Some Schurer Type q-Bernstein Operators

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ABSTRACT

In this thesis consist of six chapters. The introduction is given in the first chapter. In the second chapter, some necessary definitions, preliminaries and theorems are given. In this chapter, we also give the important theorems; by Korovkin and Volkov, Bernstein polynomials in one two variables, q-Bernstein, Bernstein-Chlodowsky and q-Bernstein Chlodowsky polynomials.

In the third chapter, q-Bernstein Schurer operators are defined. Many properties and results of these polynomials, such as Korovkin type approximation and the rate of convergence of these operators in terms of Lipschitz class functional are given.

In the fourth chapter q-Bernstein-Schurer-Chlodowsky operators are introduced. Korovkin type approximation theorem is given and the rate of convergence of this approximation is obtained by means of modulus of continuity of the function is obtained.

In the fifth chapter, Schurer-type q-Bernstein Kantorovich operators are defined. Moreover the order of convergence of the operators in terms of modulus of continuity of the derivative of the function, and elements of Lipschitz classes are discussed.

In the last chapter, Kantorovich type q-Bernstein operators are defined. Furthermore, Korovkin type approximation theorem is proved and the rate of convergence of this approximation are given.

Keywords: q-Bernstein Schurer operators, Korovkin theorem, Schurer Type q-Bernstein Polynomials, Kantorovich type q-Bernstein-Schurer-Chlodovsky operators.

ÖZ

Bu tez altı bölümden oluşmaktadır. Birinci bölüm giriş kısmı olarak verilmiştir. İkinci bölümde, tez boyunca ihtiyaç duyulacak bazı tanımlar, tanımlarla ilgili bazı temel özellikler ve teoremler verilmiştir. Ayrıca Korovkin and Volkov Teoremleri, bir ve iki değişkenli Bernstein Polinomları, q-Bernstein Polinomları ve Bernstein Chlodowsky and q-Bernstein Chlodowsky Polinomları incelenmiştir.

Üçüncü bölümde q-Bernstein Schurer Operatörleri tanımlanmıştır. q-Bernstein Schurer Operatörlerinin yakınsaklığı Korovkin Teoremi yardımıyla ve Liptsitz sınıfındaki yakınsaklığı incelenmiştir.

Dördüncü bölümde q-Bernstein Schurer-Chlodowsky Operatörü tanımlanmıştır. Korovkin tipli yakınsaklık teoremi, fonksiyonun ve fonksiyonunun türevinin süreklilik modülü yardımıyla yakınsama hızları hesaplanmıştır.

Beşinci bölümde Schurer tipli q-Bernstein Kantorovich Operatörleri tanımlanmıştır. Bu operatörlerin modüllerinin ve türevlerinin yakınsaklıkları hesaplanmıştır.

Altıncı bölümde Kantorovich tipli q-Bernstein-Schurer-Chlodowsky Operatörleri tanımlanmıştır. Bununla birlikte Korovkin tipli teorem yaklaşımı ispatlanmış ve bu yakınsamanın yakınsaklık derecesi hesaplanmıştır.

Anahtar Kelimeler: q-Bernstein Schurer Operatörleri, Korovkin Teoremi, Schurer Type q-Bernstein Operatörleri, Kantorovich Type q-Bernstein-Schurer-Chlodovsky operatörleri.

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TABLE OF CONTENS

LIST OF SYMBOLS

$\mathbb{N}$	the set of naturel number
$\mathbb{N}_0$	the set of naturel number including zero
$\mathbb{R}$	the set of real numbers
(a,b)	an open interval
[a,b]	a closed interval
$\mathbb{C}$[a,b]	the set of all real-valued and continuous functions defined on the compact interval [a,b].
(f,δ)	the first modulus of continuity
$L(f;x)$	linear operator
$B_n(f;x)$	Bernstein polynomials
$B_n(f;q;x)$	q-Bernstein polynomials
$B_n^c(f;x)$	Bernstein Chlodowsky polynomials
$C_n(f;x)$	q-Bernstein Chlodowsky polynomials
$B_n^p(f;q;x)$	q-Bernstein Schurer operators
$C_n^p(f;q;x)$	q-Bernstein-Schurer- Chlodowsky polynomials
$K_n^p(f;q;x)$	Schurer type q-Bernstein Kantorovich operators

$T_n^p(f;q;x)$	Kantorovich type q-Bernstein-Schurer-Chlodowsky operators

Chapter 1

INTRODUCTION

It was S.N. Bernstein, who proposed the operators [15]

$$B_n(f;x) = \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k},$$

called the Bernstein operators and gave simple proof of the Weierstrass famous theorem in 1912: "each continuous real valued function f on $[a, b]$ is uniformly approximable by algebraic polynomials".

Korovkin (1957) has shown that for a sequence (L_n) of positive linear operators, convergence $L_n\ (f) \to f$ in the uniform norm follows for all $f \in C(A)$, if it holds for finitely many "test functions" $f_1, f_2, \ldots f_n$ from $C(A)$, where $C(A)$ is the space of continuous functions defined on the compact domain A.

After the work by Bernstein, Chlodowsky extended the Bernstein polynomials by defining the operators, which are known as Chlodowsky polynomials, [4]

$$C_n(f;x) = \sum_{k=0}^{n} f\left(\frac{kb_n}{n}\right) \binom{n}{k} \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}, \ \ (0 \le x \le b_n)$$

where (b_n) is an increasing sequence of positive numbers satisfying the properties, $\lim\limits_{n\to\infty} b_n = 0$ and $\lim\limits_{n\to\infty} \frac{b_n}{n} = 0$. We refer the paper by Harun Karslı [13], who overviewed the results and historical developments on the Chlodowsky operators.

Among all the linear positive operators, the followings are deserved to be listed:

Laguerre type operators: For $x \in [0, \infty)$ the Laguerre type operators are defined in [3], by

$$P_n(f;x) = (1-x)^{n+1} \exp\left(\frac{xt}{1-x}\right) \sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right) L_k^{(n)}(t) x^k.$$

Letting $t = 0$ in the above operators one gets the modified form of the Meyer-König and Zeller (MKZ) operators where the MKZ operators are defined by [20]

$$M_n(f;x) = (1-x)^{n+1} \sum_{k=0}^{\infty} f\left(\frac{k}{n+k+1}\right) \binom{n+k}{k} x^k, \ (0 \leq x < 1).$$

Szasz-Mirakjan operators: For $x \in [0,1]$, the Szasz-Mirakjan operators are defined by [24]

$$S_n(f;x) = \exp(-nx) \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!}.$$

It was A.Lupaş [16], who first proposed q-based Bernstein operators. For $x \in [0,1]$ and $q > 0$, he introduced the operators

$$R_{n,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\frac{k(k-1)}{2}} x^k (1-x)^{n-k}}{(1-x+qx)\ldots(1-xq^{n-1}x)},$$

where for $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, the q-integer $[n] = [n]_q$ is defined by

$$[n] := 1 + q + \ldots + q^{n-1}; \ [0] := 0,$$

the q-factorial $[n]! = [n]_q!$ is defined by

$$[n]! = [1][2]\ldots[n]; [0]! := 1$$

and for $0 \leq k \leq n$, the q-binomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

Another q-based Bernstein operator was introduced in 1996 by Phillips [23]. He considered the operators

$$B_{n,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n \\ k \end{bmatrix} x^k \prod_{s=0}^{n-k-1} (1-q^s x)$$

where $x \in [0,1]$ and $q > 0$.

In 2008, Harun Karslı and Vijay Gupta [14] proposed the q-Chlodowsky Bernstein operators. For $0 \leq x \leq b_n$, they considered the operators

$$C_{n,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]} b_n\right) \begin{bmatrix} n \\ k \end{bmatrix} \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n-k-1} (1-q^s \frac{x}{b_n})$$

where (b_n) is a positive increasing sequence satisfying $\lim_{n\to\infty} b_n = \infty$.

On the other hand, in 2011 Carmen-Violeta Muraru [21] introduced and investigated the q-Bernstein-Schurer operators. These operators are defined for fixed $p \in \mathbb{N}_0$ and for all $x \in [0,1]$, by

$$B_{n,p}(f;q;x) = \sum_{k=0}^{n+p} f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n+p \\ k \end{bmatrix} (x)^k \prod_{s=0}^{n+p-k-1} (1-q^s x).$$

Note that the case $q = 1$ reduces to the operators considered by Schurer [25].

The q-Laguerre type linear positive operators were defined in 2007 by M. A. Özarslan. For $x \in [0,1]$, $t \in (-\infty, 0]$ and $q \in (0,1)$, he considered the operators [18]

$$P_{n,q}(f;x) = \frac{1}{F_n(x,t)} \sum_{k=0}^{\infty} f\left(\frac{[k]}{[k+n]}\right) L_k^{(n)}(t,q) x^k$$

where $L_k^{(n)}(t,q)$ are the q-Laguerre polynomials,

$$F_n(x,t) = \frac{(xq^{n+1};q)_\infty}{(x;q)_\infty} \sum_{m=0}^{\infty} \frac{q^{m^2+nm}\left[-(1-q)\,xt\right]^m}{(q,q)_m\,(xq^{n+1};q)_m},$$

$$(a;q)_\infty = \prod_{j=0}^{\infty} (1-aq^j), \quad (a \in \mathbb{C})$$

and

$$(a;q)_n = \begin{cases} 1 & , \qquad n=0 \\ (1-a)(1-aq)\ldots(1-aq^{n-1}), & (n \in \mathbb{N}, a \in \mathbb{C}). \end{cases}$$

The case $t = 0$ reduces to the q-Meyer-König and Zeller operators [26]

$$M_{n,q}(f:x) = \prod_{j=0}^{\infty}(1-q^j x) \sum_{k=0}^{\infty} f\left(\frac{[k]}{[k+n]}\right) \begin{bmatrix} n+k \\ k \end{bmatrix} x^k, \ 0 \le x < 1.$$

In the literature, there are two kinds of q-Szasz Mirakjan operators.

The Chlodowsky type q-Szasz Mirakjan operators:

These operators were defined by Aral and Gupta [2]

$$S_{n,q}(f:x) = E_q\left(-[n]\frac{x}{b_n}\right) \sum_{k=0}^{\infty} \frac{[n]^k x^k}{b_n^k\,[k]!} f\left(\frac{[k]}{[n]} b_n\right)$$

where

$$E_q(x) = \sum_{k=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{[n]!} x^n = (-(1-q)x;q)_\infty; x \in \mathbb{R}, |q| < 1,$$

and (b_n) is an increasing sequence of positive real numbers such that $\lim_{n\to\infty} b_n = \infty$.

q-Szasz Mirakjan operators: Let $x \in [0,\infty)$, $0 < q < 1$. The q-Szasz Mirakjan operators were defined in [17] by N.I. Mahmudov as follows:

$$S_{n,q}^*(f:x) = \frac{1}{E_q([n]\,x)} \sum_{k=0}^{\infty} f\left(\frac{[k]}{q^{k-2}\,[n]}\right) q^{\frac{k(k-1)}{2}} \frac{[n]^k\, x^k}{[k]!}.$$

Note that very recently, the q-Szasz Schurer operators were introduced and investigated by M.A. Özarslan in [19].

Finally, we should note that several linear positive operators are investigated in [1], [5], [6], [7],[10],[13].

This thesis organized the as follows:

In chapter 2, we present some preliminaries and auxiliary results, which are needed throughout the thesis.

In chapter 3, we consider the q-Bernstein Schurer operators. We investigate the shape properties of these operators. Furthermore, we calculate the rate of convergence of these operators in terms of Lipschitz class functions.

In chapter 4, we define q-Bernstein-Schurer-Chlodowsky operators. We give a Korovkin type approximation theorem and calculate the rate of convergence of this approximation by means of modulus of continuity of the function and the derivative of the function. Moreover, we compute the rate of convergence for Lipschitz class functionals.

In chapter 5, we introduce Schurer type q-Bernstein Kantorovich operators. We calculate the order of convergence of the operators in terms of modulus of continuity of the derivative of the function and elements of Lipschitz classes.

In chapter 6, we define Kantorovich type q-Bernstein-Schurer-Chlodowsky operators. We prove a Korovkin type approximation theorem and calculate the rate of convergence of this approximation.

Chapter 2

PRELIMINARIES AND AUXILIARY RESULTS

2.1 Linear Positive Operators

In this section we give some basic properties, definitions and elementary properties of the positive linear operators.

Definition 1. *Let X and Y be real linear spaces of functions. The mapping $L : X \to Y$ is said to be linear operator if*

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$$

$\forall f, g \in X$ and $\forall \alpha, \beta \in \mathbb{R}$.

If $f \geq 0$ implies that $Lf \geq 0$ then L is a positive operator.

If

$$X^+ = \{f \in X : f(x) \geq 0\} \text{ and } Y^+ = \{g \in Y : g(x) \geq 0\},$$

$L : X^+ \to L(X^+) \subset Y^+$ and L is linear, then we call the operator L is linear positive operator.

Remark 2. *The linear positive operators are monotone.*

Proof. Let $f(x) \leq g(x)$ then it implies that $g(x) - f(x) \geq 0$ and if L is linear positive operator then $L(g - f; x) \geq 0$. Hence $L(g; x) \geq L(f; x)$. In other words, if

$f, g \in X$ with $f \leq g$ then $Lf \leq Lg$. $\square$

Example 3. *Assume that $p_k(x)$ is a positive real valued polynomials, $k = 0, 1, 2, \cdots, n$ and $x \in I \subset \mathbb{R}$, then the sequence of operators*

$$A_n(f; x) = \sum_{k=0}^{n} f(\alpha_k) p_k(x)$$

are linear and positive, where $\alpha_k \in I$ for all $k = 0, 1, \cdots, n$. To prove this,

$$\begin{aligned} A_n(af + bg; x) &= \sum_{k=0}^{n} (af(\alpha_k) + bg(\alpha_k)) p_k(x) \\ &= a \sum_{k=0}^{n} f(\alpha_k) p_k(x) + b \sum_{k=0}^{n} g(\alpha_k) p_k(x) \\ &= aA_n(f; x) + bA_n(g; x). \end{aligned}$$

In addition, if $f(\alpha_k) \geq 0$ for all $\alpha_k \in I$ $(k = 0, 1, \cdots, n)$ then

$$A_n(f; x) = \sum_{k=0}^{n} f(\alpha_k) p_k(x) \geq 0.$$

Example 4. *The following operator*

$$L(f; x) = \int_a^b f(t) K(t, x) dt$$

is linear and positive iff $K(t, x) \geq 0$ for all $t, x \in [a, b]$, where the continuous function $K(t, x)$ is the kernel of the operator. We show that the condition $K(t, x) \geq 0$ for all $t, x \in [a, b]$ is necessary. If $K(t, x_0) < 0$ at the point $t = x_0$, then there exists an interval $[\alpha, \beta] \subset [a, b]$ such that $K(t_0, x)$ is negative on $[\alpha, \beta]$. Then for function

$$f(t) = \begin{cases} 0, & t \in [a, b] / [\alpha, \beta] \\ 1, & t \in [\alpha, \beta] \end{cases}$$

we have

$$L(f; x) = \int_\alpha^\beta K(t, x_0) dt < 0.$$

Therefore, the condition $K(t, x) \geq 0$ for all $t, x \in [a, b]$ is necessary.

The norm of the operator L is defined by

$$||L|| = ||L||_{(X\to Y)} = \sup_{||f||_X \neq 0} \frac{\|L(f;x)\|_Y}{||f||_X}.$$

The equivalent definition as:

$$||L|| = \sup_{||f||_X = 1} \|L(f;x)\|_Y .$$

Definition 5. *Assume that $L : X \to Y$ be linear operator. $L(f;x)$ is called bounded if there exists a positive number C such that*

$$||L(f;x)||_Y \leq C||f||_X.$$

From the monotonicity of the linear positive operator L,

$$f(x) \leq |f(x)|$$

implies

$$|L(f;x)| < L(|f|;x).$$

Each point of space $C\,[a,b]$ is a continuous real-valued function on $[a,b]$ and $||L||$ is norm of a linear bounded operator.

Lemma 6. *If $X = Y = C\,[a,b]$, then*

$$||L||_{C[a,b]\to C[a,b]} = ||L(1;x)||_{C[a,b]}.$$

Proof. By the definition (2.1.4), it is straight forward to show that:

$$||L||_{C[a,b]\to C[a,b]} = \sup_{||f||_{C[a,b]}=1} = ||L(f;x)||_{C[a,b]} \leq ||L(1;x)||_{C[a,b]}. \tag{2.1.1}$$

On the other hand

$$||L||_{C[a,b]\to C[a,b]} = \sup_{||f||_{C[a,b]}=1} = ||L(f;x)||_{C[a,b]} \leq ||L(1;x)||_{C[a,b]}. \tag{2.1.2}$$

The proof is (2.1.1) and (2.1.2). □

2.2 Korovkin's Theorem and Volkov's Theorem

In this section we give the Korovkin's Theorem for one and two variables.

Theorem 7. ***(Korovkin's Theorem)*** *Let* $L_n : C[a,b] \to C[a,b]$ *for* $n \in \mathbb{N} = \{1, 2, \ldots\}$*. If the sequence of operators* L_n *satisfy*

$$L_n(1;x) \rightrightarrows 1 \tag{2.2.1}$$

$$L_n(t;x) \rightrightarrows x \tag{2.2.2}$$

$$L_n(t^2;x) \rightrightarrows x^2 \tag{2.2.3}$$

then for all $f \in C[a,b]$*, we have*

$$L_n(f;x) \rightrightarrows f(x) \text{ as } n \to \infty.$$

Proof. Since $f \in C[a,b]$, then it is bounded, $\exists M \in \mathbb{R}$ such that $|f(x)| \leq M$. Because of the fact that $f \in C[a,b]$ then for all $\varepsilon > 0$ there exist a real number $\delta > 0$ such that for all $x, t \in [a,b]$, $|t-x| < \delta$ implies

$$|f(t) - f(x)| < \varepsilon.$$

Therefore, for $x, t \in [a,b]$, we have

$$|f(t) - f(x)| < \varepsilon + \frac{2M}{\delta^2}(t-x)^2. \tag{2.2.4}$$

On the other hand,

$$\begin{aligned}
&\|L_n(f;x) - f(x)\|_{C[a,b]} \\
&= \|L_n(f(t);x) - f(x)\|_{C[a,b]} \\
&= \|L_n(f(t) - f(x);x) + f(x)\left(L_n(1;x) - 1\right)\|_{C[a,b]} \\
&\leq \|L_n(|f(t) - f(x)|;x)\| + \|L_n(1;x) - 1\| \, \|f\| \, .
\end{aligned} \tag{2.2.5}$$

From (2.2.4)

$$
\begin{aligned}
L_n\left(|f(t)-f(x)|;x\right) &\le L_n(\varepsilon + \frac{2M}{\delta^2}((t-x)^2;x) \\
&= \varepsilon L_n(1;x) + \frac{2M}{\delta^2} L_n((t-x)^2;x) \\
&= \varepsilon(L_n(1;x)-1)+\varepsilon \\
&+ \frac{2M}{\delta^2}[L_n(t^2;x) - 2xL_n(t;x) + x^2 L_n(1;x)] \\
&= \varepsilon(L_n(1;x)-1)+\varepsilon + \frac{2M}{\delta^2}[(L_n(t^2;x)-x^2) \\
&- 2x(L_n(t,x)-x) + x^2(L_n(1;x)-1)].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
L_n(|f(t)-f(x)|;x) &\le \varepsilon + C_1\|L_n(1;x)-1\|_{C[a,b]} \\
&+ C_2\|L_n(t;x)-x\|_{C[a,b]} + C_3\|L_n(t^2;x)-x^2\|_{C[a,b]},
\end{aligned} \tag{2.2.6}
$$

where C_1, C_2 and C_3 are positive constants. From (2.2.5) and (2.2.6), we have

$$
\begin{aligned}
\|L_n(f;x)-f(x)\| &\le \varepsilon + C_1^*\|L_n(1;x)-1\|_{C[a,b]} \\
&+ C_2^*\|L_n(t;x)-x\|_{C[a,b]} + C_3^*\|L_n(t^2;x)-x^2\|_{C[a,b]},
\end{aligned}
$$

where C_1^*, C_2^* and C_3^* are positive constants. Thus for $n \to \infty$ we have $\|L_n(f;x) - f(x)\|_{C[a,b]} \to 0$. $\square$

Corollary 8. *If the sequence of operators $\{L_n\}$ satisfy $L_n(1;x) \rightrightarrows 1$ and $L_n((t-x)^2;x) \rightrightarrows 0$ then for all $f \in C\,[a,b]$ we have $L_n(f;x) \rightrightarrows f(x)$.*

The Korovkin's theorem in two variables is known as Volkov's theorem in the literature which is stated as follows:

