Stability of Systems of Differential Equations and Biological Applications

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ABSTRACT

In this thesis, we deal with systems of ordinary differential equations and discuss the stability properties of their solutions. We classify equilibrium points of linear systems with respect to their type and stability and discuss the methods for investigating the stability properties of nonlinear systems. Existence of periodic solutions which plays an important role in stability theory is also discussed. In addition, some important ecological applications, such as Lotka-Volterra predator-prey model, competition model and nutrient-prey-predator model with intratrophic predation, modeled by the systems of differential equations are also considered. Recent results obtained for these applications are also included.

Keywords: Stability, Periodic solution, Predator-prey model, Intratrophic predation.

ÖZ

Bu tezde, birinci dereceden denklem sistemleri ve sistemlerin çözümlerinin kararlılığı üzerinde çalıştık. Lineer sistemlerin kritik noktalarını türlerine ve kararlılıklarına göre sınıflandırdık, lineer olmayan sistemlerin kararlılık özelliklerini inceleyen metodları ele aldık. Çözümlerin kararlılık analizinde önemli rol oynayan periyodik çözümlerin varlığı üzerinde çalıştık. Bunlara ek olarak, diferansiyel denklemlerle ifade edilebilen bazı önemli ekolojik uygulamaları inceledik. Örneğin; Lotka-Volterra av-avcı ilişki modeli, türler arası rekabet modeli ve intratropik avlanma etkisindeki besin-av-avcı modeli. Bu uygulamalarla ilgili elde edilen yeni sonuçlara da yer verdik.

Anathar Kelimeler: Kararlılık, Periodik çözüm, Av-avcı ilişkisi, Intratropik avlanma.

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INTRODUCTION

Theory of differential equations has been of great interest for many years. It plays an important role in different subjects such as physics, biology, chemistry, etc. It is usually difficult to find the exact solution of a given system of differential equations. Any information about the qualitive properties of solutions of the system is essential. Consequently, stability is very important for understanding the nature of solutions of the system.

In Chapter 2, for a general system of differential equations, we introduce some definitions and theorems for the stability of the equilibrium points of the system. An alternative method for studying stability, called Liapunov method, is explained in Section 4 of this chapter. In the theory of differential equations, existence of periodic solutions of the system plays an important role. In our survey, it is discussed in the last section of Chapter 2. Several examples are included to support the theory.

In Chapter 3, the theory of factorable planar systems and the nature of their equilibrium points are discussed with a number of illustrative examples.

Chapter 4 is concerned with the applications of the stability theory. It deals with biological systems. Lotka-Volterra predator-prey model and competition model are examined in detail. The last section of this chapter studies the effect of harvesting on the system if both species are harvested.

In Chapter 5, recent results obtained for the ratio-dependent predator-prey systems are collected. In this chapter, we worked on equilibrium points of these systems and the conditions needed for the stability of these equilibrium points.

Finally, in the last chapter, Chapter 6, we discuss the equilibrium points of special model with three trophic levels, a nutrient-prey-predator model with intratrophic predation. We analyze the effect of intratrophic predation on the system.

Chapter 2

STABILITY OF DIFFERENTIAL EQUATIONS

First of all, we want to introduce a general system of differential equations and give some important definitions for stability. We consider the system of differential equations

$$\dot{\overrightarrow{x}} = \overrightarrow{X}(\overrightarrow{x}, t),$$

where

$$\overrightarrow{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \text{and} \qquad \overrightarrow{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}.$$

Definition 2.0.1 (Solution of a System)**.** *The vector*

$$\overrightarrow{x} = [x_1(t), ..., x_n(t)]^T$$

which satisfies the equations of the system

$$\dot{\overrightarrow{x}} = \overrightarrow{X}(\overrightarrow{x}, t)$$

is called a solution of the system. For a given initial value t_0,

$$\overrightarrow{x}(t_0) = \overrightarrow{x_0}$$

is called an initial solution of the system.

Theorem 2.0.1 (Existence and Uniqueness)**.** *Consider the system*

$$\dot{\overrightarrow{x}} = \overrightarrow{X}(\overrightarrow{x}, t),$$

with the initial condition

$$\overrightarrow{x}(t_0) = \overrightarrow{x_0}.$$

If the functions X_i *and* $\frac{\partial X_i}{\partial x_j}$ $(i, j = 1, ..., n)$ *are continuous over a domain* R *of* $(n+1)$*-dimensional* tx*-space and* (t_0, x_0) *is a point inside* R*, then the initial value problem has a unique solution* $\overrightarrow{x} = \overrightarrow{x}(\overrightarrow{x_0}, t, t_0)$ *in a* t*-interval* I *containing* t_0 *[1, Page 302, Theorem 6.2.1].*

Definition 2.0.2. *Consider the system*

$$\dot{\overrightarrow{x}} = \overrightarrow{X}(\overrightarrow{x}, t).$$

Suppose that $\overrightarrow{X}$ *is continuous and* $\frac{\partial X_j}{\partial x_i}, i, j = 1, 2, ..., n$ *are continuous for* $\overrightarrow{x} \in R$*, where* R *is a domain and* I *is an open interval. Then if* $\overrightarrow{x_0} \in R$ *and* $t_0 \in I$*, there exists a solution* $\overrightarrow{x}(t)$*, defined uniquely in some neighborhood of* $(\overrightarrow{x_0}, t_0)$*, which satisfies* $\overrightarrow{x}(t_0) = \overrightarrow{x_0}$*. These systems are called regular on* $R \times I$*. If a system is regular on* $-\infty < x_i < \infty$*,* $i = 1, 2, ..., n$*,* $-\infty < t < \infty$*, it is known as a regular system.*

The system is called autonomous if t, time variable, does not appear explicitly in the right-hand side. Thus, the general n-dimensional autonomous system can be written as

$$\dot{\overrightarrow{x}} = \overrightarrow{X}(\overrightarrow{x})$$

and so has the form

$$\begin{aligned}
\frac{dx_1}{dt} &= X_1\left(x_1(t), ..., x_n(t)\right), \\
\frac{dx_2}{dt} &= X_2\left(x_1(t), ..., x_n(t)\right), \\
&\vdots \\
\frac{dx_n}{dt} &= X_n(x_1(t), ..., x_n(t)),
\end{aligned}$$

where X_i are the functions of $x_1, ..., x_n$.

If the independent variable t is considered as time, the solution

$$\overrightarrow{x} = \overrightarrow{x}(t)$$

shows a phase path, or trajectory, in the phase plane $(x_1, ..., x_n)$ and the diagram of these phase paths is known as a phase diagram.

The solutions of

$$\overrightarrow{X}(\overrightarrow{x}) = \overrightarrow{0}$$

are called critical, singular, fixed or equilibrium points of the system.

Now consider the case $n = 2$. Two dimensional systems are known as planar systems. The system can be written as

$$\frac{dx}{dt} = P(x, y),$$

$$\frac{dy}{dt} = Q(x, y). \tag{2.0.1}$$

The intersection point $(\widetilde{x}, \widetilde{y})$ of the curves

$$P(x, y) = 0 \text{ and } Q(x, y) = 0$$

is the equilibrium point of the system.

For a given t_0, the parametric equations

$$x = x(t), \qquad y = y(t)$$

satisfying the initial conditions

$$x(t_0) = x_0, \qquad y(t_0) = y_0$$

show the solution curve of the system (2.0.1) in the xy-plane, which is the phase plane for the system.

The constant-valued functions

$$x(t) = \widetilde{x}, \qquad y(t) = \widetilde{y}$$

are also solutions to the system. Hence the critical point can be considered as constant-valued solution. This solution is known as an equilibrium solution, which is one single point $(\widetilde{x}, \widetilde{y})$.

There may also be periodic solutions to the system. A periodic solution is called a cycle. If $x = x(t), y = y(t)$ is the periodic solution, then

$$\begin{aligned} x(t+p) &= x(t), \\ y(t+p) &= y(t), \end{aligned}$$

where p is the period of the solution.

Now consider the general autonomous system in n-dimensions

$$\dot{\overrightarrow{x}} = \overrightarrow{X}(\overrightarrow{x}).$$

Let $\overrightarrow{x^*}(t)$ be the solution of this system. We will now introduce the stability of the phase path representing the solution $\overrightarrow{x^*}(t)$. In this case, we deal with the part of the phase path starting from a particular point $\overrightarrow{a^*}$. Thus we have a half-path H^* representing $\overrightarrow{x^*}(t)$ such that

$$\overrightarrow{x^*}(t_0) = \overrightarrow{a^*}.$$

Definition 2.0.3 (Poincaré Stability-Stability of Paths)**.** *Let H^* be the half-path for the solution $\overrightarrow{x^*}(t)$ of $\dot{\overrightarrow{x}} = \overrightarrow{X}(\overrightarrow{x})$ starting at $\overrightarrow{a^*}$. Suppose that H is the half-path which starts at $\overrightarrow{a}$. If for every $\varepsilon > 0$, there exists δ depending on ε such that*

$$|\overrightarrow{a} - \overrightarrow{a^*}| < \delta(\varepsilon) \qquad \textit{implies that} \qquad \max_{\overrightarrow{x} \in H} dist(\overrightarrow{x}, H^*) < \varepsilon,$$

then H^* is called Poincaré stable (or orbitally stable). In other words, all paths starting with a point sufficiently close to $\overrightarrow{a^*}$ remain close to the half-path H^*, i.e., small distur-

bances of the initial value lead to small changes in the half-path. Otherwise, H^* is said to be unstable.

Example 2.0.1. *Show that all the paths of*

$$\dot{x} = x,$$

$$\dot{y} = y \tag{2.0.2}$$

are Poincaré unstable.

Solution 2.0.2. *In the matrix form, the system reads as*

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

and $x(t) = Ae^t, \quad y(t) = Be^t$ are the solutions where A and B are constants. Then

$$\begin{aligned} \frac{dx}{dt} &= x, \\ \frac{dy}{dt} &= y, \end{aligned}$$

and

$$\frac{dx}{dy} = \frac{x}{y},$$

$$\int \frac{dx}{x} = \int \frac{dy}{y},$$

$$\ln |x| + \ln |c| = \ln |y| \,,$$

$$xc = y,$$

where c is any constant. The paths are given by the family of straight lines

$$y = cx,$$

where c is any constant. Consider a half-path H^* starting at $\overrightarrow{a^*} = (x_0, 0)$, $x_0 > 0$. Take $\varepsilon > 0$, the tolerance region of H^* is sketched in Figure 2.1

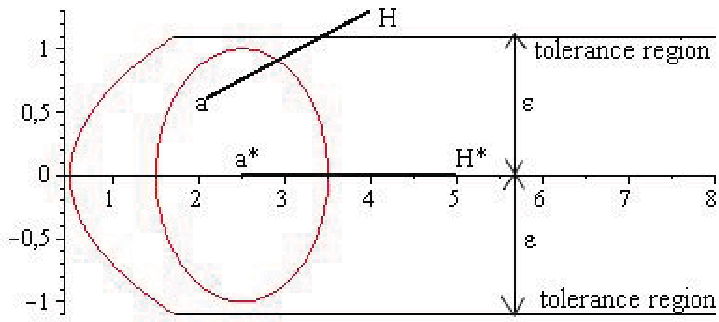

Figure 2.1: The tolerance region for the half-path H^* of the system (2.0.2).

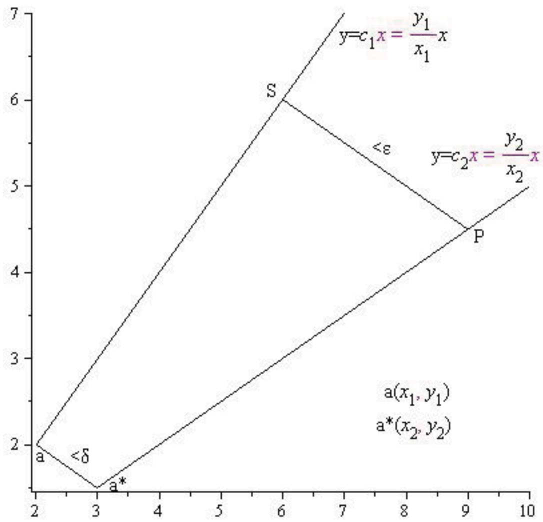

Figure 2.2: The paths of the system (2.0.2).

If

$$dist(\overrightarrow{a}, \overrightarrow{a^*}) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} < \delta,$$

will we have

$$dist(S, P) < \varepsilon?$$

$$\begin{aligned} dist(S, P) &= \sqrt{(x_1 - x_2)^2 + \left(\frac{y_1}{x_1}x - \frac{y_2}{x_2}x\right)^2} \\ &= \sqrt{(x_1 - x_2)^2 + \left(\frac{y_1}{x_1} - \frac{y_2}{x_2}\right)^2 x^2} > \varepsilon \end{aligned}$$

as $x \to \infty$, so the path is unstable, see Figure 2.2.

Definition 2.0.4 (Liapunov Stability-Stability of Equilibrium Points)**.** *Let $\overrightarrow{x^*}(t)$ be the* *solution of the system*

$$\dot{\overrightarrow{x}} = \overrightarrow{X}(\overrightarrow{x}, t).$$

If, for every $\varepsilon > 0$*, there exists* $\delta(\varepsilon, t_0) > 0$ *such that*

$$\left\|\overrightarrow{x}(t_0) - \overrightarrow{x^*}(t_0)\right\| < \delta \qquad \textit{implies that} \qquad \left\|\overrightarrow{x}(t) - \overrightarrow{x^*}(t)\right\| < \varepsilon, \forall\, t > t_0,$$

where $\overrightarrow{x}$ *is any other solution of the system, then the solution is called Liapunov stable for* $t \geq t_0$*. If the system is autonomous then we simply say that it is Liapunov stable for all* t_0*. In other words, when the initial point is sufficiently close to the critical point, the solution curves (trajectories) also remain close to the critical point. Otherwise it is called Liapunov unstable.*

Definition 2.0.5 (Uniform Stability)**.** *If the solution is stable for* $t > t_0$ *and the* δ *is independent of* t_0*, then it is uniformly stable.*

Definition 2.0.6 (Asymptotic Stability)**.** *If the solution is stable for* $t > t_0$ *and the trajectories approach the critical point as* $t \to \infty$*, then it is called asymptotically stable, i.e.,* $\exists$ $\delta(t_0) > 0$ *such that*

$$\left\|\overrightarrow{x}(t_0) - \overrightarrow{x^*}(t_0)\right\| < \delta \qquad \textit{implies that} \qquad \lim_{t\to\infty} \overrightarrow{x}(t) = \overrightarrow{x^*}(t),$$

where $\overrightarrow{x_0} = (x_0, y_0)$*,* $\overrightarrow{x^*} = (\widetilde{x}, \widetilde{y})$ *and* $\overrightarrow{x}(t) = (x(t), y(t))$*.*

2.1 Types of Equilibrium Points

Definition 2.1.1. *Let* C *be a path of the system (2.0.1) and let* $x = x(t)$*,* $y = y(t)$ *be a solution of (2.0.1) which represents* C *parametrically. Let* $(\widetilde{x}, \widetilde{y})$ *be a critical point of (2.0.1). We shall say that the path* C *approaches the critical point* $(\widetilde{x}, \widetilde{y})$ *as* $t \to +\infty$ *if*

$$\lim_{t\to+\infty} x(t) = \widetilde{x}, \qquad \lim_{t\to+\infty} y(t) = \widetilde{y}.$$

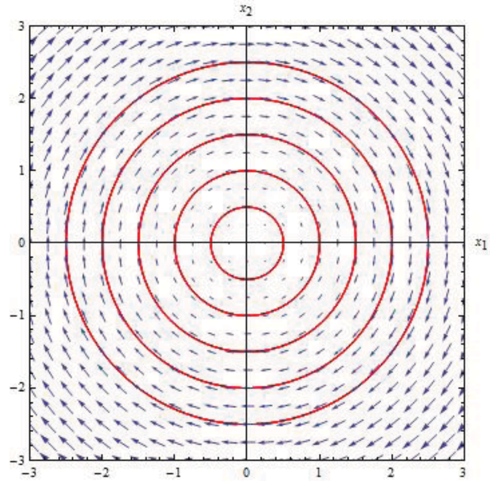

Figure 2.3: A center.

