Modified RN metric in an External Electrostatic Field (Il nuovo cimento. B 113,6 1998)

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Modified Reissner-Nordstrom metric in an external electrostatic field (*)

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Summary. — We introduce a new metric that describes a Reissner-Nordstrom black hole coupled to an external, stationary electrostatic field. The field modifies both the horizon and particle geodesics to significant degrees.

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1. – Introduction

Black holes of general relativity studied so far are known (apart from a cosmological term) to take place in a vacuum environment. A charged hole, such as the Reissner-Nordstrom (RN), is endowed with static electric charge besides its mass. Thus, the electromagnetic (em) effects originate from the charge of the Schwarzschild (S) mass. In the usual trend of physics once we solve an isolated system the next stage of study becomes to consider the system in a uniform external field. The prototype examples of quantum-mechanical H-atom and classical electrodynamical charged sphere in a uniform electric (or magnetic) field are well known. From the same token, recently we have shown that the metric of a S black hole becomes modified in coupling to an external em field [1-3]. What is done mathematically in this process is to interpolate two exact solutions of Einstein's equations. These solutions were the well-known S metric and the uniform em field solution of Bertotti and Robinson (BR) [4, 5].

In the present paper we extend the S metric to the RN metric which gives us the exact coupling of a central charge in equilibrium with an externally isotropic electric field. It is known that static configuration in two (or more) RN sources is provided by the mutual balance between gravitational and electric forces [6]. The equilibrium condition in our case, however, must not be true in a non-static coordinate system, which leads naturally to an accelerated RN source. Our external metric will be considered more generally to employ a NUT parameter. Since it is of type D it may be thought of as a member of some broader classes such as in [7]. We wish to add that in all such studies asymptotic flatness is the main emphasis, whereas our main concern here is the additional requirement of asymptotic BR. This is the crucial point that makes our metric new, which modifies the RN black-hole horizons and geodesics.

(*) The authors of this paper have agreed to not receive the proofs for correction.

2. – The space-time metric

Our space-time is endowed with electromagnetic field from two different sources, the RN charge of a central object and all pervasive field of the BR origin. In other words, these two distinct metrics each must constitute the boundary conditions of our space-time. We obtain the following stationary line element that satisfies these required boundary conditions:

$$ds^2 = H^2(r)(dt - Q_0 \cos\theta \, d\phi)^2 - H^{-2}(r)\, dr^2 - R^2(d\theta^2 + \sin^2\theta \, d\phi^2)\,. \tag{1}$$

Here, and in the sequel we use the following abbreviations:

$$\begin{cases} H^2(r) = \left(1 - \dfrac{2m}{r} + \dfrac{e^2}{r^2}\right) f^{-1}(r)\,, \\ R^2 = r^2 f(r) = c_0 m^2 - b_0 mr + \dfrac{1}{2} a_0 r^2\,, \\ Q_0 = mq(1 + a^2)\,, \end{cases} \tag{2}$$

in which the constants a_0, b_0 and c_0 are related to the external parameter a $(0 \leqslant a \leqslant 1)$ and the NUT parameter p $(p = \sqrt{1 + q^2} \geqslant 1)$ through

$$\begin{cases} c_0 = p(1 + a^2) - 2a\,, \\ b_0 = a_0 - 2a\,, \\ a_0 = p(1 + a^2) + a^2 - 1\,. \end{cases} \tag{3}$$

It is shown below that the only non-vanishing Maxwell spinor is ϕ_1, leading to the em 2-form

$$F = \sqrt{2}k \left\{ \frac{\cos\alpha}{R^2} dr \wedge (dt - Q_0 \cos\theta \, d\phi) + \sin\alpha \sin\theta \, d\theta \wedge d\phi \right\}, \tag{4}$$

where $k = [m^2(1 - a^2) + (1/2)\, a_0 e^2]^{1/2} = \text{const}$, Λ is the wedge product and $\alpha(r)$ is the phase of ϕ_1 chosen as $\alpha(r) = Q_0 \int dr/R^2$. Denoting the dual of F by $^*F$ it can easily be checked that the sourceless Maxwell equations

$$dF = 0$$

and

$$d^*F = 0\,, \tag{5}$$

are satisfied, in which d stands for the exterior derivative. In the orthonormal tetrad basis 1-forms

