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New metrics for spinning spheroids in general relativity

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The quaternionic version of the Ernst formalism is used to propose an alternative generalization of the Zipoy–Voorhees metric that represents spinning spheroids. Rotating discs and their superposition that generalize the static Curzon solution arises as a particular member of the class. Some features are different from the well-known Kerr and Tomimatsu–Sato solutions.

I. INTRODUCTION

One of the most important problems in general relativity is to find exact solutions that correspond to solid, physical sources. For once we sacrifice the physical significance of a source we run up against the well-phrased joke which states that any metric is a "solution" to the Einstein equations. In the case of material sources with exact spherical symmetry, both electrically charged and rotating, the problem is well understood and satisfactorily solved. Deviation from the spherical symmetry, however, cannot be handled at equal ease. Expectedly, the same difficulty arises in the classical Newtonian potential theory in a natural way which inherits general relativity. Collapsed stellar objects lose a number of degrees of freedom (i.e., no-hair theorems) to settle down as spherical black holes. For this reason going to the black hole limit saves us from a number of burdens in general relativity. Finding the exact gravitational fields for noncollapsed spheroidal objects has not been an easy task at all. Direct adaptation of Newtonian potentials into general relativity often caused ambiguities. From various considerations Zipoy[1] and later Voorhees[2] constructed metrics that possess interesting physical aspects with strange topologies. In particular, Zipoy introduced for the first time the prolate (and oblate) spheroidal coordinates to represent the gravitational fields of spheroidal objects. Beside other characteristics Zipoy–Voorhees (ZV) metric reduces in particular limits to the Schwarzchild (for exact spherical symmetry) and to the Curzon[3] (for flat disc) solutions.

In this paper we generalize the static spheroidal fields of ZV to the stationary ones. This class of metrics is different from the Tomimatsu–Sato[4] (TS) generalization of the Schwarzchild and Kerr[5] solutions. Finitely bounded spheroids TS gives physically satisfactory solutions, but as the source extends beyond limits, relatively either in radial or azimuthal directions, it remains open to describe such fields in closed forms. We cite as an example the gravitational field of a very large (ideally infinite) plate of thin matter.[6] Gravity of such a plate focuses itself at a finite distance and therefore it ceases to behave well asymptotically. In this sense our class of solutions has different asymptotic features compared to the TS solutions. For instance, the asymptotic values for mass quadropole expression is entirely different from those given in the TS solutions. Similar to the TS family, in our solutions separability of the Hamilton–Jacobi equation also happens to be for the particular value of the distortion parameter, namely for $\delta=1$, alone.

Our method makes use of the Ernst[7] potential formalism and its particular integrals that depend functionally on the solutions of the Laplace equation. The same method has been used in colliding electromagnetic[8]/gravitational[9] waves as well as in the cylindrical waves of Einstein and Rosen.[10] In this formalism all stationary metric components are expressed through integrability conditions as functions of a single harmonic function and its partial derivatives/integrals. Unlike the trainlike expressions that arise in the TS solutions, and their further generalizations by Cosgrove[11] and Yamazaki,[12] in a surprisingly simple mathematical procedure we construct a variety of stationary fields. This approach was initiated first in 1953 by Papapetrou.[13] The Ernst potential for the class of solutions that we find in this paper turn out to satisfy a quaternionic Ernst equation. Such a formalism was used in general relativity before.[14] Owing to the difficulty with asymptotic flatness, however, the Papapetrou solutions did not find a warm welcome and the entire arena was left to the black hole solutions. We propose that this class has promising features and it may serve to resurrect the idea of direct adaptation of Newtonian potentials to general relativity. The organization of the paper is as follows. We present our methods of solutions in Sec. II. We generalize the static Curzon solution and ZV metric in Secs. III and IV, respectively. We conclude with a brief discussion in Sec. V.

II. A CLASS OF SOLUTIONS

The general stationary axially symmetric vacuum Einstein fields are given by the metric

$$ds^2=e^{2\Psi}(dt-\omega d\phi)^2-e^{-2\Psi}[e^{2\gamma}(dr^2+dz^2)+r^2d\phi^2], \tag{1}$$

where the metric functions depend on r and z alone. We choose prolate spheroidal coordinates x and y in accordance with

$$r=\kappa(x^2-1)^{1/2}(1-y^2)^{1/2},$$
$$z=\kappa\, xy, \tag{2}$$

where κ is a positive constant parameter that measures the oblateness of the source. Let us note that although our discussion will be based on the prolate coordinates, with minor modifications it can be extended to cover the oblate coordinates as well. The basic pair of vacuum equations are expressed as a single complex Ernst equation

$$(\xi\bar{\xi}-1)\nabla^2\xi=2\bar{\xi}(\nabla\xi)^2, \tag{3}$$

where the relation of the complex potential ξ to the metric functions will be explained later.