Definition 2.1.2. *Let C be a path of the system (2.0.1) which approaches the critical point $(\widetilde{x}, \widetilde{y})$ of (2.0.1) as $t \to +\infty$, and let $x = x(t)$, $y = y(t)$ be a solution of (2.0.1) which represents C parametrically. We say that C enters the critical point $(\widetilde{x}, \widetilde{y})$ as $t \to +\infty$ if*

$$\lim_{t \to +\infty} \frac{y(t)}{x(t)} \tag{2.1.1}$$

exists or if the quotient in (2.1.1) becomes either positively or negatively infinite as $t \to +\infty$.

Definition 2.1.3 (Isolated Critical Point)**.** *A critical point is called isolated if there exists no other critical point in any neighborhood of it.*

Definition 2.1.4 (Center)**.** *The isolated equilibrium point (a, b) is called a center if there exists a neighborhood of (a, b) which contains a countably infinite number of closed paths each of which contains (a, b) in its interior and which are such that the diameters of the paths approach 0 as $n \to \infty$. But (a, b) is not approached by any path either as $t \to \infty$ or as $t \to -\infty$.*

Figure 2.3 shows an example of a center at $(0, 0)$.

Definition 2.1.5 (Saddle Point)**.** *The isolated critical point (a, b) is called a saddle point if there exists a neighborhood of (a, b) in which the following two conditions hold:*

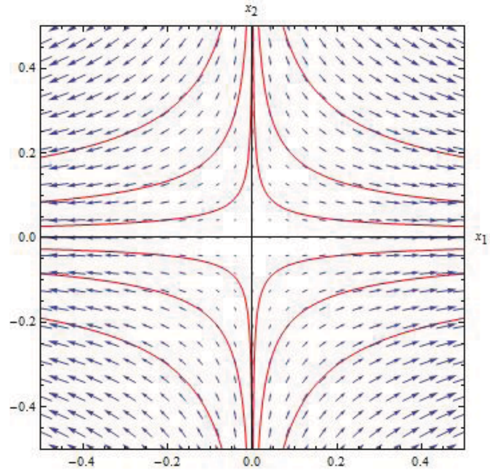

Figure 2.4: A saddle point.

1. *There exist two paths which approach and enter (a, b) from a pair of opposite directions as $t \to \infty$ and there exist two paths which approach and enter (a, b) from a different pair of opposite directions as $t \to -\infty$.*

2. *In each of the four domains between any two of the four directions in (1), there are infinitely many paths which are arbitrarily close to (a, b) but do not approach (a, b) either as $t \to \infty$ or as $t \to -\infty$.*

Figure 2.4 shows a saddle point at $(0, 0)$.

Definition 2.1.6 (Spiral)**.** *The isolated critical point (a, b) is called a spiral point (or focus) if there exists a neighborhood of (a, b) such that every path P in this neighborhood has the following properties:*

1. *P is defined for all $t > t_0$ (or for all $t < t_0$) for some number t_0;*

2. *P approaches (a, b) as $t \to \infty$ (or as $t \to -\infty$); and*

3. *P approaches (a, b) in a spiral-like manner, winding around (a, b) an infinite number of times as $t \to \infty$ (or as $t \to -\infty$).*

An example of a spiral point at the point $(0, 0)$ is demonstrated in Figure 2.5.

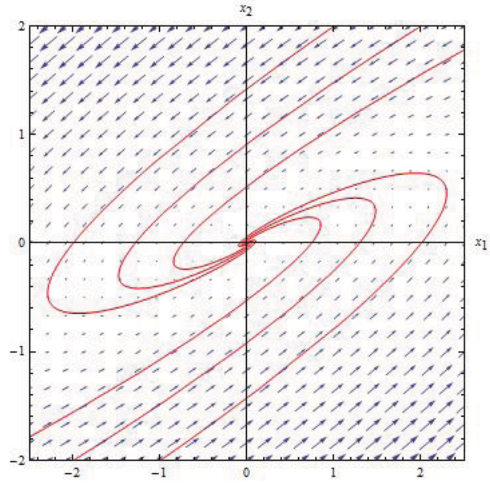

Figure 2.5: A spiral.

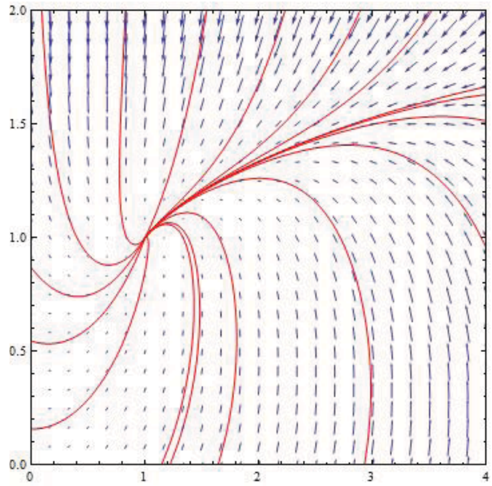

Figure 2.6: A node.

Definition 2.1.7 (Node)**.** *The isolated critical point (a, b) is called a node if there exists a neighborhood of (a, b) such that every path P in this neighborhood has the following properties:*

1. *P is defined for all $t > t_0$ (or for all $t < t_0$) for some number t_0;*
2. *P approaches (a, b) as $t \to \infty$ (or as $t \to -\infty$); and*
3. *P enters (a, b) as $t \to \infty$ (or as $t \to -\infty$).*

Figure 2.6 is an example of a node at the point $(1, 1)$.

Example 2.1.1. *Solve the system of equations*

$$\begin{aligned} \dot{x} &= -y(x^2+y^2), \\ \dot{y} &= x(x^2+y^2). \end{aligned} \tag{2.1.2}$$

Show that the zero solution is Liapunov stable and that all other solutions are stable.

Solution 2.1.1. *Using polar coordinates,*

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \end{aligned}$$

we have

$$x^2+y^2=r^2.$$

Taking derivative of both sides gives

$$\begin{aligned} 2x\frac{dx}{dt}+2y\frac{dy}{dt} &= 2r\frac{dr}{dt}, \\ x\dot{x}+y\dot{y} &= r\frac{dr}{dt}, \end{aligned}$$

where

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial x}{\partial r}\frac{dr}{dt}+\frac{\partial x}{\partial \theta}\frac{d\theta}{dt}=\cos\theta\frac{dr}{dt}+(-r\sin\theta)\frac{d\theta}{dt}, \\ \frac{dy}{dt} &= \frac{\partial y}{\partial r}\frac{dr}{dt}+\frac{\partial y}{\partial \theta}\frac{d\theta}{dt}=\sin\theta\frac{dr}{dt}+(-\cos\theta)\frac{d\theta}{dt}. \end{aligned}$$

Using these, we obtain

$$\begin{aligned} x\dot{x}+y\dot{y} &= x[-y(x^2+y^2)]+y[x(x^2+y^2)] \Rightarrow \\ r\frac{dr}{dt} &= -xy(x^2+y^2)+xy(x^2+y^2) \Rightarrow \\ r\frac{dr}{dt} &= 0 \Rightarrow \\ \frac{dr}{dt} &= 0 \Rightarrow \\ r &= c, \end{aligned}$$

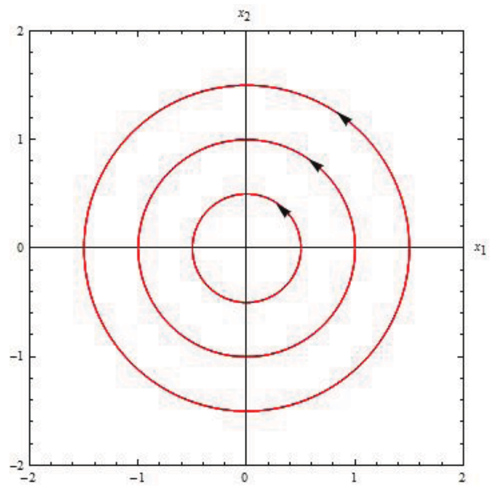

Figure 2.7: The trajectories of the system (2.1.2).

where c is any constant. Similary,

$$
\begin{aligned}
y\dot{x} - x\dot{y} &= y[-y(x^2+y^2)] - x[x(x^2+y^2)] \Rightarrow \\
-r^2\frac{d\theta}{dt} &= -y^2(x^2+y^2) - x^2(x^2+y^2) \Rightarrow \\
-r^2\frac{d\theta}{dt} &= -(x^2+y^2)^2 \Rightarrow \\
-r^2\frac{d\theta}{dt} &= -r^2 \Rightarrow \\
\frac{d\theta}{dt} &= 1,
\end{aligned}
$$

and the direction of motion along the trajectories is anti-clockwise. Therefore, the origin is a center, stable, see Figure 2.7.

2.2 Classification of Equilibrium Points in Two-Dimensional Space

Consider a two-dimensional linear autonomous system with constant coefficients

$$\frac{dx}{dt} = ax + by,$$

$$\frac{dy}{dt} = cx + dy. \tag{2.2.1}$$

The coefficient matrix is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The nature of the only critical point $(0, 0)$ is determined by the roots of the characteristic equation

$$\det(A - \lambda I) = 0, \tag{2.2.2}$$

that is,

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad - bc) \\ &= 0. \end{aligned}$$

Let $p = a + d$ and $q = ad - bc$, so we have

$$\lambda^2 - p\lambda + q = 0.$$

We assume that the critical point $(0, 0)$ of the system (2.2.1) is an isolated critical point, i.e., $ad - bc \neq 0$. Otherwise, equations

$$ax + by = 0$$

and

$$cx + dy = 0$$

define the same line and all points on the line are critical points, so $(0, 0)$ is not isolated.

Hence, we do not investigate the case where $ad - bc = 0$, so $\lambda = 0$ is not a root of the characteristic equation (2.2.2).

The roots of the characteristic equation are

$$\lambda_{1,2} = \frac{p \pm \sqrt{\triangle}}{2},$$

where $\triangle = p^2 - 4q$.

Now we will investigate the following cases for the roots:

Case I: (real unequal roots of the same sign)

a. If $\lambda_1 \neq \lambda_2 \in \mathbb{R}$ and $\lambda_1 > 0$, $\lambda_2 > 0$, the solution

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},$$

$$\tag{2.2.3}$$

$$y(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t},$$

where c_1, c_2, k_1, k_2 are arbitrary coefficients, is not bounded as $t \to \infty$. This kind of phase diagram is called a node. Since the phase paths are tending outwards from the origin, the critical point $(0, 0)$ is an unstable node. We can formulate the conditions for an unstable node as

$$\triangle > 0, \quad q > 0, \quad p > 0.$$

b. If $\lambda_1 \neq \lambda_2 \in \mathbb{R}$ and $\lambda_1 < 0$, $\lambda_2 < 0$, the solution x, y in (2.2.3) tends to zero as $t \to \infty$, hence the critical point $(0, 0)$ is a stable node which corresponds to the conditions

$$\triangle > 0, \quad q > 0, \quad p < 0.$$

It is also asymptotically stable.

Case II: (real unequal roots of the opposite sign)

If $\lambda_1 \neq \lambda_2 \in \mathbb{R}$ and $\lambda_1 < 0$, $\lambda_2 > 0$, some of the phase paths aproach the origin while the others go away from the origin, so the solution $(0, 0)$ is unstable and it is known as a saddle point. The conditions for coefficients are

$$\triangle > 0, \quad q < 0.$$

Case III: (real equal roots)

In this case, $\triangle = 0$.

a. If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$ and $\lambda > 0$, the general solution

$$x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t},$$

(2.2.4)

$$y(t) = k_1 e^{\lambda t} + k_2 t e^{\lambda t},$$

where c_1, c_2, k_1, k_2 are arbitrary coefficients, is unbounded as $t \to \infty$. Hence, $(0,0)$ is an unstable node.

b. If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$ and $\lambda < 0$, from (2.2.4), $x \to 0$ and $y \to 0$ as $t \to \infty$, so $(0,0)$ is a stable node, in fact, asymptotically stable.

Case IV: (complex roots)

When $\triangle < 0$, the characteristic equation (2.2.2) has complex conjugate roots,

$$\lambda_1 = \alpha + i\beta, \qquad \lambda_2 = \alpha - i\beta,$$

where α and β are non-zero constants. The general solution is

$$x(t) = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t),$$

(2.2.5)

$$y(t) = e^{\alpha t}(k_1 \cos \beta t + k_2 \sin \beta t),$$

where c_1, c_2, k_1, k_2 are constants.

a. If $\alpha > 0$, the solution (2.2.5) is unbounded as $t \to \infty$ and the phase paths are spirals around the origin. So $(0,0)$ is an unstable spiral (called focus).

b. If $\alpha < 0$, $x(t)$ and $y(t)$ in (2.2.5) approach the critical point $(0,0)$. Hence $(0,0)$ is a stable spiral, focus. In fact, it is asymptotically stable.

Case V: (pure imaginary roots)

In this case, $\triangle < 0$ and $p = 0$. The roots of (2.2.2) are in the form

$$\lambda_1 = i\beta, \qquad \lambda_2 = -i\beta,$$

where β is non-zero. The general solution is

$$x(t) = c_1 \cos \beta t + c_2 \sin \beta t,$$

$$y(t) = k_1 \cos \beta t + k_2 \sin \beta t.$$

The critical point $(0, 0)$ is a center, i.e., stable but not asymptotically stable since the trajectories are ellipses around $(0, 0)$.