$$\begin{cases} \omega^0 = H(dt - Q_0 \cos\theta \, d\phi)\,, \\ \omega^1 = H^{-1} dr\,, \\ \omega^2 = R\, d\theta\,, \\ \omega^3 = R \sin\theta \, d\phi\,, \end{cases} \tag{6}$$

the Ricci components can be computed and also shown to satisfy the Einstein-Maxwell equations

$$R_{ab} = -2\left(\eta^{cd} F_{ac} F_{bd} - \frac{1}{4} \eta_{ab} F_{cd} F^{cd}\right), \tag{7}$$

where η_{ab} and F_{ab} refer to the orthonormal metric and em tensors, respectively. We note that the arbitrary parameter q arises from the similarity integral of the Ernst equation [8]. Obviously for $a = 1 = p$ the coordinates (t, r, θ, ϕ) are the usual S-coordinates and the above line element reduces to the RN metric. We note also that the constants (Q_0, m, a_0, b_0, c_0) appearing in the metric are constrained to satisfy

$$Q_0^2 + m^2(b_0^2 - 2a_0 c_0) = 0 . \tag{8}$$

This is crucial for the satisfaction of the EM equations with $\phi_{11} \neq 0 \neq \psi_2$, with all other tetrad scalar invariants equal to zero.

The method to obtain the above metric is based on Ernst's formalism in which we assume a functional dependence between the em and gravitational potentials, so that the two limiting solutions (*i.e.* pure em and pure gravitational, which is the S metric in our present problem) are interpolated by the parameter a. For $a = 1$ it is rather simple to see that the metric (1) has S limit, however, for $a = 0$ one has to apply the following tedious transformation:

$$\begin{cases} \dfrac{r}{m} - 1 = \dfrac{1}{2\varrho}(\varrho^2 - T^2 + 1), \\ \tan h \dfrac{t}{m} = \dfrac{1}{2T}(\varrho^2 - T^2 - 1), \end{cases} \tag{9}$$

to obtain (after scaling $ds^2 \to (1/m^2)\, ds^2$) the BR line element

$$ds^2 = \frac{1}{\varrho^2}(dT^2 - d\varrho^2) - (d\theta^2 + \sin^2\theta\, d\phi^2). \tag{10}$$

An alternative method to obtain the metric (1) might be to start from an em potential 1-form

$$A = A_\mu(r, \theta)\, dx^\mu = \frac{\sqrt{2}k}{Q_0} \sin\alpha(r)\cdot(dt - Q_0 \cos\theta\, d\phi), \tag{11}$$

which embodies both RN and BR potentials in special limits and integrates the related metric functions from the EM equations. This method, however, is not guaranteed to be more economical or simpler than special techniques obtained by other means in different contexts.

We make the choice of the following null tetrad basis 1-forms $(l, n, m, \overline{m})$:

$$\begin{cases} \sqrt{2}\, l = H(dt - Q_0 \cos\theta\cdot d\phi) - H^{-1} dr, \\ \sqrt{2}\, n = H(dt - Q_0 \cos\theta\cdot d\phi) + H^{-1} dr, \\ \sqrt{2}\, m = R(d\theta + i \sin\theta\, d\phi), \end{cases} \tag{12}$$

and obtain the only non-vanishing Newman-Penrose scalar components

$$(13)\quad \begin{cases} \Psi_2 = -\dfrac{1}{4R^6}\{m[a_0(a_0-b_0)\,r^3 + ma_0(4c_0-b_0)\,r^2 + \\ \qquad +2m^2(b_0^2 - b_0c_0 - 3a_0c_0)\,r + 2m^3 b_0 c_0] + e^2[2m^2(a_0c_0 - b_0^2) + \\ \qquad +2a_0b_0mr - a_0^2r^2] + iQ_0[-a_0r^3 + 3a_0mr^2 + \\ \qquad +2r(m^2(1-a^2) - a_0e^2) - 2m(c_0m^2 - b_0e^2)]\}\,, \\ \Phi_{11} = \dfrac{m^2(1-a^2)}{2R^4} + \dfrac{a_0e^2}{4R^4}\,, \end{cases}$$

where we have used $R = r\sqrt{f(r)}$. Also note that if the condition (8) did not exist such simple results would not come out. For $q=0$ $(p=1)$ the relation between R and r is given by

$$(14)\qquad R = ar + (1-a)\,m\,.$$

If we employ R as the S coordinate, we can express the S metric coupled to a uniform external em field by

$$(15)\qquad \mathrm{d}s^2 = \frac{R - m(1+a)}{R + m(a-1)}\,\mathrm{d}t^2 - \frac{R + m(a-1)}{R - m(1+a)}\,\frac{\mathrm{d}R^2}{a^2} - R^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,.$$

