Quantum singularities in (2 + 1) dimensional matter coupled black hole spacetimes

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Quantum singularities considered in the 3D Banados-Teitelboim-Zanelli (BTZ) spacetime by Pitelli and Letelier [Phys. Rev. D **77**, 124030 (2008)] is extended to charged BTZ and 3D Einstein-Maxwell-dilaton gravity spacetimes. The occurrence of naked singularities in the Einstein-Maxwell extension of the BTZ spacetime both in linear and nonlinear electrodynamics as well as in the Einstein-Maxwell-dilaton gravity spacetimes are analyzed with the quantum test fields obeying the Klein-Gordon and Dirac equations. We show that with the inclusion of the matter fields, the conical geometry near $r = 0$ is removed and restricted classes of solutions are admitted for the Klein-Gordon and Dirac equations. Hence, the classical central singularity at $r = 0$ turns out to be quantum mechanically singular for quantum particles obeying the Klein-Gordon equation but nonsingular for fermions obeying the Dirac equation. Explicit calculations reveal that the occurrence of the timelike naked singularities in the considered spacetimes does not violate the cosmic censorship hypothesis as far as the Dirac fields are concerned. The role of horizons that clothes the singularity in the black hole cases is replaced by repulsive potential barrier against the propagation of Dirac fields.

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I. INTRODUCTION

In recent years, the (2 + 1) dimensional, Banados-Teitelboim-Zanelli (BTZ) [1] black hole has attracted much attention. One of the basic reasons for this attraction is that the BTZ black hole has a relatively simple tractable mathematical structure so that it provides a better understanding of investigating the general aspects of black hole physics, and since the BTZ black hole carries all the characteristic features such as the event horizon and Hawking radiation, it can be treated as a real black hole. Another motivation to study the BTZ black hole is the AdS/CFT correspondence which relates thermal properties of black holes in the AdS space to a dual CFT. In view of these points, the unresolved black hole properties belonging to (3 + 1) or higher dimensional black holes at the quantum level make the BTZ black hole an excellent background for exploring the black hole physics.

Another interesting subject is the study of naked singularities that can be considered as a threat to the cosmic censorship hypothesis. Compared to the black holes, the naked singularities are less understood. Today, there is no common consensus either on the structure or the existence of the naked singularities.

Recently, Pitelli and Letelier (PL) [2] have analyzed the occurrence of naked singularities for the BTZ spacetime from a quantum mechanical point of view. In their analysis, the criteria proposed by Horowitz and Marolf (HM) [3] is used. The classical naked singularity is studied with the quantum test particles that obey Klein-Gordon and Dirac equations. They confirmed that the naked singularity is ‘‘healed’’ when tested by massless scalar particles or fermions without introducing extra boundary conditions. However, for massive scalar particles additional information is needed. Despite the recent developments on the concept of quantum singularities [4], our understanding of naked singularities as far as quantum gravity is concerned is still far from being complete.

The purpose of this paper is to analyze the naked singularities within the context of the quantum mechanics that form in the matter coupled 2 + 1 dimensional black hole spacetimes. Our motivation here is to investigate the effect of the matter fields on the quantum singularity structure of the BTZ spacetime because the surface at $r = 0$ for the BTZ black hole is not a curvature singularity, but is a singularity in the causal structure. This situation changes when a matter field is coupled. This is precisely the case that we shall elaborate on in this article. For this purpose we consider the charged BTZ spacetime both in linear and nonlinear electrodynamics. This is analogous to a kind of Einstein-Maxwell extension of the work presented in [2]. Furthermore, we extend the analysis to cover the 2 + 1 dimensional Einstein-Maxwell-dilaton coupled black hole spacetime. The presence of charge both in the linear and nonlinear case and also the dilaton field modifies the resulting spacetime geometry significantly. Near the origin, the spacetime is not conic and true curvature singularity develops at $r = 0$. Consequently, the spacetime geometry that we have investigated in this study differs when compared with the case considered in [2].

The plan of the paper is as follows. In Sec. II, we first review the definition of quantum singularities for general static spacetimes. In Sec. III, we consider the charged BTZ black hole in nonlinear electrodynamics. Klein-Gordon and Dirac fields are used to test the quantum singularity.

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We also discuss the Sobolev norm which is used for the first time in this context by Ishibashi and Hosoya [5]. In Secs. IV and V, we consider the charged BTZ in linear electrodynamics and dilaton coupled $3D$ black hole spacetime in the Einstein-Maxwell and Einstein-Maxwell-dilaton theory, respectively. Dirac and scalar fields are used to judge the quantum singularity. The paper ends with a conclusion in Sec. VI.