As usual, a supplementary complex potential ε is introduced in accordance with

$$\xi=\frac{1-\varepsilon}{1+\varepsilon}. \tag{4}$$

The potential ε is defined by $\varepsilon=e^{2\Psi}+i\Phi$, where Ψ is one of the metric functions and Φ is called a twist potential. The metric function ω is given in terms of Φ through the integrability relations

$$\omega_z=re^{-4\Psi}\Phi_r,$$
$$\omega_r=-re^{-4\Psi}\Phi_z. \tag{5}$$

In order to find a solution to the Ernst equation (3) we assume a functional dependence of ξ on an arbitrary solution X of the Laplace equation

$$X_{rr}+(1/r)X_r+X_{zz}=0. \tag{6}$$

In the prolate spheroidal coordinates the Laplace equation becomes

$$[(x^2-1)X_x]_x+[(1-y^2)X_y]_y=0. \tag{7}$$

The solution $\xi(X)$ of (3) is given by

$$\xi(X)=\frac{(1+ia)\cosh 2X-b\sinh 2X-1}{(1-ia)\cosh 2X-b\sinh 2X+1}, \tag{8}$$

where a and b are real integration constants that are constrained to satisfy

$$b^2(1+a^2)=1. \tag{9}$$

We have two choices: (i) $b=\cos\alpha$, $a=\tan\alpha$, or (ii) $b^{-1}=\cosh\alpha$, $a=\sinh\alpha$, where α is another constant parameter. In this paper we shall choose the bounded parametrization and for brevity we prefer to use the notations $p=\cos\alpha$ and $q=\sin\alpha$, such that $p^2+q^2=1$. In the stationary axially symmetric field problem it is traditional to call p the mass parameter and q the twist parameter.

The functions $e^{2\Psi}$ and Φ are found by equating (4) and (8), while ω can be determined from (5) easily. We state our equations that give metric functions as follows (with a reference to the r, z coordinates for simplicity):

$$e^{-2\Psi}=\cosh 2X-p\sinh 2X,$$
$$\omega_r=2qrX_z,\quad \gamma_z=2rX_rX_z,$$
$$\omega_z=-2qrX_r,\quad \gamma_r=r(X_r^2-X_z^2), \tag{10}$$
$$(p^2+q^2=1).$$

It is readily seen that this class of solutions has the feature that it satisfies

$$\nabla\Psi\cdot\nabla\omega=0, \tag{11}$$

i.e., it is a subclass of the Papapetrou family of solutions. The nice feature of this class is that all metric functions are determined from a single harmonic function X that can be adopted from the classical potential theory. The corresponding asymptotic Newtonian potential is defined for the static fields by

$$\phi_N=\frac{1}{2}(g_{oo}-\eta_{oo})=\frac{-m}{R}+O\left(\frac{1}{R^3}\right)\quad (R\to\infty), \tag{12}$$

where g_{oo} and η_{oo} are the time components of the metric tensor for curved and flat metrics, respectively. Obviously, q is a measure of rotation in the above class. For $q=0$ ($p=1$) the solutions reduce to the corresponding nonrotating (static) ones, in which $\Psi=X$ and $\omega=0$. For $p=0$ ($q=1$) we obtain an extreme Kerr-like solution. One interesting aspect of the class is that the metric function γ remains the same in both static and its corresponding stationary generalization.