Theorem 9. ***(Volkov's Theorem)*** *Let $L_{n,m} : C\left([a,b]\times[c,d]\right) \to C\left([a,b]\times[c,d]\right)$ for $n,m \in \mathbb{N}$. If the double sequence of linear positive operators $L_{n,m}$ satisfy*

$$L_{n,m}(1;x,y) \rightrightarrows 1$$

$$L_{n,m}(t;x,y) \rightrightarrows x$$

$$L_{n,m}(s;x,y) \rightrightarrows y$$

$$L_{n,m}(t^2+s^2;x,y) \rightrightarrows x^2+y^2$$

then for all $f \in C([a,b]\times[c,d]) \to C([a,b]\times[c,d])$, *we have*

$$L_{n,m}(f;x,y) \rightrightarrows f(x,y) \text{ as } n,m \to \infty\ .$$

Proof. Since $f \in C([a,b]\times[c,d])$ then $\exists M \in \mathbb{R}^+$ such that $|f(x,y)| \leq M$. Furthermore, for all $\varepsilon > 0$ there exists a real number $\delta > 0$ such that for all $x,t \in [a,b]$ and $y,s \in [c,d]$,

$$\sqrt{(t-x)^2+(y-s)^2} < \delta$$

then

$$|f(t,s)-f(x,y)| < \varepsilon.$$

Accordingly, for all $x,t \in [a,b]$ and $y,s \in [c,d]$, we have

$$|f(t,s)-f(x,y)| < \varepsilon + \frac{2M}{\delta^2}[(t-x)^2+(y-s)^2]. \tag{2.2.7}$$

On the other hand

$$\begin{aligned}
&||L_{n,m}(f;x,y)-f(x,y)||_{C([a,b]\times[c,d])}\\
&= ||L_{n,m}(f(t,s);x,y)-f(x,y)||_{C([a,b]\times[c,d])}\\
&= ||L_{n,m}(f(t,s)-f(x,y);x,y)+f(x,y)(L_{n,m}(1;x,y)-1)||_{C([a,b]\times[c,d])}\\
&\leq ||L_{n,m}(|f(t,s)-f(x,y)|;x,y)||_{C([a,b]\times[c,d])} + ||f||(L_{n,m}(1;x)-1)||_{C([a,b]\times[c,d])}.
\end{aligned} \tag{2.2.8}$$

Using (2.2.7), we get that

$$\begin{aligned}
&L_{n,m}(|f(t,s)-f(x,y)|;x,y)\\
&\leq L_{n,m}(\varepsilon + \frac{2M}{\delta^2}[(t-x)^2+(y-s)^2];x,y)\\
&\leq \varepsilon L_{n,m}(1;x,y) + \frac{2M}{\delta^2}L_{n,m}\left((t-x)^2+(y-s)^2;x,y\right)
\end{aligned}$$

$$= \varepsilon(L_{n,m}(1;x,y)-1)+\varepsilon+\frac{2M}{\delta^2}L_{n,m}\big((t^2+s^2)$$

$$-2xt-2ys+(x^2+y^2);x,y\big)$$

$$= \varepsilon(L_{n,m}(1;x,y)-1)+\varepsilon+\frac{2M}{\delta^2}\big[\big\{L_{n,m}\big(\big(t^2+s^2\big);x,y\big)\big\}$$

$$-2x\left\{L_{n,m}\left(t;x,y\right)-x\right\}$$

$$-2y\{L_{n,m}(s;x,y)-y\}+(x^2+y^2)\{L_{n,m}(1;x,y)-1\}\big]$$

thus

$$||L_{n,m}(|f(t,s)-f(x,y)|;x,y)||_{C([a,b]\times[c,d])} \tag{2.2.9}$$

$$\leq \varepsilon+C_1||L_{n,m}(1;x,y)-1||+C_2||L_{n,m}(t;x,y)-x||_{C([a,b]\times[c,d])}$$

$$+C_3||L_{n,m}(s;x,y)-y||_{C([a,b]\times[c,d])}+C_4||L_{n,m}(t^2+s^2;x,y)-x^2+y^2)||_{C([a,b]\times[c,d])},$$

where C_1, C_2, C_3 and C_4 are positive constants. Combining (2.2.8) and (2.2.9), we have

$$||L_{n,m}(f;x,y)-f(x,y)||_{C([a,b]\times[c,d])}$$

$$\leq \varepsilon+C_1^*||L_{n,m}(1,x,y)-1||_{C([a,b]\times[c,d])}$$

$$+C_2||L_{n,m}(t;x,y)-x||_{C([a,b]\times[c,d])}$$

$$+C_3||L_{n,m}(s;x,y)-y||_{C([a,b]\times[c,d])}$$

$$+C_4||L_{n,m}(t^2+s^2);x,y-(x^2+y^2)||_{C([a,b]\times[c,d])},$$

where C_1^*, C_2, C_3 and C_4. Therefore

$$||L_{n,m}(f;x,y)-f(x,y)||_{C([a,b]\times[c,d])}\to 0.$$

□

2.3 Bernstein Polynomials in One and Two Variables

Definition 10. *Let $x \in [0,1]$, the Bernstein polynomials (operators) $B_n(f;x)$ are defined as follows:*

$$B_n(f;x) = \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k}.$$

They are positive linear operators, since

$$(\genfrac{}{}{0pt}{}{n}{k}) x^k (1-x)^{n-k} = \frac{n!}{k!(n-k)!} x^k (1-x)^{n-k} \geq 0.$$

First few Bernstein polynomials of degree one, two and three are given as follows:

$$\begin{aligned} B_1(f;x) &= \sum_{k=0}^{1} f(\frac{k}{1}) \binom{1}{k} x^k (1-x)^{1-k} \\ &= f(0)\binom{1}{0}(1-x) + f(1)\binom{1}{1}x \\ &= f(0)(1-x) + f(1)x. \end{aligned}$$

$$\begin{aligned} B_2(f;x) &= \sum_{k=0}^{2} f(\frac{k}{2}) \binom{2}{k} x^k (1-x)^{2-k} \\ &= f(0)\binom{2}{0}(1-x)^2 + f(\frac{1}{2})\binom{2}{1}x(1-x) \\ &+ f(1)\binom{2}{2}x^2 \\ &= f(0)(1-x)^2 + 2f(\frac{1}{2})x(1-x) + f(1)x^2. \end{aligned}$$

$$\begin{aligned} B_3(f;x) &= \sum_{k=0}^{3} f(\frac{k}{3}) \binom{3}{k} x^k (1-x)^{3-k} \\ &= f(0)\binom{3}{0}(1-x)^3 + f(\frac{1}{3})\binom{3}{1}x(1-x)^2 \\ &+ f(\frac{2}{3})\binom{3}{2}x^2(1-x) + f(\frac{3}{3})\binom{3}{3}x^3 \\ &= f(0)(1-x)^3 + 3f(\frac{1}{3})x(1-x)^2 + 3f(\frac{2}{3})x^2(1-x) \\ &+ f(1)x^3. \end{aligned}$$

The Bernstein operator is clearly linear, since

$$B_n\left(\lambda f+\mu g\right)=\lambda B_n f+\mu B_n g, \tag{2.3.1}$$

for all functions f and g on $[0,1]$ and all real numbers λ and μ.

It is known that ([15]) the Bernstein polynomials satisfy,

$$B_n(1;x)=1,\ B_n(t;x)=x$$

$$B_n(t^2;x)=x^2+\frac{x\left(1-x\right)}{n}$$

and

$$B_n((t-x)^2)=\frac{x\left(1-x\right)}{n}.$$

Since, the conditions of Korovkin's theorem are satified, then

$$\|B_n(f;x)-f(x)\|_{C[0,1]}\rightarrow 0$$

for all $f\in C\left[0,1\right]$.

Definition 11. *([22]) A function f is convex on $[a,b]$ if for any $x_1,x_2\in[a,b]$,*

$$\lambda f(x_1)+\left(1-\lambda\right)f(x_2)\geq f\left(\lambda x_1+\left(1-\lambda\right)x_2\right) \tag{2.3.2}$$

for any $\lambda\in[0,1]$. Geometrically, we can say that a chord connecting of any two points on the convex curve $y=f(x)$ is never below the curve.

In order to investigate the derivative properties of Bernstein polynomials we need some definitions and propositions. Let $f:[0,1]\rightarrow\mathbb{R}$, the divided difference of function f is defined as follows:

$$\Delta_t f(x)=f(x+t)-f(x)$$

and

$$\begin{aligned}\Delta_t^2 f(x) &= \Delta_t(\Delta_t f(x)) = \Delta_t(f(x+t) - f(x)) = \Delta_t(f(x+t)) - \Delta_t f(x) \\ &= [f(x+2t) - f(x+t)] - [f(x+t) - f(x)] \\ &= f(x+2t) - 2f(x+t) + f(x). \\ &. \\ &. \\ &. \\ \Delta_t^k f(x) &= \Delta_t(\Delta_t^{k-1} f) \\ &= f(x+kt) - \binom{k}{1} f(x+(k-1)t) + \ldots + (-1)^k f(x).\end{aligned}$$

Note that, if $\Delta_t f(x) \geq 0$ for all $x \in [0,1]$ then f is non-decreasing.

Corollary 12. *Let $f : [0,1] \to \mathbb{R}$. Then*

$$B_n^m (f, x) = \frac{n!}{(n-m)!} \sum_{k=0}^{n-m} \Delta_{1/n}^m f\left(\frac{k}{n}\right) P_{n-m,k}(x), \, m = 0, 1, ..., n, \tag{2.3.3}$$

where $P_{n-m,k}(x) = \binom{n-m}{k} x^k (1-x)^{n-m-k}$.

Remark 13. *Corollary 2.3.3 shows that, if f is monotonically increasing, then so is $B_n(f;x)$.*

Taking $x = 0$ in (2.3.3), we obtain that

$$B_n^m(f;0) = \frac{n!}{(n-m)!} \Delta_{1/n}^m f(0) = n(n-1)\ldots(n-m+1)\Delta_{1/n}^m f(0). \tag{2.3.4}$$

On the other hand, since the Maclaurin series of any function is given by

$$f(x) = \sum_{m=0}^{\infty} f^{(m)}(0) \frac{x^m}{m!},$$

then the Maclaurin expansion of the Bernstein polynomials is represented by

$$\begin{aligned}B_n(f;x) &= \sum_{m=0}^{n} B_n^m(f;0) \frac{x^m}{m!} \\ &= \sum_{m=0}^{n} n(n-1)\cdots(n-m-1)\Delta_{1/n}^m f(0) \frac{x^m}{m!} \\ &= \sum_{m=0}^{n} \binom{n}{m} \Delta_{1/n}^m f(0) x^m. \end{aligned} \tag{2.3.5}$$

Now consider the polynomial $f(x) = p_k(x)$ *of degree* k*, then* $\Delta^m_{1/n} p_k(x) = 0$ *for* $k < m$. *Therefore;* $B_n(p_k(t); x)$ *is a polynomial of degree* $\leq k$.

On the other hand, generally $B_n(p_k(t); x) \neq p_k(x)$.

Theorem 14. *If* $f \in C^k[0,1]$*, for some* $k \geq 0$*, then*

$$m \leq f^{(k)}(x) \leq M, \quad x \in [0,1] \text{ implies } c_k m \leq B_n^{(k)}(f; x) \leq c_k M,$$

for all $n \geq k$. $x \in [0,1]$ *where* $c_0 = c_1 = 1$ *and*

$$c_k = \binom{n}{k} \frac{k!}{n^k} = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) ... \left(1 - \frac{k-1}{n}\right), \; 2 \leq k \leq n.$$

Remark 15. *The coefficients* $a_{n,m} = \binom{n}{m} \Delta^m_{1/n} f(0)$*, in the expansion (2.3.3) can be re-given by*

$$\begin{aligned} a_{n,m} &= \frac{n!}{m!(n-m)!} \Delta^m_{1/n} f(0) = \frac{n(n-1)\dots(n-m+1)}{m!} \Delta^m_{1/n} \\ &= \frac{1}{m!} \left(\frac{n}{n}\right) \left(\frac{n-1}{n}\right) \dots \left(\frac{n-(m-1)}{n}\right) \Delta^m_{1/n} f(0) n^m \\ &= \frac{1}{m!} (1 - \frac{1}{n}) \dots \; (1 - \frac{m-1}{n}) \frac{\Delta^m f(0)}{(\frac{1}{n})^m}. \end{aligned}$$

Note that $a_{n,m}$ *converges to* $\dfrac{f^m(0)}{m!}$ *as* $n \to \infty$.

Therefore, the right hand side of (2.3.5) is exactly the sum of the first

$n + 1$ terms of the Taylor's expansion of the function $f(x)$, with slightly modified coefficients.

For any polynomial $p_k(x)$, it is known that

$$B_n(p_k(t); x) \rightrightarrows p_k(x).$$

Uniformly on $[0, 1]$. We choose $p_k(x)$ such a way that

$$|f - p_k| < \varepsilon.$$

Then

$$|B_n(f;x) - B_n(p_k;x)| = |B_n(f - p_k;x)| < |B_n(\varepsilon;x)|$$
$$= |\varepsilon B_n(1;x)| = \varepsilon$$

and then

$$|B_n(f;x) - B_n(p_k;x)| < \varepsilon.$$

Thus, for large n,

$$|f(x) - B_n(f;x)| \leq |f(x) - p_k(x)| + |p_k(x) - B_n(p_k;x)| + |B_n(p_k;x) - B_n(f;x)|$$
$$< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon,$$

which shows that

$$B_n(f;x) \rightrightarrows f(x)$$

on $[0,1]$. This is another proof of Korovkin's theorem for the Bernstein operators.

Now consider the operators. ([22])

$$\tilde{B}_{n-1}(f;x) = \sum_{k=0}^{n-1} f(\frac{k}{n}) \binom{n-1}{k} x^k (1-x)^{n-1-k}.$$

These operators are linear and satisfy

$$\tilde{B}_{n-1}(1;x) = 1$$

$$\tilde{B}_{n-1}(t;x) = x - \frac{x}{n}$$
$$\tilde{B}_{n-1}(t^2;x) = \frac{(n-1)(n-2)}{n^2x^2} + \frac{n-1}{n^2x},$$

for $n \geq 2$.

Therefore from Remark 2.3.6 we have that

$$\tilde{B}_{n-1}(f^{'};x) - B_n^{'}(f;x)$$
$$= \sum_{k=0}^{n-1} f'(\frac{k}{n})\binom{n-1}{k}x^k(1-x)^{n-1-k}$$
$$- \sum_{k=0}^{n-1} n\Delta_{1/n}f(\frac{k}{n})\binom{n-1}{k}x^k(1-x)^{n-1-k} \tag{2.3.6}$$
$$= \sum_{k=0}^{n-1}\{f^{'}(\frac{k}{n}) - n\Delta_{1/n}k(\frac{k}{n})\}\binom{n-1}{k}x^k(1-x)^{n-1-k}. \tag{2.3.7}$$

Now, let's take into account the curly bracet.

$$f^{'}(\frac{k}{n}) - n\Delta_{1/n}f(\frac{k}{n}) = f^{'}(\frac{k}{n}) - n(f(\frac{k+1}{n}) - f(\frac{k}{n})$$
$$= f(k/n) - \frac{f(\frac{k}{n}) - f(\frac{k}{n})}{\frac{1}{n}}. \tag{2.3.8}$$

On the other hand, from the mean value theorem, there exists number θ, where $0 \leq \theta < 1$, such that

$$\frac{f(\frac{k+1}{n}) - f(\frac{k}{n})}{\frac{1}{n}} \simeq f^{'}(\frac{k+\theta}{n}).$$

Then, from (2.3.8), we have that

$$f^{'}(\frac{k}{n}) - n\Delta_{1/n}f(\frac{k}{n}) = f^{'}(\frac{k}{n}) - f^{'}(\frac{k+\theta}{n}).$$

Thus for large n, the above difference tends to zero. So, for all $\varepsilon > 0$, there exsists $N > 0$, such that, $\forall n \geq N$.

$$f^{'}(\frac{k}{n}) - n\Delta_{1/n}f(\frac{k}{n}) < \varepsilon.$$

Therefore from (2.3.7), we have that

$$\tilde{B}_{n-1}(f^{'};x) - B_n^{'}(f;x) < \varepsilon\sum_{k=0}^{n-1}\binom{n-1}{k}x^k(1-x)^{n-1-k} = \varepsilon$$

The above inequality shows that, given any $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that

$$||\tilde{B}_{n-1}(f^{'};\cdot) - \tilde{B}_n^{'}(f;\cdot)||_{C[0,1]} < \varepsilon,$$

for all $n \geq N$.

Now, for any given $\varepsilon > 0$, there exists $\mathbb{N} = \mathbb{N}(\varepsilon)$ such that

$$\begin{aligned} ||B_n^{'}(f;\cdot) - f'||_\infty &= ||B_n^{'}(f;\cdot) - \tilde{B}_{n-1}(f^{'};\cdot) + \tilde{B}_{n-1}(f^{'};\cdot) - f^{'}||_{C[0,1]} \\ &\leq ||B_n^{'}(f;\cdot) - \tilde{B}_{n-1}(f^{'};\cdot)||_{C[0,1]} + ||\tilde{B}_{n-1}(f^{'};\cdot) - f^{'}||_{C[0,1]} \\ &< \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

for all $n \geq \mathbb{N}(n \in \mathbb{N})$.

This shows that, $B_n^{'}(f;x) \rightrightarrows f'(x)$ for all $f^{'} \in C\,[a,b]$.

Theorem 16. *([22]) A function f is convex on $[a,b]$ if and only if all second order divided differences of f are nonnegative.*

Theorem 17. *([22]) If $f(x)$ is convex on $[0,1]$, then*

$$B_n(f;x) \geq f(x), \quad 0 \leq x \leq 1, \tag{2.3.9}$$

for all $n \geq 1$.

Theorem 18. *([22]) If $f(x)$ is convex on $[0,1]$,*

$$B_{n-1}(f;x) \geq f(x) \quad 0 \leq x \leq 1, \tag{2.3.10}$$

for all $n \geq 2$. The Bernstein polynomials are equal at $x = 0$ and $x = 1$, since they interpolate f at these points. If $f \in C\,[0,1]$, the inequality in (2.3.9) is strict for $0 < x < 1$, for a given value of n, unless f is linear in each of the intervals $\left[\frac{r-1}{n-1}, \frac{r}{n-1}\right]$, for $1 \leq r \leq n-1$, when we have simply $B_{n-1}(f;x) = B_n(f;x)$.

Theorem 19. *([22]) Let $f(x)$ be bounded on $[0,1]$. Then for any $x \in [0,1]$ at which $f^{''}(x)$ exists,*

$$\lim_{n\to\infty} n\left(B_n(f;x) - f(x)\right) = \frac{1}{2}x\left(1-x\right)f^{''}\left(x\right). \tag{2.3.11}$$

Let $x, y \in [0,1]$, the Bernstein polynomials in two variables are defined by

$$B_{n,m}(f;x,y) = \sum_{k=0}^{n}\sum_{l=0}^{m} f(\frac{k}{n},\frac{l}{m})\binom{n}{k}\binom{m}{l}x^k(1-x)^{n-k}(1-y)^{m-l}; n,m \in \mathbb{N}$$

such that $B_{n,m} : C\left([0,1] \times [0,1]\right) \to C\left([0,1] \times [0,1]\right)$. These polynomials are positive linear operators.

Furthermore, these polynomials satisfy the conditions of Volkov's theorem ([7]) since

$$B_{n,m}\left(1, x, y\right) = 1$$

$$B_{n,m}\left(t, x, y\right) = x$$

$$B_{n,m}\left(s, x, y\right) = t$$

$$B_{n,m}(t^2 + s^2; x, y) = x^2 + y^2 + \frac{x(1-x)}{n} + \frac{y(1-y)}{m}.$$

Therefore, from the Volkov's theorem, we have

$$||B_{n,m}(f; x, y) - f(x, y)||_{C([o,1]\times[0,1])} \to 0.$$

Theorem 20. *Let*

$$B_n(f; x) = \sum_{k=0}^{\infty} f(\frac{k}{n}, \cdot)\binom{n}{k} x^k (1-x)^{n-k}$$

$$B_m(f; y) = \sum_{l=0}^{m} (\cdot, \frac{l}{m})\binom{m}{l} y^l (1-y)^{m-l}$$

such that $B_n : C[0,1] \to C[0,1]$ *and* $B_m : C[0,1] \to C[0,1]$ *for all* $n, m \in \mathbb{N}$. *Then*

(i) $B_n[B_m(f; y); x] = B_{n,m}(f; x, y)$

(ii) $B_m[B_n(f; x); y] = B_{n,m}(f; x, y)$.