II. A BRIEF REVIEW OF QUANTUM SINGULARITIES

In classical general relativity, the spacetime is said to be singular if the evolution of timelike or null geodesics is not defined after a proper time. Horowitz and Marolf, based on the pioneering work of Wald [6], have proposed the criteria to test the classical singularities with quantum test particles that obey the Klein-Gordon equation for static spacetime having timelike singularities. According to this criteria, the singular character of the spacetime is defined as the ambiguity in the evolution of the wave functions. That is to say, the singular character is determined in terms of the ambiguity when attempting to find a self-adjoint extension of the operator to the entire space. If the extension is unique, it is said that the space is quantum mechanically regular. The brief review is as follows:

Consider a static spacetime $(M, g_{\mu\nu})$ with a timelike Killing vector field ξ^{μ}. Let t denote the Killing parameter and Σ denote a static slice. The Klein-Gordon equation on this space is

$$(\nabla^{\mu}\nabla_{\mu} - M^2)\psi = 0. \tag{1}$$

This equation can be written in the form of

$$\frac{\partial^2 \psi}{\partial t^2} = \sqrt{f}D^i(\sqrt{f}D_i\psi) - fM^2\psi = -A\psi, \tag{2}$$

in which $f = -\xi^{\mu}\xi_{\mu}$ and D_i is the spatial covariant derivative on Σ. The Hilbert space $(L^2(\Sigma))$ is the space of square integrable functions on Σ. The domain of the operator A, $D(A)$ is taken in such a way that it does not enclose the spacetime singularities. An appropriate set is $C_0^{\infty}(\Sigma)$, the set of smooth functions with compact support on Σ. Operator A is real, positive and symmetric therefore its self-adjoint extensions always exist. If it has a unique extension A_E, then A is called essentially self-adjoint [7]. Accordingly, the Klein-Gordon equation for a free particle satisfies

$$i\frac{d\psi}{dt} = \sqrt{A_E}\psi, \tag{3}$$

with the solution

$$\psi(t) = \exp[-it\sqrt{A_E}]\psi(0). \tag{4}$$

If A is not essentially self-adjoint, the future time evolution of the wave function (Eq. (4)) is ambiguous. Then, Horowitz and Marolf define the spacetime as quantum mechanically singular. However, if there is only one self-adjoint extension, the operator A is said to be essentially self-adjoint and the quantum evolution described by Eq. (4) is uniquely determined by the initial conditions. According to the Horowitz and Marolf criterion, this spacetime is said to be quantum mechanically nonsingular. In order to determine the number of self-adjoint extensions, the concept of deficiency indices is used. The deficiency subspaces $N_{\pm}$ are defined by (see Ref. [5] for a detailed mathematical background),

$$\begin{aligned} N_+ &= \{\psi \in D(A^*), \quad A^*\psi = Z_+\psi, \quad \mathrm{Im}Z_+ > 0\} \\ &\quad \text{with dimension } n_+ \\ N_- &= \{\psi \in D(A^*), \quad A^*\psi = Z_-\psi, \quad \mathrm{Im}Z_- > 0\} \\ &\quad \text{with dimension } n_-. \end{aligned} \tag{5}$$

The dimensions (n_+, n_-) are the deficiency indices of the operator A. The indices $n_+(n_-)$ are completely independent of the choice of $Z_+(Z_-)$ depending only on whether Z lies in the upper (lower) half complex plane. Generally one takes $Z_+ = i\lambda$ and $Z_- = -i\lambda$, where λ is an arbitrary positive constant necessary for dimensional reasons. The determination of deficiency indices then reduces to counting the number of solutions of $A^*\psi = Z\psi$; (for $\lambda = 1$),

$$A^*\psi \pm i\psi = 0 \tag{6}$$

that belong to the Hilbert space $\mathcal{H}$. If there are no square integrable solutions (i.e. $n_+ = n_- = 0$), the operator A possesses a unique self-adjoint extension and it is essentially self-adjoint. Consequently, a sufficient condition for the operator A to be essentially self-adjoint is to investigate the solutions satisfying Eq. (6) that do not belong to the Hilbert space.

III. (2 + 1)—DIMENSIONAL BTZ SPACETIME COUPLED WITH NONLINEAR ELECTRODYNAMICS

A. Solutions and spacetime structure

The action describing (2 + 1)—dimensional Einstein theory coupled with nonlinear electrodynamics is given by [8],

$$S = \int \sqrt{g}\left(\frac{1}{16\pi}(R - 2\Lambda) + L(F)\right)d^3x. \tag{7}$$

The field equations via variational principle read as

$$G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}, \tag{8}$$

$$T_{ab} = g_{ab}L(F) - F_{ac}F_b{}^cL_{,F}, \tag{9}$$

$$\nabla_a(F^{ab}L_{,F}) = 0 \tag{10}$$

in which $L_{,F}$ stands for the derivative of $L(F)$ with respect to $F = \frac{1}{4}F_{ab}F^{ab}$. The nonlinear field is chosen so that the energy momentum tensor (9) has a vanishing trace. The trace of the tensor gives

$$T = T_{ab}g^{ab} = 3L(F) - 4FL_{,F}. \qquad (11)$$

Hence, to have a vanishing trace, the electromagnetic Lagrangian is obtained as

$$L = c|F|^{3/4}, \qquad (12)$$

where c is an integration constant. With reference to the paper [8], the complete solution to the above action is given by the metric,

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\theta^2, \qquad (13)$$

where the metric function $f(r)$ is given by

$$f(r) = -m + \frac{r^2}{l^2} + \frac{4q^2}{3r}. \qquad (14)$$

Here $m > 0$ is the mass, $l^2 = -\Lambda^{-1}$ the case $\Lambda > 0$ ($\Lambda < 0$), that corresponds with an asymptotically de Sitter (anti–de Sitter) spacetime, and q is the electric charge. This metric represents the BTZ spacetime in nonlinear electrodynamics. If $\Lambda = 0$, we have an asymptotically flat solution coupled with a Coulomb-like field. The Kretschmann scalar which indicates the occurrence of curvature singularity is given by

$$\mathcal{K} = \frac{12}{l^4} + 6\frac{\beta^2}{r^6}, \qquad (15)$$

in which $\beta = \frac{4q^2}{3}$. It is clear that $r = 0$ is a typical central curvature singularity. According to the values of Λ, m and q, this singularity may be clothed by single or double horizons. (See Ref. [8] for details).