We can define now the corresponding Ernst potential ξ for the metric functions. For this purpose we introduce the quaternionic potential

$$\varepsilon=re^{-2\Psi}+\hat{e}\omega, \tag{13}$$

where $\hat{e}$ is the quaternionic unit that satisfies $\hat{e}^2=1$. Here, $re^{-2\Psi}$ and ω are defined as the scalar and vector parts of the quaternion, respectively. We recall that in the standard Ernst formalism we have complex ε in which $\hat{e}$ is replaced by the complex unit i. However, it can be checked by direct calculation that once we choose a quaternionic potential the correct Einstein field equations become satisfied. The equation satisfied by ε now is

$$(\varepsilon+\varepsilon^*)\nabla^2\varepsilon=2\nabla\varepsilon\cdot\nabla\varepsilon, \tag{14}$$

where * denotes the quaternionic conjugation (i.e., $\varepsilon^*=re^{-2\Psi}-\hat{e}\omega$). Similar to the Ernst's complex formulation we can introduce another potential by

$$\xi=\frac{1-\varepsilon}{1+\varepsilon}, \tag{15}$$

which satisfies the quaternionic Ernst equation

$$(\xi\xi^*-1)\nabla^2\xi=2\xi^*(\nabla\xi)^2. \tag{16}$$

In brief, the Ernst potential corresponding to our class of solutions is a quaternionic function instead of complex. In the sequel we apply our method to particular problems.

III. THE ROTATING CURZON SOLUTION

By separating the solution of Laplace equation in the variables r and z and integrating over all continuum values of the separation constant k we obtain the following particular solution for X:

$$2X=-m\int_0^\infty e^{-kz}J_0(kr)dk=\frac{-m}{\sqrt{r^2+z^2}}. \tag{17}$$

In this solution m is a constant that is interpreted as the mass of the source. By applying the method of Sec. II we obtain the following metric functions for the rotating Curzon solution:

$$e^{-2\Psi}=\cosh 2X-p\sinh 2X,$$
$$\omega=\frac{-qmz}{\sqrt{r^2+z^2}},\quad \gamma=\frac{-m^2r^2}{8(r^2+z^2)^2}, \tag{18}$$

where $2X$ is to be substituted from (17). As we stated before the Ernst potential for the Curzon solution is a quaternion that is given by

$$\varepsilon=R\sin\theta\left[\cosh\left(\frac{m}{R}\right)-p\sinh\left(\frac{m}{R}\right)\right]-(mq\cos\theta)\hat{e}, \tag{19}$$

where we have introduced the coordinates (R,θ) by

$$r=R\sin\theta,\quad z=R\cos\theta. \tag{20}$$

In the extreme Curzon limit ε takes the form

$$\varepsilon=R\sin\theta\cosh(m/R)-\hat{e}m\cos\theta. \tag{21}$$

Letting $X\to\beta X$, with $\beta=$const, leads to the changes $\omega\to\beta\omega$ and $\gamma\to\beta^2\gamma$; therefore the mass may be considered as resulting from a scale change of the function X. Let us note that we can add a second arbitrary integration constant in the general integral (8) as a scale factor of X. However, we shall omit such a constant. The asymptotic flatness of the Curzon solution is to be understood for $r\to\infty$ $(z<\infty)$ and $z\to\pm\infty$, accompanied with a redefinition of the time. As it can be seen from the asymptotic metric

$$ds^2=(1-O(1/R))(dt+qm\cos\theta\,d\phi)^2$$
$$-(1-O(1/R))[dR^2+R^2(d\theta^2+\sin^2\theta\,d\phi^2)], \tag{22}$$

there is an ambiguity in its asymptotic behavior. The cross term in this metric is reminiscent of the Taub-NUT[15] metric. The determinant of the metric and the curvature components reveal that there are essential singularities for $r=0$ and $z=0$. Possible extensions of the static Curzon metric were done by Szekeres and Morgan,[16] and a similar analysis can be extended to the present case.

The Hamilton–Jacobi (HJ) equation for the Curzon metric does not separate. To see this, we express the HJ functional in the following form:

$$S=\mu^2\lambda+Et+L_z\phi+\Sigma(r,z), \tag{23}$$

where E and L_z are the constants due to the (t,ϕ) symmetries, μ is the rest mass of the test particle, and λ is an affine parameter. By a change of variables according to (20) the HJ equation becomes

$$R^2(e^{-2\Psi}E^2-\mu^2)e^{-2\Psi}-(1/\sin^2\theta)(L_z+mqE\cos\theta)^2$$
$$-e^{-m^2\sin^2\theta/R^2}(R^2\Sigma_R^2+\Sigma_\theta^2)=0, \tag{24}$$

where $e^{-2\Psi}$ is given in (18) and is a function of R alone. We see that due to the $e^{2\gamma}(=e^{-m\sin\theta/R})$ term the HJ functional does not separate in R and θ. Even the static Curzon solution, for which $q=0$, does not separate due to the same term. If we make the concession that $e^{2\gamma}\simeq 1+O(1/R)$ and we keep terms up to the order $O(1/R)$, it can be seen that $\Sigma(R,\theta)$, and therefore S becomes separable.