Proof. Consider

$$B_n(B_m(f, y); x) = \sum_{k=0}^{n} B_m f(\frac{k}{n}, s); y)\binom{n}{k} x^k (1-x)^{n-k}$$

$$= \sum_{k=0}^{n}\sum_{l=0}^{m} f(\frac{k}{n}, \frac{l}{m})\binom{m}{l} y^l(1-y)^{m-l}\binom{n}{k} x^k(1-x)^{n-k}$$
$$= \sum_{k=0}^{n}\sum_{l=0}^{m} f(\frac{k}{n}, \frac{l}{m})\binom{n}{k}\binom{m}{p} x^k(1-x)^{n-k} y^l(1-y)^{m-l}$$
$$= B_{n,m}(f; x, y).$$

The proof (i) is completed. Similarly proof of (ii) can be given in a similar way. □

2.4 Modulus of Continuity and Lipschitz Class Functions

Definition 21. *For $\delta > 0$, we define the $r-th$ order modulus of continuity of f on the interval I, by*

$$\omega(f; \delta) = \max_{\substack{|h|\leq\delta \\ t,x\in I}} |\Delta_h f(x)| = \max_{\substack{|h|\leq\delta \\ t,x\in I}} |\Delta_h f(x+h) - f(x)|$$

or equivalently,

$$\omega(f; \delta) = \max_{\substack{|t-x|\leq\delta \\ t,x\in I}} |f(t) - f(x)|.$$

Theorem 22. *([8]) Let $f, g, h \in C[a, b]$, $\delta > 0$, $\delta_2 \geq \delta_1 > 0$, $\lambda > 1$, $n \geq 1$ be an integer, $\alpha \in \mathbb{R}$. Then*

(i) $\omega(f; \delta)$ *is nondecreasing in* δ.

(ii) $\omega(\alpha f + g; \delta) \leq |\alpha|\omega(f; \delta) + \omega(g; \delta)$

(iii) $\omega(f; n\delta) \leq n\omega(f; \delta)$

(iv) $\omega(f; \lambda\delta) \leq (1 + \lambda)\omega(f; \delta)$

(v) $\omega(f; \delta) \leq \dfrac{2\delta_2}{\delta_1}\omega(f; \delta_1)$

(vi) $\lim_{\delta\to 0} \omega(f; \delta) = 0.$

Proof. (i) Let $0 < \delta_1 \leq \delta_2$, then

$$\omega(f;\delta_1) = \max_{|h|\leq\delta_1} |f(x+h) - f(x)|$$

$$\leq \max_{|h|\leq\delta_2} |f(x+h) - f(x)| = \omega(f;\delta_2).$$

Hence, $\omega(f;\delta)$ is nondecreasing in δ.

(ii) Direct computations yield

$$\begin{aligned}
\omega(\alpha f_1 + f_2;\delta) &= \max_{|h|\leq\delta} |(\alpha f_1 + f_2)(x+h) - (\alpha f_1 + f_2)(x)| \\
&\leq \max_{|h|\leq\delta} \{|(\alpha f_1(x+h) - \alpha f_1(x)| + |f_2(x+h) - f_2(x)|\} \\
&= \max_{|h|\leq\delta)} |\alpha|(f_1(x+h) - f_1(x)| + \max_{|h|\leq\delta} |f_2(x+h) - f_2(x)| \\
&= |\alpha|\omega(f_1;\delta) + \omega(f_2;\delta).
\end{aligned}$$

(iii) Since

$$\begin{aligned}
\sum_{k=0}^{n-1} \Delta_t f(x+t) &= \Delta_t f(x) + \Delta_t f(x+kt) + \cdots + \Delta_t f(x+(n-1)t) \\
&= [f(x+t) - f(x)] + [f(x+2t) - f(x+t)] \\
&+ \cdots + [f(x+nt) - f(x+(n-1)t)] \\
&= f(x+nt) - f(x) = \Delta_{nt} f(x).
\end{aligned}$$

Therefore, taking $nt = h$ then $t = \dfrac{h}{n}$

$$\begin{aligned}
\omega(f;n\delta) &= \max_{|h|\leq n\delta} |\Delta_h f(x)| \\
&= \max_{h|\leq n\delta} |\sum_{k=0}^{n-1} \Delta_{h/n} f(x+k\frac{h}{n})| \\
&\leq \max_{|h|\leq n\delta)} \left\{ |\Delta_{h/n} f(x)| + |\Delta_{h/n} f(x+\frac{h}{n})| + \cdots + |\Delta_{h/n} f(x+(n-1)\frac{h}{n}| \right\} \\
&= \max_{|h|/n\leq\delta} \left\{ \left|\Delta_{h/n} f(x)\right| + \max_{|h|/n\leq\delta} \left|\Delta_{h/n} f\left(x+\frac{h}{n}\right)\right| \right. \\
&\left. + \cdots + \max_{|h|/n\leq\delta} \left|\Delta_{h/n} f\left(x+\frac{n-1}{n}h\right)\right| \right\}.
\end{aligned}$$

Letting $\dfrac{h}{n} = h_1$, we get

$$\begin{aligned}
&\omega(f; n\delta) \\
&\le \max_{|h|\le\delta} |\Delta_{h_1} f(x)| + \max_{|h_1|} |h_1| \\
&\le \delta \left|\Delta_{h_1} f(x + \frac{h}{n}\right| + \cdots + \max_{|h_1|\le\delta} |\Delta_{h_1} f(x + (n-1)h_1| \\
&\le \omega(f;\delta) + \omega(f;\delta) + \cdots + \omega(f;\delta) = n\omega(f;\delta).
\end{aligned}$$

(iv) Using (i) and (iii), we obtain that

$$\begin{aligned}
\omega(f;\lambda\delta) &\le \omega(f; (|\lambda| + 1)\delta) \\
&\le (|\lambda + 1|)\omega(f;\delta) \\
&\le (\lambda + 1)\omega(f;\delta).
\end{aligned}$$

(v) Direct calculations give, since $0 < \delta_1 \le \delta_2$.

$$\omega(f;\delta_2) = \omega(f; \frac{\delta_2}{\delta_1}\delta_1) \le (1 + \frac{\delta_2}{\delta_1})\omega(f;\delta_1)$$

$$\begin{aligned}
&= \frac{\delta_1 + \delta_2}{\delta_1}\omega(f;\delta_1) \\
&= \frac{\delta_2}{\delta_1}(1 + \frac{\delta_1}{\delta_2})\omega(f;\delta_1) \\
&< 2\frac{\delta_2}{\delta_1}\omega(f;\delta_1).
\end{aligned}$$

□

Corollary 23. *([8]) If f is continuous on $[0,1]$ and $\omega(f;\delta)$ is the modulus of continuity of $f(x)$, then*

$$|B_n(f;x) - f(x)| \le 2\omega\left(f; \sqrt{\frac{x(1-x)}{n}}\right).$$

Definition 24. *Let's call that a function $f \in C[0,1]$ belongs to $Lip_M(\alpha)$*

$(0 < \alpha \le 1)$

if the inequality

$$|f(t)-f(x)| \leq M|t-x|^{\alpha}; \;\; (t,x \in [0,1])$$

holds.

Theorem 25. *([8]) Let $f \in Lip_M(\alpha)$, then*

$$|B_n(f;x)-f(x)| \leq M(\frac{x(1-x)}{n})^{\alpha/2}.$$

2.5 The q-Integers

This section partially taking by ([12]).

Definition 26. *For any real number $q>0$ and $r>0$, the q-integer of the number r is defined by*

$$[r] = \begin{cases} (1-q^r)/(1-q), & q \neq 1 \\ r \qquad\qquad\quad , & q=1, \end{cases}$$

q-factorial is defined by

$$[r]! = \begin{cases} [r]\,[r-1]\ldots[1], & r=1,2,3,..., \\ 1 \qquad\qquad\quad , & r=0 \end{cases}$$

and q-binomial coefficient defined by

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[n-r]!\,[r]!}$$

where $n \geq 0$, $r \geq 0$.

Definition 27. *The following expression*

$$D_q f(x) = \frac{f(qx)\,f(x)}{(q-1)\,x}$$

is called the q-derivative of the function $f(x)$.

Definition 28. *The q-analogue of the integration is defined as follows*

$$\int_0^b f(t)\,d_q t = (1-q)\,b\sum_{j=0}^{\infty} f\left(q^j b\right) q^j \qquad 0<q<1,$$

where $t \in [0,b]$ and $f(x)$ is continuous on $[0,b]$.

Theorem 29. *(q-**binomial theorem**) For $0 \leq r \leq n$, $\begin{bmatrix} n \\ k \end{bmatrix}$ is a coefficient of of $q-$binomial, then we have*

$$\prod_{k=1}^{n} \left(1 + q^{k-1}x\right) = \sum_{k=0}^{n} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} x^k$$

and for $q = 1$, the above relation gives

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

2.6 q-Bernstein Polynomials

In this section we give the generalization of Bernstein polynomials ([22]) based on the q-integers. Let us

$$B_n^q(f; x) = \sum_{r=0}^{n} f\left(\frac{[r]}{[n]}\right) \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1 - q^s x) \qquad (2.6.1)$$

for each positive integer n, q is fixed and $\begin{bmatrix} n \\ r \end{bmatrix}$ denotes a q-binomial coefficient. In particular setting $q = 1$ in equation (2.6.1), gives Bernstein polynomials. It is clear that

$$B_n^q(f; 0) = f(0) \qquad B_n^q(f; 1) = f(1). \qquad (2.6.2)$$

On the other hand B_n^q, defined by (2.6.1), is a linear positive operator for $0 < q < 1$.

Theorem 30. *([22])The generalized Bernstein polynomial can be stated in the form*

$$B_n^q(f; x) = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} \Delta_q^r f_0 x^r, \qquad (2.6.3)$$

where

$$\Delta_q^r f_j = \Delta_q^{r-1} f_{j+1} - q^{r-1} \Delta^{r-1} f_j, \quad r \geq 1$$

with $\Delta_q^0 f_j = f_j = f\left([j] / [n]\right)$.

Note that q-differences of the monomial x^k of order greater than k is zero, and we know that for all $n \geq k$, $B_n\left(x^k; x\right)$ is a polynomial of degree k.

Furthermore, q-Bernstein polynomial satisfy

$$B_n^q(1;x) = 1. \tag{2.6.4}$$

$$B_n^q(t;x) = x. \tag{2.6.5}$$

$$B_n^q\left(t^2;x\right) = x^2 + \frac{x(1-x)}{[n]}. \tag{2.6.6}$$

The above expressions for $B_n(1;x)$, $B_n(t;x)$, and $B_n(t^2;x)$ generalize their counterparts given earlier for the case $q = 1$.

Theorem 31. *([22]) If $f(x)$ is convex on $[0,1]$, then*

$$B_n^q(f;x) \geq f(x), \quad 0 \leq x \leq 1, \tag{2.6.7}$$

for all $n \geq 1$ and for $0 < q \leq 1$.

Theorem 32. *([22]) If $f(x)$ is convex on $[0,1]$,*

$$B_{n-1}^q(f;x) \geq B_n^q(f;x), \quad 0 \leq x \leq 1, \tag{2.6.8}$$

for all $n \geq 2$, where $B_{n-1}^q(f;x)$ and $B_n^q(f;x)$ are computed using the same value of the parameter q.
If $f \in C[0,1]$, the inequality in (2.6.8) is strict for $0 < x < 1$ unless, for a given value of n, the function f is linear in each of the intervals $\left[\frac{[r-1]}{[n-1]}, \frac{[r]}{[n-1]}\right]$, for $1 \leq r \leq n-1$, and $B_{n-1}^q(f;x) = B_n^q(f;x)$.

2.7 Bernstein Chlodowsky and q-Bernstein Chlodowsky Polynomials

The classical Bernstein-Chlodowsky polynomials are defined by ([4])

$$B_n^c(f,x) = \sum_{r=0}^{n} f\left(\frac{r}{n}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}\right)^r\left(1-\frac{x}{b_n}\right)^{n-r}, \tag{2.7.1}$$

where $0 \leq x \leq b_n$ and b_n is the sequence of positive numbers such that
$\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} \frac{b_n}{n} = 0$.
These operators are also studied in ([9]) and ([11]).

Lemma 33. *For the Bernstein-Chlodowsky polynomials, we have*

(i) $B_n^c(1,x) = 1.$

(ii) $B_n^c(t,x) = x.$

(iii) $B_n^c(t^2,x) = x^2 + \frac{x(b_n - x)}{n}.$

Proof. (i) Direct calculation yields

$$\begin{aligned} B_n^c(1,x) &= \sum_{r=0}^{n} \binom{n}{r} \left(\frac{x}{b_n}\right)^r \left(1 - \frac{x}{b_n}\right)^{n-r} \\ &= \left(\frac{x}{b_n} + \left(1 - \frac{x}{b_n}\right)\right)^n \\ &= 1^n \\ &= 1. \end{aligned}$$

(ii) We have

$$\begin{aligned} B_n^c(t,x) &= \sum_{r=0}^{n} f\left(\frac{r}{n} b_n\right) \binom{n}{r} \left(\frac{x}{b_n}\right)^r \left(1 - \frac{x}{b_n}\right)^{n-r} \\ &= x \sum_{r=1}^{n} \frac{(n-1)!}{(r-1)!\,(n-1-r)!} \left(\frac{x}{b_n}\right)^{r-1} \left(1 - \frac{x}{b_n}\right)^{n-r} \\ &= x \sum_{r=0}^{n-1} \frac{(n-1)!}{r!\,(n-r)!} \left(\frac{x}{b_n}\right)^r \left(1 - \frac{x}{b_n}\right)^{n-1-r} \\ &= x \left(\frac{x}{b_n} + \left(1 - \frac{x}{b_n}\right)\right)^{n-1} \\ &= x. \end{aligned}$$

(iii) Finally,

$$
\begin{aligned}
& B_n^c\left(t^2, x\right) \\
& =\sum_{r=1}^n f\left(\frac{r}{n} b_n\right)\binom{n}{r}\left(\frac{x}{b_n}\right)^r\left(1-\frac{x}{b_n}\right)^{n-r} \\
& =\sum_{r=1}^n\left(\frac{r}{n} b_n\right)^2 \frac{n!}{(n-r)!r!}\left(\frac{x}{b_n}\right)^r\left(1-\frac{x}{b_n}\right)^{n-r} \\
& =x b_n \sum_{r=2}^n \frac{r-1}{n} \frac{(n-1)!}{(r-1)!\,(n-r)!}\left(\frac{x}{b_n}\right)^{r-1}\left(1-\frac{x}{b_n}\right)^{n-r} \\
& +\frac{x b_n}{n} \sum_{r=1}^n \frac{(n-1)!}{(r-1)!\,(n-r)!}\left(\frac{x}{b_n}\right)^{r-1}\left(1-\frac{x}{b_n}\right)^{n-r}
\end{aligned}
$$

$$
\begin{aligned}
& =x^2 \frac{(n-1)}{n} \sum_{r=2}^n \frac{(n-2)!}{(r-2)!\,(n-r)!}\left(\frac{x}{b_n}\right)^{r-2}\left(1-\frac{x}{b_n}\right)^{n-r} \\
& +\frac{x b_n}{n} \sum_{r=0}^{n-1} \frac{(n-1)!}{r!\,(n-1-r)!}\left(\frac{x}{b_n}\right)^r\left(1-\frac{x}{b_n}\right)^{n-1-r} \\
& =x^2 \frac{(n-1)}{n} \sum_{r=0}^{n-2} \frac{(n-2)!}{(r-2)!\,(n-2-r)!}\left(\frac{x}{b_n}\right)^r\left(1-\frac{x}{b_n}\right)^{n-2-r} \\
& +\frac{x b_n}{n} \\
& =x^2-\frac{x^2}{n}+\frac{x b_n}{n} \\
& =x^2+\frac{x\left(b_n-x\right)}{n} .
\end{aligned}
$$

Whence the result. □

Remark 34. *It is obvious that*

$$
\begin{aligned}
B_n^c\left((t-x)^2 ; x\right) & =B_n^c(t^2 ; x)-2 x(B_n^c(t ; x))+x^2(B_n^c(1 ; x)) \\
& =x^2+\frac{x\left(b_n-x\right)}{n}-2 x^2+x^2 \\
& =\frac{x\left(b_n-x\right)}{n} .
\end{aligned}
$$

H. Karslı and V. Grupta ([14]) introduced the $q-$Bernstein Chlodowsky polynomials as follow:

$$
C_n(f ; q ; x)=\sum_{k=0}^n f\left(\frac{[k]}{[n]} b_n\right)\begin{bmatrix} n \\ k \end{bmatrix}\left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n-k-1}\left(1-q^s \frac{x}{b_n}\right), \quad 0 \leq x \leq b_n, \tag{2.7.2}
$$

where b_n is a positive increasing sequence with the property $\lim_{n\to\infty} b_n = \infty$. It is easily verified that $C_n(f;q;x)$ are linear and positive operators for $0 < q < 1$.

Lemma 35. (i) $C_n(1;q;x) = 1$.
(ii) $C_n(t;q;x) = x$.
(iii) $C_n(t^2;q;x) = x^2 + \dfrac{x(b_n - x)}{[n]}$.

Proof. (i) It is clear that

$$\begin{aligned} C_n(1;q;x) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n-k-1} \left(1 - q^s \frac{x}{b_n}\right) \\ &= 1. \end{aligned}$$

(ii) We have

$$\begin{aligned} C_n(t;q;x) &= \sum_{k=0}^{n} \frac{[k]_q}{[n]_q} b_n \begin{bmatrix} n \\ k \end{bmatrix} \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n-k-1} \left(1 - q^s \frac{x}{b_n}\right) \\ &= b_n \sum_{k=0}^{n} \frac{[k]_q}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix} \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n-k-1} \left(1 - q^s \frac{x}{b_n}\right) \\ &= b_n \frac{x}{b_n} \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \left(\frac{x}{b_n}\right)^{k-1} \prod_{s=0}^{n-k-1} \left(1 - q^s \frac{x}{b_n}\right) \\ &= x \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} \left(\frac{x}{b_n}\right)^{k-1} \prod_{s=0}^{n-k-2} \left(1 - q^s \frac{x}{b_n}\right) \\ &= x C_{n-1}(1;q;x) = x. \end{aligned}$$

(iii) Finally we have

$$\begin{aligned}
&C_n\left(t^2;q;x\right)\\
&=\sum_{k=0}^{n}\left(\frac{[k]}{[n]}b_n\right)^2\begin{bmatrix}n\\k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n-k-1}\left(1-q^s\frac{x}{b_n}\right)\\
&=b_n^2\sum_{k=1}^{n}\frac{[k]}{[n]}\begin{bmatrix}n-1\\k-1\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n-k-1}\left(1-q^s\frac{x}{b_n}\right)\\
&=b_n^2\sum_{k=1}^{n}\frac{q\,[k-1]}{[n]}\begin{bmatrix}n-1\\k-1\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n-k-1}\left(1-q^s\frac{x}{b_n}\right)\\
&+\frac{b_n^2}{[n]}\sum_{k=1}^{n}\begin{bmatrix}n-1\\k-1\end{bmatrix}_q\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n-k-1}\left(1-q^s\frac{x}{b_n}\right)\\
&=\frac{qb_n^2\,[n-1]}{[n]}\sum_{k=2}^{n}\begin{bmatrix}n-2\\k-2\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n-k-1}\left(1-q^s\frac{x}{b_n}\right)\\
&+\frac{b_n^2}{[n]}\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}\left(\frac{x}{b_n}\right)^{k+1}\prod_{s=0}^{n-k-2}\left(1-q^s\frac{x}{b_n}\right)
\end{aligned}$$

$$\begin{aligned}
&=\frac{qb_n^2\,[n-1]}{[n]}\left(\frac{x}{b_n}\right)^2\sum_{k=0}^{n-2}\begin{bmatrix}n-2\\k-2\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n-k-2}\left(1-q^s\frac{x}{b_n}\right)\\
&+\frac{b_n^2}{[n]}\frac{x}{b_n}\sum_{k=0}^{n-1}\begin{bmatrix}n-1\\k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)_q^{n-k-1}\\
&=\frac{q\,[n-1]}{[n]}x^2C_{n-2}\left(1;q;x\right)+\frac{b_n}{[n]}xC_{n-1}\left(1;q;x\right)\\
&=x^2+\frac{x\left(b_n-x\right)}{[n]}.
\end{aligned}$$

This completes the proof. □

Lemma 36. *For the q-Bernstein Chlodowsky polynomials, we have*

$$\begin{aligned}
C_n\left(\left(t-x\right);q;x\right)&=C_n\left(t;q;x\right)-xC_n\left(1;q;x\right)\\
&=x-x\\
&=0
\end{aligned}$$

and

$$
\begin{aligned}
C_n\left((t-x)^2;q;x\right) &= C_n(t^2;q;x) - 2x(C_n(t;q;x) + x^2(C_n(1;q;x) \\
&= x^2 + \frac{x\,(b_n - x)}{[n]} - 2x^2 + x^2 \\
&= \frac{x\,(b_n - x)}{[n]}.
\end{aligned}
$$

Chapter 3

Q-BERNSTEIN SCHURER OPERATORS

3.1 Construction of the Operators

In this section we discuss the q-Bernstein Schurer operators defined by Muraru C. M. ([21]); and given by

$$B_n^p(f;q;x) = \sum_{r=0}^{n+p} f\left(\frac{[r]}{[n]}\right) \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1} (1-q^s x) \qquad (3.1.1)$$

for each $n \in \mathbb{N}$, $f \in C([0,p+1])$, p is fixed positive integer and $0 < q < 1$. It is clear that this operators are linear and positive.