The general homogenous linear system in $n-$dimensions is

$$\dot{\overrightarrow{x}} = A(t)\overrightarrow{x}, \tag{2.2.6}$$

where $A(t)$ is an $n \times n$ matrix with entries $a_{ij}(t)$, which are continuous functions of time. It can also be written as

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(t)x_j, \qquad i = 1, 2, ..., n.$$

Let $\overrightarrow{\phi_1}(t), \overrightarrow{\phi_2}(t), ..., \overrightarrow{\phi_n}(t)$ be linearly independent solutions of the system. Then the matrix

$$\Phi(t) = \left[\overrightarrow{\phi_1}(t), \overrightarrow{\phi_2}(t), ..., \overrightarrow{\phi_n}(t)\right]$$

is called a fundamental matrix of the homogenous system (2.2.6). Every solution can be written as a linear combination of these solution vectors.

Example 2.2.1. *Construct a fundamental matrix for the system*

$$\dot{x}_1 = -x_1,$$

$$\dot{x}_2 = x_1 + x_2 + x_3,$$

$$\dot{x}_3 = -x_2.$$

Solution 2.2.1. *Write the system in the matrix notation*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} | \quad A - \lambda I \mid &= \begin{vmatrix} -1-\lambda & 0 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & -1 & -\lambda \end{vmatrix} \\ &= (-1-\lambda)(-1)^{1+1} \begin{vmatrix} 1-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} \\ &= (-1-\lambda)[-\lambda(1-\lambda)+1] = 0, \end{aligned}$$

$$(-1-\lambda)(\lambda^2 - \lambda + 1) = 0,$$

$$\lambda_1 = -1, \quad \lambda_{2,3} = \frac{1 \pm \sqrt{3}i}{2}.$$

For $\lambda = -1$,

$$(A+I)\overrightarrow{u} = \overrightarrow{0},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or

$$\begin{aligned} u_1 + 2u_2 + u_3 &= 0, \\ -u_2 + u_3 &= 0. \end{aligned}$$

Using these equations, we find the solution of $(A+I)\overrightarrow{u} = \overrightarrow{0}$,

$$\overrightarrow{u} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} k, \quad k \in \mathbb{R},$$

and

$$\overrightarrow{y_1}(t) = e^{\lambda t}\overrightarrow{u} = e^{-t}\overrightarrow{u} = \begin{bmatrix} -3e^{-t} \\ e^{-t} \\ e^{-t} \end{bmatrix}.$$

For $\lambda = \frac{1}{2} + \frac{\sqrt{3}}{2}i$,

$$\left(A - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i \right) I \right) \overrightarrow{v} = \overrightarrow{0},$$

in the matrix notation

$$\begin{bmatrix} -\frac{3}{2}-\frac{\sqrt{3}}{2}i & 0 & 0 \\ 1 & \frac{1}{2}-\frac{\sqrt{3}}{2}i & 1 \\ 0 & -1 & -\frac{1}{2}-\frac{\sqrt{3}}{2}i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and componentwise

$$\begin{aligned} \left(-\frac{3}{2}-\frac{\sqrt{3}}{2}i\right) v_1 &= 0, \\ v_1 + \left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) v_2 + v_3 &= 0, \\ -v_2 + \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) v_3 &= 0. \end{aligned}$$

The solution of $\left(A - \left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) I\right) \overrightarrow{v} = \overrightarrow{0}$ *is*

$$\overrightarrow{v} = \begin{bmatrix} 0 \\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i \\ 1 \end{bmatrix} l, \quad l \in \mathbb{R},$$

or

$$\overrightarrow{v} = \underbrace{\begin{bmatrix} 0 \\ \frac{-1}{2} \\ 1 \end{bmatrix}}_{\overrightarrow{a}} + i \underbrace{\begin{bmatrix} 0 \\ \frac{-\sqrt{3}}{2} \\ 0 \end{bmatrix}}_{\overrightarrow{b}}.$$

$$
\begin{aligned}
\overrightarrow{y_2}(t) &= e^{\lambda t}\overrightarrow{v} \\
&= e^{(\frac{1}{2}+\frac{\sqrt{3}}{2}i)t}(\overrightarrow{a}+i\overrightarrow{b}) \\
&= e^{\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t + i\sin\frac{\sqrt{3}}{2}t\right)\left(\overrightarrow{a}+i\overrightarrow{b}\right) \\
&= e^{\frac{1}{2}t}\left[\left(\cos\frac{\sqrt{3}}{2}t\right)\overrightarrow{a} - \left(\sin\frac{\sqrt{3}}{2}t\right)\overrightarrow{b}\right] + \\
&\qquad i\left[\left(\sin\frac{\sqrt{3}}{2}t\right)\overrightarrow{a} + \left(\cos\frac{\sqrt{3}}{2}t\right)\overrightarrow{b}\right] \\
&= \left\{e^{\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t\begin{bmatrix}0\\ \frac{-1}{2}\\ 1\end{bmatrix} - e^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t\begin{bmatrix}0\\ \frac{-\sqrt{3}}{2}\\ 0\end{bmatrix}\right\} + \\
&\qquad i\left\{e^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t\begin{bmatrix}0\\ \frac{-1}{2}\\ 1\end{bmatrix} + e^{\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t\begin{bmatrix}0\\ \frac{-\sqrt{3}}{2}\\ 0\end{bmatrix}\right\} \\
&= \begin{bmatrix}0\\ e^{\frac{1}{2}t}\left(-\frac{1}{2}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t\right)\\ e^{\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t\end{bmatrix} + \\
&\qquad i\begin{bmatrix}0\\ e^{\frac{1}{2}t}\left(-\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{2}\sin\frac{\sqrt{3}}{2}t\right)\\ e^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t\end{bmatrix}.
\end{aligned}
$$

The fundamental matrix is

$$
\begin{bmatrix}
-3e^{-t} & 0 & 0 \\
e^{-t} & e^{\frac{1}{2}t}\left(-\frac{1}{2}\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t\right) & e^{\frac{1}{2}t}\left(-\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t - \frac{1}{2}\sin\frac{\sqrt{3}}{2}t\right) \\
e^{-t} & e^{\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t & e^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t
\end{bmatrix}.
$$

Finally, let us consider a general non-homogenous linear system

$$
\dot{\overrightarrow{x}} = A(t)\overrightarrow{x} + \overrightarrow{f}(t), \tag{2.2.7}
$$

where $\overrightarrow{f}(t)$ is a column vector. Suppose $\overrightarrow{x^*}(t)$ is a solution of the equation (2.2.7). To be able to investigate the stability of $\overrightarrow{x^*}(t)$, define

$$
\overrightarrow{\xi}(t) = \overrightarrow{x}(t) - \overrightarrow{x^*}(t),
$$

where $\overrightarrow{x}(t)$ is any other solution. Then we obtain the following homogenous equation

$$
\dot{\overrightarrow{\xi}} = A(t)\overrightarrow{\xi}. \tag{2.2.8}
$$

Theorem 2.2.2. *All solutions of the linear system (2.2.7) have the same stability properties with the zero solution of (2.2.8).*

Theorem 2.2.3. *The zero solution of the system (2.2.6) is stable iff every solution is bounded as $t \to \infty$. In fact, from Theorem 2.2.2, it is also true for all solutions of the system. If A is a constant matrix and every solution is bounded, then the solutions are uniformly stable.*

2.3 Stability of Homogenous Systems

Now let's investigate the stability of different types of homogeneous systems one by one.

2.3.1 Stability of Linear Systems with Constant Coefficients

Consider the system

$$\dot{\overrightarrow{x}} = A\overrightarrow{x},$$

where A is an $n \times n$ matrix with real elements. As in the two-dimensional case, the characteristic equation is

$$\det(A - \lambda I) = 0, \tag{2.3.1}$$

or

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn}-\lambda \end{vmatrix} = 0.$$

The roots λ_i of the characteristic equation (2.3.1) are the eigenvalues of A and the vectors $\overrightarrow{v_i}$ satisfying

$$(A - \lambda_i I)\overrightarrow{v_i} = \overrightarrow{0}$$

are the corresponding eigenvectors of λ_i.

If A has n distinct eigenvalues $\lambda_1, ..., \lambda_n$, then there exist n linearly independent eigenvectors $\overrightarrow{v_1}, ..., \overrightarrow{v_n}$ and the fundamental matrix is in the form

$$\Phi(t) = \left[\overrightarrow{v_1}e^{\lambda_1 t}, \overrightarrow{v_2}e^{\lambda_2 t}, ..., \overrightarrow{v_n}e^{\lambda_n t}\right].$$

Theorem 2.3.1. *Let $\dot{x} = Ax$ be an n-dimensional linear system with constant coefficients, i.e., A is an $n \times n$ real matrix. Suppose that $\lambda_i, i = 1, ..., n$ are the eigenvalues of A.*

i. *If either $Re\{\lambda_i\} < 0$, $i = 1, 2, ..., n$, or if $Re\{\lambda_i\} \leq 0$, $i = 1, 2, ..., n$, and there is no repeated zero eigenvalue, then all solutions of the system are uniformly stable.*

ii. *All solutions of the system are asymptotically stable iff $Re\{\lambda_i\} < 0$, $i = 1, 2, ..., n$.*

iii. *If all solutions of the system are stable, then $Re\{\lambda_i\} \leq 0$, $i = 1, 2, ..., n$.*

iv. *If $Re\{\lambda_i\} > 0$ for any i, then the solution is unstable.*

2.3.2 Stability of Linear Non-Autonomous Systems

The system considered is in the form

$$\dot{\overrightarrow{x}} = A(t)\overrightarrow{x}$$

and can be written as

$$\dot{\overrightarrow{x}} = \{B + C(t)\}\overrightarrow{x},$$

where B is an $n \times n$ constant matrix.

Theorem 2.3.2. *Assume that*

i. *B is an $n \times n$ matrix and the eigenvalues of B have negative real parts;*

ii. *$C(t)$ is continuous for $t \geq t_0$ and*

$$\int_{t_0}^{t} \|C(t)\| \, dt$$

is bounded for $t > t_0$.

Then all solutions of the system $\dot{\overrightarrow{x}} = \{B + C(t)\}\overrightarrow{x}$ are asymptotically stable.

Corollary 2.3.1. *If the solutions of $\dot{\overrightarrow{x}} = B\overrightarrow{x}$ are only bounded and $C(t)$ satisfies the conditions of Theorem 2.3.2, then all solutions of $\dot{\overrightarrow{x}} = \{B+C(t)\}\overrightarrow{x}$ are bounded, hence stable.*

2.3.3 Stability of Autonomous Non-Linear Systems

A general non-linear system has the form

$$\dot{\overrightarrow{x}} = \overrightarrow{X}(\overrightarrow{x}).$$

Linearization at fixed points is used to determine the stability. Suppose that $\overrightarrow{x} = \overrightarrow{x^*}$ is the equilibrium point of the system. Let $\overrightarrow{\xi}$, small, be the magnitude of the perturbation about the equilibrium point. As a result of perturbation, we have

$$\overrightarrow{x} = \overrightarrow{x^*} + \overrightarrow{\xi}.$$

Substituting this into our system gives

$$\dot{\overrightarrow{x}} = \dot{\overrightarrow{\xi}} = \overrightarrow{X}(\overrightarrow{x^*} + \overrightarrow{\xi}).$$

Taylor series expansion of $\overrightarrow{X}$ about the point $\overrightarrow{x^*}$ is

$$\begin{aligned}\dot{\overrightarrow{\xi}} &= \overrightarrow{X}(\overrightarrow{x^*} + \overrightarrow{\xi}) \\ &= \overrightarrow{X}(\overrightarrow{x^*}) + J\overrightarrow{\xi} + o(\left\|\overrightarrow{\xi}\right\|) \\ &= J\overrightarrow{\xi} + o(\left\|\overrightarrow{\xi}\right\|),\end{aligned}$$

where J is the Jacobian matrix of $\overrightarrow{X}$ evaluated at the critical point $\overrightarrow{x^*}$, i.e.,

$$J = \left[\frac{\partial X_i(\overrightarrow{x})}{\partial x_j}\right]_{\overrightarrow{x}=\overrightarrow{x^*}}.$$

As a result, we obtain a homogenous linear system

$$\dot{\overrightarrow{\xi}} = J\overrightarrow{\xi} \tag{2.3.2}$$

whose zero solution is the only equilibrium point.

In this case, Jacobian matrix at $\overrightarrow{x} = \overrightarrow{x^*}$ is a constant $n \times n$ matrix, so now we have a linear system with constant coefficients. Therefore, Theorem 2.3.1 can be used for the stability analysis of the zero solution of (2.3.2).

Example 2.3.1. *Consider the system*

$$\begin{aligned}\dot{x} &= -y,\\ \dot{y} &= x + \lambda(1 - y^2 - z^2)y,\\ \dot{z} &= -y + \mu(1 - x^2 - y^2)z.\end{aligned}$$

Classify the linear approximation of equilibrium point at the origin in terms of parameters λ *and* μ*. Verify that the system has a periodic solution*

$$\begin{aligned}x &= \cos(t - t_0),\\ y &= \sin(t - t_0),\\ z &= \cos(t - t_0),\end{aligned}$$

for any t_0.

Solution 2.3.3. *This is a non-linear system with a single equilibrium point* $(0, 0, 0)$*. We have*

$$\begin{aligned}J &= \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix}_{(0,0,0)}\\ &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & \lambda - 3\lambda y^2 - \lambda z^2 & -\lambda z^2 \\ -2\mu xz & -1 - 2\mu yz & \mu - \mu x^2 - \mu y^2 \end{bmatrix}_{(0,0,0)}\\ &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & \lambda & 0 \\ 0 & -1 & \mu \end{bmatrix}.\end{aligned}$$

The linear approximation at the origin is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & \lambda & 0 \\ 0 & -1 & \mu \end{bmatrix} \begin{bmatrix} x - 0 \\ y - 0 \\ z - 0 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & \lambda & 0 \\ 0 & -1 & \mu \end{bmatrix},$$

then

$$\mid A - \xi I \mid = \begin{vmatrix} -\xi & -1 & 0 \\ 1 & \lambda - \xi & 0 \\ 0 & -1 & \mu - \xi \end{vmatrix} = (\mu - \xi)(-1)^{3+3} \begin{vmatrix} -\xi & -1 \\ 1 & \lambda - \xi \end{vmatrix} = 0,$$

$$(\mu - \xi)(\xi^2 - \lambda\xi + 1) = 0,$$

$$\xi_1 = \mu, \quad \xi_{2,3} = \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}.$$

We have the following cases:

i. *If* $\mu < 0$ *and* $\lambda < 0$*;* $Re\{\xi_i\} < 0$*, for all* i*, the origin is uniformly stable.*

ii. *If both* $\mu > 0$*,* $\lambda > 0$*;* $Re\{\xi_i\} > 0$*, for all* i*, the origin is unstable.*

iii. *If either* $\mu > 0$ *or* $\lambda > 0$*;* $Re\{\xi_i\} > 0$*, for some* i*, the origin is unstable.*

iv. *If* $\mu = 0$ *and* $\lambda < 0$*;* $Re\{\xi_i\} \leq 0, i = 1, 2, 3,$ *the origin is uniformly stable.*

v. *If* $\lambda = 0$ *and* $\mu < 0$*; we have imaginary roots for the linearized system. Thus, the eigenvalues do not give us an idea about the stability of the zero solution.*

For the second part of the question, direct verification

$$\begin{aligned}
\dot{x} &= -\sin(t - t_0) = -y, \\
\dot{y} &= \cos(t - t_0) = \cos(t - t_0) + \lambda[1 - \sin^2(t - t_0) - \cos^2(t - t_0)]\sin(t - t_0) \\
&= x + \lambda(1 - y^2 - z^2)y, \\
\dot{z} &= -\sin(t - t_0) = -\sin(t - t_0) + \mu[1 - \cos^2(t - t_0) - \sin^2(t - t_0)]\cos(t - t_0) \\
&= -y + \mu(1 - x^2 - y^2)z,
\end{aligned}$$

shows that

$$x = \cos(t - t_0), \quad y = \sin(t - t_0), \quad z = \cos(t - t_0)$$

is a solution for the given system.