However, for specific values of Λ, m and q the central curvature singularity becomes naked and it deserves to be investigated within the framework of quantum mechanics. To find the condition for naked singularities the metric function is written in the following form,

$$f(r) = -\frac{m}{r}\left(r + \tilde{\Lambda}r^3 - \frac{4\tilde{q}^2}{3}\right), \qquad (16)$$

where $\tilde{\Lambda} = \frac{\Lambda}{m}$ and $\tilde{q}^2 = \frac{q^2}{m}$. Since the range of coordinate r varies from 0 to infinity, the negative root will indicate the condition for a naked singularity. In order to find the roots, we set $f(r) = 0$ which yields $r^3 + \frac{r}{\tilde{\Lambda}} - \frac{4\tilde{q}^2}{3\tilde{\Lambda}} = 0$. The standard procedure is followed for a solution via a new variable defined by $r = z - \frac{1}{3\tilde{\Lambda}z}$ that transforms the equation to $27\tilde{\Lambda}^3z^6 - 36\tilde{\Lambda}^2\tilde{q}^2z^3 - 1 = 0$. This equation can be solved easily and the final answer is

$$r = u^{1/3} - \frac{1}{3\tilde{\Lambda}u^{1/3}}, \qquad (17)$$

in which $u = \frac{12\tilde{q}^2\tilde{\Lambda}\pm 2\sqrt{3\tilde{\Lambda}(12\tilde{q}^4\tilde{\Lambda}+1)}}{18\tilde{\Lambda}^2}$, with a constraint condition $3\tilde{\Lambda}(12\tilde{q}^4\tilde{\Lambda} + 1) > 0$. After some algebra, we end up with the following equation,

$$r = a^{1/3}\left\{\left(1 \pm \frac{b}{a}\right)^{1/3} + \left(1 \mp \frac{b}{a}\right)^{1/3}\right\}, \qquad (18)$$

where $a = \frac{2\tilde{q}^2}{3\tilde{\Lambda}}$ and $b = \frac{\sqrt{3\tilde{\Lambda}(12\tilde{q}^4\tilde{\Lambda}+1)}}{9\tilde{\Lambda}^2}$. It can be verified easily that the expression inside the curly bracket in Eq. (18) is always positive. Hence, the only possibility for a negative root is $a < 0$. This implies $\tilde{\Lambda} < 0$. Therefore, the condition $12\tilde{q}^4\tilde{\Lambda} + 1 < 0$ is imposed from the constraint condition. As a result, for a naked singularity, $\tilde{\Lambda} < -\frac{1}{12\tilde{q}^4}$ or $\Lambda < -\frac{m^3}{12q^4}$ should be satisfied.

Our aim now is to investigate the quantum singularity structure of the naked singularity that may arise if the constant coefficients satisfy $\Lambda < -\frac{m^3}{12q^4}$.

B. Klein-Gordon fields

Using separation of variables, $\psi = R(r)e^{in\theta}$, we obtain the radial portion of Eq. (6) as

$$R_n'' + \frac{(fr)'}{fr}R_n' - \frac{n^2}{fr^2}R_n - \frac{M^2}{f}R_n \pm \frac{i}{f^2}R_n = 0, \qquad (19)$$

where a prime denotes the derivative with respect to r.

1. The case of $r \to \infty$

The Coulomb-like field in metric function (14) becomes negligibly small and hence the metric takes the form

$$ds^2 \simeq -\left(\frac{r^2}{l^2}\right)dt^2 + \left(\frac{r^2}{l^2}\right)^{-1}dr^2 + r^2d\theta^2. \qquad (20)$$

This particular case overlaps with the results already reported in [2]. Hence, no new result arises for this particular case. This is expected because the effect of the source term vanishes for large values of r.

2. The case of $r \to 0$

The case near the origin is topologically different compared to the analysis reported in [2]. Here, the spacetime is not conic. The approximate metric near the origin is given by

$$ds^2 \simeq -\left(\frac{\beta}{r}\right)dt^2 + \left(\frac{\beta}{r}\right)^{-1}dr^2 + r^2d\theta^2. \qquad (21)$$

This metric can also be interpreted as the 2 + 1 dimensional topological Schwarzchild-like black hole geometry.

For the solution of the radial equation (19), we assume a massless case (i.e. $M = 0$), and ignore the term $\pm i\frac{R_n}{f^2}$ (since it is negligible near the origin). Then it takes the form

$$R_n'' - \frac{n^2}{\beta r} R_n = 0, \quad (22)$$

whose solution is

$$R_n(r) = C_{1n}\sqrt{r} I_1(k) + C_{2n}\sqrt{r} K_1(k), \quad (23)$$

where $I_1(k)$ and $K_1(k)$ are the first and second kind modified Bessel functions and $k = \sqrt{\frac{4n^2 r}{\beta}}$. The behavior of the modified Bessel functions for real $\nu \geq 0$ as $r \to 0$ are given by

$$I_\nu(x) \simeq \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu, \qquad K_\nu(x) \simeq \begin{cases} -\left[\ln\left(\frac{x}{2}\right) + 0.5772\ldots\right], & \nu = 0 \\ \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu, & \nu \neq 0 \end{cases}, \quad (24)$$