For $a=1$ we recover the S metric, whereas for $0<a<1$ we have the geometry of a mass m coupled to an external em field. Note that this form of the metric is inappropriate to consider the limit $a=0$. We note also that $R>0$, unless we assume extreme conditions (such as $a=1$, $r=0$). This provides us with the horizons on the condition

$$(16)\qquad r^2 - 2mr + e^2 = 0\,.$$

In terms of R the roots of this equation give us the radii of the horizons

$$(17)\qquad R_\pm^2 = p(1+a^2)\,m^2 - \frac{1}{2}e^2[p(1+a^2) + a^2 - 1] \pm 2am\sqrt{m^2 - e^2}\,.$$

For $q=0$ this expression simplifies to

$$(18)\qquad R_\pm = m \pm a\sqrt{m^2 - e^2}\,.$$

This expression shows that an external em field shrinks the outer horizon but expands the inner one. For $e=0$, we have to add also that $a=1$ reduces to the standard S horizon $R_+ = 2m$, $(R_- = 0)$. By analysis of the general expression for $R_\pm$ we conclude that an external rotation of the em field $(p>1)$ pushes the horizon outward. In a simpler case we choose the extreme case $m=e$ and obtain

$$(19)\qquad R_\pm = \frac{m}{\sqrt{2}}[p(1+a^2) + 1 - a^2]^{1/2} \geqslant m\,.$$

3. – Modified geodesics

In this section we shall consider the simpler case $q=0$, so that the geodesics Lagrangian L of the system is

$$2L = \frac{\Delta}{R^2}\dot{t}^2 - \frac{R^2}{\Delta}\dot{R}^2 - R^2(\dot{\theta}^2 + \sin^2\theta\cdot\dot{\phi}^2), \tag{20}$$

where the dot denotes d/ds, differentiation with respect to the proper distance. Also we have introduced

$$\Delta = R^2 - 2mR + Q^2 m^2 \tag{21}$$

in which

$$Q^2 = 1 + \left(\frac{e^2}{m^2} - 1\right) a^2 \tag{22}$$

and scaled t by the parameter $1/a$. First integrals of motion follow as

$$\frac{\Delta}{R^2}\dot{t} = E, \qquad R^2\dot{\phi} = h, \tag{23}$$

where E and h are energy and angular momentum constants, respectively. From now on the results of our geodesics follow as in ref. [3] except the meaning of Q^2. It is seen from (22) that the electric charge e of the RN source amplifies Q^2 and this has a direct effect on detectable planetary precession and light deflection. From [3], the total planetary precession

$$\delta \approx \frac{6\pi m^2}{h^2}\left(1 - \frac{Q^2}{6}\right) \tag{24}$$

and the deflection of light grazing the source

$$\Delta\phi_\infty \approx \frac{4m}{R_0}\left(1 - \frac{9m^2 Q^2}{16 R_0^2}\right), \tag{25}$$

where R_0 is the nearest approach, both tend to reduce when e is taken into account.

4. – Discussion and conclusion

A significant amount of theoretical physics papers concern exclusively black holes. In this vast literature it was a missing link to consider black holes embedded in external sources. This gap has been filled for the case of a RN black hole in an external em field. The metric that we present is asymptotically BR. In this sense it is analogous to the problem of a charged sphere in an external electric field in classical electrodynamics.

The physical conclusion drawn out is that for a marginally formed RN black hole the black-hole property will be lost if the collapsed star radius R_s (take $q=0$) satisfies

$$m + a\sqrt{m^2 - e^2} < R_s < m + \sqrt{m^2 - e^2}\,. \tag{26}$$

The analysis similar to ours can be carried out for different sources such as scalar field or perfect fluid. Finally, one naturally asks: Can external sources prevent black-hole formation completely?

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