Furthermore,

$$
\begin{aligned}
x(t+T) &= \cos(t+T-t_0) = \cos(t-t_0) = x(t),\\
y(t+T) &= \sin(t+T-t_0) = \sin(t-t_0) = y(t),\\
z(t+T) &= \cos(t+T-t_0) = \cos(t-t_0) = z(t),
\end{aligned}
$$

where $T = 2k\pi$, $k = 1, 2, ...$ So the solution is periodic with a period T.

Example 2.3.2. *Test the stability of the linear system*

$$
\begin{aligned}
\dot{x}_1 &= t^{-2}x_1 + 4x_2 - 2x_3 + t^2,\\
\dot{x}_2 &= -x_1 + t^{-2}x_2 + x_3 + t,\\
\dot{x}_3 &= t^{-2}x_1 - 9x_2 - 4x_3 + 1.
\end{aligned}
$$

Solution 2.3.4. *Write the system in the matrix form as*

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 4 & -2 \\ -1 & 0 & 1 \\ 0 & -9 & -4 \end{bmatrix} + \begin{bmatrix} t^{-2} & 0 & 0 \\ 0 & t^{-2} & 0 \\ t^{-2} & 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}.
$$

Let

$$
B = \begin{bmatrix} 0 & 4 & -2 \\ -1 & 0 & 1 \\ 0 & -9 & -4 \end{bmatrix}, \quad C(t) = \begin{bmatrix} t^{-2} & 0 & 0 \\ 0 & t^{-2} & 0 \\ t^{-2} & 0 & 0 \end{bmatrix}, \quad \overrightarrow{f}(t) = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix},
$$

then

$$
\dot{\overrightarrow{x}} = \{B + C(t)\}\overrightarrow{x} + \overrightarrow{f}(t).
$$

Let

$$
A(t) = B + C(t).
$$

Then, by Theorem 2.2.2 $\dot{\overrightarrow{x}} = A(t)\overrightarrow{x} + \overrightarrow{f}(t)$ has the same stability properties as a homogeneous equation $\dot{\overrightarrow{x}} = A(t)\overrightarrow{x}$. We have

$$
| B - \lambda I | = \begin{bmatrix} -\lambda & 4 & -2 \\ -1 & -\lambda & 1 \\ 0 & -9 & -4-\lambda \end{bmatrix} = \lambda^3 + 4\lambda^2 + 13\lambda + 34 = 0. \qquad (2.3.3)
$$

Solving equation (2.3.3), we find eigenvalues as

$$\begin{aligned}\lambda_1 &= -3.232345867,\\ \lambda_2 &= -0.3838270651 - 3.220458527i,\\ \lambda_3 &= -0.3838270651 + 3.220458527i,\end{aligned}$$

so we conclude that all eigenvalues have negative real parts. On the other hand, $C(t)$ is continuous for $t > 0$ and

$$\begin{aligned}\int_{t_0}^{\infty} \parallel C(s) \parallel ds &= \lim_{t\to\infty} \int_{t_0}^{t} \left|2s^{-2}\right| ds\\ &= \lim_{t\to\infty} \left|-2s^{-1}\right| \Big|_{t_0}^{t}\\ &= \lim_{t\to\infty} \left|-2\left(\frac{1}{t} - \frac{1}{t_0}\right)\right|\\ &= \frac{2}{t_0} < \infty, \qquad t_0 > 0,\end{aligned}$$

therefore it is bounded, where

$$\parallel C(s) \parallel = \mid s^{-2} \mid + \mid s^{-2} \mid + 0 = 2s^{-2}.$$

According to Theorem 2.3.2, all solutions of $\dot{\overrightarrow{x}} = \{B + C(t)\}\overrightarrow{x}$ are asymptotically stable. Hence, the solutions of $\dot{\overrightarrow{x}} = \{B+C(t)\}\overrightarrow{x} + \overrightarrow{f}(t)$ are also asymptotically stable.

An n-th order differential equation can be converted to an n-dimensional system. Consider the following differential equation

$$x^{(n)} + a_1(t)x^{(n-1)} + ... + a_n(t)x = f(t).$$

The equivalent system is obtained by introducing new variables

$$\begin{aligned}x &= x_1,\\ \dot{x}_1 &= x_2,\\ \dot{x}_2 &= x_3,\end{aligned}$$

$$\vdots$$

$$\dot{x}_{n-1} = x_n,$$

so that

$$\dot{x}_n = -a_1(t)x_n - ... - a_n(t)x_1 + f(t).$$

Now, using this system, we can discuss the stability of the differential equation.

Example 2.3.3. *Determine the stability of the solutions of*

a.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^t;$$

b.

$$\ddot{x} + e^{-t}\dot{x} + x = e^t.$$

Solution 2.3.5. **a.** *A corresponding homogeneous system is*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix},$$

then

$$\mid A - \lambda I \mid = \begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (-2-\lambda)^2 - 1 = 0,$$

$$\lambda^2 + 4\lambda + 3 = 0,$$

$$\lambda_1 = -3 < 0, \quad \lambda_2 = -1 < 0.$$

Therefore, the origin is a stable node for the homogeneous system. Thus, all solutions of the non-homogeneous system are also stable.

b. *Let*

$$\begin{aligned} x &= x_1, \\ \dot{x} &= \frac{dx_1}{dt} = x_2, \end{aligned}$$

then

$$\ddot{x} = \frac{dx_2}{dt} = -e^{-t}x_2 - x_1 + e^t,$$

and

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -e^{-t} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^t \end{bmatrix}$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -e^{-t} \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ e^t \end{bmatrix}.$$

Let

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$C(t) = \begin{bmatrix} 0 & 0 \\ 0 & -e^{-t} \end{bmatrix}.$$

Consider the system

$$\dot{x} = Bx,$$

that is,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then

$$| B - \lambda I | = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

and

$$\lambda_{1,2} = \pm i.$$

For $\lambda = i$,

$$(B - iI)\overrightarrow{u} = \overrightarrow{0},$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$-iu_1 + u_2 = 0.$$

The solution of $(B - iI)\overrightarrow{u} = \overrightarrow{0}$ *is*

$$\overrightarrow{u} = \begin{bmatrix} 1 \\ i \end{bmatrix} c, \quad c \in \mathbb{R},$$

$$\overrightarrow{u} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\overrightarrow{a}} + i \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\overrightarrow{b}}.$$

Then

$$\begin{aligned}
\overrightarrow{y}(t) &= e^{\lambda t}\overrightarrow{u} \\
&= e^{it}\overrightarrow{u} \\
&= e^{it}(\overrightarrow{a} + i\overrightarrow{b}) \\
&= (\cos t + i \sin t)(\overrightarrow{a} + i\overrightarrow{b}) \\
&= \{(\cos t)\overrightarrow{a} - (\sin t)\overrightarrow{b}\} + i\{(\sin t)\overrightarrow{a} + (\cos t)\overrightarrow{b}\} \\
&= \left\{ \cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} + i \left\{ \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.
\end{aligned}$$

The fundamental matrix is

$$\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix},$$

and the solution is

$$\begin{aligned}
x_1(t) &= c_1 \cos t + c_2 \sin t, \\
x_2(t) &= -c_1 \sin t + c_2 \cos t,
\end{aligned}$$

where $c_1, c_2 \in \mathbb{R}$*.*

$$\begin{aligned}
|x_1(t)| &= |c_1 \cos t + c_2 \sin t| \leq |c_1| + |c_2| = K, \\
|x_2(t)| &= |-c_1 \sin t + c_2 \cos t| \leq |c_1| + |c_2| = K,
\end{aligned}$$

where K *is a constant.*

Therefore, all solutions of the system $\dot{x} = B\overrightarrow{x}$ *are bounded.*

Also,

$$\int_{t_0}^{\infty} \parallel C(s) \parallel ds = \int_{t_0}^{\infty} \left|-e^{-s}\right| ds$$
$$= \lim_{t\to\infty} \int_{t_0}^{t} \left|-e^{-s}\right| ds$$
$$= \lim_{t\to\infty} \left|e^{-s}\right| \Big|_{t_0}^{t}$$
$$= \lim_{t\to\infty} \left|e^{-t} - e^{-t_0}\right| = \left|-e^{-t_0}\right| < \infty$$

is bounded. Using Corollary 2.3.1, we conclude that all solutions of $\dot{\overrightarrow{x}} = \{B + C(t)\}\overrightarrow{x}$ *are bounded and stable. Since the solutions of the homogenous part are stable, the solutions of the given non-homogeneous system are also stable.*

2.4 Stability Analysis by Liapunov Method

For autonomous systems, we can introduce another method to determine the stability of the zero solution. It is called Liapunov method. We will investigate a general autonomous system

$$\dot{x} = X(x, y),$$
$$\dot{y} = Y(x, y) \tag{2.4.1}$$

with the equilibrium point $(0, 0)$.

Definition 2.4.1 (Topographic System)**.** *Define a family of curves*

$$V(x, y) = \alpha, \qquad \alpha > 0$$

with the following properties:

i. $V(x, y)$ *is continuous on a connected neighborhood* D *of the origin and* $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ *are continuous on* D *except possibly at the origin.*

ii. $V(0,0) = 0$ *and* $V(x,y) > 0$ *for all* $(x,y) \in D$.

iii. *There exists* $\mu > 0$ *such that for all* α, $0 < \alpha < \mu$,

$$V(x,y) = \alpha, \qquad (x,y) \in D$$

uniquely determines a simple closed curve τ_α *around the origin.*

These curves are known as a topographic system.

2.4.1 Geometrical Meaning of Liapunov Stability

First of all, let's introduce some important theorems.

Theorem 2.4.1 (Poincarè-Bendixson)**.** *Let the system*

$$\begin{aligned} \dot{x} &= X(x,y), \\ \dot{y} &= Y(x,y) \end{aligned}$$

be regular on a closed bounded region R. *If a positive half-path* H *lies entirely in* R, *then one of the following holds*

i. H *itself is a closed phase path in* R;

ii. H *approaches a closed phase path in* R;

iii. H *approaches an equilibrium point in* R.

Theorem 2.4.2. *Consider the topographic curve* τ *defined by*

$$V(x,y) = \alpha, \qquad \alpha > 0$$

in D. *Suppose that*

$$\dot{V}(x,y) \leq 0$$

in this domain. If H *is a half-path starting at a point* P *inside* τ, *then* H *can never escape from this closed region determined by* τ.

Here

$$\dot{V}(x,y) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = X\frac{\partial V}{\partial x} + Y\frac{\partial V}{\partial y}.$$

Hence, Poincaré-Bendixson Theorem guarantees the stability of the zero solution.

Let H be a phase path and τ be the topographic curve passing through the point P. The sign of the function $\dot{V}(x,y)$ determines the direction of H.

i. If $\dot{V} > 0$ at P, H points outward from τ.

ii. If $\dot{V} < 0$ at P, H points inward through τ.

iii. If $\dot{V} = 0$ at P, H is tangent to τ.

Theorem 2.4.3 (Liapunov Stability of the Zero Solution)**.** *Let the function $V(x,y)$ satisfy the conditions of the Definition 2.4.1.*

i. *If $\dot{V}(x,y) \leq 0$ on D with the origin excluded, the zero solution of the system (2.4.1) is uniformly stable and $V(x,y)$ is called a weak Liapunov function.*

ii. *If $\dot{V}(x,y) < 0$ on D with the origin excluded, the zero solution of the system (2.4.1) is uniformly stable and asymptotically stable. In this case, $V(x,y)$ is called a strong Liapunov function.*

The domain D, from which all half-paths approach the origin as $t \to \infty$, is known as the domain of asymptotic stability. If D is the whole xy-plane, the system is globally asymptotically stable.

Example 2.4.1. *Using $V(x,y) = x^2 + y^2$, find the domain of asymptotic stability for the following system,*

$$\begin{aligned}\dot{x} &= -\frac{1}{2}x(1-y^2),\\ \dot{y} &= -\frac{1}{2}y(1-x^2).\end{aligned}$$

Solution 2.4.4. *We have*

$$V(x,y) = x^2 + y^2 \geq 0,$$

and

$$\begin{aligned}\dot{V}(x,y) &= 2x\left[-\frac{1}{2}x(1-y^2)\right] + 2y\left[-\frac{1}{2}y(1-x^2)\right] \\ &= -x^2(1-y^2) - y^2(1-x^2) < 0\end{aligned}$$

holds when $-1 < x < 1$, $-1 < y < 1$. *Hence, the domain of asymptotic stability is*

$$D = \{x, y \in \mathbb{R} \mid -1 < x < 1 \, and -1 < y < 1\}.$$

2.4.2 Determining Stability by Weak Liapunov Function

It is also possible to show asymptotic stabilty by extending weak Liapunov functions.

Theorem 2.4.5. *Let* $V(x,y)$ *satisfy the conditions for a topographic system for the regular system (2.4.1). If*

i. $\dot{V}(x,y) \leq 0$ *on* D *with the origin excluded,*

ii. *none of the topographic curves in* D *is also a phase path,*

then there exists no closed phase path in D.

Theorem 2.4.6. *Let* $V(x,y)$ *satisfy the conditions in the Definition 2.4.1 and* $V(x,y) = \alpha$, $\alpha > 0$ *be a topographic system in* D *for the regular system (2.4.1), which has* $(0,0)$ *as the only equilibrium point. Assume that*

i. $\dot{V}(x,y) \leq 0$ *in* D *with the origin excluded;*

ii. *no closed curve of topographic system is also a phase path.*

Then the zero solution is uniformly and asymptotically stable.

(The result follows from Theorem 2.4.5 and the Poincaré-Bendixson Theorem).

We can also state Liapunov stability without using Poincaré-Bendixson Theorem. The following definition is necessary for this approach.