Next, we can apply the principle of superposition to the Curzon solution. An interesting property of this class of solutions is that since the Laplace equation is linear, different solutions can be superimposed in a simple manner. Let

$$2X=-\sum_i^N\frac{m_i}{[r^2+(z-a_i)^2]^{1/2}}\quad (a_i=\text{const}) \tag{25}$$

be a solution of the Laplace equation. This may be interpreted as N number of planes (or discs), each one located at $z_i=a_i$. Then, the superimposed metric functions corresponding to this configuration can be obtained in the following closed form:

$$e^{-2\Psi}=\cosh 2X-p\sinh 2X,$$

$$\omega=-q\sum^{N}\left(\frac{m_i(z-a_i)}{[r^2+(z-a_i)^2]^{1/2}}\right), \tag{26}$$

$$\gamma=2r\int^{z}X_rX_z\,dz+\int^{r}r(X_r^2-X_z^2)dr,$$

where X is to be substituted from (25).

IV. THE GENERALIZED ZV METRIC

The procedure that aided in obtaining the rotating Curzon solution can be generalized to the whole class of ZV metric, which is given by

$$ds^2=\left(\frac{x-1}{x+1}\right)^{\delta}dt^2-\kappa^2\left(\frac{x+1}{x-1}\right)^{\delta}$$
$$\times\left[(x^2-y^2)\left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2}\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)\right.$$
$$\left.+(x^2-1)(1-y^2)d\phi^2\right]. \tag{27}$$

The prolate spheroidal coordinates x, y in this metric are connected to the spherical Schwarzchild coordinates (R,θ) as follows. Take

$$r=m(u^2-1)^{1/2}(1-v^2)^{1/2}=\kappa\,(x^2-1)^{1/2}(1-y^2)^{1/2},$$
$$z=muv=\kappa\,xy, \tag{28}$$

where (u,v) is a special set of prolate coordinates that is related to the Schwarzchild coordinates (R,θ) by

$$u=R/m-1,\quad v=\cos\theta. \tag{29}$$

If we solve x and y in terms of u and v the result is

$$\left.\begin{matrix}x\\y\end{matrix}\right\}=\frac{\delta}{\sqrt{2}}\left\{u^2+v^2+\frac{1-\delta^2}{\delta^2}\right.$$
$$\left.\pm\left[\left(u^2+v^2+\frac{1-\delta^2}{\delta^2}\right)^2-\frac{4u^2v^2}{\delta^2}\right]^{1/2}\right\}^{1/2}, \tag{30}$$

where $\delta=m/\kappa$ is the distortion parameter and $\pm$ correspond to x or y. The physical meaning of the coordinates (u,v) is that they are special coordinates for which $\delta=1$ (or $m=\kappa$). The coordinates (x,y) are a set of new prolate coordinates for an arbitrary source configuration characterized by the parameter δ. For $\delta=1$, the ZV metric reduces to the Schwarzchild solution. For $\delta>1$ $(\delta<1)$ it represents discs (rods) that are the extreme cases of the spheroids.

In the limit $\kappa\to 0$ (or $\delta\to\infty$ with $m=$finite) the same metric reduces to the Curzon solution. The proper usage of the metric should be in the following manner: substitute (x,y) in terms of (u,v) which are to be substituted further in terms of the Schwarzchild coordinates.

The integrability equations for the metric functions γ and ω in terms of (x,y) take the following forms

$$\gamma_x=\frac{1-y^2}{x^2-y^2}\{x(x^2-1)X_x^2$$
$$-x(1-y^2)X^2y-2y(x^2-1)X_xX_y\},$$
$$\gamma_y=\frac{x^2-1}{x^2-y^2}\{y(x^2-1)X_x^2 \tag{31}$$
$$-y(1-y^2)X^2y+2x(1-y^2)X_xX_y\},$$

and

$$\omega_x=2\,q\kappa(1-y^2)X_y,$$
$$\omega_y=-2\,q\kappa(x^2-1)X_x. \tag{32}$$

Using the ZV metric as seed, in which

$$e^{2X}=\left(\frac{x-1}{x+1}\right)^{\delta}, \tag{33}$$

we obtain the following rotating metric functions:

$$e^{-2\Psi}=\cosh 2\,X-p\sinh 2\,X,$$
$$e^{2\gamma}=\left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2}, \tag{34}$$
$$\omega=-2q\kappa\delta y.$$