Note that in the special case $p = 0$, we have the q-Bernstein operator

$$B_n^0(f;q;x) = B_n(f;q;x).$$

Lemma 37. *Let $B_n^p(f;q;x)$ be given in (3.1.1). Then*

(i) $B_n^p(1;q;x) = 1.$

(ii) $B_n^p(t;q;x) = \frac{[n+p]}{[n]}x.$

(iii) $B_n^p(t^2;q;x) = \frac{[n+p-1][n+p]}{[n]^2}qx^2 + \frac{[n+p]}{[n]^2}x.$

Proof. (i) Using the binomial identity, we have

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)^{n-k} = 1.$$

Hence

$$B_n^p(1;q;x) = \sum_{r=0}^{n+p} \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1} (1-q^s x)$$
$$= 1.$$

(ii) It is easy to show that

$$\begin{aligned}
B_n^p(t;q;x) &= \sum_{r=1}^{n+p} \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1} (1-q^s x) \frac{[r]}{[n]} \frac{[n+p]}{[n+p]} \\
&= x \frac{[n+p]}{[n]} \sum_{r=0}^{n+p-1} \frac{[n+p-1]!}{[n+p-r-1]!\,[r]!} x^r \prod_{s=0}^{n+p-r-2} (1-q^s x) \\
&= x \frac{[n+p]}{[n]} \sum_{r=0}^{n+p-1} \begin{bmatrix} n+p-1 \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-2} (1-q^s x) \\
&= \frac{[n+p]}{[n]} x.
\end{aligned}$$

Whence the result.

(iii) Finally we calculate $B_n^p(t^2;q;x)$

$$\begin{aligned}
B_n^p\left(t^2;q;x\right) &= \sum_{r=1}^{n+p} \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1} (1-q^s x) \frac{[r]^2}{[n]^2} \\
&= \sum_{r=1}^{n+p} \frac{[r]}{[n]} \frac{[r]}{[n]} \frac{[n+p]!}{[n+p-r]!\,[r]!} x^r \prod_{s=0}^{n+p-r-1} (1-q^s x)
\end{aligned}$$

and then, multiplying by $\frac{[n+p]^2}{[n+p]^2}$, we get

$$\begin{aligned}
&B_n^p\left(t^2;q;x\right) \\
&= \frac{[n+p]}{[n]^2} \sum_{r=2}^{n+p} \frac{q\,[r-1]\,[n+p-1]!}{[r-1]!\,[n+p-r]!} x^r \prod_{s=0}^{n+p-r-1} (1-q^s x) \\
&+ \frac{[n+p]}{[n]^2} \sum_{r=1}^{n+p} \frac{[n+p-1]!}{[r-1]!\,[n+p-r]!} x^r \prod_{s=0}^{n+p-r-1} (1-q^s x) \\
&= \frac{[n+p-1]\,[n+p]}{[n]^2} q \sum_{r=0}^{n+p-2} \frac{[n+p-2]!}{[r]!\,[n+p-r-2]!} x^{r+2} \prod_{s=0}^{n+p-r-3} (1-q^s x) \\
&+ \frac{[n+p]}{[n]} \sum_{r=0}^{n+p-1} \frac{[n+p-1]!}{[r]!\,[n+p-r-1]!} x^{r+1} \prod_{s=0}^{n+p-r-2} (1-q^s x) \\
&= \frac{[n+p-1]\,[n+p]}{[n]^2} q x^2 + \frac{[n+p]}{[n]^2} x.
\end{aligned}$$

This completes the proof. □

3.2 Shape Properties

In this subsection we investigate the shape preserving properties of q-Bernstein Schurer operators.

Theorem 38. *The generalized q-Bernstein Schurer operator can be stated in the form*

$$B_n^p(f;q;x) = \sum_{r=0}^{n+p} \begin{bmatrix} n+p \\ r \end{bmatrix} \Delta_q^r f_0 x^r, \tag{3.1.2}$$

where

$$\Delta_q^r f_j = \Delta_q^{r-1} f_{j+1} - q^{r-1}\Delta^{r-1} f_j, \quad r \geq 1$$

with $\Delta_q^0 f_j = f_j = f([j]/[n+p])$.

Proof. Consider the identity ([22])

$$\prod_{s=0}^{n+p-r-1} (1-q^s x) = \sum_{s=0}^{n+p-r} (-1)^s q^{s(s-1)/2} \begin{bmatrix} n+p-r \\ s \end{bmatrix} x^s. \tag{3.1.3}$$

Note that for the case $q=1$, it is equivalent to binomial expansion. Considering (3.1.3) in the definition (3.1.1), we get

$$B_n^p(f;q;x) = \sum_{r=0}^{n+p} \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \sum_{s=0}^{n+p-r} (-1)^s q^{s(s-1)/2} \begin{bmatrix} n+p-r \\ s \end{bmatrix} x^s.$$

Let us set $t = r+s$. Then, since

$$\begin{bmatrix} n+p \\ r \end{bmatrix} \begin{bmatrix} n+p-r \\ s \end{bmatrix} = \begin{bmatrix} n+p \\ t \end{bmatrix} \begin{bmatrix} t \\ r \end{bmatrix},$$

we get

$$\sum_{t=0}^{n+p} \begin{bmatrix} n+p \\ t \end{bmatrix} x^t \sum_{r=0}^{t} (-1)^{t-r} q^{(t-r)(t-r-1)/2} \begin{bmatrix} t \\ r \end{bmatrix} f_r = \sum_{t=0}^{n} \begin{bmatrix} n+p \\ t \end{bmatrix} \Delta_q^t f_0 x^t.$$

This completes the proof. □

Theorem 39. *If $f(x)$ is convex and nondecreasing on $[0,1]$, then*

$$B_n^p(f;q;x) \geq f(x), \qquad 0 \leq x \leq 1, \tag{3.1.4}$$

for all $n+p \geq 1$ and for $0 < q \leq 1$.

Proof. For each $x \in [0,1]$, let us define

$$x_r = \frac{[r]}{[n]} \text{ and } \lambda_r = \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r+1} (1 - q^s x), 0 \le r \le n+p.$$

where x_r is the quotient of the q-integers $[r]$ and $[n]$, and $\begin{bmatrix} n+p \\ r \end{bmatrix}$ denotes the q-binomial coefficients. Also, it is clear that $\lambda_r \ge 0$.

It is known that

$$B_n^p(1; q; x) = 1.$$

So

$$\lambda_0 + \lambda_1 + \cdots + \lambda_{n+p} = 1.$$

Also, it is proved that

$$B_n^p(t; q; x) = \frac{[n+p]}{[n]} x,$$

so

$$\lambda_0 x_0 + \lambda_1 x_1 + \cdots + \lambda_{n+p} x_{n+p} = \frac{[n+p]}{[n]} x.$$

Therefore, since $f(x)$ is a convex function, we have the following inequlity

$$\begin{aligned} B_n^p(f; q; x) &= \sum_{r=0}^{n+p} \lambda_r f(x_r) \ge f\left(\sum_{r=0}^{n+p} \lambda_r x_r\right) \\ &= f\left(\frac{[n+p]}{[n]} x\right) \ge f(x). \end{aligned}$$

□

Theorem 40. *If $f(x)$ is convex on $[0,1]$,*

$$B_{n-1}^p(f; q; x) \ge B_n(f; q; x), \quad 0 \le x \le 1, \tag{3.1.5}$$

for all $n \ge 2$, where $B_{n-1}^p(f; q; x)$ and $B_n^p(f; q; x)$ are estimated using the same value of the parameter q.

Proof. For $0 < q < 1$, let us write

$$\left(B_{n-1}^p(f; q; x) - B_n^p(f; q; x)\right) \prod_{s=0}^{n+p-1} (1 - q^s x)^{-1}$$

$$= \sum_{r=0}^{n+p-1} f\left(\frac{[r]}{[n-1]}\right) \begin{bmatrix} n+p-1 \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-2} (1-q^s x) \prod_{s=0}^{n+p-1} (1-q^s)^{-1}$$
$$- \sum_{r=0}^{n+p} f\left(\frac{[r]}{[n]}\right) \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1} (1-q^s x) \prod_{s=0}^{n+p-1} (1-q^s)^{-1}$$
$$= \sum_{r=0}^{n+p-1} f\left(\frac{[r]}{[n-1]}\right) \begin{bmatrix} n+p-1 \\ r \end{bmatrix} x^r \prod_{s=n+p-r-1}^{n+p-1} (1-q^s x)^{-1}$$
$$- \sum_{r=0}^{n+p} f\left(\frac{[r]}{[n]}\right) \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=n-r}^{n+p-1} (1-q^s x)^{-1}$$

Now, let

$$x^r \prod_{s=n-r-1}^{n-1} (1-q^s x)^{-1} = \psi_r(x) + q^{n-r-1} \psi_{r+1}(x),$$

where

$$\psi_r(x) = x^r \prod_{s=n-r}^{n+p-1} (1-q^s x)^{-1}. \tag{3.1.6}$$

Restating results in terms of $\psi_0(x)$ and $\psi_n(x)$ yields

$$\left(B_{n-1}^p(f;q;x) - B_n^p(f;q;x)\right) \prod_{s=0}^{n+p-1} (1-q^s x)^{-1} = \sum_{r=1}^{n+p-1} \begin{bmatrix} n+p \\ r \end{bmatrix} a_r \psi_r(x), \tag{3.1.7}$$

where

$$a_r = \frac{[n-r]}{[n]} f\left(\frac{[r]}{[n-1]}\right) + q^{n+p-r} \frac{[r]}{[n+p]} f\left(\frac{[r-1]}{[n-1]}\right) - f\left(\frac{[r]}{[n]}\right). \tag{3.1.8}$$

It is clear from (3.1.7) that each $\psi_r(x)$ is nonnegative on $[0,1]$ for $0 \le q \le 1$, and thus from (3.1.8), it will suffice to show that a_r is nonnegative. Let us state

$$\lambda = \frac{[n-r]}{[n]}, \quad x_1 = \frac{[r]}{[n-1]}, \quad x_2 = \frac{[r-1]}{[n-1]}.$$

It follows that

$$1-\lambda = q^{n-r} \frac{[r]}{[n]} \text{ and } \lambda x_1 + (1-\lambda) x_2 = \frac{[r]}{[n]},$$

and we see immediately, on comparing (3.1.7) and (3.1.8), that

$$a_r = \lambda f(x_1) + (1-\lambda) f(x_2) - f(\lambda x_1 + (1-\lambda) f(x_2)) \ge 0,$$

and so $B_{n-1}^p(f;q;x) \ge B_n^p(f;q;x)$. The inequality will be strict for $0 < x < 1$ unless every a_r is zero; this can happpen only when f is linear in each of the intervals between consecutive points $[r] / [n+p-1]$, $0 < r \le n+p-1$, then we have $B_{n-1}^p(f;q;x) = B_n^p(f;q;x)$ for $0 < x < 1$. This completes the proof. □

3.3 Rate of Convergence

Theorem 41. *([14]) If $f(x)$ is continuous on $[0,1]$ and $\omega(f;\delta)$ is the modulus of continuity of $f(x)$, then*

$$|B_n^p(f;q;x) - f(x)| \leq 2\omega\left(f; \sqrt{\lambda_n(x)}\right)$$

where $\lambda_n(x) = x^2\left(\frac{[n+p-1]\,[n+p]}{[n]^2}q - 2\frac{[n+p]}{[n]} + 1\right) + \frac{[n+p]}{[n]^2}x.$

Proof. Using linearity and monotonicity of the operator B_n^p, we get

$$\begin{aligned}
&|B_n^p(f;q;x) - f(x)| \\
&= |\sum_{r=0}^{n+p} f(\frac{[r]}{[n]})\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x) - f(x)| \\
&= |\sum_{r=0}^{n+p}(f(\frac{[r]}{[n]}) - f(x))| \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x) \\
&\leq \sum_{r=0}^{n+p}|f(\frac{[r]}{[n]}) - f(x)| \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x)
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{r=0}^{n+p}\omega(f;|\frac{[r]}{[n]} - x|)\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x) \\
&= \sum_{r=0}^{n+p}\omega(f;\frac{\left|\frac{[r]}{[n]} - x\right|}{\delta}\delta)\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x) \\
&\leq \sum_{r=0}^{n+p}\left[(1+\frac{\left|\frac{[r]}{[n]} - x\right|}{\delta})\omega(f;\delta)\right]\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x).
\end{aligned}$$

Then using Cauchy-Schwarz Bunyakowsky inequality, we have

$$
\begin{aligned}
&|B_n^p(f;q;x) - f(x)| \\
&\leq \sum_{r=0}^{n+p} \left((1 + \frac{\left|\frac{[r]}{[n]} - x\right|}{\delta})\omega(f;\delta) \right) \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1} (1 - q^s x) \\
&= \omega(f;\delta) \left[\sum_{r=0}^{n+p} \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1} (1 - q^s x) \right. \\
&\left. + \frac{1}{\delta} \sum_{r=0}^{n+p} |\frac{[r]}{[n]} - x| \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1} (1 - q^s x) \right] \\
&= \omega(f;\delta) \left[1 + \frac{1}{\delta} \sum_{r=0}^{n+p} \{ (\frac{[r]}{[n]} - x)^2 \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1} (1 - q^s x) \}^{1/2} \right. \\
&\left. \{ \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1} (1 - q^s x) \}^{1/2} \right].
\end{aligned}
$$

Hence

$$
\begin{aligned}
&|B_n^p(f;q;x) - f(x)| \\
&\leq \omega(f;\delta)[1 + \frac{1}{\delta^2} [\sum_{r=0}^{n+p} (\frac{[r]}{[n]} - x)^2 \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1} (1 - q^s x)]^{1/2} \\
&\times [\sum_{r=0}^{n+p} \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1} (1 - q^s x)]^{1/2} \\
&= \omega(f;\delta)[1 + \frac{1}{\delta^2} \sqrt{B_n(t-x)^2;q;x)}]. \qquad (3.1.9)
\end{aligned}
$$

On the other hand, since

$$
\begin{aligned}
B_n^p(t-x)^2;q;x) &= B_n^p(t^2;q;x) - 2xB_n^p(t;q;x) + x^2 B_n^p(1;q;x) \\
&= x^2 \left(\frac{[n+p-1]\,[n+p]}{[n]^2} q - 2\frac{[n+p]}{[n]} + 1 \right) + \frac{[n+p]}{[n]^2} x.
\end{aligned}
$$

By (3.1.9), we get

$$
|B_n^p(f;q;x) - f(x)| \leq \omega(f;\delta)(1 + \frac{1}{\delta^2} \sqrt{\lambda_n(x)}),
$$

where

$$
\lambda_n = x^2 \left(\frac{[n+p-1]\,[n+p]}{[n]^2} q - 2\frac{[n+p]}{[n]} + 1 \right) + \frac{[n+p]}{[n]^2} x. \qquad (3.1.10)
$$

Choosing $\delta = \sqrt{\lambda_n(x)}$, we find

$$
|B_n^p(f;q;x) - f(x)| \leq 2\omega(f;\sqrt{\lambda_n(x)}).
$$

$\square$

Theorem 42. *Let $f \in Lip_M(\alpha)$, then*

$$|B_n^p(f;q;x)-f(x)| \leq M\left[x^2\left(\frac{[n+p-1]\,[n+p]}{[n]^2}q-2\frac{[n+p]}{[n]}+1\right)+\frac{[n+p]}{[n]^2}x\right]^{\alpha/2}$$

where $\lambda_n(x)$ is given by (3.1.10).

Proof. Considering the monotonicity and the lineariy of the operators, and taking into account that $f \in Lip_M(\alpha)$ $(0<\alpha\leq 1)$

$$\begin{aligned}
&|B_n^p(f;q;x)-f(x)| \\
&=|\sum_{r=p}^{n+p}(f(\frac{[r]}{[n]})-f(x)\begin{bmatrix}n+p\\ r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx)\,| \\
&\leq \sum_{r=0}^{n+p}|f(\frac{[r]}{[n]})-f(x)\begin{bmatrix}n+p\\ r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx)\,| \\
&\leq M\sum_{r=0}^{n+p}|\frac{[r]}{[n]}-x|^{\alpha}\begin{bmatrix}n+p\\ r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx)\,.
\end{aligned}$$

Using Hölder's inequality, we get

$$\begin{aligned}
&|B_n^p(f;q;x)-f(x)| \\
&\leq M\sum_{r=0}^{n+p}[(\frac{[r]}{[n]}-x)^2\begin{bmatrix}n+p\\ r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx)]^{\frac{\alpha}{2}}[\begin{bmatrix}n+p\\ r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx)]^{\frac{2-\alpha}{2}} \\
&\leq M\left[\{\sum_{r=0}^{n+p}([(\frac{[r]}{[n]}-x)^2\begin{bmatrix}n+p\\ r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx)])^{\frac{\alpha}{2}}\}\right. \\
&\left.\times\{\sum_{r=0}^{n+p}[\begin{bmatrix}n+p\\ r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx)])^{\frac{2-\alpha}{\alpha}}\}\right] \\
&=M[B_n^p((t-x)^2;q;x)]^{\frac{\alpha}{2}} \\
&=M[(B_n^p(t^2;q;x)-2x(B_n^p(t;q;x)+x^2(B_n^p(1;q;x)]^{\frac{\alpha}{2}} \\
&=M(\lambda_n(x))^{\frac{\alpha}{2}}.
\end{aligned}$$

$\square$

Chapter 4

Q-BERNSTEIN-SCHURER-CHLODOWSKY POLYNOMIALS

4.1 Construction of the Operators

We introduce the q-Bernstein-Schurer-Chlodowsky Polynomials by

$$C_n^p(f;q;x)=\sum_{r=0}^{n+p} f\left(\frac{[r]}{[n]}b_n\right)\begin{bmatrix} n+p \\ r \end{bmatrix}\left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right) \tag{4.1.1}$$

where $p \in \mathbb{N}_0$, (b_n) is a positive increasing sequence and $0 \leq x \leq b_n$. These operators are linear and positive provided that $0 < q < 1$.

This operator satisfy Korovkin's Theorem conditions as follows:

Lemma 43. *For the q-Bernstein-Schurer-Chlodowsky Polynomials we have*

(i) $C_{n,}^p(1;q;x)=1.$

(ii) $C_n^p(t;q;x)=\dfrac{[n+p]}{[n]}x.$

(iii) $C_n^p(t^2;q;x)=\dfrac{[n+p-1]\,[n+p]}{[n]^2}qx^2+\dfrac{x\,(b_n-x)}{[n]}.$

Proof. (i) Consider the Binomial identity

$$(1-x)^{n+p-r}=\prod_{s=0}^{n+p-r-1}(1-q^s x),$$

then we have

$$\begin{aligned} C_n^p(1;q;x) &= \sum_{r=0}^{n+p}\begin{bmatrix} n+p \\ r \end{bmatrix}\left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right) \\ &= 1. \end{aligned}$$

(ii) Direct calculations yield,

$$\begin{aligned}
C_n^p(t;q;x) &= \sum_{r=0}^{n+p} \frac{[r]}{[n]} b_n \begin{bmatrix} n+p \\ r \end{bmatrix} \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1} \left(1 - q^s \frac{x}{b_n}\right) \\
&= b_n \sum_{r=0}^{n+p} \frac{[r]}{[n]} \begin{bmatrix} n+p \\ r \end{bmatrix} \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1} \left(1 - q^s \frac{x}{b_n}\right) \\
&= \frac{[n+p]}{[n]} b_n \frac{x}{b_n} \sum_{r=1}^{n+p} \begin{bmatrix} n+p-1 \\ r-1 \end{bmatrix} \left(\frac{x}{b_n}\right)^{r-1} \prod_{s=0}^{n+p-r-1} \left(1 - q^s \frac{x}{b_n}\right) \\
&= x \frac{[n+p]}{[n]} \sum_{r=0}^{n+p-1} \begin{bmatrix} n+p-1 \\ r \end{bmatrix} \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-2} \left(1 - q^s \frac{x}{b_n}\right) \\
&= x \frac{[n+p]}{[n]},
\end{aligned}$$

which completes the proof of (ii).

