

In ref. (1) we had integrated these equations under the assumption that Φ^A is only a function of v which satisfied $\nabla^2 v = 0$, *i.e.* v is harmonic. Employing nonharmonic variables transforms the problem naturally into a much complicated one (4). We show now that when Φ^A depends on two independent harmonic functions the problem can be stated as a

Theorem. Let Φ^A be parametrized by two harmonic functions v and $\tilde{v}$ such that $\Phi^A(v,\tilde{v})$ is geodesic with respect to each of them independently. The field equations from the action principle then are satisfied, provided the constraint condition $\nabla v\cdot\nabla\tilde{v}=0$ holds.

Proof. The variational principle $\delta E[\Phi]=0$ yields the covariant equation

$$\partial_\mu\left(\sqrt{g}\,g^{\mu\nu}\frac{\partial\Phi^A}{\partial x^\nu}\right)+\sqrt{g}\,g^{\mu\nu}\,\Gamma^A_{BC}\frac{\partial\Phi^B}{\partial x^\mu}\,\frac{\partial\Phi^C}{\partial x^\nu}=0\,.$$

Using the fact now that $\Phi^A=\Phi^A(v,\tilde{v})$, one obtains

$$g^{\mu\nu}v_{,\mu}v_{,\nu}\left(\frac{\partial^2\Phi^A}{\partial v^2}+\Gamma^A_{BC}\frac{\partial\Phi^B}{\partial v}\,\frac{\partial\Phi^C}{\partial v}\right)+g^{\mu\nu}\tilde{v}_{,\mu}\tilde{v}_{,\nu}\left(\frac{\partial^2\Phi^A}{\partial\tilde{v}^2}+\Gamma^A_{BC}\frac{\partial\Phi^B}{\partial\tilde{v}}\,\frac{\partial\Phi^C}{\partial\tilde{v}}\right)+$$
$$+\frac{\partial\Phi^A}{\partial v}\,\nabla^2 v+\frac{\partial\Phi^A}{\partial\tilde{v}}\,\nabla^2\tilde{v}+g^{\mu\nu}(v_{,\mu}\tilde{v}_{,\nu}+v_{,\nu}\tilde{v}_{,\mu})\left(\frac{\partial^2\Phi^A}{\partial v\,\partial\tilde{v}}+\Gamma^A_{BC}\frac{\partial\Phi^B}{\partial v}\,\frac{\partial\Phi^C}{\partial\tilde{v}}\right)=0\,.$$

Harmonicity and the geodesic condition makes all foregoing expressions vanish, whereas the last term is made vanish by virtue of the constraint

$$\nabla v\cdot\nabla\tilde{v}=0\,. \tag{5}$$

Let us note that Kerr solution can be reformulated in terms of two harmonic functions in conform to the above theorem. Indeed, in prolate spheroidal co-ordinates x, y, eq. (5) reads

$$(x^2-1)\,v_x\tilde{v}_x+(1-y^2)\,v_y\tilde{v}_y=0\,,$$

and is solved by the harmonic pair of functions $v=\mathrm{tgh}^{-1}x$ and $\tilde{v}=\mathrm{tgh}^{-1}y$.

Although the expression for $\Phi^A(v,\tilde{v})$ can be more general in terms of v and $\tilde{v}$, we shall proceed by making a particular, separable dependence, namely, $\Phi^A(v,\tilde{v})=f^A(v)\exp[ic\tilde{v}]$. In this choice c is an arbitrary, real constant and the results of ref. (1) will be obtained in the limit $c=0$. In order to formulate our objective in a covariant language, we consider the harmonic map, Φ^A: $M_0\to M'$, between the manifolds, M_0: $ds_0^2=\exp[2v]\,dv^2+d\tilde{v}^2+\exp[2v]\,d\varphi^2$, and M', given by expression (2).

(4) M. Halilsoy: *J. Eng. Appl. Sci.*, in press.

Our further parametrization will be

$$(6)\qquad \begin{cases} \Phi^1 = \xi = y\cos\Psi \exp[i(\alpha + c\tilde{v})]\,, & \Phi^2 = \bar{\xi}\,, \\ \Phi^3 = \eta = y\sin\Psi \exp[i(\beta + c\tilde{v})]\,, & \Phi^4 = \bar{\eta}\,, \end{cases}$$

which casts the action functional into

$$(7)\qquad E[y, \Psi, \alpha, \beta] = \int \mathrm{d}v (y^2 - 1)^{-2}\Big[y'^2 + y^2(1-y^2)\Psi'^2 + \\ + y^2(\cos^2\Psi\cdot\alpha'^2 + \sin^2\Psi\cdot\beta'^2) - \frac{1}{4}y^4\sin^2 2\Psi(\alpha' - \beta')^2 + c^2y^2e^{2v}\Big]\,, \qquad \left(' \equiv \frac{\partial}{\partial v}\right).$$

Note that $\tilde{v}$-dependence washes out from the variational principle, its overall effect being to bring into the action such a c-dependent term. It is readily observed that Ψ, α and β equations remain invariant and correspond to eqs. (10), (12) and (13) of ref. (1), respectively. The y equation modifies due to the c-term, and reads as

$$(8)\qquad y'' - \frac{2yy'^2}{y^2-1} + (y^2-1)^2\left[y(a+b)^2 - \frac{k^2}{y^3}\right] + \frac{c^2y(1+y^2)}{y^2-1}\exp[2v] = 0\,,$$

where, the constant k, a and b are the same constants by integrations adopted in ref. (1). Changing variables by $v = \ln\sigma$ and $y^2 = Y$, transforms this equation into

$$(9)\qquad Y'' + \frac{1}{\sigma}Y' - \left(\frac{1}{2Y} + \frac{1}{Y-1}\right)Y'^2 + \frac{2}{\sigma^2}Y(Y-1)^2\left[(a+b)^2 - \frac{k^2}{Y^2}\right] + \\ + 2c^2Y\left(\frac{Y+1}{Y-1}\right) = 0 \qquad \left(' \equiv \frac{\partial}{\partial\sigma}\right),$$

which is identified as a particular Painleve's fifth transcendent (5). Once Y (and therefore y) is known, all the remaining Ψ, α and β equations can be reduced to quadratures, as functions of y. From the one-to-one correspondence between the Ernst system and Einstein-Maxwell's theory, the foregoing solution provides a transcendental solution to the latter.

The constraint condition $\nabla v\cdot\nabla\tilde{v} = 0$ $(= v_\varrho\tilde{v}_\varrho + v_z\tilde{v}_z)$ admits the simplest solution as $v = \log\varrho$, $\tilde{v} = z$, which corresponds to the solution of ref. (3). A new solution is given by the choice of harmonic functions,

$$(10)\qquad v = (\varrho^2 + z^2)^{-\frac{1}{2}}\,, \qquad \tilde{v} = \log\left[\frac{z}{\varrho} + \left(1 + \frac{z^2}{\varrho^2}\right)^{\frac{1}{2}}\right].$$

We remark that it is not clear about how large this set of harmonic functions satisfying the above constraint is. In the complex plane it is well-known that this reduces to the class of harmonic and conjugate harmonic functions. Further, the Kerr solution does not admit such a transcendental extension, since its Φ-function is not in the separable form which we have assumed.

In conclusion, we note that the foregoing procedure to obtain transcendental solutions can directly be adopted in any completely integrable systems that are expressible in terms of harmonic maps. The self-dual SU_N gauge field problem is one such example.

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(5) E. INCE: *Ordinary Differential Equations* (Dover Publ., New York, N. Y., 1956), p. 317.