thus $I_1(k) \sim \frac{1}{\Gamma(2)}(\frac{k}{2})$ and $K_1(k) \sim \frac{\Gamma(1)}{2}(\frac{2}{k})$. Checking for the square integrability of the solution (23) requires the behavior of the integral for $I_1(k) \approx \int r^4 dr$ and $K_1(k) \approx \int dr$ which are both convergent as $r \to 0$. Any linear combination is also square integrable. It follows the solution (23) belonging to the Hilbert space $\mathcal{H}$ and therefore the operator A described in Eq. (6) is not essentially self-adjoint. So, the naked singularity at $r = 0$ is quantum mechanically singular if it is probed with quantum particles.

Another approach to remove the quantum singularity is to choose the function space to be the Sobolev space (H^1) which is used for the first time in this context by Ishibashi and Hosoya [5]. Here, the function space is defined by $\mathcal{H} = \{R \| \|R\| < \infty\}$, where the norm defined in 2 + 1 dimensional geometry as

$$\|R\|^2 \sim \int r f^{-1} |R|^2 dr + \int r f \left|\frac{\partial R}{\partial r}\right|^2 dr, \quad (25)$$

which involves both the wave function and its derivative to be square integrable. The failure in the square integrability indicates that the operator A is essentially self-adjoint and thus, the spacetime is "wave regular." According to this norm, the first integral is square integrable while the second integral behaves for the functions $I_1(k)$ as $\approx \int_0 dr$ and $K_1(k)$ integral vanishes. As a result, the wave functions are square integrable and thus the spacetime is quantum mechanically wave singular. It should be noted that the Sobolev space is not the natural quantum mechanical Hilbert space.

C. Dirac fields

We apply the same methodology as in [2] for finding a solution to the Dirac equation. Since the fermions have only one spin polarization in 2 + 1 dimensions [9], Dirac matrices are reduced to Pauli matrices [10] so that

$$\gamma^{(j)} = (\sigma^{(3)}, i\sigma^{(1)}, i\sigma^{(2)}), \quad (26)$$

where Latin indices represent internal (local) indices. In this way,

$$\{\gamma^{(i)}, \gamma^{(j)}\} = 2\eta^{(ij)} I_{2\times 2}, \quad (27)$$

where $\eta^{(ij)}$ is the Minkowski metric in 2 + 1 dimensions and $I_{2\times 2}$ is the identity matrix. The coordinate dependent metric tensor $g_{\mu\nu}(x)$ and matrices $\sigma^\mu(x)$ are related to the triads $e_\mu^{(i)}(x)$ by

$$g_{\mu\nu}(x) = e_\mu^{(i)}(x) e_\nu^{(j)}(x) \eta_{(ij)}, \qquad \sigma^\mu(x) = e_{(i)}^\mu \gamma^{(i)}, \quad (28)$$

where μ and ν are the external (global) indices.

The Dirac equation in 2 + 1 dimensional curved spacetime for a free particle with mass M becomes

$$i\sigma^\mu(x)[\partial_\mu - \Gamma_\mu(x)]\Psi(x) = M\Psi(x), \quad (29)$$

where $\Gamma_\mu(x)$ is the spinorial affine connection and is given by

$$\Gamma_\mu(x) = \frac{1}{4} g_{\lambda\alpha}[e_{\nu,\mu}^{(i)}(x) e_{(i)}^\alpha(x) - \Gamma_{\nu\mu}^\alpha(x)] s^{\lambda\nu}(x), \quad (30)$$

$$s^{\lambda\nu}(x) = \frac{1}{2}[\sigma^\lambda(x), \sigma^\nu(x)]. \quad (31)$$

The causal structure of the spacetime indicates that there are two singular cases to be investigated. The asymptotic case $r \to \infty$ has already been analyzed by PL. The case of $r \to 0$ is not conical so there is a topological difference in the spacetime near $r = 0$. Hence, the suitable triads for the metric (21) are given by

$$e_\mu^{(i)}(t, r, \theta) = \mathrm{diag}\left(\left(\frac{\beta}{r}\right)^{1/2}, \left(\frac{r}{\beta}\right)^{1/2}, r\right). \quad (32)$$

The coordinate dependent gamma matrices and the spinorial affine connection are given by

$$\sigma^\mu(x) = \left(\left(\frac{r}{\beta}\right)^{1/2}\sigma^{(3)}, i\left(\frac{\beta}{r}\right)^{1/2}\sigma^{(1)}, \frac{i\sigma^{(2)}}{r}\right), \qquad \Gamma_\mu(x) = \left(\frac{-\beta\sigma^{(2)}}{4r^2}, 0, \frac{i}{2}\left(\frac{\beta}{r}\right)^{1/2}\sigma^{(3)}\right). \quad (33)$$