Definition 2.4.2. *Let $f(x)$ be a scalar function such that $f(0) = 0$. If, for $x \neq 0$,*

i. $f(x) > 0$, *then it is called positive definite;*

ii. $f(x) \geq 0$, *then it is called positive semidefinite;*

iii. $f(x) < 0$, *then it is called negative definite;*

iv. $f(x) \leq 0$, *then it is called negative semidefinite.*

Now consider a general system

$$\dot{\overrightarrow{x}} = \overrightarrow{X}(\overrightarrow{x}). \tag{2.4.2}$$

Theorem 2.4.7 (Liapunov Stability)**.** *If, in a neighborhood D of the origin,*

i. *the system (2.4.2) is regular and $\overrightarrow{X}(\overrightarrow{0}) = \overrightarrow{0}$,*

ii. $V(x)$ *is continuous and positive definite,*

iii. $\dot{V}(x)$ *is continuous and negative semidefinite,*

then the zero solution is uniformly stable.

Theorem 2.4.8 (Asymptotic Stability)**.** *Suppose that*

i. *the system (2.4.2) is regular and $\overrightarrow{X}(\overrightarrow{0}) = \overrightarrow{0}$,*

ii. $V(x)$ *is continuous and positive definite,*

iii. $\dot{V}(x)$ *is continuous and negative definite*

in a neighborhood D of the origin. Then the zero solution is uniformly and asymptotically stable.

Theorem 2.4.9 (Liapunov Instability). *Let $\overrightarrow{x}(t) = \overrightarrow{0}$ be the zero solution of the regular autonomous system (2.4.2), where $\overrightarrow{X}(\overrightarrow{0}) = \overrightarrow{0}$. If there exists a function $U(x)$ such that in some neighborhood $\|\overrightarrow{x}\| \leq k$ of the origin*

i. $U(x)$ *and its partial derivatives are continuous,*

ii. $U(0) = 0,$

iii. $\dot{U}(x)$ *is positive definite,*

iv. *in every neighborhood of the origin, there exists at least one point x at which* $U(x) > 0,$

then the zero solution is unstable.

Example 2.4.2. *Find a simple V or U function to establish the stability or instability of the zero solution of the following system of equations*

a.

$$\begin{aligned} \dot{x} &= -x^3 + y^4, \\ \dot{y} &= -y^3 + y^4; \end{aligned}$$

b.

$$\begin{aligned} \dot{x} &= e^x - \cos y, \\ \dot{y} &= x. \end{aligned}$$

Solution 2.4.10. **a.** *Let*

$$V(x, y) = x^2 + y^2 > 0,$$

then

$$\begin{aligned}\dot{V}(x.y) &= 2x(-x^3+y^4)+2y(-y^3+y^4)\\ &= -2(x^4+y^4)+2y^4(x+y)\\ &\leq -2(x^4+y^4)+2y^4 \mid x+y \mid\\ &\leq -2(x^4+y^4)+2y^4(\mid x \mid + \mid y \mid)\\ &< 0\end{aligned}$$

in the neighborhood of the origin, defined by $\mid x \mid + \mid y \mid < 1$. *Hence, the zero solution is stable.*

b. *Let*

$$U(x,y) = x^2 + \sin^2 y > 0,$$

then

$$\begin{aligned}\dot{U}(x,y) &= 2x(e^x - \cos y) + 2x \sin y \cos y\\ &= 2x[e^x + \cos y(-1+\sin y)] > 0\end{aligned}$$

in the neighborhood of the origin, defined by $0 \leq x \leq \dfrac{\pi}{4},\ 0 \leq y \leq \dfrac{\pi}{4}$. *Here we have used the inequality;*

$$-1 \leq \cos y(-1+\sin y) \leq \frac{\sqrt{2}}{2}\left(-1+\frac{\sqrt{2}}{2}\right). \tag{2.4.3}$$

For the inequality (2.4.3), consider the function

$$\begin{aligned}f(y) &= \cos y(-1+\sin y),\\ f'(y) &= -\sin y(-1+\sin y)+\cos y \cos y\\ &= \sin y - \sin^2 y + (1-\sin^2 y)\\ &= -2\sin^2 y + \sin y + 1.\end{aligned}$$

Let's find the minimum and maximum values of the function f

$$f'(y) = 0 \Leftrightarrow$$

$$-2\sin^2 y + \sin y + 1 = 0.$$

Let $\sin y = t$;

$$-2t^2 + t + 1 = 0 \Rightarrow$$

$$t_1 = -\frac{1}{2}, \quad t_2 = 1.$$

Consider

$$\sin y = -\frac{1}{2}, \quad \sin y = 1;$$

$$y_1 = -\frac{\pi}{6}, \quad y_2 = \frac{\pi}{2};$$

$$f'(t) = -2t^2 + t + 1;$$

t		$-\frac{1}{2}$		1	
$f'(t)$	$-$	0	$+$	0	$-$

The function $f(t)$ *increases on the interval* $\left[-\frac{1}{2}, 1\right]$. *Thus the function* $f(y)$ *increases on the interval* $\left[-\frac{\pi}{6}, \frac{\pi}{2}\right]$, *i.e. f increases on* $\left[0, \frac{\pi}{4}\right]$.

Note that

$$f(0) = -1, \quad f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(-1 + \frac{\sqrt{2}}{2}\right).$$

On $\left[0, \frac{\pi}{4}\right]$, *we have*

$$-1 \leq f(y) = \cos y(-1 + \sin y) \leq \frac{\sqrt{2}}{2}\left(-1 + \frac{\sqrt{2}}{2}\right).$$

Thus, using Theorem 2.4.9, we conclude that the zero solution is unstable.

2.4.3 Linear Approximation and Stability

In some cases, it is appropriate to use the linear approximation of the given system to determine the asymptotic stability and the instability of the zero solution, that is Liapunov functions for the linearized system are also applicable for the original system.

We will give the theory for two-dimensional autonomous systems in the form

$$\dot{x} = \overrightarrow{X}(\overrightarrow{x}) = A\overrightarrow{x} + \overrightarrow{f}(\overrightarrow{x}),$$

where A is a constant 2×2 matrix.

Theorem 2.4.11. *Let* $(0,0)$ *be the equilibrium point of the regular system*

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}+\begin{bmatrix}f_1(x,y)\\ f_2(x,y)\end{bmatrix}, \tag{2.4.4}$$

where

$$f_1(x,y)=O(x^2+y^2) \text{ and } f_2(x,y)=O(x^2+y^2)$$

as $x^2+y^2\to 0$. *If the linear approximation of the system (2.4.4) is asymptotically stable, then the zero solution of (2.4.4) is asymptotically stable.*

Theorem 2.4.12. *Let* $(0,0)$ *be the equilibrium point of the system (2.4.4). If the eigenvalues of* A *are different, nonzero and at least one has positive real part, then the zero solution is unstable.*

Example 2.4.3. *Prove that the equation*

$$\ddot{x}-\dot{x}^2 sign(\dot{x})+x=0$$

has an unstable zero solution.

Solution 2.4.13. *Write equation as a system:*

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= y^2 sign(y)-x,\end{aligned}$$

or

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}+\begin{bmatrix}0\\ y^2 sign(y)\end{bmatrix}.$$

Let

$$A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}.$$

Consider

$$|A-\lambda I|=\begin{vmatrix}-\lambda & 1\\ -1 & -\lambda\end{vmatrix}=\lambda^2+1=0,$$

then

$$\lambda_{1,2} = \pm i.$$

Therefore, the zero solution of the linear system is a center, i.e. stable. In this case, Theorem 2.4.11 cannot be applied to determine the stability of the original system. Now consider the following function,

$$U(x, y) = x^2 + y^2 > 0,$$

then

$$\begin{aligned} \dot{U}(x, y) &= 2x(y) + 2y(y^2 sign(y) - x) \\ &= 2y^3 sign(y) \geq 0, \quad for\ every\ x, y \in \mathbb{R}. \end{aligned}$$

According to Theorem 2.4.9, the zero solution is unstable.

Example 2.4.4. *Show that the origin is a stable spiral for the system*

$$\begin{aligned} \dot{x} &= -y - x\sqrt{x^2 + y^2}, \\ \dot{y} &= x - y\sqrt{x^2 + y^2}, \end{aligned}$$

and a centre for the linear approximation. Find a Liapunov function for the zero solution.

Solution 2.4.14. *We have*

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -x\sqrt{x^2 + y^2} \\ -y\sqrt{x^2 + y^2} \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

then

$$|A - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0,$$

$$\lambda_{1,2} = \pm i.$$

Thus, a linear approximation is a center, i.e. stable. For the stability of the non-linear system consider

$$\begin{aligned}
V(x,y) &= (x^2+y^2)^{\frac{3}{2}} \geq 0, \\
\dot{V}(x,y) &= \frac{3}{2}(x^2+y^2)^{\frac{1}{2}}2x\left(-y-x\sqrt{x^2+y^2}\right)+\frac{3}{2}(x^2+y^2)^{\frac{1}{2}}2y\left(x-y\sqrt{x^2+y^2}\right) \\
&= -3xy(x^2+y^2)^{\frac{1}{2}}-3x^2(x^2+y^2)+3xy(x^2+y^2)^{\frac{1}{2}}-3y^2(x^2+y^2) \\
&= -3(x^2+y^2)^2 < 0.
\end{aligned}$$

So the zero solution of the original system is asymptotically stable, i.e. stable spiral.

All this theory for two-dimensional systems can be extended to $n-$dimensions.

2.4.4 Stability for n-dimensional Systems

Let

$$\dot{\vec{x}} = A\vec{x} + \vec{f}(\vec{x}) \tag{2.4.5}$$

be an n-dimensional regular system, where A is a constant $n \times n$ matrix.

Theorem 2.4.15. *Assume that*

i. the zero solution of the linear approximation

$$\dot{\vec{x}} = A\vec{x}$$

is asymptotically stable;

ii. $f(0) = 0$ and

$$\lim_{\|x\|\to 0} \frac{\|f(x)\|}{\|x\|} = 0.$$

Then the zero solution of (2.4.5) is asymptotically stable.

Theorem 2.4.16. *Suppose that*

i. *the eigenvalues of A are distinct, nonzero and at least one has a positive real part;*

ii. $f(0) = 0$ *and*

$$\lim_{\|x\| \to 0} \frac{\|f(x)\|}{\|x\|} = 0.$$

Then the zero solution of (2.4.5) is unstable.

2.5 Periodic Solutions

In this section, we deal with the existence of periodic solutions of the planar systems in the form

$$\begin{aligned} \dot{x} &= X(x,y), \\ \dot{y} &= Y(x,y). \end{aligned}$$

Definition 2.5.1 (Periodic Solution)**.** *A solution of the system such that*

$$\begin{aligned} x(t+T) &= x(t), \\ y(t+T) &= y(t), \end{aligned}$$

where T is constant, is called periodic. The phase paths of periodic solutions are closed curves.

Periodic solution can occur as a part of a family of closed curves or as an isolated closed curve, which is known as a limit cycle. So the limit cycle can be defined as an isolated periodic solution.

Let's state some theorems about the existence and non-existence of periodic solutions of planar systems.

Theorem 2.5.1. *If*

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}$$

is of one sign for a connected domain D, then the system has no periodic solutions in D.

Theorem 2.5.2. *Every closed curve representing periodic solution surrounds at least one critical point.*

2.5.1 Existence of Periodic Solutions

We want to find a closed region that contains a limit cycle. Consider two closed curves C_1 and C_2 surrounding the equilibrium point of the system, with C_2 inside C_1. There must be no critical points in the closed region R between C_1 and C_2. If we can also guarantee that all trajectories crossing C_2 head out and all trajectories passing C_1 head in, then according to Poincaré Bendixson Theorem, any path entering R will not be able to get out of the region. Therefore, R has at least one closed path, i.e., a periodic solution. In fact, the matter is to find the narrowest region R that contains a periodic solution. In most cases, it is not that easy to find this closed region.

Example 2.5.1. *Show that there exists a limit cycle for the system*

$$\begin{aligned} \dot{x} &= x + y - x^3 - 6xy^2, \\ \dot{y} &= -\frac{1}{2}x + 2y - 8y^3 - x^2 y. \end{aligned}$$

Solution 2.5.3. *Consider the function*

$$V(x, y) = x^2 + 2y^2.$$

Then the total derivative

$$\begin{aligned} \dot{V}(x, y) &= 2x(x + y - x^3 - 6xy^2) + 4y(-\frac{1}{2}x + 2y - 8y^3 - x^2 y) \\ &= 2x^2 + 2xy - 2x^4 - 12x^2y^2 - 2xy + 8y^2 - 32y^4 - 4x^2y^2 \\ &= 2x^2 + 8y^2 - 2x^4 - 16x^2y^2 - 32y^4 \\ &= 2(x^2 + 4y^2) - 2(x^4 + 8x^2y^2 + 16y^4) \\ &= 2(x^2 + 4y^2) - 2(x^2 + 4y^2)^2. \end{aligned}$$

If $x^2 + 4y^2 < 1$, then

$$\begin{aligned} x^2 + 4y^2 &> (x^2 + 4y^2)^2, \\ 2(x^2 + 4y^2) &> 2(x^2 + 4y^2)^2, \end{aligned}$$

$$2(x^2+4y^2)-2(x^2+4y^2)^2>0,$$

and

$$\dot{V}(x,y)>0.$$

So all trajectories are directed outwards on the curve

$$C_1 : x^2+4y^2=c, \qquad \textit{for any } c \textit{ such that} \qquad 0<c<1.$$

If $x^2+4y^2>2$*, then*

$$2(x^2+4y^2)<(x^2+4y^2)^2<2(x^2+4y^2)^2,$$

$$2(x^2+4y^2)-2(x^2+4y^2)^2<0,$$

and

$$\dot{V}(x,y)<0.$$

So all trajectories are directed inwards on the curve

$$C_2 : x^2+4y^2=c, \qquad \textit{for any } c \textit{ such that} \qquad c>2.$$

Since all paths enter the annular region R *between* C_1 *and* C_2*, it is guaranteed that there exists a limit cycle in* R*,*

$$R=\{(x,y)|-1\leq x^2+4y^2\leq 2\}.$$

Chapter 3

GEOMETRIC PROPERTIES OF FACTORABLE PLANAR SYSTEMS OF DIFFERENTIAL EQUATIONS

In this chapter, we deal with factorable planar systems that are defined below.

Definition 3.0.2 (Factorable Planar System)**.** *A two dimensional system with separable phase equations*

$$\dot{x} = f(x)h(y), \qquad (3.0.1)$$

$$\dot{y} = k(x)g(y),$$

where f, h, k, g are continuously differentiable on $(-\infty, \infty)$, is called a factorable planar system [8].

Consider the phase equation of (3.0.1)

$$\frac{dy}{dx} = \frac{k(x)g(y)}{f(x)h(y)},$$

or

$$-\frac{h(y)}{g(y)}dy + \frac{k(x)}{f(x)}dx = 0.$$

Taking integrals of both sides gives

$$\int_a^x \frac{k(u)}{f(u)}du - \int_b^y \frac{h(v)}{g(v)}dv = C,$$

where C is an integration constant and $(a, b) \in \mathbb{R}^2$. The first integral of (3.0.1) is obtained as

$$H(x, y) = F(x) - G(y) = C,$$

where

$$F(x) = \int_a^x \frac{k(u)}{f(u)} du \qquad \text{and} \qquad G(x) = \int_b^y \frac{h(v)}{g(v)} dv.$$

If we compare factorable planar system with the Hamiltonian system

$$\dot{x} = -\frac{h(y)}{g(y)}, \qquad \dot{y} = -\frac{k(x)}{f(x)}, \tag{3.0.2}$$

we can easily conclude that they both have the same phase equation. In this case, the first integral $H(x, y)$ of (3.0.1) is known as a Hamiltonian function of (3.0.2).