We show now that for $\delta=1$ (i.e., exact spherical symmetry) we obtain a HJ separable metric. Letting

$$S=\mu^2\lambda+Et+L_z\phi+\Sigma_1(x)+\Sigma_2(y) \tag{35}$$

leads to

$$\kappa^2(x^2-1)e^{-2\Psi}(E^2e^{-2\Psi}-\mu^2)-(x^2-1)\Sigma_1'^2$$
$$=\frac{(\omega E+L_z)^2}{1-y^2}+(1-y^2)\Sigma_2'^2$$
$$=\lambda_0,$$

where λ_0 is a separation constant. Solutions for Σ_1 and Σ_2 are obtained as follows:

$$\Sigma_1(x)=\pm\int^x\left\{\kappa^2e^{-2\Psi}(E^2e^{-2\Psi}-\mu^2)-\frac{\lambda_0}{x^2-1}\right\}^{1/2}dx,$$
$$\Sigma_2(y)=\pm\int^y\left\{\frac{\lambda_0}{1-y^2}-\frac{(L_z-2q\kappa Ey)^2}{(1-y^2)^2}\right\}^{1/2}dy. \tag{36}$$

For $\delta\neq1$, similar to the case of Curzon solution, we face difficulty in the separation of HJ equation. The same property is shared also by the TS family, where separability occurs only for $\delta=1$ (the Kerr solution).

Using the ZV metric we can obtain superimposed solution of a spherical source (Schwarzchild) and a disc (Curzon) in an easy process. To this end, choose

$$e^{2X}=\left(\frac{x-1}{x+1}\right)^\delta e^{-m/\sqrt{x^2+y^2-1}} \tag{37}$$

to represent a solution of the Laplace equation. Note that the exponential factor represents the Curzon term, because $x^2+y^2-1=r^2+z^2$ (for $\kappa=1$). The first factor obviously represents for $\delta=1$ the Schwarzchild term and what we are doing physically is to add a Curzon term to a Schwarzchild term. In this process we preserve the Schwarzchild solution by taking the limit, $m\to0$, for the disk. The remaining metric functions Ψ, γ, and ω are given as follows:

$$e^{-2\Psi}=\cosh 2X-p\sinh 2X$$
$$e^{2\gamma}=\left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2}\times e^{-\{m^2(x^2-1)(1-y^2)/4(x^2+y^2-1)^2+2m\delta x/\sqrt{x^2+y^2-1}\}},$$
$$\omega=-2q\kappa y\left(1+\frac{mx}{2\sqrt{x^2+y^2-1}}\right), \tag{38}$$

where X is given in (37). It can easily be checked that in the limits $m=0$, $\delta=1$ (and $q=0$) we recover the Schwarzchild metric and in the limit $\delta=0$, we obtain the Curzon solution. The ZV metric can be generalized further by making the choice

$$e^{2X}=\left(\frac{x-1}{x+1}\right)^{\delta_1}\left(\frac{1-y}{1+y}\right)^{\delta_2}, \tag{39}$$

where δ_1 and δ_2 may be interpreted as arbitrary distortion parameters for the different coordinate lines. The rotating version of this metric has the following metric functions:

$$e^{-2\Psi}=\cosh 2X-p\sinh 2X,$$
$$e^{2\gamma}=(x^2-1)^{\delta_1^2}(1-y^2)^{\delta_2^2}(x+y)^{-(\delta_1-\delta_2)^2}\times(x-y)^{-(\delta_1+\delta_2)^2}, \tag{40}$$
$$\omega=2\kappa q(x\delta_2-y\delta_1),$$

which have a more symmetric appearance with respect to the coordinates x and y. It reduces to (34) by choosing $\delta_1=\delta$ and $\delta_2=0$, and in the limit $q=0$ we recover the spheroidally symmetric static solution of ZV. The metric functions (40), however, must be devoid of a physical content, because, as it can easily be seen, ω diverges for $x\to\infty$. (Note that $e^{2\Psi}\to1$, and $e^{2\gamma}\to1$ as $x\to\infty$.) In the problem of colliding gravitational waves where asymptotic flatness is not a requirement this latter class proved to result in a significant family of solutions.