Now, for the spinor

$$\Psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}, \quad (34)$$

the Dirac equation can be written as

$$i\left(\frac{r}{\beta}\right)^{1/2}\frac{\partial\psi_1}{\partial t} - \left(\frac{\beta}{r}\right)^{1/2}\frac{\partial\psi_2}{\partial r} + \frac{i}{r}\frac{\partial\psi_2}{\partial\theta} - \frac{1}{4}\left(\frac{\beta}{r^3}\right)^{1/2}\psi_2 - M\psi_1 = 0,$$
$$-i\left(\frac{r}{\beta}\right)^{1/2}\frac{\partial\psi_2}{\partial t} - \left(\frac{\beta}{r}\right)^{1/2}\frac{\partial\psi_1}{\partial r} - \frac{i}{r}\frac{\partial\psi_1}{\partial\theta} - \frac{1}{4}\left(\frac{\beta}{r^3}\right)^{1/2}\psi_1 - M\psi_2 = 0. \quad (35)$$

The following ansatz will be employed for the positive frequency solutions:

$$\Psi_{n,E}(t,x) = \begin{pmatrix} R_{1n}(r) \\ R_{2n}(r)e^{i\theta} \end{pmatrix} e^{in\theta} e^{-iEt}. \quad (36)$$

The radial parts of the Dirac equation for investigating the behavior as $r \to 0$, are

$$R''_{1n} + \frac{\alpha_1}{\sqrt{r}} R'_{1n} + \frac{\alpha_2}{r^{3/2}} R_{1n} = 0,$$
$$R''_{2n} + \frac{\alpha_3}{\sqrt{r}} R'_{2n} + \frac{\alpha_4}{r^{3/2}} R_{2n} = 0, \quad (37)$$

where $\alpha_1 = \frac{2M-E}{2M\sqrt{\beta}}$, $\alpha_2 = \frac{-7E+4M(4n+1)}{16M\sqrt{\beta}}$, $\alpha_3 = \frac{2M+E}{2M\sqrt{\beta}}$ and $\alpha_4 = \frac{7E-4M(4n+3)}{16M\sqrt{\beta}}$. Then, the solutions are given by

$$R_1(r) = e^{-(b/2)\rho}\{C_1\sqrt{\rho}\text{Whittaker}_M(a,1,b\rho) + C_2\sqrt{\rho}\text{Whittaker}_W(a,1,b\rho)\},$$
$$R_2(r) = e^{-(b'/2)\rho}\{C_3\sqrt{\rho}\text{Whittaker}_M(a',1,b'\rho) + C_4\sqrt{\rho}\text{Whittaker}_W(a',1,b'\rho)\}WM$$

where $\rho = \sqrt{r}$, $a = \frac{-9E+8M(1+2n)}{4(2M-E)}$, $a' = \frac{9E-8M(1+2n)}{4(2M+E)}$, $b = 2\alpha_1$, and $b' = 2\alpha_3$.

When we look for the square integrability of the above solutions, we obtained that both functions Whittaker_M and Whittaker_W are square integrable near $\rho = 0$ (or $r = 0$) for both $R_1(r)$ and $R_2(r)$. One has

$$\int r f^{-1}|R|^2 dr \approx \int \rho^6 e^{-b\rho}[\text{Whittaker}_M(a,1,b\rho)]^2 d\rho < \infty, \quad (38)$$

and

$$\approx \int \rho^6 e^{-b\rho}[\text{Whittaker}_W(a,1,b\rho)]^2 d\rho < \infty. \quad (39)$$

We note that these results are verified first by expanding the Whittaker functions in series form up to the order of $\mathcal{O}(\rho^6)$ and then by integrating term by term in the limit as $r \to 0$.

The set of solutions for the Dirac equation for the spacetime (21) is given by

$$\Psi_{n,E}(t,\mathbf{x}) = \begin{pmatrix} e^{-(b/2)\rho}\{C_{1n}\sqrt{\rho}\text{Whittaker}_M(a,1,b\rho) + C_{2n}\sqrt{\rho}\text{Whittaker}_W(a,1,b\rho)\} \\ e^{-(b'/2)\rho}\{C_{3n}\sqrt{\rho}\text{Whittaker}_M(a',1,b'\rho) + C_{4n}\sqrt{\rho}\text{Whittaker}_W(a',1,b'\rho)\}e^{i\theta} \end{pmatrix} e^{in\theta} e^{-iEt},$$

and an arbitrary wave packet can be written as

$$\Psi(t,\mathbf{x}) = \sum_{n=-\infty}^{+\infty} C_n \begin{pmatrix} e^{-(b/2)\rho}\sqrt{\rho}(\text{Whittaker}_M(a,1,b\rho) + \text{Whittaker}_W(a,1,b\rho)) \\ e^{-(b'/2)\rho}\sqrt{\rho}(\text{Whittaker}_M(a',1,b'\rho) + \text{Whittaker}_W(a',1,b'\rho))e^{i\theta} \end{pmatrix} e^{in\theta} e^{-iEt} \quad (40)$$

where C_n is an arbitrary constant. Hence, initial condition $\Psi(0,\mathbf{x})$ is sufficient to determine the future time evolution of the particle. The spacetime is then quantum regular when tested by fermions.