Lemma 3.0.1. *Let H be a first integral of a planar C^1dynamical system. If H is not constant on any open set, then there are no limit cycles.*

Theorem 3.0.4. *Factorable planar systems have no limit cycles.*

Proof. Suppose to the contrary that there exists a limit cycle γ, contained in the closure of an open set $U \in \mathbb{R}^2$. By Lemma 3.0.1, H is constant on U, i.e.,

$$\frac{\partial H}{\partial x} = F'(x) = \frac{k(x)}{f(x)} = 0, \quad \text{and} \quad \frac{\partial H}{\partial y} = -G'(y) = \frac{-h(y)}{g(y)} = 0,$$

for every $(x, y) \in U$. Consequently, $k^{-1}(0) \times h^{-1}(0)$ contains the set U and also the closure of U. That is, the limit cycle γ is contained in $k^{-1}(0) \times h^{-1}(0)$. On $k^{-1}(0) \times h^{-1}(0)$,

$$k(x) = 0 \quad \text{and} \quad h(y) = 0.$$

Therefore,

$$\dot{x} = 0,$$

$$\dot{y} = 0,$$

that is there exists a critical point in $k^{-1}(0) \times h^{-1}(0)$. This contradicts Poincarè Bendixson Theorem since γ is in U. Thus, there are no limit cycles. $\square$

Although factorable planar systems do not have limit cycles, they can have periodic solutions. This is guaranteed by the following theorem.

Lemma 3.0.2. *A critical point* (a, b) *of a Hamiltonian system (3.0.2) is a center if it is a strict local minimum or maximum of the Hamiltonian function* $H(x, y)$.

Theorem 3.0.5. *Assume that* $k(a) = h(b) = 0$. *If* $f(a)g(b)k'(a)h'(b) < 0$, *then the equilibrium point* (a, b) *is a center and nearby solutions of (3.0.1) form closed orbits around* (a, b).

Proof. Jacobian matrix at the point (a, b) is

$$J_{(a,b)} = \begin{bmatrix} f'(a)h(b) & f(a)h'(b) \\ k'(a)g(b) & k(a)g'(b) \end{bmatrix} = \begin{bmatrix} 0 & f(a)h'(b) \\ k'(a)g(b) & 0 \end{bmatrix},$$

and the linearized system becomes

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J_{(a,b)} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The characteristic equation is

$$\lambda^2 - f(a)g(b)k'(a)h'(b) = 0,$$

with the roots

$$\lambda_{1,2} = \pm i\sqrt{f(a)g(b)k'(a)h'(b)}.$$

Now let us apply the second derivative test to the Hamiltonian function at the point (a, b).

$$H_{xx}(a, b)H_{yy}(a, b) - H_{xy}^2(a, b) = \frac{k'(a)h'(b)}{f(a)g(b)} = \frac{-f(a)g(b)k'(a)h'(b)}{[f(a)g(b)]^2} > 0.$$

If

$$H_{xx}(a, b) = \frac{k'(a)}{f(a)} > 0,$$

then (a, b) is a strict local minimum of H. If

$$H_{xx}(a, b) = \frac{k'(a)}{f(a)} < 0,$$

then (a, b) is a strict local maximum of H. In any case, (a, b) is a center of (3.0.1) by Lemma 3.0.2. □

Consider now a general second order differential equation

$$\ddot{x} = \varphi(x, \dot{x}),$$

which can be represented as a system

$$\dot{x} = y,$$

$$\dot{y} = \varphi(x, y). \tag{3.0.3}$$

Notice that all equilibrium points, if exist, are on the x-axis since equating right-hand side of the system (3.0.3) to zero gives $y = 0$, $\varphi(x, 0) = 0$.

Suppose that φ is factorable,

$$\varphi(x, y) = k(x)g(y).$$

Corollary 3.0.1. *Let g, k in φ be as in (3.0.1) and assume that a is an isolated zero of k (i.e. $k(a) = 0$) such that $g(0)k'(a) < 0$. Then, the second order equation*

$$\ddot{x} - g(\dot{x})k(x) = 0$$

has periodic solutions around a which are not limit cycles, that is, the equilibrium point $(a, 0)$ is a center.

Proof. Notice that in the system (3.0.3), $f(x) = 1$ and $h(y) = y$. Thus, the result follows directly from Theorem 3.0.4 and Theorem 3.0.5. □

Theorem 3.0.6. *Assume that, for all $u \in (-\infty, \infty)$, $f'(u)g'(u) \geq 0$, $f'(u) \neq 0$ (or $g'(u) \neq 0$) and also $h(u)k(u) \geq 0$, $h(u) \neq 0$ (or respectively $k(u) \neq 0$). Then (3.0.1) has no periodic solutions.*

Proof. Let $f'(u)h(u) \neq 0$, for all $u \in (-\infty, \infty)$, so $f'(u) \neq 0$ and $h(u) \neq 0$. Both f' and h are continuous, thus each is either always positive or always negative for all $u \in (-\infty, \infty)$. By the hypothesis, k and h have the same sign and similarly f' and g' have the same sign. In any case,

$$f'(x)h(y) + k(x)g'(y) \neq 0$$

does not change sign. According to Theorem 2.5.1, there exist no periodic solutions. □

3.1 Properties of Equilibrium Points of Factorable Planar Systems

An equilibrium point (a, b) of the system (3.0.1) makes at least one of the following pairs $(0, 0)$:

$$(f(a), k(a)), \quad (f(a), g(b)), \quad (h(b), k(a)), \quad (h(b), g(b)).$$

If $(f(a), k(a)) = (0, 0)$, the equilibrium point is (a, y), where y is any real number. If $(h(b), g(b)) = (0, 0)$, the equilibrium point is (x, b), where x is any real number. Hence, in both cases above, we have dense set of points. An isolated equilibrium point can only be obtained if one of the points

$$(f(a), g(b)), \quad (h(b), k(a))$$

coincides with the origin.

Consider now the Jacobian matrix of the system (3.0.1) evaluated at (a, b),

$$J_{(a,b)} = \begin{bmatrix} f'(a)h(b) & f(a)h'(b) \\ k'(a)g(b) & k(a)g'(b) \end{bmatrix}.$$

The characteristic equation is

$$[\lambda - f'(a)h(b)]\,[\lambda - k(a)g'(b)] - f(a)g(b)k'(a)h'(b) = 0.$$

If $(f(a), k(a)) = (0, 0)$, we have

$$\lambda_1 = 0, \qquad \lambda_2 = f'(a)h(b).$$

If $(f(a), g(b)) = (0, 0)$, the roots are

$$\lambda_1 = f'(a)h(b), \qquad \lambda_2 = k(a)g'(b).$$

If $(h(b), k(a)) = (0, 0)$, we have

$$\lambda_{1,2} = \pm\sqrt{f(a)g(b)k'(a)h'(b)}.$$

If $(h(b), g(b)) = (0, 0)$, we have

$$\lambda_1 = 0, \qquad \lambda_2 = k(a)g'(b).$$

It is clear that in any case we cannot have a focus. There are only two possibilities for this point; a saddle point or a node. The following theorem states this result clearly.

Theorem 3.1.1. *Every hyperbolic equilibrium* (a, b) *(i.e., the eigenvalues of* $J(a, b)$ *both have nonzero real parts) of (3.0.1) is either a saddle point or a node. Furthermore,* (a, b) *is a node if* $f(a) = g(b) = 0$ *and* $f'(a)h(b)$ *has the same sign as* $k(a)g'(b)$*, or a saddle point otherwise.*

Proof. We suppose that (a, b) is hyperbolic, so the pairs

$$(f(a), k(a)) \quad \text{and} \quad (h(b), g(b))$$

are ignored. We have two middle cases remaining.

For $(f(a), g(b)) = (0, 0)$, the eigenvalues are

$$\lambda_1 = f'(a)h(b), \qquad \lambda_2 = k(a)g'(b).$$

Note that λ_1 and λ_2 have the same sign by the hypothesis. Hence, (a, b) is a node, stable if $\lambda_1, \lambda_2 < 0$ and unstable if $\lambda_1, \lambda_2 > 0$.

On the other hand, if $(h(b), k(a)) = (0, 0)$, the eigenvalues

$$\lambda_{1,2} = \pm\sqrt{f(a)g(b)k'(a)h'(b)}$$

are real and nonzero since (a, b) is hyperbolic. Since $\lambda_1 < 0$, $\lambda_2 > 0$, (a, b) is a saddle point. □

Corollary 3.1.1. *Every hyperbolic equilibrium* $(a, 0)$ *of*

$$\ddot{x} - g(\dot{x})k(x) = 0$$

is a saddle point.

Proof. In this system, $k(a) = 0$. Hence, the only zero pair is

$$(h(b), k(a)).$$

The point $(a, 0)$ is hyperbolic, so the eigenvalues

$$\lambda_{1,2} = \pm\sqrt{f(a)g(b)k'(a)h'(b)}$$

are nonzero and real, i.e., $g(0)k'(a) > 0$. This leads to a saddle point since $\lambda_1 < 0$ and $\lambda_2 > 0$. □

From Corollary 3.0.1 and Corollary 3.1.1, we conlude that the equilibrium point $(a, 0)$ of the system

$$\ddot{x} - g(\dot{x})k(x) = 0$$

is either a center or a saddle point if all eigenvalues are nonzero.

3.1.1 Special Cases

More complicated results come up if at least one eigenvalue of $J(a, b)$ is zero, i.e., (a, b) is a nonhyperbolic equilibrium point.

Example 3.1.1. *Consider the system*

$$\ddot{x} = \alpha x^m (\dot{x})^n, \qquad \alpha \neq 0, \quad n, m \geq 0, \quad m + n > 1,$$

which in the plane has the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \alpha x^m y^n. \end{aligned}$$

This system has zero eigenvalues.

Let $n = 0$. The system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \alpha x^m, \end{aligned}$$

is known as Newtonian, and $(0, 0)$ is the only equilibrium point.

If $n = 0$ and $m = 1$,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \alpha x. \end{aligned}$$

Then the coefficient matrix is

$$\begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix}.$$

The eigenvalues are $\lambda_{1,2} = \pm\sqrt{\alpha}$.

Assume that $\alpha < 0$. Then we have ellipses around $(0, 0)$ and

$$\frac{dy}{dx} = \frac{\alpha x}{y},$$

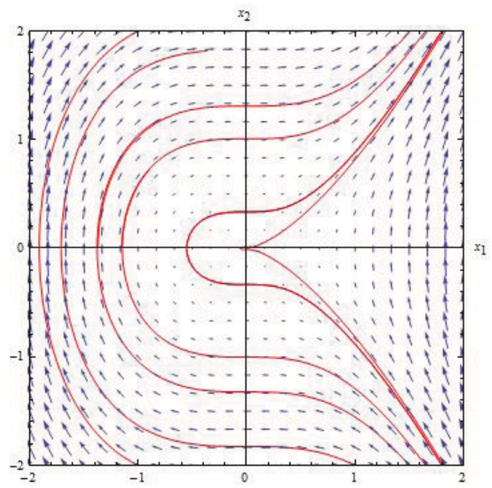

Figure 3.1: A phase diagram of a cusp.

or

$$ydy = \alpha x dx.$$

Taking integral of both sides,

$$\int ydy = \int \alpha x dx,$$

gives

$$\frac{y^2}{2} - \alpha \frac{x^2}{2} = C, \tag{3.1.1}$$

where C is an integration constant. It is clear from the phase equation (3.1.1) that $(0, 0)$ is a center.

Assume that $\alpha > 0$. Then we have $\lambda_1 > 0$ and $\lambda_2 < 0$. The phase diagram has hyperbolas and $(0, 0)$ is a saddle point.

If $n = 0$ and $m = 2$, the phase equation is

$$\frac{y^2}{2} - \alpha \frac{x^3}{3} = C,$$

where C is an integration constant. In this case, the origin is a cusp for all α. See Figure 3.1.

Let $n \geq 1$, we have a nonisolated set of equilibrium points on the x-axis. The phase *equation can be derived as*

$$\frac{dy}{dx} = \frac{\alpha x^m y^n}{y}.$$

Then

$$\int \frac{1}{y^{n-1}} dy = \int \alpha x^m dx,$$

which yields

$$\frac{y^{2-n}}{2-n} = \alpha \frac{x^{m+1}}{m+1} + C_0,$$

or

$$(m+1)y^{2-n} + (n-2)\alpha x^{m+1} = C,$$

where C_0 and C are constants of integration.

If $n = 1$,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \alpha x^m y. \end{aligned}$$

Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{\alpha x^m y}{y}, \\ \int dy &= \int \alpha x^m dx, \end{aligned}$$

and

$$y = \alpha \frac{x^{m+1}}{m+1} + C,$$

where C is an integration constant. The equilibrium point at the origin can not be classified as a node, saddle point, etc.

For $n = 2$,

$$\begin{aligned} \frac{dy}{dx} &= \frac{\alpha x^m y^2}{y}, \\ \int \frac{1}{y} dy &= \int \alpha x^m dx, \\ \ln|y| &= \alpha \frac{x^{m+1}}{m+1} + C_1, \end{aligned}$$

$$(m+1)\ln|y| - \alpha x^{m+1} = C,$$

where C_1 and C are constants of integration.

If $n > 2$, the phase equation is

$$(m+1)\frac{1}{y^{n-2}} + (n-2)\alpha x^{m+1} = C,$$

where C is constant of integration. In this case, we cannot have $y = 0$, i.e., trajectories do not cross the x-axis, and it is impossible to have periodic solutions.

Example 3.1.2. *Consider the system*

$$\begin{aligned}\dot{x} &= \frac{\sin x - x}{y^2+1},\\ \dot{y} &= \frac{-y}{x^2+1}.\end{aligned}$$

Clearly, $(0,0)$ is a nonhyperbolic equilibrium point of the system. Liapunov method can be used to determine the stability of this equilibrium point.

Take a Liapunov function

$$V(x,y) = x^2 + y^2.$$

Then

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = \frac{-2x(x-\sin x)}{y^2+1} - \frac{2y^2}{x^2+1} < 0, \qquad for\ all\ (x,y) \neq (0,0).$$

Hence, all trajectories point towards the origin, that is, $(0,0)$ is asymptotically stable. By Theorem 3.0.6, there exist no periodic solutions.

