Scalar Plane Waves in General Relativity.

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Summary. – A more general class of scalar plane waves is presented in general relativity. It is shown that no matter how the scalar field is modified, space-time possesses an essential singularity.

Scalar plane waves in general relativity can be described either by the longitudinal waves of the Brinkmann line element

$$ds^2 = 2du'\,dv' - dx'^2 - dy'^2 - 2(x'^2 + y'^2)\,h(u')\,du'^2\,, \tag{1}$$

or by transverse waves in the Rosen form

$$ds^2 = 2\exp[-M(u)]\,du\,dv - \exp[-U(u)]\,(dx^2 + dy^2)\,. \tag{2}$$

While BRINKMANN form is suitable to handle superposition of plane waves, Rosen form is more convenient to study the collision of plane waves. Such a space-time supports suitably a massless scalar field, which without loss of generality, we shall denote by the real function $\Phi(u)$.

Pure gravitational plane waves on the other hand are represented by the Rosen line element

$$ds^2 = 2\exp[-M(u)]\,du\,dv - \exp[-U(u)]\,(\exp[V(u)]\,dx^2 + \exp[-V(u)]\,dy^2)\,. \tag{3}$$

Collision between gravitational plane [1,2] and scalar plane waves [3] were formulated

[1] K. KHAN and R. PENROSE: *Nature*, **229**, 185 (1971).
[2] P. SZEKERES: *J. Math. Phys.* (*N.Y.*), **13**, 286 (1972).
[3] W. Z. CHAO: *J. Phys. A: Math., Nucl. Gen.*, **15**, 2429 (1982).

before, so that we shall make use of the field equations directly. The Einstein field equations for colliding gravitational waves are

$$
\left\{
\begin{aligned}
& U_{uv} = U_u U_v \,, \\
& 2V_{uv} - U_u V_v - U_v V_u = 0 \,, \\
& 2U_{uu} + 2M_u U_u - U_u^2 - V_u^2 = 0 \,, \\
& 2U_{vv} + 2M_v U_v - U_v^2 - V_v^2 = 0 \,, \\
& 2M_{uv} + U_u U_v - V_u V_v = 0 \,,
\end{aligned}
\right. \tag{4}
$$

whose general solution was presented by SZEKERES [2]. Field equations for colliding real scalar fields are

$$
\left\{
\begin{aligned}
& U_{uv} = U_u U_v \,, \\
& 2\Phi_{uv} - U_u \Phi_v - U_v \Phi_u = 0 \,, \\
& 2U_{uu} + 2M_u U_u - U_u^2 = \Phi_u^2 \,, \\
& 2U_{vv} + 2M_v U_v - U_v^2 = \Phi_v^2 \,, \\
& 2M_{uv} + U_u U_v = \Phi_u \Phi_v \,.
\end{aligned}
\right. \tag{5}
$$

The striking similarity between the sets of equations (4) and (5), prompts one to construct scalar field solutions from a given vacuum solution. It can readily be observed that,if (U, V, M) is a know solution for set (4), then by identifying, $\Phi = V$, one obtains a solution for colliding scalar plane waves, namely for set (5). This observation was given in [3].

A more general class of scalar wave solution can be obtained by the choice

$$
\tilde{\Phi} = V + \alpha U \,, \qquad \tilde{U} = U \,, \qquad \tilde{M} = M + \alpha V + \tfrac{1}{2}\alpha^2 U \,, \tag{6}
$$

where, $\alpha =$ real, arbitrary constant. It can be checked that $(\tilde{\Phi}, \tilde{U}, \tilde{M})$ constitutes another solution of (5), whenever (U, V, M) is a known solution for (4). Note that the class of solution (6) has no $V = 0$, limit, but it has, $\alpha = 0$ limit which reduces in the real domain to the solution in [3]. In order to have no-scalar-wave case, we must have both $V = 0$ and $\alpha = 0$, simultaneously, and the resulting space-time then is flat, as it should.

Using the solutions (U, V, M) of Szekeres, (6) reads explicitly,

$$
\left\{
\begin{aligned}
& \exp[\tilde{\Phi}] = \left(\frac{w+p}{w-p}\right)^{k_1/2} \left(\frac{r+q}{r-q}\right)^{k_2/2} t^{-2\alpha} \,, \\
& \exp[-\tilde{U}] = t^2 \,, \\
& \exp[-\tilde{M}] = t^{(\alpha+(k_1+k_2)/2)^2-1}\, w^{-k_1^2/4}\, r^{-k_2^2/4} (pq + rw)^{-k_1 k_2/2} (w+p)^{-\alpha k_1} (r+q)^{-\alpha k_2}
\end{aligned}
\right. \tag{7}
$$

where, the standard notations are, as in ([2])

$$t^2 = w^2 - p^2 = r^2 - q^2 = 1 - u^{n_1} - v^{n_2}, \qquad r^2 = 1 - p^2, \qquad w^2 = 1 - q^2,$$

$$p = u^{n_1/2}\theta(u), \qquad b = v^{n_2/2}\theta(v), \qquad k_i^2 = 8\left(1 - \frac{1}{n_i}\right), \qquad i = 1, 2.$$

Also note that scaling of the null co-ordinates, which is at our disposal, does not change the feature of the problem.

Employing α as a «screening» parameter we proceed now to construct regular scalar fields, at least locally, since, as it stands in (7), the scalar field has undesired features.

We consider two particular cases.

1) Letting $\alpha = -k_1/2(-k_2/2)$, yields in the incoming regions well-behaved scalar plane waves. The nonvanishing component $T_{uu} = \Phi_u^2$ $(T_{vv} = \Phi_v^2)$ of the energy-momentum tensor, $T_{\mu\nu} = \Phi_\mu \Phi_\nu - \frac{1}{2} g_{\mu\nu} \Phi_\alpha \Phi^\alpha$ is also finite. For the $n = 2$ case (impulsive waves) one gets, $\int du\, T_{uu} = 4$.

The space-time metric, however, possesses the $p(u) = 1(q(v) = 1)$ singularity. Moreover, the regular natures of Φ and $T_{\mu\nu}$ in the incoming regions is not shared in the scattering region.

2) Let, $\alpha = -(k_1 + k_2)/2$ (for simplicity, also let $k_1 = k_2$), one has

$$\exp[\tilde{\Phi}] = (w + p)^{-\alpha}(r + q)^{-\alpha},$$

which is regular at the space-time singularity $p^2 + q^2 = 1$ of the interaction region. Unfortunately energy-momentum tensor of this scalar field diverges for $(p, q) = (1, 0)$ and $(0, 1)$. Moreover, the incoming wave limit of this wave diverges for $p = 1$ $(= q)$.

The conclusion drawn out is that using freedoms at our disposal it is impossible to obtain an everywhere regular scalar field with a physical energy-momentum tensor. Locally this can be attained, but globally, for scattering scalar waves not. Space-time emerges irrespective of the chosen scalar field, always singular. This reflects once more the singular nature of colliding gravitational waves since, after all, we have employed Szekeres singular solutions. This result is conform with the singularity theorem proved before ([4]).

([4]) F. J. TIPLER: *Phys. Rev. D*, **22**, 2929 (1980).