Now we return to the metric described by the functions (34) and we proceed by making an asymptotic expansion for component g_{00}. We obtain

$$e^{2\Psi}=1-\frac{2mp}{r}+\frac{2m^2}{r^2}(p-1)(2p+1)+\frac{2m^3}{r^3}\left[2(p-1)\times(1-2p^2)+\frac{p}{3}\left(1-\frac{1}{\delta^2}\right)P_2(\cos\theta)\right]+0\left(\frac{1}{r^4}\right), \tag{41}$$

where $P_2(\cos\theta)=\frac{3}{2}\cos^2\theta-\frac{1}{2}$. We make the following successive transformations,

$$r=p\tilde{r}-(m/p)(p-1)(p+2),$$
$$\tilde{r}^2=R^2-2m^2\left(1-\frac{1}{p}\right)\left[\frac{1}{p^2}-2+\left(1-\frac{1}{p}\right)\left(\frac{1}{p}+2\right)^2\right], \tag{42}$$

to obtain the asymptotic expression

$$e^{2\Psi}=1-\frac{2m}{R}+\frac{2m^3}{R^3}\cdot\frac{1}{3p^2}\left(1-\frac{1}{\delta^2}\right)P_2(\cos\theta)+O\left(\frac{1}{R^4}\right). \tag{43}$$

Recall that for the TS metrics the asymptotic expansion of $e^{2\Psi}$ is given by

$$e^{2\Psi}=1-\frac{2m}{R}+\frac{2m^3}{R^3}P_2(\cos\theta)\left[\frac{1}{3}\left(1-\frac{1}{\delta^2}\right)p^2+q^2\right]+O\left(\frac{1}{R^4}\right), \tag{44}$$

where $p^2+q^2=1$. We see the striking difference between our expression (43) and this one. In the latter, for $\delta=1$

there is still a nonzero quadrupole term, namely $Q=m^3q^2$, whereas in our case $Q=0$. This result arises from the fact that in our solution g_{00} is only a function of x and it doesn't depend on y. This situation compells us to adopt the interpretation that $\delta=1$ corresponds to a spinning point mass. Since a point mass (or mass monopole) has no quadrupole term and it does not differ whether such a mass is rotating or not, this interpretation seems plausible. For $\delta\neq1$, no such ambiguity arises, and as in the TS family we have $Q(\delta_2)>Q(\delta_1)$, whenever $\delta_2>\delta_1$. As $\delta\to\infty$ the expression (43) may be compared with the asymptotic expression for a Newtonian potential for a disc:

$$\phi_N \sim \frac{-m}{R}+\frac{m^3}{3\epsilon^2R^3}P_2(\cos\theta)+O\left(\frac{1}{R^4}\right), \tag{45}$$

where the parameter ϵ becomes equal to p.

Following the analysis by Voorhees we can study the intrinsic geometry of the surfaces $\Psi=$const, in order to explore the source configuration. This implies $x=a=$const, and results in the same equipotential surface for the static case, namely

$$R=m\left[1+\frac{a}{\delta}\left(\frac{a^2-1+\delta^2\sin^2\theta}{a^2-\cos^2\theta}\right)^{1/2}\right], \tag{46}$$

in the spherical coordinates (R,θ). The horizon, $g_{00}=0$ implies that $a=1$, which gives $R=2m$, irrespective of the distortion parameter. This result also is in contrast to the TS family where the horizon becomes automatically modified by the rotation of the gravitating sources.

V. DISCUSSION

We have presented an alternative class within the Papapetrou family of solutions to describe spinning spheroids. Our solutions can be obtained directly from harmonic functions which abound in the classical potential theory. Asymptotically this class does not behave well, therefore we interpret them to represent very large deformed spheroids (i.e., rods and discs). The particular case $\delta=1$ corresponds to a spinning point mass. The solutions given are valid for arbitrary distortion parameter $\delta>0$, in a closed form. One useful aspect of this class of solutions is that different solutions can be superimposed easily. In particular, we have given the exact rotating solutions that describe superposed Curzon and Schwarzchild sources. We remind the reader that in the case of TS an exact superimposed solution is a rather difficult task. We conclude by stating that electrically charged spheroids and their topological implication may be the next stage of study in this line of work.

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