Example 3.1.3. *Translations and reflections are transformations which preserve factorability. More general transformations of the plane can affect the factorability of the system. For instance, factorability is not preserved under polar coordinate transformations. A factorable system in polar coordinates is usually not factorable when it is transformed to* *rectangular coordinates. The phase plane of a factorable system in polar coordinates can have foci and limit cycles. This is modelled in the following system*

$$\begin{aligned} \dot{r} &= r(1-r), \\ \dot{\theta} &= 1. \end{aligned}$$

Note that $\dfrac{dr}{dt} > 0$ *when* $r < 1$*, so the trajectories inside the circle* $r = 1$ *point outwards towards the circle and* $\dfrac{dr}{dt} < 0$ *when* $r > 1$*, so trajectories outside the circle* $r = 1$ *approach the circle. Thus,* $r = 1$ *is a stable limit cycle, whereas* $(0, 0)$ *is an unstable focus. When the system is transformed to rectangular coordinates, it can be noticed that it is not factorable in rectangular coordinates.*

Therefore, we conclude that all theorems in this chapter are applicable only to systems which are factorable in rectangular coordinates.

Chapter 4

ECOLOGICAL APPLICATIONS

One of the most important applications of stability theory is in biology. Mathematical models can be used to discuss interactions between two species in the same environment. We use autonomous systems with linear first degree polynomials of x and y ignoring the time variable t. Although the models are defined for all x and y, they are only logical in the population quadrant $x \geq 0$, $y \geq 0$, since the number of population members cannot be negative.

Let's try to model some important interactions providing corresponding examples.

4.1 Lotka-Volterra Predator-Prey Model

If one species (predators) feed on the other (prey), the model is called predator-prey model. The population density of each species depends on the population density of the other species. Interactions between foxes (predators) and rabbits (prey), sharks (predators) and fish (prey) can be considered as examples of this model.

Let us consider $x(t)$ as the number of prey and $y(t)$ as the number of predators in the same environment. If there exists no pedators, the number of prey grows with a constant rate such that

$$\frac{dx}{dt} = ax, \qquad a > 0.$$

If the prey are absent, the number of predators decreases with a constant rate such that

$$\frac{dy}{dt} = -cy, \qquad c > 0.$$

Therefore, predator population becomes extinct. On the other hand, if both prey and predator are present, the encounters between two species, which are directly proportional to xy, cause a decline in the prey population and a growth in the predator population.

Considering the facts above, a general predator-prey model is obtained as

$$\frac{dx}{dt} = ax - bxy = x(a - by),$$

$$\frac{dy}{dt} = -cy + dxy = y(-c + dx), \tag{4.1.1}$$

where a, b, c, d are all positive.

This system has two equilibrium points, $(0, 0)$ and $(c/d, a/b)$. Here the stability of the equilibrium points gives us an idea about the interaction between two species. The system is nonlinear so let us linearize it to examine the stability of each critical point. The Jacobian matrix is

$$J = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix} = \begin{bmatrix} a - by & -bx \\ dy & -c + dx \end{bmatrix},$$

where $f_1(x, y) = ax - bxy$ and $f_2(x, y) = -cy + dxy$.

For the point $(0, 0)$,

$$J = \begin{bmatrix} a - by & -bx \\ dy & -c + dx \end{bmatrix}_{(0,0)} = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix},$$

so the linearized system becomes

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

with the characteristic equation $(a - \lambda)(-c - \lambda) = 0$. The eigenvalues are $\lambda_1 = a > 0$ and $\lambda_2 = -c < 0$, i.e., $(0, 0)$ is a saddle point. But this case is not that much important since the equilibrium solution $x(t) \equiv 0$, $y(t) \equiv 0$ corresponds to the extinction of both species.

The important case is a coexistence of the species which depends on the stability of the nonzero equilibrium point $(c/d, a/b)$. So let's check the stability of $(c/d, a/b)$. We have

$$J = \begin{bmatrix} a - by & -bx \\ dy & -c + dx \end{bmatrix}_{(c/d, a/b)} = \begin{bmatrix} 0 & \frac{-bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix}.$$

Using the following substitution

$$\begin{aligned} u &= x - \frac{c}{d}, \\ v &= y - \frac{a}{b}, \end{aligned}$$

we obtain a corresponding critical point $(0, 0)$ for $(c/d, a/b)$. Hence, the corresponding linearized system is

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-bc}{d} \\ \frac{ad}{b} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

The characteristic equation is $\lambda^2 + ac = 0$, i.e., the eigenvalues are $\lambda_1 = i\sqrt{ac}$ and $\lambda_2 = -i\sqrt{ac}$. Since the roots are pure imaginary, we cannot be sure about the stability of $(c/d, a/b)$. We must find the trajectories in the phase plane. Dividing second equation in (4.1.1) by the first gives

$$\frac{dy}{dx} = \frac{y(-c + dx)}{x(a - by)}.$$

We can separate variables to obtain

$$\int \frac{a - by}{y} dy = \int \frac{-c + dx}{x} dx,$$

so we have

$$a \ln y - by = -c \ln x + dx + c_1,$$

$$c \ln x - dx + a \ln y - by = c_1,$$

$$(x^c e^{-dx})(y^a e^{-by}) = e^{c_1} = c_0,$$

where c_1 and c_0 are constants. These equations define closed curves around the critical point $(c/d, a/b)$. This guarantees that the critical point is a stable center. Hence, the corresponding equilibrium solution $x(t) \equiv \frac{c}{d}$ and $y(t) \equiv \frac{a}{b}$ shows that both populations, the prey and predators, coexist without extinction in the same environment.

4.2 Lotka-Volterra Competition Model

If two species compete for resources such as food, water, light, etc., it is known as a competitive interaction. If only one species uses the resources, the other species can hardly survive. So our question here is, how can these two competing species coexist?

Let $x(t)$ and $y(t)$ denote the population densities of each species. Assume that in the absence of one species, the other population becomes limited and the competition effects the number in such a way that each population is inversely proportional to the product xy. Therefore, a model for competitive interaction is

$$\begin{aligned}\frac{dx}{dt} &= a_1x - b_1x^2 - c_1xy = x(a_1 - b_1x - c_1y),\\ \frac{dy}{dt} &= a_2y - b_2y^2 - c_2xy = y(a_2 - b_2y - c_2x),\end{aligned}$$

where a_1, b_1, c_1, a_2, b_2, c_2 are all positive constants.

The system has four critical points; $(0, 0)$, $(0, a_2/b_2)$, $(a_1/b_1, 0)$, $(\tilde{x}, \tilde{y})$, where the last critical point is defined as the intersection of the lines

$$\begin{aligned} b_1x + c_1y &= a_1,\\ c_2x + b_2y &= a_2. \end{aligned} \tag{4.2.1}$$

From the analysis of the first three equilibrium points, it can be easily concluded that:

i. For $(0, 0)$, the equilibrium solution $x(t) \equiv 0$, $y(t) \equiv 0$ shows a decline in both species.

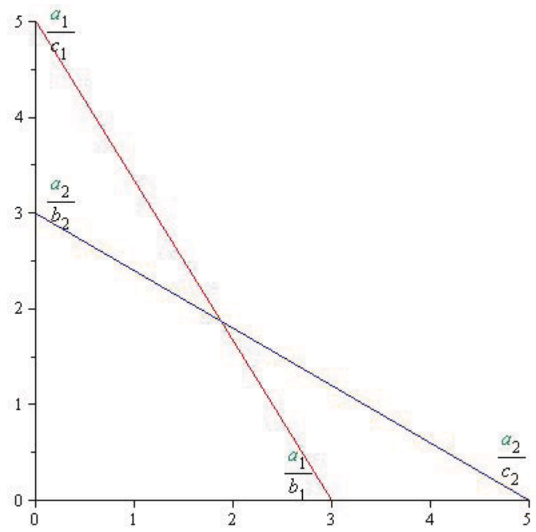

Figure 4.1: The graph of lines (4.2.1) if $c_1c_2 < b_1b_2$.

ii. For $(0, a_2/b_2)$, the equilibrium solution $x(t) \equiv 0$, $y(t) \equiv a_2/b_2$ shows that, the second species wins the competiton and the first species goes extinct.

iii. For $(a_1/b_1, 0)$, the equilibrium solution $x(t) \equiv a_1/b_1$, $y(t) \equiv 0$ shows that, the first species uses all resources and the lack of resources causes the extinction of the second species.

The coexistence is our primary interest, so the stability of the last critical point $(\tilde{x}, \tilde{y})$ is more important than the other obvious cases mentioned above. The stability of $(\tilde{x}, \tilde{y})$ can be discussed by comparing the slopes of two lines in (4.2.1). We have two possibilities demonstrated in Figures 4.1 and 4.2. Note that the red line in the figures shows the first line in (4.2.1), while the blue line shows the second line in (4.2.1).

In Figure 4.1,

$$\frac{a_2/b_2}{a_2/c_2} < \frac{a_1/c_1}{a_1/b_1}, \qquad \text{i.e.,} \qquad c_1c_2 < b_1b_2,$$

and in Figure 4.2,

$$\frac{a_2/b_2}{a_2/c_2} > \frac{a_1/c_1}{a_1/b_1}, \qquad \text{i.e.,} \qquad c_1c_2 > b_1b_2.$$

We know that b_1, b_2 restrict the growth rate of each population and c_1, c_2 are the constants

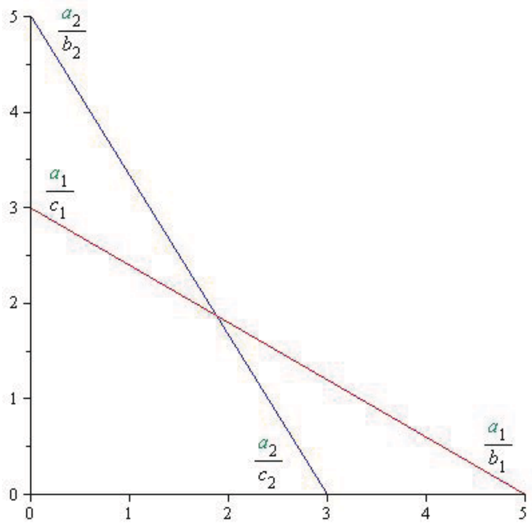

Figure 4.2: The graph of lines (4.2.1) if $c_1c_2 > b_1b_2$.

representing competition between two species. Using these, the following interpretations can be done.

i. If $c_1c_2 < b_1b_2$, that is, if competition effect is smaller that the limitation effect, then the phase paths approach the critical point $(\tilde{x}, \tilde{y})$ as $t \to \infty$. So $(\tilde{x}, \tilde{y})$ is asymptotically stable. This guarantees the coexistence of both species.

ii. If $c_1c_2 > b_1b_2$, i.e., if the competition effect is greater than the inhibition effect, then one of the species goes extinct because either $x(t)$ or $y(t)$ approaches zero as $t \to \infty$, corresponding to the unstable critical point $(\tilde{x}, \tilde{y})$. In this case, coexistence is impossible.

Example 4.2.1. *Discuss the prey-predator system that is modeled by the equations*

$$\begin{aligned} \frac{dx}{dt} &= 5x - x^2 - xy = x(5 - x - y), \\ \frac{dy}{dt} &= xy - 2y = y(x - 2). \end{aligned} \tag{4.2.2}$$

Solution 4.2.1. *Equating the right-hand sides of the system to zero,*

$$f_1(x, y) = x(5 - x - y) = 0,$$

$$f_2(x, y) = y(x - 2) = 0,$$

we obtain fixed points $(0,0)$, $(5,0)$, *and* $(2,3)$. *We linearize the system to find the Jacobian matrix*

$$J = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix} = \begin{bmatrix} 5-2x-y & -x \\ y & x-2 \end{bmatrix}.$$

For $(0,0)$, *the linearized system is*

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J_{(0,0)} \begin{bmatrix} x \\ y \end{bmatrix},$$

that is,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The characteristic equation is $(5-\lambda)(-2-\lambda) = 0$, *i.e.,* $\lambda_1 = 5 > 0$ *and* $\lambda_2 = -2 < 0$. *Hence,* $(0,0)$ *is a saddle point, and it is unstable.*

For $(5,0)$, *we use the substitution*

$$\begin{aligned} u_1 &= x - 5, \\ u_2 &= y, \end{aligned}$$

to obtain a linearized system with the critical point $(0,0)$,

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = J_{(5,0)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

so that

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} -5 & -5 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

The characteristic equation is $(-5-\lambda)(3-\lambda) = 0$. *The roots are* $\lambda_1 = -5 < 0$ *and* $\lambda_2 = 3 > 0$, *i.e.,* $(5,0)$ *is an unstable saddle point.*

For $(2,3)$, *the suitable substitution is*

$$\begin{aligned} v_1 &= x - 2, \\ v_2 &= y - 3. \end{aligned}$$

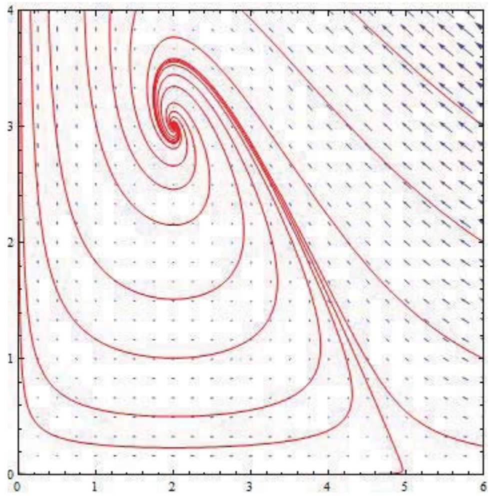

Figure 4.3: The trajectories of the system (4.2.2).