(iii) We have

$$\begin{aligned}
C_n^p(t^2;q;x) &= \sum_{r=0}^{n+p} \left(\frac{[r]}{[n]} b_n\right)^2 \begin{bmatrix} n+p \\ r \end{bmatrix} \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1} \left(1 - q^s \frac{x}{b_n}\right) \\
&= b_n^2 \sum_{r=1}^{n+p} \frac{[r]}{[n]} \begin{bmatrix} n+p-1 \\ r-1 \end{bmatrix} \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1} \left(1 - q^s \frac{x}{b_n}\right) \\
&= b_n^2 \sum_{r=1}^{n+p} \frac{q\,[r-1]}{[n+p]} \begin{bmatrix} n+p-1 \\ r-1 \end{bmatrix} \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1} \left(1 - q^s \frac{x}{b_n}\right) \\
&+ \frac{b_n^2}{[n+p]} \sum_{r=1}^{n+p} \begin{bmatrix} n+p-1 \\ r-1 \end{bmatrix} \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1} \left(1 - q^s \frac{x}{b_n}\right)
\end{aligned}$$

then we get

$$\begin{aligned}
&= \frac{qb_n^2\,[n+p-1]}{[n+p]} \begin{bmatrix} n+p-2 \\ r-2 \end{bmatrix} \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1} \left(1 - q^s \frac{x}{b_n}\right) \\
&+ \frac{b_n^2}{[n+p]} \sum_{r=0}^{n+p-1} \begin{bmatrix} n+p-1 \\ r \end{bmatrix} \left(\frac{x}{b_n}\right)^{r+1} \prod_{s=0}^{n+p-r-1} \left(1 - q^s \frac{x}{b_n}\right) \\
&= \frac{qb_n^2\,[n+p-1]}{[n]} \left(\frac{x}{b_n}\right)^2 \sum_{r=0}^{n+p-2} \begin{bmatrix} n+p-2 \\ r-2 \end{bmatrix} \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-2} \left(1 - q^s \frac{x}{b_n}\right) \\
&+ \frac{b_n^2}{[n+p]} \frac{x}{b_n} \sum_{r=0}^{n+p-1} \begin{bmatrix} n+p-1 \\ r \end{bmatrix} \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1} \left(1 - q^s \frac{x}{b_n}\right) \\
&= \frac{q\,[n+p-1]}{[n+p]} x^2 C_{n-2}^p\,(1;q;x) + \frac{b_n}{[n+p]_q} x C_{n-1}^p\,(1;q;x) \\
&= \frac{[n+p-1]\,[n+p]}{[n]^2} q x^2 + \frac{x\,(b_n - x)}{[n]}.
\end{aligned}$$

Thus the proof is completed.

□

For the first two central moments, we have the following:

Lemma 44. *Let $p \in \mathbb{N}_0$, (b_n) is a increasing sequence of positive real numbers. Then for the q-Bernstein-Schurer-Chlodowsky operators we have*
(i) $C_n^p\left((t-x);q;x\right) = x\left(\frac{[n+p]}{[n]} - 1\right)$.
(ii) $C_n^p\left((t-x)^2;q;x\right) = x^2\left(\frac{[n+p-1][n+p]}{[n]^2}q - 2\frac{[n+p]}{[n]} + 1\right) + \frac{x(b_n - x)}{[n]}$.

Proof. (i) Using the linearity of the operators and taking into account lemma (4.0.11), we have

$$\begin{aligned} C_n^p\left((t-x);q;x\right) &= C_n^p\left(t;q;x\right) - xC_n^p\left(1;q;x\right) \\ &= x\left(\frac{[n+p]}{[n]} - 1\right). \end{aligned}$$

(ii) Consider

$$\begin{aligned} C_n^p\left((t-x)^2;q;x\right) &= C_n^p\left(t^2;q;x\right) - 2xC_n^p\left(t;q;x\right) + x^2C_n^p\left(1;q;x\right) \\ &= \frac{[n+p-1][n+p]}{[n]^2}qx^2 + \frac{x(b_n-x)}{[n]} - 2x^2\frac{[n+p]}{[n]} + x^2, \end{aligned} \tag{4.1.2}$$

then we have

$$C_n^p\left((t-x)^2;q;x\right) = x^2\left(\frac{[n+p-1][n+p]}{[n]^2}q - 2\frac{[n+p]}{[n]} + 1\right) + \frac{x(b_n-x)}{[n]}.$$

Whence the result. □

Lemma 45. *For the second central moment we have the following inequality:*

$$\sup_{0\le x\le b_n} C_n^p\left((t-x)^2;q;x\right) \le \frac{b_n^2}{[n]^2}\left([p]^2 + \frac{[n]}{4}\right).$$

Proof. We can write

$$\begin{aligned}
C_n^p\left((t-x)^2;q;x\right) &= x^2\left(\frac{[n+p-1]\,[n+p]}{[n]^2}q-2\frac{[n+p]}{[n]}+1\right)+\frac{x\,(b_n-x)}{[n]} \\
&\le x^2\left(\frac{[n+p]}{[n]}-1\right)^2+\frac{x\,(b_n-x)}{[n]} \\
&= \frac{x^2}{[n]^2}q^{2n}\,[p]^2+\frac{x\,(b_n-x)}{[n]} \\
&\le \frac{x^2}{[n]^2}\,[p]^2+\frac{x\,(b_n-x)}{[n]}. \qquad (4.1.3)
\end{aligned}$$

Now taking supremum over the inteval $x\in[0,b_n]$ on both sides of the inequality (4.1.3), we get

$$\begin{aligned}
\sup_{0\le x\le b_n} C_n^p\left((t-x)^2;q;x\right) &\le \sup_{0\le x\le b_n}\left\{\frac{x^2}{[n]^2}\,[p]^2+\frac{x\,(b_n-x)}{[n]}\right\} \\
&= \frac{b_n^2}{[n]^2}\left([p]^2+\frac{[n]}{4}\right).
\end{aligned}$$

□

4.2 Korovkin Type Approximation Theorem

In this subsection we prove a Korovkin type approximation theorem for the q-Bernstein-Schurer-Chlodowsky operators

Lemma 46. *Let A be a positive real number independent of n and f be a continuous function which vanishes on $[A,\infty)$. Assume that $q:=q_n$ with $0<q\le 1$ and $\lim_{n\to\infty}\frac{b_n}{[n]}=0$, then we have*

$$\lim_{n\to\infty}\sup_{0\le x\le b_n}\left|\tilde{C}_n^p\left(f;q;x\right)-f\left(x\right)\right|=0.$$

Proof. By hypothesis, f is bounded say $|f(x)|\le M\ (M>0)$. For arbitrary small $\varepsilon>0$, we have

$$\left|f\left(\frac{[k]}{[n]}b_n\right)-f\left(x\right)\right|<\varepsilon+\frac{2M}{\delta^2}\left(\frac{[k]}{[n]}b_n-x\right)^2,$$

where $x \in [0, b_n]$ and $\delta = \delta(\varepsilon)$ are independent of n. Thus,

$$\sum_{r=0}^{n+p} \left(\frac{[r]}{[n]} b_n - x \right)^2 \begin{bmatrix} n+p \\ r \end{bmatrix} \left(\frac{x}{b_n} \right)^r \prod_{s=0}^{n+p-r-1} \left(1 - q^s \frac{x}{b_n} \right)$$
$$= x^2 \left(\frac{[n+p-1][n+p]}{[n]^2} q - 2 \frac{[n+p]}{[n]} + 1 \right) + \frac{x(b_n - x)}{[n]}.$$

Therefore by Lemma 4.1.3

$$\sup_{0 \le x \le b_n} \left| \tilde{C}_n^p(f; q; x) - f(x) \right| = \varepsilon + 2M \frac{b_n^2}{[n]^2} \left([p]^2 + \frac{[n]}{4} \right).$$

Since $\dfrac{b_n}{[n]} \to 0$ as $n \to \infty$, the proof is completed. □

Theorem 47. *Let f be a continuous function on the semiaxis $[0, \infty)$ and*

$$\lim_{x \to \infty} f(x) = k_f < \infty.$$

Assume that $q := q_n$ with $0 < q \le 1$, $\lim_{n \to \infty} q_n = 1$ and $\lim_{n \to \infty} \dfrac{b_n}{[n]} = 0$. Then

$$\lim_{n \to \infty} \sup_{0 \le x \le b_n} \left| \tilde{C}_n^p(f; q; x) - f(x) \right| = 0.$$

Proof. For any $\varepsilon > 0$ we can find a point x_0 such that

$$|f(x)| < \varepsilon, \quad x \ge x_0. \tag{4.1.4}$$

Define a function g as follows

$$g(x) = \begin{cases} f(x) & , \quad 0 \le x \le x_0 \\ y = 2f(x_0)(x - x_0) + f(x_0), & x_0 \le x \le x_0 + \frac{1}{2} \\ 0 & , \quad x \ge x_0 + \frac{1}{2}. \end{cases}$$

Then

$$\sup_{0 \le x \le b_n} |f(x) - g(x)| \le \sup_{x_0 \le x \le x_0 + \frac{1}{2}} |f(x) - g(x)| + \sup_{x \ge x_0 + \frac{1}{2}} |f(x)|.$$

Since

$$\max_{x_0 \le x \le x_0 + \frac{1}{2}} |g(x)| = |f(x_0)|$$

we have, from (4.1.4) that

$$\sup_{0\leq x\leq b_n} |f(x) - g(x)| \leq 3\varepsilon.$$

Now we can write

$$\begin{aligned}
&\sup_{0\leq x\leq b_n} \left|\tilde{C}_n^p(f;q_n;x) - f(x)\right| \\
&\leq \sup_{0\leq x\leq b_n} \tilde{C}_n^p(|f-g|;q_n;x) + \sup_{0\leq x\leq b_n} \left|\tilde{C}_n^p(g;q_n;x) - g(x)\right| + \sup_{0\leq x\leq b_n} |f(x) - g(x)| \\
&\leq 6\varepsilon + \sup_{0\leq x\leq b_n} \left|\tilde{C}_n^p(g;q_n;x) - g(x)\right|.
\end{aligned}$$

where $g(x) = 0$ for $x_0 + \frac{1}{2} \leq x \leq b_n$. By the lemma 4.2.1, we obtain the result. □

4.3 Order of Convergence

In this subsection we obtain the rate of convergence of the approximation, given in the previous subsection, by means of modulus of continuity of the function, elements of the Lipschits classes and the modulus of continuity of the derivative of the function.

Theorem 48. *Let (q_n) be a sequence of real numbers such that $q := q_n$; $0 < q_n < 1$ and $[n] := [n]_q$. If $f \in C_B[0,\infty)$, we have*

$$|C_n^p(f;q;x) - f(x)| \leq 2\omega\left(f, \sqrt{\delta_{n,q}(x)}\right),$$

where $\omega(f,.)$ is modulus of continuity of f and

$$\delta_{n,q}(x) = x^2\left(\frac{[n+p-1][n+p]}{[n]^2}q - 2\frac{[n+p]}{[n]} + 1\right) + \frac{x(b_n - x)}{[n]}.$$

Proof. By using the positivity and linearity of the operators, we have

$$\begin{aligned}
&|C_n^p(f;q;x) - f(x)| \\
&= |\sum_{r=0}^{n+p} f(\frac{[r]}{[n]}b_n)\begin{bmatrix} n+p \\ r \end{bmatrix}\left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1}\left(1 - q^s\frac{x}{b_n}\right) - f(x)| \\
&\leq \sum_{r=0}^{n+p} |f(\frac{[r]}{[n]}b_n) - f(x)|\begin{bmatrix} n+p \\ r \end{bmatrix}\left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1}\left(1 - q^s\frac{x}{b_n}\right).
\end{aligned}$$

Now using the properties of the modulus of continuity, we can write

$$
\begin{aligned}
&\left|C_n^p(f;q;x)-f(x)\right| \\
&\leq \sum_{r=0}^{n+p}\left|f\left(\frac{[r]}{[n]}b_n\right)\begin{bmatrix}n+p\\ r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)-f(x)\right| \\
&\leq \sum_{r=0}^{n+p}\left|f\left(\frac{[r]}{[n]}b_n\right)-f(x)\right|\begin{bmatrix}n+p\\ r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{r=0}^{n+p}\left(\frac{\left|\frac{[r]}{[n]}b_n-x\right|}{\delta}+1\right)\omega(f,\delta)\begin{bmatrix}n+p\\ r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right) \\
&=\omega(f,\delta)\sum_{r=0}^{n+p}\begin{bmatrix}n+p\\ r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right) \\
&+\frac{\omega(f,\delta)}{\delta}\sum_{r=0}^{n+p}\left|\frac{[r]}{[n]}b_n-x\right|\begin{bmatrix}n+p\\ r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right) \\
&=\omega(f,\delta)+\frac{\omega(f,\delta)}{\delta}\left\{\sum_{r=0}^{n+p}\left(\frac{[r]}{[n]}b_n-x\right)^2\begin{bmatrix}n+p\\ r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)\right\} \\
&=\omega(f,\delta)+\frac{\omega(f,\delta)}{\delta}\left\{C_n^p\left((t-x)^2;q;x\right)\right\}^{1/2},
\end{aligned}
$$

where $C_n^p\left((t-x)^2;q;x\right)=x^2\left(\frac{[n+p-1]\,[n+p]}{[n]^2}q-2\frac{[n+p]}{[n]}+1\right)+\frac{x\,(b_n-x)}{[n]}$.

Now choosing $\delta_{n,q}(x)=x^2\left(\frac{[n+p-1]\,[n+p]}{[n]^2}q-2\frac{[n+p]}{[n]}+1\right)+\frac{x\,(b_n-x)}{[n]}$, we have

$$\left|C_n^p(f;q;x)-f(x)\right|\leq 2\omega\sqrt{\delta_{n,q}(x)}.$$

Whence the result. □

Theorem 49. *Let (q_n) be a sequence of real numbers such that $0<q_n<1$ and $\lim_{n\to\infty}q_n=1$. If $f\in Lip_M(\alpha)$ and $x\in[0,A]>0$,*

$$\left\|C_n^p(f;q;x)-f\right\|_{C[0,b_n]}\leq M\left\{AC_n^p\left((t-x)^2;q;x\right)\right\}^{\frac{\alpha}{2}}.$$

Proof. Consider

$$
\begin{aligned}
\left|C_n^p(f;q;x)-f(x)\right| &\leq \sum_{r=0}^{n+p}\left|f\left(\frac{[r]}{[n]}b_n\right)-f(x)\right|\begin{bmatrix}n+p\\ r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right) \\
&\leq M\sum_{r=0}^{n+p}\left|\frac{[r]}{[n]}b_n-x\right|^{\alpha}\begin{bmatrix}n+p\\ r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right).
\end{aligned}
$$

Choosing $p_1 = \dfrac{2}{\alpha}$ and $p_2 = \dfrac{2}{2-\alpha}$ then $\dfrac{1}{p_1} + \dfrac{1}{p_2} = 1$. We can write

$$\left|C_n^p(f;q;x) - f(x)\right| \leq \sum_{r=0}^{n+p} \left\{ \left|\frac{[r]}{[n]}b_n - x\right|^2 \begin{bmatrix} n+p \\ r \end{bmatrix} \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1} \left(1 - q^s \frac{x}{b_n}\right) \right\}^{\frac{\alpha}{2}}$$
$$\times \left\{ \begin{bmatrix} n+p \\ r \end{bmatrix} \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1} \left(1 - q^s \frac{x}{b_n}\right) \right\}^{\frac{2-\alpha}{2}}.$$

Using Hölder inequlity, we get

$$\left|C_n^p(g;q;x) - f(x)\right| \leq M \left\{ \sum_{r=0}^{n+p} \left|\frac{[r]}{[n]}b_n - x\right|^2 \begin{bmatrix} n+p \\ r \end{bmatrix} \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1} \left(1 - q^s \frac{x}{b_n}\right) \right\}^{\frac{\alpha}{2}}.$$

From (4.1.2) we can write

$$\left|C_n^p(f;q;x) - f(x)\right| \leq M \left\{ C_n^p\left((t-x)^2; q; x\right) \right\}^{\frac{\alpha}{2}}.$$

This implies that

$$\left\|C_n^p(f;q;x) - f(x)\right\|_{C[0,b_n]} \leq M \left\{ A C_n^p\left((t-x)^2; q; x\right) \right\}^{\frac{\alpha}{2}}$$

where $x \in [0, A]$. □

Theorem 50. *Let (q_n) be a sequence of real numbers such that $q := q_n$, $0 < q_n < 1$ and $\lim_{n\to\infty} q_n = 1$. If $f(x)$ have continuous derivative $f'(x)$ and $\omega(f', \delta)$ is the modulus of continuity of $f'(x)$ in $[0, A]$, then*

$$\begin{aligned} &\left|f(x) - C_n^p(f;q;x)\right| \\ &\leq MA\frac{[p]}{[n]} + 2\sqrt{\frac{A^2}{[n]^2}[p]^2 + \frac{Ab_n}{[n]}}\,\omega\left(f', \sqrt{\frac{A^2}{[n]^2}[p]^2 + \frac{Ab_n}{[n]}}\right). \end{aligned}$$

where M is a positive constant such that $|f'(x)| \leq M$ $(0 \leq x \leq A)$.

Proof. Using the mean value theorem we have

$$\begin{aligned} f\left(\frac{[r]}{[n]}b_n\right) - f(x) &= \left(\frac{[r]}{[n]}b_n - x\right) f'(\xi) \\ &= \left(\frac{[r]}{[n]}b_n - x\right) f'(x) + \left(\frac{[r]}{[n]}b_n - x\right)\left(f'(\xi) - f'(x)\right), \end{aligned}$$

where $x<\xi<\frac{[r]}{[n]}b_n$. By using last equality we can write the following inequality,

$$\begin{aligned}
&\left|C_n^p(f;q;x)-f(x)\right| \\
&= f'(x)\sum_{r=0}^{n+p}\left(\frac{[r]}{[n]}b_n-x\right)\begin{bmatrix} n+p \\ r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right) \\
&+\sum_{r=0}^{n+p}\left(\frac{[r]}{[n]}b_n-x\right)\left(f'(\xi)-f'(x)\right)\begin{bmatrix} n+p \\ r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right) \\
&\le \left|f'(x)\right| C_n^p\left((t-x);q;x\right) \\
&+\sum_{r=0}^{n+p}\left(\frac{[r]}{[n]}b_n-x\right)\left(f'(\xi)-f'(x)\right)\begin{bmatrix} n+p \\ r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right) \\
&\le MA\left(\frac{[n+p]}{[n]}-1\right) \\
&+\sum_{r=0}^{n+p}\left(\frac{[r]}{[n]}b_n-x\right)\left(f'(\xi)-f'(x)\right)\begin{bmatrix} n+p \\ r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right) \\
&\le MA\frac{[p]}{[n]} \\
&+\sum_{r=0}^{n+p}\left(\frac{[r]}{[n]}b_n-x\right)\left(f'(\xi)-f'(x)\right)\begin{bmatrix} n+p \\ r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right) \\
&\le MA\frac{[p]}{[n]}+\sum_{r=0}^{n+p}\omega\left(f',\delta\right)\left(\frac{\left|\frac{[r]}{[n]}b_n-x\right|}{\delta}+1\right) \\
&\times\left(\frac{[r]}{[n]}b_n-x\right)\begin{bmatrix} n+p \\ r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right),
\end{aligned}$$

since

$$|\xi-x|\le\left|\frac{[r]}{[n]}b_n-x\right|.$$

Therefore, we can write the following inequality

$$\begin{aligned}
&\left|C_n^p(f;q;x)-f(x)\right| \\
&\le MA\frac{[p]}{[n]}+\sum_{r=0}^{n+p}\omega\left(f',\delta\right)\left(\frac{\left|\frac{[r]}{[n]}b_n-x\right|}{\delta}+1\right) \\
&\times\left(\frac{[r]}{[n]}b_n-x\right)\begin{bmatrix} n+p \\ r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right).
\end{aligned}$$

Using the Cauchy-Schwarz inequality for the first term we get

$$
\begin{aligned}
&\left|C_n^p(f;q;x)-f(x)\right| \\
&\leq MA\frac{[p]}{[n]}+\omega\left(f',\delta\right)\sum_{r=0}^{n+p}\left|\frac{[r]}{[n]}b_n-x\right|\begin{bmatrix}n+p\\r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right) \\
&+\frac{\omega\left(f',\delta\right)}{\delta}\sum_{r=0}^{n+p}\left(\frac{[r]}{[n]}b_n-x\right)^2\begin{bmatrix}n+p\\r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right) \\
&\leq MA\frac{[p]}{[n]}+\omega\left(f',\delta\right)\left(\sum_{r=0}^{n+p}\left(\frac{[r]}{[n]}b_n-x\right)^2\begin{bmatrix}n+p\\r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)\right)^{1/2} \\
&+\frac{\omega\left(f',\delta\right)}{\delta}\sum_{r=0}^{n+p}\left(\frac{[r]}{[n]}b_n-x\right)^2\begin{bmatrix}n+p\\r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right) \\
&= MA\frac{[p]}{[n]}+\omega\left(f',\delta\right)\sqrt{C_n^p\left((t-x)^2;q;x\right)}+\frac{\omega\left(f',\delta\right)}{\delta}C_n^p\left((t-x)^2;q;x\right).
\end{aligned}
$$

On the other hand, using (4.1.3), we get

$$
\begin{aligned}
\sup_{0\leq x\leq A}C_n^p\left((t-x)^2;q;x\right) &\leq \sup_{0\leq x\leq A}\left(\frac{x^2}{[n]^2}[p]^2+\frac{x\left(b_n-x\right)}{[n]}\right) \\
&\leq \frac{A^2}{[n]^2}[p]^2+\frac{Ab_n}{[n]}.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
&\left|C_n^p(f;q;x)-f(x)\right| \\
&\leq MA\frac{[p]}{[n]}+\omega\left(f',\delta\right)\left\{\sqrt{\frac{A^2}{[n]^2}[p]^2+\frac{Ab_n}{[n]}}+\frac{1}{\delta}\left(\frac{A^2}{[n]^2}[p]^2+\frac{Ab_n}{[n]}\right)\right\}.
\end{aligned}
$$

Putting $\delta=\sqrt{\frac{A^2}{[n]^2}[p]^2+\frac{Ab_n}{[n]}}$

$$
\begin{aligned}
&\left|C_n^p(f;q;x)-f(x)\right| \\
&\leq MA\frac{[p]}{[n]}+\omega\left(f',\sqrt{\frac{A^2}{[n]^2}[p]^2+\frac{Ab_n}{[n]}}\right)\left\{\sqrt{\frac{A^2}{[n]^2}[p]^2+\frac{Ab_n}{[n]}}+\sqrt{\frac{A^2}{[n]^2}[p]^2+\frac{Ab_n}{[n]}}\right\} \\
&= MA\frac{[p]}{[n]}+2\sqrt{\frac{A^2}{[n]^2}[p]^2+\frac{Ab_n}{[n]}}\,\omega\left(f',\sqrt{\frac{A^2}{[n]^2}[p]^2+\frac{Ab_n}{[n]}}\right).
\end{aligned}
$$

Whence the result. □

Chapter 5

SCHURER TYPE Q-BERNSTEIN KANTOROVICH OPERATORS

5.1 Construction of the Operators

In this chapter we introduce Schurer type q-Bernstein Kantorovich operators by

$$K_n^p(f;q;x)=\sum_{r=0}^{n+p}\begin{bmatrix}n+p\\r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx)\int_0^1 f\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}\right)d_qt$$

where $0<q<1$ and $p\in\mathbb{N}_0$ is fixed.