IV. (2 + 1)—DIMENSIONAL BTZ SPACETIME WITH LINEAR ELECTRODYNAMICS

A. Solutions and spacetime structure

The metric for the charged BTZ spacetime in linear electrodynamics is given by [11]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\theta^2, \quad (41)$$

with the metric function

$$f(r) = -m + \frac{r^2}{l^2} - 2q^2 \ln\left(\frac{r}{l}\right), \quad (42)$$

where q is the electric charge and $m > 0$ is the mass and $l^2 = \Lambda^{-1}$. The Kretschmann scalar is given by

$$\mathcal{K} = \frac{12}{l^4} - \frac{8q^2}{r^2 l^2} + \frac{4q^4}{r^4}, \quad (43)$$

which displays a power-law central curvature singularity at $r = 0$. According to the values of m, l and q, this central singularity is clothed by horizons or it remains naked. Our interest here is to investigate the quantum mechanical behavior of the naked singularity. In order to find the condition for naked singularity, we set $f(r_h) = 0$ and the solution for $l = 1$ is

$$r_h = \exp\left\{-\frac{m}{2q^2} - \frac{1}{2}\text{Lambertw}\left(-\frac{1}{q^2}e^{-m/q^2}\right)\right\},$$

in which Lambertw represents the Lambert function [12]. Figure 1 displays (unmarked region) the possible values of m and q that result in naked singularity.

The causal structure is similar to the case considered in the previous section. There are two singular cases to be investigated. The case for $r \to \infty$ is approximately the same case considered in [2]. Therefore, the results reported by PL are valid for this case as well. For small r values, the approximate metric can be written in the following form,

$$ds^2 \approx -(2q^2|\ln(\tilde{r})|)dt^2 + (2q^2|\ln(\tilde{r})|)^{-1}dr^2 + r^2 d\theta^2, \quad (44)$$

in which $\tilde{r} = \frac{r}{l} \ll 1$.

B. Klein-Gordon fields

The radial equation for the metric (44) is obtained for the massless case as

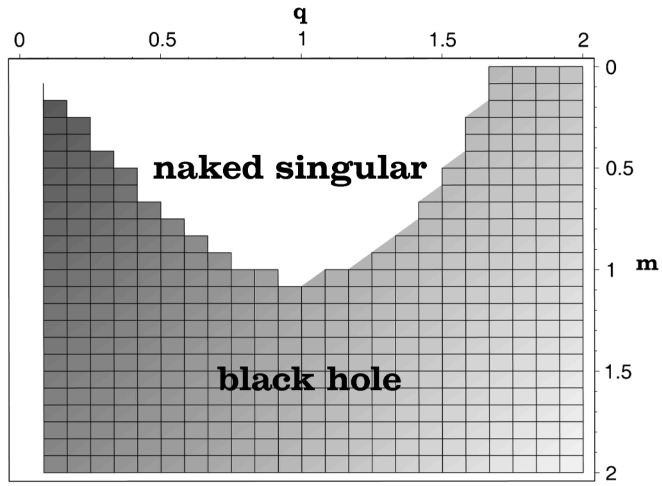

FIG. 1. Plot of r_h for different values of m and q. Marked region displays the formation of the black hole, unmarked region shows the formation of naked singularity.

$$R_n'' + \frac{1}{\tilde{r}}\left(1 + \frac{1}{\ln\tilde{r}}\right)R_n' + \frac{n^2}{2q^2r^2\ln\tilde{r}}R_n = 0. \tag{45}$$

Since $\frac{r}{l} \ll 1$, the solution can be written in terms of zeroth order first and second kind modified Bessel functions,

$$R_n(x) = C_{1n}I_0\left(\frac{\sqrt{2}n}{q}x\right) + C_{2n}K_0\left(\frac{\sqrt{2}n}{q}x\right), \tag{46}$$

where $-x^2 = \ln\tilde{r}$. As $\tilde{r} \rightarrow 0$, $x \rightarrow \infty$. The behavior of the modified Bessel functions for $x \gg 1$ are $I_0(x) \simeq \frac{e^x}{\sqrt{2\pi x}}$ and $K_0(x) \simeq \sqrt{\frac{\pi}{2x}}e^{-x}$. These functions are always square integrable for $x \rightarrow \infty$, that is

$$\int rf^{-1}|R|^2 dr \approx \int xe^{-2x^2}f^{-1}|R|^2 dx < \infty.$$

These results indicate that the charged BTZ black hole in linear electrodynamics is quantum mechanically singular when probed with quantum test particles that obey the Klein-Gordon equation.

If we use the Sobolev norm (25), the second integral which involves the derivative of the wave function $I_0(x) \simeq \frac{e^x}{\sqrt{2\pi x}}$ becomes $\approx \int x^{-2}e^{2x}(2x-1)^2 dx$. Numerical integration has revealed that as $x \rightarrow \infty$, $\sim \int x^{-2}e^{2x}(2x-1)^2 dx \rightarrow \infty$. On the other hand for the wave function $K_0(x) \simeq \sqrt{\frac{\pi}{2x}}e^{-x}$, the second integral in the Sobolev norm is solved numerically as $x \rightarrow \infty$, $\sim \int x^{-2}e^{-2x}(2x+1)^2 dx < \infty$ which is square integrable. As a result, the charged coupled BTZ black hole in linear electrodynamics is quantum mechanically wave regular if and only if the arbitrary constant parameter is $C_{2n} = 0$ in Eq. (46).

Consequently, if the naked singularity both in linear and nonlinear electrodynamics is probed with quantum test particles, the following results are obtained:

(1) In the classical point of view, the Kretschmann scalar in the nonlinear case diverges faster than in the linear case.

(2) In the quantum mechanical point of view, if the chosen function space is Sobolev space, the spacetime remains singular for the nonlinear case, but the spacetime can be made wave regular for the linear case.

From these results we may conclude that the structure of the naked singularity in the nonlinear electrodynamics is deeper rooted than the singularity in the linear case.