Lemma 51. *For the Schurer type q-Bernstein Kantorovich operators we have*

(i) $K_n^p(1;q;x)=1.$

(ii) $K_n^p(u;q;x)=\dfrac{1}{[n+1]}\left(\dfrac{1}{[2]}+[n+p]\,qx\right).$

(iii) $K_n^p(u^2;q;x)=\dfrac{1}{[n+1]^2}\left(\dfrac{1}{[3]}+\dfrac{2[n+p]}{[2]}qx+[n+p-1]\,[n+p]\,q^3x^2+[n+p]\,q^2x\right).$

Proof. (i) From the definition of the q-integral and $\sum_{s=0}^{\infty}q^j=\dfrac{1}{1-q}$ we have

$$\begin{aligned}\int_0^1 d_qt&=(1-q)\sum_{j=0}^{\infty}q^j\\&=(1-q)\frac{1}{(1-q)}\\&=1.\end{aligned}$$

As a result

$$K_n^p(1;q;x) = \sum_{r=0}^{n+p} \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1} (1-q^s x)$$
$$= 1.$$

(ii) Again using the definition of the q-integral we can calculate

$$\int_0^1 \left(\frac{t}{[n+1]} + \frac{q[r]}{[n+1]} \right) d_q t$$
$$= \frac{1}{[n+1]} \int_0^1 t d_q t + \frac{q[r]}{[n+1]} \int_0^1 d_q t$$
$$= \frac{1}{[n+1]} (1-q) \sum_{j=0}^{\infty} q^{2j} + \frac{q[r]}{[n+1]}$$
$$= \frac{1}{[n+1]} (1-q) \frac{1}{1-q^2} + \frac{q[r]}{[n+1]}$$
$$= \frac{1}{[n+1]} \frac{1}{1+q} + \frac{q[r]}{[n+1]}.$$

Hence we have

$$K_n^p(u;q;x) = \sum_{r=0}^{n+p} \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1} (1-q^s x) \left(\frac{1}{[n+1]} \frac{1}{1+q} + \frac{q[r]}{[n+1]} \right)$$
$$= \frac{1}{[n+1]} \frac{1}{1+q} \sum_{r=0}^{n+p} \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1} (1-q^s x)$$
$$+ q \sum_{r=0}^{n+p} \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1} (1-q^s x) \frac{[r]}{[n+1]} \frac{[n]}{[n]}$$
$$= \frac{1}{[n+1]} \frac{1}{1+q} + \frac{q[n]}{[n+1]} B_n^q(t;q;x)$$
$$= \frac{1}{[n+1]} \frac{1}{1+q} + \frac{q[n]}{[n+1]} \frac{[n+p]}{[n]} x = \frac{1}{[n+1]} \left(\frac{1}{1+q} + [n+p]\, qx \right).$$

(iii) From the definition of the q-integral, we get

$$\int_0^1 t^2 d_q t = (1-q) \sum_{j=0}^{\infty} q^{2j} q^j$$
$$= (1-q) \frac{1}{1-q^3}$$
$$= \frac{1}{1+q+q^2}$$
$$= \frac{1}{[3]}.$$

We have

$$\begin{aligned}
&\int_0^1 \left(\frac{t}{[n+1]}+\frac{q\,[r]}{[n+1]}\right)^2 d_qt \\
&= \int_0^1 \left(\frac{t^2}{[n+1]^2}+2\frac{t}{[n+1]}\frac{q\,[r]}{[n+1]}+\frac{q^2\,[r]^2}{[n+1]^2}\right) d_qt \\
&= \frac{1}{[n+1]^2}\left(\int_0^1 t^2 d_qt+2q\,[r]\int_0^1 t d_qt+q^2\,[r]^2\int_0^1 d_qt\right) \\
&= \frac{1}{[n+1]^2}\left(\frac{1}{1+q+q^2}+2\frac{q\,[r]}{1+q}+q^2\,[r]^2\right).
\end{aligned}$$

Thus

$$\begin{aligned}
K_n^p\left(u^2;q;x\right) = \frac{1}{[n+1]^2}\Bigg\{&\sum_{r=0}^{n+p}\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^sx)\,\frac{1}{1+q+q^2} \\
&+\sum_{r=0}^{n+p}\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^sx)\,2\frac{q\,[r]}{1+q} \\
&+\sum_{r=0}^{n+p}\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^sx)\,q^2\,[r]^2\Bigg\}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{[n+1]^2}\frac{1}{1+q+q^2}\sum_{r=0}^{n+p}\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^sx) \\
&+ \frac{1}{[n+1]}\frac{2}{1+q}\sum_{r=0}^{n+p}\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^sx)\,\frac{[r]}{[n+1]}\frac{[n]}{[n]} \\
&+ \sum_{r=0}^{n+p}\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^sx)\,\frac{q^2\,[r]^2}{[n+1]^2}\frac{[n]^2}{[n]^2} \\
&= \frac{1}{[n+1]^2}\frac{1}{1+q+q^2} \\
&+ \frac{2q\,[n]}{(1+q)\,[n+1]^2}\sum_{r=0}^{n+p}\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^sx)\,\frac{[r]}{[n]} \\
&+ \frac{q^2\,[n]^2}{[n+1]^2}\sum_{r=0}^{n+p}\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^sx)\,\frac{[r]^2}{[n]^2} \\
&= \frac{1}{[n+1]^2}\frac{1}{1+q+q^2} \\
&+ \frac{2q\,[n]}{(1+q)\,[n+1]^2}B_n^p\left(t;q;x\right)+\frac{q^2\,[n]^2}{[n+1]^2}B_n^p\left(t^2;q;x\right).
\end{aligned}$$

Finally we get

$$\begin{aligned}
&K_n^p\left(u^2;q;x\right)\\
&=\frac{1}{[n+1]^2}\left(\frac{1}{[3]}+2\frac{[n+p]}{[2]}qx+[n+p-1]\,[n+p]\,q^3x^2+[n+p]\,q^2x\right),
\end{aligned}$$

where $B_n^p\left(f;q;x\right)$ is the q-Bernstein Schurer operator. □

Remark 52. *Taking limits in Lemma 5.1.1, when $q\to1^-$, we get*

$$\begin{aligned}
K_n^p\left(1;x\right)&=1,\\
K_n^p\left(u;x\right)&=\frac{n+p}{n+1}x+\frac{1}{2n+2},\\
K_n^p\left(u^2;x\right)&=\frac{1}{3\left(n+1\right)^2}+\frac{\left(n+p\right)\left[\left(n+p-1\right)x^2+2x\right]}{\left(n+1\right)^2}.
\end{aligned}$$

Lemma 53. *For the operator $K_n^p\left(f;q;x\right)$, we have*

$$\begin{aligned}
K_n^p\left(\left(u-x\right);q;x\right)&=x\left(\frac{[n+p]}{[n+1]}q-1\right)+\frac{1}{[2]\,[n+1]}\\
K_n^p\left(\left(u-x\right)^2;q;x\right)&=x^2\left(\frac{[n+p-1]\,[n+p]}{[n+1]^2}q^3-2\frac{[n+p]}{[n+1]}q+1\right)\\
&\quad+\frac{x}{[n+1]^2}\left(2\frac{[n+p]}{[2]}q+[n+p]\,q^2-2\frac{[n+1]}{[2]}\right)\\
&\quad+\frac{1}{[3]\,[n+1]^2}.
\end{aligned}\tag{5.1.1}$$

Proof. It is obvious that

$$\begin{aligned}
K_n^p\left(\left(u-x\right);q;x\right)&=K_n^p\left(u;q;x\right)-xK_n^p\left(1;q;x\right)\\
&=x\left(\frac{[n+p]}{[n+1]}q-1\right)+\frac{1}{[2]\,[n+1]}.
\end{aligned}$$

Direct calculations yield,

$$
\begin{aligned}
& K_n^p\left((u-x)^2;q;x\right) \\
& = K_n^p\left(u^2;q;x\right) - 2xK_n^p\left(u;q;x\right) + x^2K_n^p\left(1;q;x\right) \\
& = \frac{1}{[n+1]^2}\left(\frac{1}{[3]} + 2\frac{[n+p]}{[2]}qx + [n+p-1]\,[n+p]\,q^3x^2 + [n+p]\,q^2x\right) \\
& - 2x\frac{1}{[n+1]}\left(\frac{1}{1+q} + [n+p]\,qx\right) + x^2 \\
& = x^2\left(\frac{[n+p-1]\,[n+p]}{[n+1]^2}q^3 - 2\frac{[n+p]}{[n+1]}q + 1\right) \\
& + \frac{x}{[n+1]^2}\left(2\frac{[n+p]}{[2]}q + [n+p]\,q^2 - 2\frac{[n+1]}{[2]}\right) \\
& + \frac{1}{[3]\,[n+1]^2}.
\end{aligned}
$$

□

By the Korovkin's theorem, we can state the following theorem:

Theorem 54. *For all $f \in C\,[0,p+1]$, we have*

$$\lim_{n\to\infty}\left\|K_n^p\left(f;q_n,x\right) - f\left(x\right)\right\|_{C[0,p+1]} = 0$$

provided that $q := q_n$ with $\lim_{n\to\infty} q_n = 1$ and that $\lim_{n\to\infty}\frac{1}{[n]} = 0$.

5.2 Rate of Convergence

Theorem 55. *Let (q_n) be a sequence of real numbers such that $q := q_n$; $0 < q < 1$ and $\lim_{n\to\infty} q_n = 1$. If $f \in C[0,p+1)$, we have*

$$\left|K_n^p\left(f;q;x\right) - f\left(x\right)\right| \leq 2\omega\left(f, \sqrt{\delta_{n,q}\left(x\right)}\right),$$

where $\omega\left(f,.\right)$ is the modulus of continuity of f. Also f is continuos function.

Proof. Using the linearity and positivity of the operator, the property of the modulus of continuity and finally the Cauchy-Schwarz Bunyakowsky inequality we can write that

$$
\begin{aligned}
&\left|K_n^p(f;q;x)-f(x)\right| \\
&\leq \left|\sum_{r=0}^{n+p}\begin{bmatrix}n+p\\r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^s x)\int_0^1\left(f\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}\right)-f(x)\right)d_qt\right| \\
&\leq \sum_{r=0}^{n+p}\begin{bmatrix}n+p\\r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^s x)\int_0^1\left|f\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}\right)-f(x)\right|d_qt
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{r=0}^{n+p}\int_0^1\left(\frac{\left|\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right|}{\delta}+1\right)\omega(f,\delta)\begin{bmatrix}n+p\\r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^s x)\,d_qt \\
&=\omega(f,\delta)\sum_{r=0}^{n+p}\begin{bmatrix}n+p\\r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^s x) \\
&+\frac{\omega(f,\delta)}{\delta}\sum_{r=0}^{n+p}\int_0^1\left|\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right|\begin{bmatrix}n+p\\r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^s x)\,d_qt \\
&=\omega(f,\delta) \\
&+\frac{\omega(f,\delta)}{\delta}\left\{\sum_{r=0}^{n+p}\int_0^1\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right)^2\begin{bmatrix}n+p\\r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^s x)\right\}^{1/2}d_qt
\end{aligned}
$$

We know that from the Hölder's inequality $\frac{1}{p}+\frac{1}{q}=1$; $q=2$ and $p=2$.

$$
\begin{aligned}
&\int_0^1\left|f\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}\right)-f(x)\right|d_qt \\
&\leq \int_0^1\left|\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right|^2 d_qt \\
&\leq \left\{\int_0^1\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right)^2 d_qt\right\}^{\frac{1}{2}}\left\{\int_0^1 1d_qt\right\}^{\frac{1}{2}} \\
&=\left\{\int_0^1\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right)^2 d_qt\right\}^{\frac{1}{2}}=\{a_{n,r}(x)\}^{\frac{1}{2}}.
\end{aligned}
$$

Now we have

$$\left|K_n^p(f;q;x)-f(x)\right|=\sum_{r=0}^{n+p}\{a_{n,r}\}^{\frac{1}{2}}\,p_{n,r}(q;x)$$

where $p_{n,r}(q;x) = \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x)$. Again applying the Hölder's inequality with; $q=2$ and $p=2$, we get

$$\begin{aligned}
&\left|K_n^p(f;q;x)-f(x)\right| \\
&\leq \left\{\sum_{r=0}^{n+p} a_{n,r}p_{n,r}(x)\right\}^{\frac{1}{2}}\left\{\sum_{r=0}^{n+p} p_{n,r}(x)\right\}^{\frac{1}{2}} \\
&=\left\{\sum_{r=0}^{n+p} p_{n,r}(x)\int_0^1\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right)^2 d_qt\right\}^{\frac{1}{2}} \\
&=\left[\delta_{n,q}(x)\right]^{\frac{1}{2}}.
\end{aligned}$$

Now we have,

$$\omega(f,\delta)+\frac{\omega(f,\delta)}{\delta}\left\{K_n^p(\delta_{n,q}(x);q;x)\right\}^{1/2}.$$

Choosing $\delta_{n,q}(x)=K_n^p\left((u-x)^2;q;x\right)$, we have

$$\left|K_n^p(f;q;x)-f(x)\right|\leq 2\omega\left(f,\sqrt{K_n^p\left((u-x)^2;q;x\right)}\right).$$

□

Theorem 56. *Let $f\in Lip_M(\alpha)$, then*

$$\left|K_n^p(f;q;x)-f(x)\right|\leq M\left(K_n^p\left((u-x)^2;q;x\right)\right)^{\frac{\alpha}{2}}$$

where $K_n^p\left((u-x)^2;q;x\right)=x^2\left(\frac{[n+p-1][n+p]}{[n+1]^2}q^3-2\frac{[n+p]}{[n+1]}q+1\right)$
$+\frac{x}{[n+1]^2}\left(2\frac{[n+p]}{[2]}q+[n+p]q^2-2\frac{[n+1]}{[2]}\right)+\frac{1}{[3][n+1]^2}$.

Proof. By the linearity and positivity, we have

$$\begin{aligned}
&\left|K_n^p(f;q;x)-f(x)\right| \\
&=\sum_{r=0}^{n+p}\begin{bmatrix} n+p \\ r \end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx)\int_0^1\left|f\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}\right)-f(x)\right|d_qt.
\end{aligned}$$

We know that from the Hölder's inequality $\frac{1}{p}+\frac{1}{q}=1$; $q=\frac{2}{2-\alpha}$ and $p=\frac{2}{\alpha}$.

$$\begin{aligned}
&\int_0^1 \left| f\left(\frac{t}{[n+1]}+\frac{q\,[r]}{[n+1]}\right)-f(x)\right| d_qt \\
&\leq \int_0^1 \left|\frac{t}{[n+1]}+\frac{q\,[r]}{[n+1]}-x\right|^{\alpha} d_qt \\
&\leq \left\{\int_0^1 \left(\frac{t}{[n+1]}+\frac{q\,[r]}{[n+1]}-x\right)^2 d_qt\right\}^{\frac{\alpha}{2}} \left\{\int_0^1 1 d_qt\right\}^{\frac{2-\alpha}{2}} \\
&= \left\{\int_0^1 \left(\frac{t}{[n+1]}+\frac{q\,[r]}{[n+1]}-x\right)^2 d_qt\right\}^{\frac{\alpha}{2}} = \{a_{n,r}(x)\}^{\frac{\alpha}{2}}.
\end{aligned}$$

Now we have

$$|K_n^p(f;q;x)-f(x)| = M\sum_{r=0}^{n+p}\{a_{n,r}\}^{\frac{\alpha}{2}}\,p_{n,r}(q;x)$$

where $p_{n,r}(q;x) = \begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x)$. Again applying the Hölder's inequality with; $q=\frac{2}{2-\alpha}$ and $p=\frac{2}{\alpha}$, we get

$$\begin{aligned}
&|K_n^p(f;q;x)-f(x)| \\
&\leq M\left\{\sum_{r=0}^{n+p} a_{n,r}p_{n,r}(x)\right\}^{\frac{\alpha}{2}}\left\{\sum_{r=0}^{n+p} 1.p_{n,r}(x)\right\}^{\frac{2-\alpha}{2}} \\
&= M\left\{\sum_{r=0}^{n+p} p_{n,r}(x)\int_0^1\left(\frac{t}{[n+1]}+\frac{q\,[r]}{[n+1]}-x\right)^2 d_qt\right\}^{\frac{\alpha}{2}} \\
&= M\left[K_n^p\left((u-x)^2;q;x\right)\right]^{\frac{\alpha}{2}}.
\end{aligned}$$

□

Theorem 57. *Let (q_n) be a sequence of real numbers such that $q:=q_n$; $0<q<1$ and $\lim_{n\to\infty} q_n = 1$. If $f(x)$ have a continuous derivative $f'(x)$ and $\omega(f',\delta)$ is the modulus* *of continuity of $f'(x)$ in $[0,1]$, then*

$$\begin{aligned}
&\left|f(x)-K_n^p(f;q;x)\right| \\
&\leq MA\frac{[p]}{[n+1]}+2\left(\frac{1}{[n+1]^2}[p]^2+\frac{1}{[n+1]^2}\left(2\frac{[n+p]}{[2]}+[n+p]\right)+\frac{1}{[3][n+1]^2}\right)^{1/2} \\
&\times\omega\left(f',\left(\frac{1}{[n+1]^2}[p]^2+\frac{1}{[n+1]^2}\left(2\frac{[n+p]}{[2]}+[n+p]\right)+\frac{1}{[3][n+1]^2}\right)^{1/2}\right),
\end{aligned}$$

where M is a positive constant such that $|f'(x)| \leq M$ $(0 \leq x \leq 1)$.

Proof. Using the mean value theorem we have

$$\begin{aligned}
&f\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}\right)-f(x) \\
&=\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right)f'(\xi) \\
&=\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right)f'(x)+\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right)\left(f'(\xi)-f(x)\right),
\end{aligned}$$

where $x<\xi<\dfrac{t}{[n+1]}+\dfrac{q[r]}{[n+1]}$. Hence, we have

$$\begin{aligned}
&\left|K_n^p(f;q;x)-f(x)\right| \\
&=f'(x)\sum_{r=0}^{n+p}\int_0^1\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right)\begin{bmatrix}n+p\\ r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx)\,d_qt \\
&+\sum_{r=0}^{n+p}\int_0^1\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right)\left(f'(\xi)-f(x)\right)\begin{bmatrix}n+p\\ r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx)\,d_qt \\
&\leq\left|f'(x)\right|K_n^p((u-x);q;x) \\
&+\sum_{r=0}^{n+p}\int_0^1\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right)\left(f'(\xi)-f(x)\right)\begin{bmatrix}n+p\\ r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx)\,d_qt \\
&\leq MA\left(\frac{[n+p]}{[n+1]}-1\right)
\end{aligned}$$

$$+\sum_{r=0}^{n+p}\int_0^1\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right)\left(f'(\xi)-f(x)\right)\begin{bmatrix}n+p\\r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx)\,d_qt$$

$$\leq MA\frac{[p]}{[n+1]}$$

$$+\sum_{r=0}^{n+p}\int_0^1\left(\frac{t}{[n+1]}+\frac{[r]}{[n+1]}-x\right)\left(f'(\xi)-f(x)\right)\begin{bmatrix}n+p\\r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx)\,d_qt$$

$$\leq MA\frac{[p]}{[n+1]}$$

$$+\sum_{r=0}^{n+p}\int_0^1\omega(f',\delta)\left(\frac{\left|\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right|}{\delta}+1\right)$$

$$\times\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right)\begin{bmatrix}n+p\\r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx)\,d_qt,$$

since

$$|\xi-x|\leq\left|\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right|.$$

Therefore we can write the following inequality,

$$|K_n^p(f;q;x)-f(x)|$$

$$\leq MA\frac{[p]}{[n+1]}$$

$$+\sum_{r=0}^{n+p}\int_0^1\omega(f',\delta)\left(\frac{\left|\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right|}{\delta}+1\right)$$

$$\times\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right)\begin{bmatrix}n+p\\r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx)\,d_qt.$$

From the Cauchy-Schwarz inequality for the first term we get

$$\begin{aligned}
&\left|K_n^p(f;q;x) - f(x)\right| \\
&\leq MA\frac{[p]}{[n+1]} \\
&+\omega(f',\delta)\sum_{r=0}^{n+p}\int_0^1\left|\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right|\begin{bmatrix} n+p \\ r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^s x)\,d_qt \\
&+\frac{\omega(f',\delta)}{\delta}\sum_{r=0}^{n+p}\int_0^1\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right)^2\begin{bmatrix} n+p \\ r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^s x)\,d_qt \\
&\leq MA\frac{[p]}{[n+1]} \\
&+\omega(f',\delta)\left(\sum_{r=0}^{n+p}\int_0^1\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right)^2\begin{bmatrix} n+p \\ r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^s x)\,d_qt\right)^{1//2} \\
&+\frac{\omega(f',\delta)}{\delta}\sum_{r=0}^{n+p}\int_0^1\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right)^2\begin{bmatrix} n+p \\ r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^s x)\,d_qt \\
&= MA\frac{[p]}{[n+1]}+\omega(f',\delta)\sqrt{K_n^p\left((u-x)^2;q;x\right)}+\frac{\omega(f',\delta)}{\delta}K_n^p\left((u-x)^2;q;x\right).
\end{aligned}$$

Therefore using (5.1.1), we see that

$$\begin{aligned}
\sup_{0\leq x\leq 1}K_n^p\left((u-x)^2;q;x\right) &\leq \sup_{0\leq x\leq 1}\frac{x^2}{[n+1]^2}[p]^2 \\
&+\frac{x}{[n+1]^2}\left(2\frac{[n+p]}{[2]}q+[n+p]\,q^2-2\frac{[n+1]}{[2]}\right) \\
&+\frac{1}{[3]\,[n+1]^2} \\
&\leq \frac{1}{[n+1]^2}[p]^2 \\
&+\frac{1}{[n+1]^2}\left(2\frac{[n+p]}{[2]}q+[n+p]\,q^2-2\frac{[n+1]}{[2]}\right) \\
&+\frac{1}{[3]\,[n+1]^2}.
\end{aligned}$$

Thus

$$|K_n^p(f;q;x)-f(x)|$$
$$\leq MA\frac{[p]}{[n+1]}$$
$$\omega(f',\delta)\left\{\left(\frac{1}{[n+1]^2}[p]^2+\frac{1}{[n+1]^2}\left(2\frac{[n+p]}{[2]}q+[n+p]q^2-2\frac{[n+1]}{[2]}\right)\right.\right.$$
$$\left.+\frac{1}{[3][n+1]^2}\right)^{\frac{1}{2}}$$
$$+\frac{1}{\delta}\left(\frac{1}{[n+1]^2}[p]^2+\frac{1}{[n+1]^2}\left(2\frac{[n+p]}{[2]}q+[n+p]q^2-2\frac{[n+1]}{[2]}\right)\right.$$
$$\left.\left.+\frac{1}{[3][n+1]^2}\right)\right\}.$$

Choosing $\delta=\left(\frac{1}{[n+1]^2}[p]^2+\frac{1}{[n+1]^2}\left(2\frac{[n+p]}{[2]}q+[n+p]q^2-2\frac{[n+1]}{[2]}\right)\right.$ we get

$$\left.+\frac{1}{[3][n+1]^2}\right)^{1/2}$$

$$|K_n^p(f;q;x)-f(x)|$$
$$\leq MA\frac{[p]}{[n+1]}$$
$$+\omega\left(f',\left(\frac{1}{[n+1]^2}[p]^2+\frac{1}{[n+1]^2}\left(2\frac{[n+p]}{[2]}q+[n+p]q^2-2\frac{[n+1]}{[2]}\right)\right.\right.$$
$$\left.\left.+\frac{1}{[3][n+1]^2}\right)^{\frac{1}{2}}\right)$$
$$\times\left\{\left(\frac{1}{[n+1]^2}[p]^2+\frac{1}{[n+1]^2}\left(2\frac{[n+p]}{[2]}q+[n+p]q^2-2\frac{[n+1]}{[2]}\right)\right)^{1/2}\right.$$
$$\left.+\frac{1}{[3][n+1]^2}\right)^{\frac{1}{2}}\right)$$
$$\left.+\left(\frac{1}{[n+1]^2}[p]^2+\frac{1}{[n+1]^2}\left(2\frac{[n+p]}{[2]}q+[n+p]q^2-2\frac{[n+1]}{[2]}\right)+\frac{1}{[3][n+1]^2}\right)^{1/2}\right\}$$

$$
\begin{aligned}
&= MA\frac{[p]}{[n+1]} \\
&+ 2\left(\frac{1}{[n+1]^2}[p]^2 + \frac{1}{[n+1]^2}\left(2\frac{[n+p]}{[2]}q + [n+p]\,q^2 - 2\frac{[n+1]}{[2]}\right) + \frac{1}{[3]\,[n+1]^2}\right)^{1/2} \\
&\times \omega\left(f', \left(\frac{1}{[n+1]^2}[p]^2 + \frac{1}{[n+1]^2}\left(2\frac{[n+p]}{[2]}q + [n+p]\,q^2 - 2\frac{[n+1]}{[2]}\right) + \frac{1}{[3]\,[n+1]^2}\right)^{1/2}\right) \\
&. + \frac{1}{[3]\,[n+1]^2}\Bigg)^{\frac{1}{2}}\Bigg).
\end{aligned}
$$

$\square$

Chapter 6

KANTOROVICH TYPE Q-BERNSTEIN-SCHURER-CHLODOWSKY OPERATORS

6.1 Construction of the Operators

In this chapter we introduce the Kantorovich Type q-Bernstein-Schurer-Chlodowsky Operators. It is defined as follows

$$T_n^p(f;q;x) = \sum_{r=0}^{n+p} \begin{bmatrix} n+p \\ r \end{bmatrix} \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1} \left(1 - q^s \frac{x}{b_n}\right) \times \int_0^1 f\left(\frac{t}{[n+1]} b_n + \frac{q[r]}{[n+1]} b_n\right) d_q t, \tag{6.1.1}$$

where $q \in (0,1)$, $n \in \mathbb{N}$ and $f \in C([0,p+1])$, here $p \in \mathbb{N}_0$ is fixed. Also it is clear that this operator is linear and positive.

Lemma 58. *Let $T_n^p(f;q;x)$ be given by (6.1.1) we can write the following properties*

(i) $T_n^p(1;q;x) = 1.$

(ii) $T_n^p(u;q;x) = \dfrac{1}{[n+1]}\left(\dfrac{b_n}{[2]} + [n+p]\,qx\right).$

(iii) $T_n^p(u^2;q;x) = \dfrac{1}{[n+1]^2}\left(\dfrac{b_n^2}{[3]} + 2\dfrac{[n+p]\,b_n}{[2]}qx + [n+p-1]\,[n+p]\,q^3x^2 + (b_n - x)\,q^2x\right).$

Proof. (i) We know that [22]

$$\prod_{s=1}^{n}\left(1+q^{s-1}x\right) = \sum_{s=0}^{n} q^{s(s-1)/2} \begin{bmatrix} n \\ s \end{bmatrix} x^s$$

and

$$\sum_{r=0}^{n+p} \begin{bmatrix} n+p \\ r \end{bmatrix} x^r (1-x)_q^{n-r} = 1.$$

Therefore

$$\sum_{r=0}^{\infty} \begin{bmatrix} n+p \\ r \end{bmatrix} \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1} \left(1 - \frac{x}{b_n} q^s\right) = 1.$$

(ii) First of all we must calculate

$$\begin{aligned}
\int_0^1 \left(\frac{t}{[n+1]} b_n + \frac{q[r]}{[n+1]} b_n\right) d_q t &= \frac{b_n}{[n+1]} \int_0^1 t d_q t + \frac{q[r] b_n}{[n+1]} \int_0^1 d_q t \\
&= \frac{b_n}{[n+1]} (1-q) \sum_{j=0}^{\infty} q^j q^j + \frac{q[r] b_n}{[n+1]} \\
&= \frac{b_n}{[n+1]} (1-q) \frac{1}{1-q^2} + \frac{q[r] b_n}{[n+1]} \\
&= \frac{b_n}{[n+1]} \frac{1}{1+q} + \frac{q[r] b_n}{[n+1]} \\
&= \frac{b_n}{[2][n+1]} + \frac{q[r] b_n}{[n+1]}.
\end{aligned}$$

Now we calculate the $T_n^p(u; q; x)$,

$$\begin{aligned}
T_n^p(u; q; x) &= \sum_{r=0}^{n+p} \begin{bmatrix} n+p \\ r \end{bmatrix} \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1} \left(1 - q^s \frac{x}{b_n}\right) \left(\frac{b_n}{[2][n+1]} + \frac{q[r]}{[n+1]} b_n\right) \\
&= \frac{b_n}{[2][n+1]} \sum_{r=0}^{n+p} \begin{bmatrix} n+p \\ r \end{bmatrix} \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1} \left(1 - q^s \frac{x}{b_n}\right) \\
&+ \sum_{r=0}^{n+p} \begin{bmatrix} n+p \\ r \end{bmatrix} \left(\frac{x}{b_n}\right)^r \prod_{s=0}^{n+p-r-1} \left(1 - q^s \frac{x}{b_n}\right) \frac{q[r]}{[n+1]} b_n \frac{[n]}{[n]} \\
&= \frac{b_n}{[2][n+1]} + \frac{q[n]}{[n+1]} C_n^p(t; q; x) \\
&= \frac{b_n}{[2][n+1]} + \frac{q[n]}{[n+1]} \frac{[n+p]}{[n]} x \\
&= \frac{1}{[n+1]} \left(\frac{b_n}{[2]} + [n+p] qx\right),
\end{aligned}$$

where $C_n^p(t; q; x)$ is q-Bernstein-Schurer-Chlodowsky operator defined in Chapter 4.

(iii) Finally let's calculate $T_n^p(u^2;q;x)$,

$$\int_0^1 \left(\frac{t}{[n+1]}b_n + \frac{q[r]}{[n+1]}b_n\right)^2 d_qt$$
$$= \frac{b_n^2}{[n+1]^2}\int_0^1 t^2 d_qt + 2\frac{q[r]}{[n+1]^2}b_n^2\int_0^1 td_qt + \frac{q^2[r]^2}{[n+1]^2}b_n^2\int_0^1 d_qt.$$

On the other hand,

$$\int_0^1 t^2 d_qt = (1-q)\sum_{j=0}^{\infty} q^{2j}q^j$$
$$= (1-q)\frac{1}{1-q^3}$$
$$= (1-q)\frac{1}{(1-q)(1+q+q^2)}$$
$$= \frac{1}{[3]}.$$

Hence,

$$\int_0^1 \left(\frac{t}{[n+1]}b_n + \frac{q[r]}{[n+1]}b_n\right)^2 d_qt$$
$$= \frac{b_n^2}{[3][n+1]^2} + 2\frac{q[r]}{[2][n+1]^2}b_n^2 + \frac{q^2[r]^2}{[n+1]^2}b_n^2.$$

Since

$$
\begin{aligned}
&T_n^p\left(u^2;q;x\right)\\
&=\sum_{r=0}^{n+p}\begin{bmatrix}n+p\\r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)\\
&\times\left(\frac{b_n^2}{[3]\,[n+1]^2}+2\frac{q\,[r]}{[2]\,[n+1]^2}b_n^2+\frac{q^2\,[r]^2}{[n+1]^2}b_n^2\right)\\
&=\frac{b_n^2}{[3]\,[n+1]^2}\sum_{r=0}^{\infty}\begin{bmatrix}n+p\\r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)\\
&+2\frac{qb_n}{[2]\,[n+1]^2}\sum_{r=0}^{\infty}\begin{bmatrix}n+p\\r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)[r]\,\frac{[n]}{[n]}b_n\\
&+\frac{q^2}{[n+1]^2}\sum_{r=0}^{\infty}\begin{bmatrix}n+p\\r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)[r]^2\,\frac{[n]^2}{[n]^2}b_n^2\\
&=\frac{b_n^2}{[3]\,[n+1]^2}+2\frac{b_n\,[n]\,q}{[2]\,[n+1]^2}C_n^p\left(t;q;x\right)+\frac{q^2\,[n]^2}{[n+1]^2}C_n^p\left(t^2;q;x\right)\\
&=\frac{b_n^2}{[3]\,[n+1]^2}+2\frac{qb_n\,[n]}{[2]\,[n+1]^2}\frac{[n+p]}{[n]}x\\
&+\frac{q^2\,[n]^2}{[n+1]^2}\left(\frac{[n+p-1]\,[n+p]}{[n]^2}qx^2+\frac{x\left(b_n-x\right)}{[n]^2}\right)\\
&=\frac{1}{[n+1]^2}\left(\frac{b_n^2}{[3]}+2\frac{[n+p]\,b_n}{[2]}qx+[n+p-1]\,[n+p]\,q^3x^2+\left(b_n-x\right)q^2x\right).
\end{aligned}
$$

This completes the proof. □

Remark 59. *Taking limits in Lemma (6.1.1) as $q\to 1^-$, we have*

$$
\begin{aligned}
T_n^p\left(1;x\right)&=1.\\
T_n^p\left(u;x\right)&=\frac{1}{n+1}\left(\frac{b_n}{2}+\left(n+p\right)x\right).\\
T_n^p\left(u^2;x\right)&=\frac{1}{\left(n+1\right)^2}\left(\frac{b_n^2}{3}+\left(n+p\right)xb_n+\left(n+p-1\right)\left(n+p\right)x^2+x\left(b_n-x\right)\right).
\end{aligned}
$$

Lemma 60. *For the first two moments we have*

$$
\begin{aligned}
T_n^p\left(\left(u-x\right);q;x\right)&=\frac{1}{[n+1]}\left(\frac{b_n}{[2]}+[n+p]\,qx\right)-x\\
&+\left(\frac{[n+p]}{[n+1]}q-1\right)x+\frac{b_n}{[2]\,[n+1]}
\end{aligned}
$$

and

$$
\begin{aligned}
&T_n^p\left((u-x)^2;q;x\right)\\
&=x^2\left(\frac{[n+p-1]\,[n+p]\,q^3}{[n+1]^2}-2\frac{[n+p]}{[n+1]}q+1\right)\\
&+x\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}q+\frac{(b_n-x)}{[n+1]^2}q^2-2\frac{b_n}{[2]\,[n+1]}\right)\\
&+\frac{b_n^2}{[3]\,[n+1]^2}.
\end{aligned}
\tag{6.1.2}
$$

Proof. It is clear that

$$
\begin{aligned}
T_n^p\left((u-x);q;x\right)&=T_n^p\left(u;q;x\right)-xT_n^p\left(1;q;x\right)\\
&=\frac{1}{[n+1]}\left(\frac{b_n}{[2]}+[n+p]\,qx\right)-x\\
&=\left(\frac{[n+p]}{[n+1]}q-1\right)x+\frac{b_n}{[2]\,[n+1]}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&T_n^p\left((u-x)^2;q;x\right)\\
&=T_n^p\left(u^2;q;x\right)\\
&-2xT_n^p\left(u;q;x\right)+x^2T_n^p\left(1;q;x\right)\\
&=\frac{1}{[n+1]^2}\left(\frac{b_n^2}{[3]}+2\frac{[n+p]\,b_n}{[2]}qx+[n+p-1]\,[n+p]\,q^3x^2+q^2x\left(b_n-x\right)\right)\\
&-2x\frac{1}{[n+1]}\left(\frac{b_n}{[2]}+[n+p]\,qx\right)+x^2
\end{aligned}
$$

$$
\begin{aligned}
&=x^2\left(\frac{[n+p-1]\,[n+p]\,q^3}{[n+1]^2}-2\frac{[n+p]}{[n+1]}q+1\right)\\
&+x\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}q+\frac{(b_n-x)}{[n+1]^2}q^2-2\frac{b_n}{[2]\,[n+1]}\right)\\
&+\frac{b_n^2}{[3]\,[n+1]^2}.
\end{aligned}
$$

□

Theorem 61. *For the second central moment we have the following inequality:*

$$\sup_{0\le x\le b_n} T_n^p\left((u-x)^2;q;x\right)$$
$$\le \frac{b_n^2}{[n+1]^2}[p]^2 + b_n\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}q + \frac{b_n}{[n+1]^2}q^2\right)$$
$$+\frac{b_n^2}{[3]\,[n+1]^2}.$$

Proof. We can write

$$T_n^p\left((u-x)^2;q;x\right)$$
$$= x^2\left(\frac{[n+p-1]\,[n+p]\,q^3}{[n+1]^2} - 2\frac{[n+p]}{[n+1]}q + 1\right)$$
$$+ x\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}q + \frac{(b_n-x)}{[n+1]^2}q^2 - 2\frac{b_n}{[2]\,[n+1]}\right)$$
$$+\frac{b_n^2}{[3]\,[n+1]^2}$$

$$\le x^2\left(\frac{[n+p]}{[n+1]}-1\right)^2 + x\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}q + \frac{(b_n-x)}{[n+1]^2}q^2 - 2\frac{b_n}{[2]\,[n+1]}\right)$$
$$+\frac{b_n^2}{[3]\,[n+1]^2}$$
$$= \frac{x^2q^{2n}}{[n+1]^2}[p]^2 + x\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}q + \frac{(b_n-x)}{[n+1]^2}q^2 - 2\frac{b_n}{[2]\,[n+1]}\right)$$
$$+\frac{b_n^2}{[3]\,[n+1]^2}$$
$$\le \frac{x^2}{[n+1]^2}[p]^2 + x\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}q + \frac{(b_n-x)}{[n+1]^2}q^2 - 2\frac{b_n}{[2]\,[n+1]}\right)$$
$$+\frac{b_n^2}{[3]\,[n+1]^2}.$$

Now taking supremum over the interval $x\in[0,b_n]$ on both sides of the above inequality, we get

$$
\begin{aligned}
\sup_{0\le x\le b_n} T_n^p\left((u-x)^2;q;x\right) &\le \sup_{0\le x\le b_n}\left[\frac{x^2}{[n+1]^2}[p]^2\right.\\
&+x\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}q+\frac{(b_n-x)}{[n+1]^2}q^2\right)\\
&\left.+\frac{b_n^2}{[3]\,[n+1]^2}\right]\\
&\le \frac{b_n^2}{[n+1]^2}[p]^2+b_n\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}q+\frac{b_n}{[n+1]^2}q^2\right)\\
&+\frac{b_n^2}{[3]\,[n+1]^2}.
\end{aligned}
$$

□

6.2 Korovkin Type Approximation Theorem

In this subsection we prove a Korovkin type approximation theorem for the Kantorovich type q-Bernstein-Schurer-Chlodowsky Operators.