C. Dirac fields

The effect of the charge when $r \rightarrow \infty$ does not contribute as much as the term that contains the cosmological constant. Therefore, we ignore the mass and the charged terms in the metric function (42). This particular case has already been analyzed in [2]. The contribution of the charge is dominant when $r \rightarrow 0$. The Dirac equation for the metric (44) is solved by using the same method demonstrated in the previous section. We obtain the radial equation in the limit $r \rightarrow 0$ as

$$R_j'' + \frac{1}{r}R_j' - \frac{R_j}{4r^2} = 0, \qquad j = 1, 2 \tag{47}$$

whose solution is given by

$$R_j(r) = C_{1j}\sqrt{r} + \frac{C_{2j}}{\sqrt{r}}, \tag{48}$$

where C_{1j} and C_{2j} are arbitrary constants. The solution given in Eq. (48) is square integrable. The arbitrary wave packet can be written as

$$\Psi(t, \mathbf{x}) = \sum_{n=-\infty}^{+\infty}\left(\begin{Bmatrix} R_1(r) \\ R_2(r)e^{i\theta} \end{Bmatrix}\right)e^{in\theta}e^{-iEt}. \tag{49}$$

Thus, the spacetime is quantum mechanically regular when probed with fermions.

V. (2 + 1)—DIMENSIONAL EINSTEIN-MAXWELL-DILATON GRAVITY

A. Solutions and spacetime structure

In this section, we consider $3D$ black holes described by the Einstein-Maxwell-dilaton action,

$$S = \int d^3x\sqrt{-g}\left(R - \frac{B}{2}(\nabla\phi)^2 - e^{-4a\phi}F_{\mu\nu}F^{\mu\nu} + 2e^{b\phi}\Lambda\right), \tag{50}$$

where R is the Ricci scalar, ϕ is the dilaton field, $F_{\mu\nu}$ is the Maxwell field and Λ, a, b, and B are arbitrary couplings.

The general solution to this action is given by [13]

$$ds^2 = -f(r)dt^2 + \frac{4r^{(4/N)-2}dr^2}{N^2\gamma^{4/N}f(r)} + r^2 d\theta^2, \qquad (51)$$

where

$$f(r) = Ar^{(2/N)-1} + \frac{8\Lambda r^2}{(3N-2)N} + \frac{8Q^2}{(2-N)N}. \qquad (52)$$

Here, A is a constant of integration which is proportional to the quasilocal mass ($A = \frac{-2m}{N}$), γ is an integration constant and Q is the charge. The dilaton field is given by

$$\phi = \frac{2k}{N}\ln\left(\frac{r}{\beta(\gamma)}\right) \qquad (53)$$

in which $\beta(\gamma)$ is a γ related constant parameter. Note that the above solution for $N = 2$ contains both the vacuum BTZ metric if one takes $Q = A = 0$ and the BTZ black hole if $A < 0$, $Q = 0$. However, if the constant parameters are chosen appropriately, the resulting metric represents black hole solutions with prescribed properties. For example, when $N = \frac{6}{5}$, $A = -\frac{5m}{3}$, the metric function given in Eq. (52) becomes

$$f(r) = -\frac{5m}{3}r^{2/3} + \frac{25\Lambda}{6}r^2 + \frac{25Q^2}{3}, \qquad (54)$$

and therefore the corresponding metric is

$$ds^2 = -f(r)dt^2 + \frac{\alpha r^{4/3}dr^2}{f(r)} + r^2 d\theta^2, \qquad (55)$$

where $\alpha = \frac{25}{9\gamma^{10/3}}$ is a constant parameter.

The Kretschmann scalar for this solution is given by

$$\mathcal{K} = \frac{25\{12m^2r^{5/3} + 5\Lambda r^3[55\Lambda r^{4/3} - 4m] + 40r^{1/3}Q^2[2(5Q^2 - mr^{2/3}) - 5\Lambda r^2]\}}{81\alpha^2 r^7}, \qquad (56)$$

which indicates a central curvature singularity at $r = 0$ that is clothed by the event horizon. To find the location of horizons, g_{tt} is set to zero and we have

$$r^2 - \frac{2m}{5\Lambda}r^{2/3} + \frac{2Q^2}{\Lambda} = 0. \qquad (57)$$

There are three possible cases to be considered.

Case 1: If $\frac{Q^2}{\Lambda} < (\frac{2m}{15\Lambda})^{3/2}$, the equation admits two positive roots indicating inner and outer horizons of the black hole.

Case 2: If $\frac{Q^2}{\Lambda} = (\frac{2m}{15\Lambda})^{3/2}$, this is an extreme case and Eq. (57) has one real positive root. This means that there is only one horizon.

Case 3: If $\frac{Q^2}{\Lambda} > (\frac{2m}{15\Lambda})^{3/2}$, there is no real positive root and the solution does not admit the black hole so that the singularity at $r = 0$ is naked. With reference to the detailed analysis given in [13], the Penrose diagram of the solution illustrates the timelike character of the singularity at $r = 0$. Our aim in this section is to investigate the behavior of this naked singularity when probed with Klein-Gordon and Dirac fields in the framework of quantum mechanics.