Lemma 62. *Let A be a positive real number independent of n and f be a continuous function which vanishes on $[A,\infty)$. Assume that $q:=q_n$ with $0<q\le 1$ and $\lim\limits_{n\to\infty}\frac{b_n}{[n]}=0$, then we have*

$$
\lim_{n\to\infty}\sup_{0\le x\le b_n}\left|\tilde{T}_n^p(f;q;x)-f(x)\right|=0.
$$

Proof. By hypothesis since f is bounded we have $|f(x)|\le M$; $(M>0)$. For arbitrary small $\varepsilon>0$, we have

$$
\begin{aligned}
&\left|f\left(\frac{t}{[n+1]}b_n+\frac{q\,[r]}{[n+1]}b_n\right)-f(x)\right|\\
&<\varepsilon+\frac{2M}{\delta^2}\left(\frac{t}{[n+1]}b_n+\frac{q\,[r]}{[n+1]}b_n-x\right)^2,
\end{aligned}
$$

where $x \in [0, b_n]$ and $\delta = \delta(\varepsilon)$ are independent of n. Thus,

$$\sum_{r=0}^{n+p} \left(\frac{t}{[n+1]} b_n + \frac{q[r]}{[n+1]} b_n - x \right)^2 \begin{bmatrix} n+p \\ r \end{bmatrix} \left(\frac{x}{b_n} \right)^r \prod_{s=0}^{n+p-r-1} \left(1 - q^s \frac{x}{b_n} \right)$$
$$= \frac{b_n^2}{[n+1]^2} [p]^2 + b_n \left(2 \frac{[n+p] b_n}{[2][n+1]^2} q + \frac{b_n}{[n+1]^2} q^2 \right)$$
$$+ \frac{b_n^2}{[3][n+1]^2}.$$

Therefore by lemma 6.1.4

$$\sup_{0 \le x \le b_n} \left| \tilde{T}_n^p(f; q; x) - |f(x)| \right|$$
$$= \varepsilon + 2M \frac{b_n^2}{[n+1]^2} [p]^2$$
$$+ \frac{b_n^2}{[n+1]^2} [p]^2 + b_n \left(2 \frac{[n+p] b_n}{[2][n+1]^2} q + \frac{b_n}{[n+1]^2} q^2 \right)$$
$$+ \frac{b_n^2}{[3][n+1]^2}.$$

Since $\dfrac{b_n}{[n]} \to 0$ as $n \to \infty$, the proof is completed. □

Theorem 63. *Let f be a continuous function on the semiaxis $[0, \infty)$, for which*

$$\lim_{x \to \infty} f(x) = k_f < \infty.$$

Then

$$\lim_{n \to \infty} \sup_{0 \le x \le b_n} \left| \tilde{T}_n^p(f; q; x) - f(x) \right| = 0.$$

Proof. It is enough to prove the case $k_f = 0$. Then, for any $\varepsilon > 0$ we can find a point x_0 such that

$$|f(x)| < \varepsilon, \quad x \ge x_0. \tag{6.1.4}$$

Define a function g as follows

$$g(x) = \begin{cases} f(x) & , \quad 0 \le x \le x_0 \\ y = 2f(x_0)(x - x_0) + f(x_0), & x_0 \le x \le x_0 + \frac{1}{2} \\ 0 & , \quad x \ge x_0 + \frac{1}{2}. \end{cases}$$

Then

$$\sup_{0\le x\le b_n}|f(x)-g(x)|\le\sup_{x_0\le x\le x_0+\frac{1}{2}}|f(x)-g(x)|+\sup_{x\ge x_0+\frac{1}{2}}|f(x)|.$$

Since

$$\max_{x_0\le x\le x_0+\frac{1}{2}}|g(x)|=|f(x_0)|$$

we have from (6.1.4) that

$$\sup_{0\le x\le b_n}|f(x)-g(x)|\le 3\varepsilon.$$

Now we can write

$$\begin{aligned}
&\sup_{0\le x\le b_n}\left|\tilde{T}_n^p(f;q;x)-f(x)\right|\\
&\le\sup_{0\le x\le b_n}\tilde{T}_n^p(|f-g|;q;x)+\sup_{0\le x\le b_n}\left|\tilde{T}_n^p(g;q_n;x)-g(x)\right|+\sup_{0\le x\le b_n}|f(x)-g(x)|\\
&\le 6\varepsilon+\sup_{0\le x\le b_n}\left|\tilde{T}_n^p(g;q;x)-g(x)\right|
\end{aligned}$$

where $g(x)=0$ for $x_0+\frac{1}{2}\le x\le b_n$. By the lemma 6.2.1, we obtain the result. □

6.3 Order of Convergence

In this subsection we obtain the rate of convergenceof the approximation, given in the previous subsection, by means of modulus of continuity of the function, elements of Lipschitz classes and the modulus of continuity of the derivetive of the function.

Theorem 64. *Let (q_n) be a sequence of real numbers such that $q:=q_n$; $0<q_n<1$. If $f\in C_B[0,\infty)$, we have*

$$|T_n^p(f;q;x)-f(x)|\le 2\omega\left(f,\sqrt{\delta_{n,q}(x)}\right),$$

where $\delta_{n,q}(x)=T_n^p\left((t-x)^2;q;x\right)$ is defined by equation (6.1.2) $\omega(f,.)$ is modulus of continuity of f. Also f is continuous function.

Proof. We can write the following inequality from the $T_n^p(f;q;x)$ operator;

$$\begin{aligned}
&|T_n^p(f;q;x)-f(x)| \\
&\le \left|\sum_{r=0}^{n+p}\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x)\int_0^1 \left(f\left(\frac{t}{[n+1]}b_n+\frac{q[r]}{[n+1]}b_n\right)-f(x)\right)d_qt\right| \\
&\le \sum_{r=0}^{n+p}\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x)\int_0^1 \left|f\left(\frac{t}{[n+1]}b_n+\frac{q[r]}{[n+1]}b_n\right)-f(x)\right|d_qt \\
&\le \sum_{r=0}^{n+p}\int_0^1\left(\frac{\left|\frac{t}{[n+1]}b_n+\frac{q[r]}{[n+1]}b_n-x\right|}{\delta}+1\right) \\
&\times\omega(f,\delta)\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x)\,d_qt
\end{aligned}$$

$$\begin{aligned}
&=\omega(f,\delta)\sum_{r=0}^{n+p}\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x) \\
&+\frac{\omega(f,\delta)}{\delta}\sum_{r=0}^{n+p}\int_0^1\left|\frac{t}{[n+1]}b_n+\frac{q[r]}{[n+1]}b_n-x\right|\begin{bmatrix} n+p \\ r \end{bmatrix} x^r \prod_{s=0}^{n+p-r-1}(1-q^s x)\,d_qt \\
&=\omega(f,\delta) \\
&+\frac{\omega(f,\delta)}{\delta}\left\{\sum_{r=0}^{n+p}\int_0^1\left(\frac{t}{[n+1]}b_n+\frac{q[r]}{[n+1]}b_n-x\right)^2\begin{bmatrix} n+p \\ r \end{bmatrix} x^r\right. \\
&\times\left.\prod_{s=0}^{n+p-r-1}(1-q^s x)\,d_qt\right\}^{1/2}.
\end{aligned}$$

We know that from the Hölder's inequality $\frac{1}{p}+\frac{1}{q}=1$; $q=2$ and $p=2$, we get

$$\begin{aligned}
&\int_0^1\left|f\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}\right)-f(x)\right|d_qt \\
&\le\int_0^1\left|\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right|^2 d_qt \\
&\le\left\{\int_0^1\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right)^2 d_qt\right\}^{\frac{1}{2}}\left\{\int_0^1 1d_qt\right\}^{\frac{1}{2}} \\
&=\left\{\int_0^1\left(\frac{t}{[n+1]}+\frac{q[r]}{[n+1]}-x\right)^2 d_qt\right\}^{\frac{1}{2}}=\{a_{n,r}(x)\}^{\frac{1}{2}}.
\end{aligned}$$

Let choosing $\delta_{n,q}(x) = T_n^p\left((u-x)^2; q; x\right)$, we have

$$\left|T_n^p(f;q;x) - f(x)\right| \leq 2\omega\sqrt{T_n^p\left((t-x)^2;q;x\right)}.$$

Whence the result. □

Theorem 65. *Let (q_n) be a sequence of real numbers such that $0 < q_n < 1$ and $\lim_{n\to\infty} q_n = 1$. If $f \in Lip_M(\alpha)$ and $x \in [0, A] > 0$,*

$$\left\|T_n^p(f;q;x) - f\right\|_{C[0,b_n]} \leq M\left\{AT_n^p\left((t-x)^2;q;x\right)\right\}^{\frac{\alpha}{2}}.$$

Proof. By the linearity and monotonicity of the operators, we have,

$$\begin{aligned}
&\left|T_n^p(f;q;x) - f(x)\right| \\
&\leq \sum_{r=0}^{n+p}\left|f\left(\frac{t}{[n+1]}b_n + \frac{q[r]}{[n+1]}b_n\right) - f(x)\right|\begin{bmatrix} n+p \\ r \end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right) \\
&\leq M\sum_{r=0}^{n+p}\int_0^1\left|\frac{t}{[n+1]}b_n + \frac{q[r]}{[n+1]}b_n - x\right|^{\alpha}\begin{bmatrix} n+p \\ r \end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)d_qt.
\end{aligned}$$

Let's choose $p_1 = \dfrac{2}{\alpha}$ and $p_2 = \dfrac{2}{2-\alpha}$ then $\dfrac{1}{p_1} + \dfrac{1}{p_2} = 1$. We can write

$$\begin{aligned}
&\left|T_n^p(f;q;x) - f(x)\right| \\
&\leq \sum_{r=0}^{n+p}\left\{\int_0^1\left|\frac{t}{[n+1]}b_n + \frac{q[r]}{[n+1]}b_n - x\right|^2\begin{bmatrix} n+p \\ r \end{bmatrix}\left(\frac{x}{b_n}\right)^r\right. \\
&\times\left.\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)\right\}^{\frac{\alpha}{2}}d_qt \\
&\times\left\{\begin{bmatrix} n+p \\ r \end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)\right\}^{\frac{2-\alpha}{2}}.
\end{aligned}$$

Using Hölder's inequlity, we have

$$\begin{aligned}
&\left|T_n^p(g;q;x) - f(x)\right| \\
&\leq M\left\{\sum_{r=0}^{n+p}\int_0^1\left|\frac{t}{[n+1]}b_n + \frac{q[r]}{[n+1]}b_n - x\right|^2\right. \\
&\times\left.\begin{bmatrix} n+p \\ r \end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)\right\}^{\frac{\alpha}{2}}d_qt.
\end{aligned}$$

From (6.1.2) we can write

$$\left|T_n^p(f;q;x)-f(x)\right| \le M\left\{T_n^p\left((u-x)^2;q;x\right)\right\}^{\frac{\alpha}{2}}.$$

This implies that

$$\left\|T_n^p(f;q;x)-f(x)\right\|_{C[0,b_n]} \le M\left\{AT_n^p\left((u-x)^2;q;x\right)\right\}^{\frac{\alpha}{2}},$$

where $x\in[0,A]$ □

Theorem 66. *Let (q_n) be a sequence of real numbers such that $q:=q_n$, $0<q_n<1$ and $\lim_{n\to\infty} q_n=1$. If $f(x)$ have continuous derivative $f'(x)$ and $\omega\left(f',\delta\right)$ is the modulus of continuity of $f'(x)$ in $[0,1]$. Then*

$$\begin{aligned}
&|f(x)-T_n^p(f;q;x)| \\
&\le MA\frac{[p]}{[n+1]} \\
&+2\left\{\frac{1}{[n+1]^2}[p]^2+\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}+\frac{(b_n-1)}{[n+1]^2}-2\frac{b_n}{[2]\,[n+1]}\right)+\frac{b_n^2}{[3]\,[n+1]^2}\right\}^{1/2} \\
&\times\omega\left(f',\left\{\frac{1}{[n+1]^2}[p]^2+\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}+\frac{(b_n-1)}{[n+1]^2}-2\frac{b_n}{[2]\,[n+1]}\right)+\right.\right. \\
&\left.\left.+\frac{b_n^2}{[3]\,[n+1]^2}\right\}^{1/2}\right)
\end{aligned}$$

where M is a positive constant such that $|f'(x)|\le M\,(0\le x\le 1)$.

Proof. From the mean value theorem we have

$$\begin{aligned}
&f\left(\frac{t}{[n+1]}b_n+\frac{q\,[r]}{[n+1]}b_n\right)-f(x) \\
&=\left(\frac{t}{[n+1]}b_n+\frac{q\,[r]}{[n+1]}b_n\right)f'(\xi) \\
&=\left(\frac{t}{[n+1]}b_n+\frac{q\,[r]}{[n+1]}b_n-x\right)f'(x)+\left(\frac{t}{[n+1]}b_n+\frac{q\,[r]}{[n+1]}b_n-x\right)\left(f'(\xi)-f'(x)\right),
\end{aligned}$$

where $x<\xi<\frac{t}{[n+1]}b_n+\frac{q\,[r]}{[n+1]}b_n$. By using last equality we can write the following inequality,

$$
\begin{aligned}
& T_n^p\left(f;q;x\right)-f\left(x\right) \\
& =f^{'}\left(x\right)\sum_{r=0}^{n+p}\int_0^1\left(\frac{t}{[n+1]}b_n+\frac{q\left[r\right]}{[n+1]}b_n-x\right)\begin{bmatrix}n+p\\r\end{bmatrix}\left(\frac{x}{b_n}\right)^r \\
& \times\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)d_qt \\
& +\sum_{r=0}^{n+p}\int_0^1\left(\frac{t}{[n+1]}b_n+\frac{q\left[r\right]}{[n+1]}b_n-x\right)\left(f^{'}\left(\xi\right)-f^{'}\left(x\right)\right)\begin{bmatrix}n+p\\r\end{bmatrix}\left(\frac{x}{b_n}\right)^r \\
& \times\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)d_qt
\end{aligned}
$$

$$
\begin{aligned}
& \leq\left|f^{'}\left(x\right)\right|T_n^p\left(\left(u-x\right);q;x\right) \\
& +\sum_{r=0}^{n+p}\int_0^1\left(\frac{t}{[n+1]}b_n+\frac{q\left[r\right]}{[n+1]}b_n-x\right)\left(f^{'}\left(\xi\right)-f^{'}\left(x\right)\right)\begin{bmatrix}n+p\\r\end{bmatrix}\left(\frac{x}{b_n}\right)^r \\
& \times\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)d_qt \\
& \leq MA\left(\frac{[n+p]}{[n+1]}-1\right) \\
& +\sum_{r=0}^{n+p}\int_0^1\left(\frac{t}{[n+1]}b_n+\frac{q\left[r\right]}{[n+1]}b_n-x\right)\left(f^{'}\left(\xi\right)-f^{'}\left(x\right)\right)\begin{bmatrix}n+p\\r\end{bmatrix}\left(\frac{x}{b_n}\right)^r \\
& \times\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)d_qt \\
& \leq MA\frac{[p]}{[n+1]} \\
& +\sum_{r=0}^{n+p}\int_0^1\left(\frac{t}{[n+1]}b_n+\frac{q\left[r\right]}{[n+1]}b_n-x\right)\left(f^{'}\left(\xi\right)-f^{'}\left(x\right)\right)\begin{bmatrix}n+p\\r\end{bmatrix}\left(\frac{x}{b_n}\right)^r \\
& \times\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)d_qt
\end{aligned}
$$

$$\leq MA\frac{[p]}{[n+1]}$$

$$+\sum_{r=0}^{n+p}\omega\left(f'\right)\int_{0}^{1}\left(\frac{\left|\frac{t}{[n+1]}b_n+\frac{[r]}{[n+1]}b_n-x\right|}{\delta}+1\right)\left(\frac{t}{[n+1]}b_n+\frac{[r]}{[n+1]}b_n-x\right)$$

$$\times\begin{bmatrix}n+p\\ r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)d_qt$$

$$\leq MA\frac{[p]}{[n+1]}$$

$$+\sum_{r=0}^{n+p}\omega\left(f'\right)\int_{0}^{1}\left(\frac{\left|\frac{t}{[n+1]}b_n+\frac{q\,[r]}{[n+1]}b_n-x\right|}{\delta}+1\right)\left(\frac{t}{[n+1]}b_n+\frac{q\,[r]}{[n+1]}b_n-x\right)$$

$$\times\begin{bmatrix}n+p\\ r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)d_qt$$

$$\leq MA\frac{[p]}{[n+1]}$$

$$+\sum_{r=0}^{n+p}\omega\left(f'\right)\int_{0}^{1}\left(\frac{\left|\frac{t}{[n+1]}b_n+\frac{q\,[r]}{[n+1]}b_n-x\right|}{\delta}+1\right)\left(\frac{t}{[n+1]}b_n+\frac{q\,[r]}{[n+1]}b_n-x\right)$$

$$\times\begin{bmatrix}n+p\\ r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)d_qt.$$

Since

$$|\xi-x|\leq\left|\frac{t}{[n+1]}b_n+\frac{q\,[r]}{[n+1]}b_n-x\right|$$

Therefore, we can write the following inequality

$$|T_n^p\left(f;q;x\right)-f(x)|$$

$$\leq MA\frac{[p]}{[n+1]}$$

$$+\sum_{r=0}^{n+p}\omega\left(f^{'},\delta\right)\int_{0}^{1}\left(\frac{\left|\frac{t}{[n+1]}b_n+\frac{q\,[r]}{[n+1]}b_n-x\right|}{\delta}+1\right)\left|\frac{t}{[n+1]}b_n+\frac{q\,[r]}{[n+1]}b_n-x\right|$$

$$\times\begin{bmatrix}n+p\\ r\end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)d_qt.$$

Using the Cauchy-Schwarz inequality for the first term we get

$$
\begin{aligned}
&\left|T_n^p(f;q;x)-f(x)\right| \\
&\leq MA\frac{[p]}{[n+1]} \\
&+\omega\left(f',\delta\right)\sum_{r=0}^{n+p}\int_0^1\left|\frac{t}{[n+1]}b_n+\frac{q[r]}{[n+1]}b_n-x\right| \\
&\times\begin{bmatrix} n+p \\ r \end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)d_qt \\
&+\frac{\omega\left(f',\delta\right)}{\delta}\sum_{r=0}^{n+p}\int_0^1\left(\frac{t}{[n+1]}b_n+\frac{q[r]}{[n+1]}b_n-x\right)^2 \\
&\times\begin{bmatrix} n+p \\ r \end{bmatrix}\left(\frac{x}{b_n}\right)^r\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)d_qt \\
&\leq MA\frac{[p]}{[n+1]} \\
&+\omega\left(f',\delta\right)\left\{\sum_{r=0}^{n}\int_0^1\left(\frac{t}{[n+1]}b_n+\frac{q[r]}{[n+1]}b_n-x\right)^2\begin{bmatrix} n+p \\ r \end{bmatrix}\left(\frac{x}{b_n}\right)^r\right. \\
&\left.\times\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)d_qt\right\}^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\omega\left(f',\delta\right)}{\delta}\sum_{r=0}^{n+1}\int_0^1\left(\frac{t}{[n+1]}b_n+\frac{q[r]}{[n+1]}b_n-x\right)^2\begin{bmatrix} n+p \\ r \end{bmatrix}\left(\frac{x}{b_n}\right)^r \\
&\times\prod_{s=0}^{n+p-r-1}\left(1-q^s\frac{x}{b_n}\right)d_qt \\
&\leq MA\frac{[p]}{[n+1]}+\omega\left(f',\delta\right)\sqrt{T_n^p\left((u-x)^2;q;x\right)}+\frac{\omega\left(f',\delta\right)}{\delta}T_n^p\left((u-x)^2;q;x\right)
\end{aligned}
$$

On the other hand Using the (6.1.3) and for the second term we have

$$\begin{aligned}
\sup_{0\leq x\leq A} T_n^p\left((u-x)^2;q;x\right) &\leq \sup_{0\leq x\leq A}\frac{x^2}{[n+1]^2}[p]^2 \\
&+ x\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}q+\frac{(b_n-x)}{[n+1]^2}q^2\right) \\
&+\frac{b_n^2}{[3]\,[n+1]^2} \\
&\leq \frac{A^2}{[n+1]^2}[p]^2 \\
&+ A\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}q+\frac{b_n}{[n+1]^2}q^2\right) \\
&+\frac{b_n^2}{[3]\,[n+1]^2}.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\left|T_n^p(f;q;x)-f(x)\right| \\
&\leq MA\frac{[p]}{[n+1]} \\
&+\omega\left(f',\delta\right)\left\{\frac{1}{[n+1]^2}[p]^2+\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}q+\frac{(b_n-1)}{[n+1]^2}q^2\right)\right. \\
&\left.+\frac{b_n^2}{[3]\,[n+1]^2}\right\}^{1/2} \\
&+\frac{1}{\delta}\left\{\frac{1}{[n+1]^2}[p]^2+\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}q+\frac{(b_n-1)}{[n+1]^2}q^2\right)\right. \\
&\left.+\frac{b_n^2}{[3]\,[n+1]^2}\right\}.
\end{aligned}$$

Using $\delta=\left\{\frac{1}{[n+1]^2}[p]^2+\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}q+\frac{(b_n-1)}{[n+1]^2}q^2\right)\right.$

$$+\frac{b_n^2}{[3]\,[n+1]^2}\Bigg\}^{1/2}$$

$$\begin{aligned}
&\left|T_n^p(f;q;x)-f(x)\right| \\
&\leq MA\frac{[p]}{[n+1]} \\
&+\omega\left(f',\left\{\frac{1}{[n+1]^2}[p]^2+\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}q+\frac{(b_n-1)}{[n+1]^2}q^2\right)\right.\right. \\
&\left.\left.+\frac{b_n^2}{[3]\,[n+1]^2}\right\}^{1/2}\right) \\
&\times\left[\left\{\frac{1}{[n+1]^2}[p]^2+\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}q+\frac{(b_n-1)}{[n+1]^2}q^2\right)+\frac{b_n^2}{[3]\,[n+1]^2}\right\}^{1/2}\right. \\
&+\left\{\frac{1}{[n+1]^2}[p]^2+\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}q+\frac{(b_n-1)}{[n+1]^2}q^2\right)\right. \\
&\left.\left.+\frac{b_n^2}{[3]\,[n+1]^2}\right\}^{1/2}\right] \\
&=MA\frac{[p]}{[n+1]} \\
&+2\left\{\frac{1}{[n+1]^2}[p]^2+\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}q+\frac{(b_n-1)}{[n+1]^2}q^2\right)\right. \\
&\left.+\frac{b_n^2}{[3]\,[n+1]^2}\right\}^{1/2} \\
&\times\omega\left(f',\left\{\frac{1}{[n+1]^2}[p]^2+\left(2\frac{[n+p]\,b_n}{[2]\,[n+1]^2}q+\frac{(b_n-1)}{[n+1]^2}q^2\right)\right.\right. \\
&\left.\left.+\ \frac{b_n^2}{[3]\,[n+1]^2}\right\}^{1/2}\right).
\end{aligned}$$

Whence the result. □

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