B. Klein-Gordon fields

The radial equation for the metric (55) is obtained for the massles case ($M = 0$) as

$$R_n'' + \frac{(fr^{1/3})'}{fr^{1/3}}R_n' - \frac{\alpha n^2}{fr^{2/3}}R_n \pm \frac{i\alpha r^{4/3}}{f^2}R_n = 0. \qquad (58)$$

The behavior of the radial equation as $r \to 0$ is

$$R_n'' + \frac{1}{3r}R_n' - \frac{k^2}{r^{2/3}}R_n = 0, \qquad (59)$$

where $k = \frac{3\alpha n^2}{25Q^2}$. The solution is given by

$$R_n(r) = C_{1n}\cosh\left(\frac{3k}{2}r^{2/3}\right) + iC_{2n}\sinh\left(\frac{3k}{2}r^{2/3}\right). \qquad (60)$$

Both solutions are square integrable in Hilbert space, that is, $\int rg_{rr}|R|^2dr < \infty$. Therefore, the spacetime is quantum mechanically singular when probed with quantum particles obeying the Klein-Gordon equation.

If we use the Sobolev norm,

$$\|R\|^2 \sim \int rg_{rr}|R|^2dr + \int rg_{rr}^{-1}\left|\frac{\partial R}{\partial r}\right|^2 dr,$$

although the first integral of the solution is square integrable, the second integral for $C_{1n} = 0$ fails to be square integrable and the spacetime is quantum mechanically wave regular.

C. Dirac fields

The Dirac equation can be written as

$$\frac{i}{\sqrt{f}}\psi_{1,t} - \frac{\sqrt{f}}{\sqrt{\alpha}r^{2/3}}\psi_{2,r} + \frac{i}{r}\psi_{2,\theta} - \left\{\frac{5(-2m + 15\Lambda r^{4/3})}{36\sqrt{\alpha}r\sqrt{f}} + \frac{\sqrt{f}}{2\sqrt{\alpha}r^{5/3}}\right\}\psi_2 - M\psi_1 = 0,$$
$$\frac{-i}{\sqrt{f}}\psi_{2,t} - \frac{\sqrt{f}}{\sqrt{\alpha}r^{2/3}}\psi_{1,r} - \frac{i}{r}\psi_{1,\theta} - \left\{\frac{5(-2m + 15\Lambda r^{4/3})}{36\sqrt{\alpha}r\sqrt{f}} + \frac{\sqrt{f}}{2\sqrt{\alpha}r^{5/3}}\right\}\psi_1 - M\psi_2 = 0 \qquad (61)$$

where f is given in (54). By using the same ansatz as in (36), the radial part of the Dirac equation becomes

$$R_n'' + \frac{a_1}{r^{1/3}} R_n' + \frac{a_2}{r^{2/3}} R_n = 0, \qquad n = 1, 2 \tag{62}$$

in which $a_1 = \frac{3Q\sqrt{3\alpha} - m}{15Q^2}$, $a_2 = \frac{-108Q^2\alpha n(1+n) + m(m - 6Q\sqrt{3\alpha})}{900Q^4}$. The solution becomes

$$R_n(r) = r^{1/6} e^{-(3a_1/4)r^{2/3}} \times \left\{ \begin{array}{l} C_{1n} \mathrm{Whittaker}_M\left(\frac{a_1}{4\sqrt{a_1^2 - 4a_2}}, \frac{3}{4}, \frac{3}{2} r^{2/3} \sqrt{a_1^2 - 4a_2}\right) + \\ C_{2n} \mathrm{Whittaker}_W\left(\frac{a_1}{4\sqrt{a_1^2 - 4a_2}}, \frac{3}{4}, \frac{3}{2} r^{2/3} \sqrt{a_1^2 - 4a_2}\right) \end{array} \right\}, \tag{63}$$

which is square integrable. This is verified first by expanding the Whittaker functions in series and then by integrating term by term in the limit as $r \to 0$. Consequently, the spacetime is quantum mechanically regular when probed with Dirac fields.

VI. CONCLUSION

Matter coupled $2+1$ dimensional black hole spacetimes are shown to share similar quantum mechanical singularity structure as in the case of the pure BTZ black hole. The inclusion of matter fields changes the topology and creates true curvature singularity at $r = 0$. The effect of the matter fields allows only specific frequency modes in the solution of Klein-Gordon and Dirac fields. If the quantum singularity analysis is based on the natural Hilbert space of quantum mechanics which is the linear function space with square integrability L^2, the singularity at $r = 0$ turns out to be quantum mechanically singular for particles obeying the Klein- Gordon equation and regular for fermions obeying the Dirac equation. We have proved that the quantum singularity structure of $2+1$ dimensional black hole spacetimes is generic for Dirac particles and the character of the singularity in the quantum mechanical point of view is irrespective of whether the matter field is coupled or not. This result suggests that the Dirac fields preserve the cosmic censorship hypothesis in the considered spacetimes that exhibit timelike naked singularities. Instead of horizons (that clothe the singularity in the black hole cases) the repulsive barrier is replaced against the propagation of Dirac fields. However, for particles obeying Klein-Gordon fields, the singularity becomes worse when a matter field is coupled.

However, we have also shown that in the charged BTZ (in linear electrodynamics) and dilaton coupled black hole spacetimes a specific choice of waves exhibit quantum mechanical wave regularity when probed with waves obeying the Klein-Gordon equation, if the function space is Sobolev with the norm defined in (25). The singularity at $r = 0$ is stronger in the nonlinear electrodynamic case. It should be reminded that one may not feel comfortable using the Sobolev norm in place of natural linear function space of quantum mechanics.

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