Then $(0,0)$ *is the corresponding critical point for* $(2,3)$, *and the corresponding system is*

$$\begin{bmatrix}\dot{v}_1\\ \dot{v}_2\end{bmatrix} = J_{(2,3)}\begin{bmatrix}v_1\\ v_2\end{bmatrix},$$

that is,

$$\begin{bmatrix}\dot{v}_1\\ \dot{v}_2\end{bmatrix} = \begin{bmatrix}-2 & -2\\ 3 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}.$$

The characteristic equation is $\lambda^2 + 2\lambda + 6 = 0$ *with the complex roots* $\lambda_1 = -1 + i\sqrt{5}$ *and* $\lambda_2 = -1 - i\sqrt{5}$. *Since* $Re(\lambda) < 0$, *the critical point* $(2,3)$ *is an asymptotically stable spiral.*

We can conclude that, for any pair of initial values x_0, y_0, *both species coexist with the population densities approaching the constant values* $x(t) \equiv 2$ *and* $y(t) \equiv 3$. *The result is shown graphically in Figure 4.3.*

Example 4.2.2. *Discuss the prey-predator system*

$$\begin{aligned}\frac{dx}{dt} &= x^2 - 2x - xy = x(x-y-2), \qquad (4.2.3)\\ \frac{dy}{dt} &= y^2 - 4y + xy = y(x+y-4).\end{aligned}$$

Solution 4.2.2. *The equilibrium points are found from the system of equations*

$$\begin{aligned} x(x-y-2) &= 0, \\ y(x+y-4) &= 0. \end{aligned}$$

We obtain four equilibrium points: $(0,0)$, $(0,4)$, $(2,0)$, $(3,1)$. *The Jacobian matrix is*

$$J = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix} = \begin{bmatrix} 2x-2-y & -x \\ y & 2y-4+x \end{bmatrix}.$$

For $(0,0)$*, the linearized system is*

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J_{(0,0)} \begin{bmatrix} x \\ y \end{bmatrix},$$

or

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The characteristic equation is $(-2-\lambda)(-4-\lambda) = 0$ *and the roots are* $\lambda_1 = -2 < 0$, $\lambda_2 = -4 < 0$. *Hence,* $(0,0)$ *is an asymptotically stable node.*

For $(0,4)$, *we can use the substitution*

$$\begin{aligned} u_1 &= x, \\ u_2 &= y-4. \end{aligned}$$

The corresponding linearized system with the critical point shifted to the origin is

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = J_{(0,4)} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

or

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

The eigenvalues of the characteristic equation $(-6-\lambda)(4-\lambda) = 0$ *are* $\lambda_1 = -6 < 0$, $\lambda_2 = 4 > 0$. *Therefore, the critical point* $(0,4)$ *is an unstable saddle point.*

For $(2,0)$, *the suitable substitution reducing to the corresponding system with the critical point* $(0,0)$ *is*

$$v_1 = x - 2,$$
$$v_2 = y.$$

The linearized system is

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = J_{(2,0)} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

so that,

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

The characteristic equation is $(2-\lambda)(-2-\lambda) = 0$. *The roots are* $\lambda_1 = 2 > 0$, $\lambda_2 = -2 < 0$. *Hence* $(2,0)$ *is the unstable saddle point.*

Finally, let us investigate the most important critical point $(3,1)$. *The substitution is*

$$w_1 = x - 3,$$
$$w_2 = y - 1.$$

The linearized system with the critical point $(0,0)$ *is*

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = J_{(3,1)} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

The characteristic equation is $\lambda^2 - 4\lambda + 9 = 0$, *and the roots are* $\lambda_1 = 2 + i\sqrt{5}$ *and* $\lambda_2 = 2 - i\sqrt{5}$. $Re(\lambda) > 0$ *so* $(3,1)$ *is an unstable spiral. Figure 4.4 demonstrates the result.*

Therefore, we can conclude that the phase plane is divided into two regions. If the initial point is in the first region which is close to $(0,0)$, *both species go extinct, but if it is in the second region, both populations increase without any limitations.*

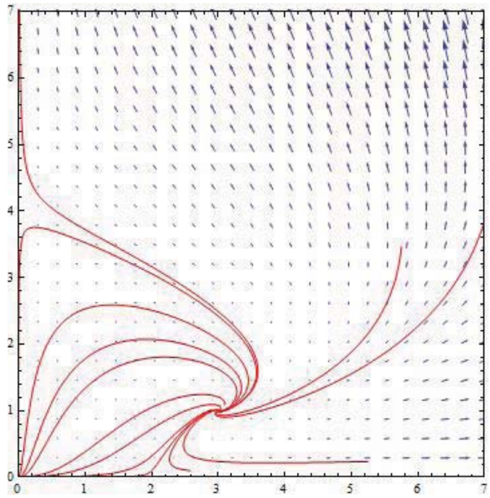

Figure 4.4: The trajectories of the system (4.2.3).

4.3 Harvesting

One or two species may be harvested. How does this affect the populations of the species? The easiest model is a constant-effort harvesting. Suppose that both prey and predator are harvested at a constant rate. In this model, the amount harvested is proportional to the population density, which leads to the system

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy - H_1 x, \\ \frac{dy}{dt} &= -cy + dxy - H_2 y, \end{aligned}$$

where $H_1 \geq 0$ and $H_2 \geq 0$ are harvesting coefficients. The equilibrium point is

$$\left(\frac{c + H_2}{d}, \frac{a - H_1}{b} \right).$$

Here it is assumed that $a > H_1$, since otherwise the equilibrium point is not in the population quadrant. Notice that without harvesting, the equilibrium point is

$$\left(\frac{c}{d}, \frac{a}{b} \right).$$

In the presence of harvesting, this equilibrium point moves to the right and towards the x-axis. We know that x and y coordinates of the equilibrium point show the populations of the prey and predator respectively. Therefore, we conclude that, the density of the prey population increases, while the density of the predator population decreases, that is, harvesting helps the prey and harms the predator [1, Page 326].

Chapter 5

RECENT RESULTS ON THE QUALITATIVE BEHAVIOR OF A RATIO-DEPENDENT PREDATOR-PREY SYSTEM

Consider the system

$$\begin{aligned} \dot{x} &= x\alpha(x) - yV(x), \\ \dot{y} &= yK(x), \end{aligned}$$

where x denotes the density of the prey population and y denotes the density of the predator population. It is assumed that

i. α is smooth with $\alpha'(x) < 0$, $x \geq 0$ and $\alpha(0) > 0 > \lim_{x\to+\infty} \alpha(x)$;

ii. K and V are nonnegative and increasing, $K(0) = 0 = V(0)$.

Consider now the model

$$\begin{aligned} \dot{x} &= f_1(x, y), \\ \dot{y} &= f_2(x, y), \end{aligned} \tag{5.0.1}$$

where

$$f_1(x, y) = \alpha x \left(1 - \frac{x}{K}\right) - \frac{\beta xy}{\varepsilon y + x}, \qquad x^2 + y^2 > 0,$$

$$f_2(x, y) = -\frac{y(\gamma + \delta y)}{1 + y} + \frac{\beta xy}{\varepsilon y + x}, \qquad x^2 + y^2 > 0,$$

$\alpha > 0$, $\varepsilon > 0$ are the growth rates of prey population in the absence of predators and environmental limitations (i.e. in the absence of predators, prey population grows). On the other hand, in the absence of prey population, predator population decreases. The mortality (death rate) of predator population depends on y and can be expressed by the formula

$$E(y) = \frac{\gamma + \delta y}{1 + y}.$$

Here it is assumed that $\gamma < \delta$.

This model differs from other predator-prey systems by the predator mortality rate since mortality is not a constant or unbounded function. It increases with the predator population.

The system (5.0.1) can be written in polar coordinates using transformations

$$x = r(\theta)\cos\theta \qquad \text{and} \qquad y = r(\theta)\sin\theta.$$

Then we notice that $f_i \in C^1(\mathbb{R}_0^+, \mathbb{R})$ for $i = 1, 2$. Hence, the Existence and Uniqueness Theorem guarantees that the solution of (5.0.1) exists and is unique. If we consider f_1 and f_2 in the form

$$f_1(x, y) = xM_1(x, y) = x\left[\alpha\left(1 - \frac{x}{K}\right) - \frac{\beta y}{\varepsilon y + x}\right],$$

$$f_2(x, y) = yM_2(x, y) = y\left[-\frac{(\gamma + \delta y)}{1 + y} + \frac{\beta x}{\varepsilon y + x}\right],$$

the following can be derived easily [6].

i. M_1 and M_2 are smooth functions so the positive quadrant is an invariant region.

ii. $\frac{\partial M_1}{\partial y} = -\frac{\beta x}{(\varepsilon y + x)^2} < 0$ and $\frac{\partial M_2}{\partial x} = \frac{\beta \varepsilon y}{(\varepsilon y + x)^2} > 0$, where $x, y > 0$. That is, it is a predator-prey system with the prey x and the predator y.

Theorem 5.0.1. *All solutions of the system (5.0.1) are bounded.*

Proof. We have

$$\dot{x} \leq \alpha x \left(1 - \frac{x}{K}\right).$$

Therefore,

$$\lim_{t \to +\infty} \sup x(t) \leq K, \quad \text{i.e.,} \quad x(t) < K + c, \quad \text{for some } 0 < c < 1.$$

Adding two equations in (5.0.1) gives

$$\dot{x} + \dot{y} = \alpha x \left(1 - \frac{x}{K}\right) - \frac{y(\gamma + \delta y)}{1 + y}.$$

Hence, there exists a constant $C > 0$ such that all trajectories starting at the initial point $(x_0, y_0) \in \mathbb{R}^2_+$, enter the region

$$R = \{(x, y) \in \mathbb{R}^2_+ : x + y \leq C + \varepsilon, \quad \varepsilon > 0\}.$$

□

The system (5.0.1) can be rewritten using the following transformations

$$x = Ku, \qquad y = \frac{K}{\varepsilon} v, \qquad t = \frac{s}{\alpha}.$$

A new system with the new variables u and v is

$$\dot{u} = f(u, v) = u(1 - u) - \frac{auv}{v + u},$$
$$\tag{5.0.2}$$
$$\dot{v} = g(u, v) = -\frac{v(b + cv)}{1 + ev} + \frac{duv}{v + u},$$

where $a = \frac{\beta}{\alpha\varepsilon}$, $b = \frac{\gamma}{\alpha}$, $k = \frac{\delta}{\alpha}$, $d = \frac{\beta}{\alpha}$, $e = \frac{K}{\varepsilon}$, $c = ke$, and the derivatives are with respect to the variable s.

Criterion 5.0.2 (Dulac's Negative Criterion)**.** *For the system*

$$\begin{aligned} \dot{x} &= X(x, y), \\ \dot{y} &= Y(x, y), \end{aligned}$$

there are no closed paths in a simply-connected region in which

$$\frac{\partial(\rho X)}{\partial x} + \frac{\partial(\rho Y)}{\partial y}$$

is of one sign, where $\rho(x, y)$ is any function having continuous first partial derivatives.

Theorem 5.0.3. *Suppose that $d > a$. Then the system (5.0.2) has no limit cycles in $\mathbb{R}^2_+$.*

Proof. Let

$$F = \begin{bmatrix} f(u,v) \\ g(u,v) \end{bmatrix}.$$

Define a function h such that

$$h(u,v) = \frac{1}{uv}, \qquad \text{for } u > 0 \text{ and } v > 0.$$

From the assumption $d > a$ (i.e. $\varepsilon > 1$) and from the natural assumption $\gamma < \delta$ (i.e. $be > c$), we have

$$\begin{aligned}(div(hF))(u,v) &= \left(\frac{\partial}{\partial u}(hf)\right)(u,v) + \left(\frac{\partial}{\partial v}(hg)\right)(u,v) \\ &= -\frac{1}{v} - \frac{d-a}{(v+u)^2} - \frac{c-be}{u(1+ev)^2} < 0, \qquad (u > 0,\ v > 0).\end{aligned}$$

Hence, by Dulac's negative criterion, there exist no limit cycles in $\mathbb{R}^2_+$. □

5.1 Equilibrium Points and Their Stability

The equilibrium points of the system (5.0.2) are $(0,0)$ and $(1,0)$ for all parameter values. Besides this, there may exists another equilibrium point (u_*, v_*) for some parameter values.

For the point $(1,0)$, the Jacobian matrix is

$$J_{(1,0)} = \begin{bmatrix} -1 & -a \\ 0 & -b+d \end{bmatrix}.$$

The characteristic equation is obtained as

$$(-1-\lambda)(-b+d-\lambda) = 0,$$

and the eigenvalues are

$$\lambda_1 = -1 \qquad \text{and} \qquad \lambda_2 = -b + d.$$

If $b < d$ (i.e. $\gamma < \beta$), then we have $\lambda_1 < 0$, $\lambda_2 > 0$. Hence $(1, 0)$ is a saddle point.

If $b > d$ (i.e. $\gamma > \beta$), then $\lambda_1 < 0$, and $\lambda_2 < 0$. So $(1, 0)$ is a stable node.

For the point $(0, 0)$, the Jacobian matrix cannot be calculated directly since $\frac{u}{v}$ is not defined. We can use the transformation $w = \frac{u}{v}$ to rewrite the system (5.0.2) as follows:

$$\begin{aligned} \dot{w} &= w\left[1 - wv + \frac{b + cv}{1 + ev} - \frac{a + dw}{1 + w}\right], \\ \dot{v} &= v\left[\frac{dw}{1 + w} - \frac{b + cv}{1 + ev}\right]. \end{aligned}$$

A new system has no singularities. Therefore, we can evaluate the Jacobian matrix at $(0, 0)$:

$$J_{(0,0)} = \begin{bmatrix} 1 - a + b & 0 \\ 0 & -b \end{bmatrix}.$$

The roots of the characteristic equation are $\lambda_1 = 1 - a + b$ and $\lambda_2 = -b$.

If $a < 1 + b$, then $\lambda_1 > 0$ and $\lambda_2 < 0$. Thus, $(0, 0)$ is a saddle point.

If $a > 1 + b$, then $\lambda_1 < 0$ and $\lambda_2 < 0$, leading to a stable node at the point $(0, 0)$.

For the point (u_*, v_*), assume that

$$b < d < k \qquad i.e, \qquad \gamma < \beta < \delta. \tag{5.1.1}$$

The prey nullcline is found as follows:

$$\begin{aligned} u(1 - u) - \frac{auv}{v + u} &= 0 \Rightarrow \\ u(1 - u)(v + u) - auv &= 0 \Rightarrow \\ u^2(1 - u) + u(1 - u)v - auv &= 0 \Rightarrow \\ (u(1 - u) - au)\, v &= -u^2(1 - u). \end{aligned}$$

$$v = \frac{-u^2(1 - u)}{u(1 - u) - au},$$

$$v = h_1(u) := \frac{u(1-u)}{u+a-1},$$

and similarly the predator nullcline is

$$\begin{aligned}
\frac{-v(b+cv)}{1+ev} + \frac{duv}{v+u} &= 0 \Rightarrow \\
-v(b+cv)(v+u) + duv(1+ev) &= 0 \Rightarrow \\
(-bv - cv^2)(v+u) + duv + deuv^2 &= 0 \Rightarrow \\
-bv^2 - cv^3 - buv - cuv^2 + duv + deuv^2 &= 0 \Rightarrow \\
-cv^2 + (-b - cu + deu)v + (-bu + du) &= 0.
\end{aligned}$$

The roots of this equation are

$$v_{1,2} = \frac{(b + cu - deu) \pm \sqrt{(-b - cu + deu)^2 + 4c(-bu + du)}}{-2c}.$$

We take the positive one as the predator nullcline

$$v = h_2(u) := \frac{(de - c)u - b + \sqrt{(b + (c - de)u)^2 - 4c(b - d)u}}{2c}.$$

The equilibrium point (u_*, v_*) occurs at the intersection of prey and predator nullclines, if it exists.

There are three cases for prey nullcline depending on the sign of $a - 1$.

Case I: ($a < 1$)

i. For $u \in [0, 1 - a)$,

$$h_1(u) < 0.$$

ii.

$$
\begin{aligned}
h_1'(u) &= \frac{(1-2u)(u+a-1)-u(1-u)}{(u+a-1)^2} \\
&= \frac{u+a-1-2u^2-2au+2u-u+u^2}{(u+a-1)^2} \\
&= \frac{-u^2-2au+2u+a-1+a^2-a^2+a-a}{(u+a-1)^2} \\
&= \frac{a^2-a-(u^2+2au-2u+a^2-2a+1)}{(u+a-1)^2} \\
&= \frac{a^2-a-(u+a-1)^2}{(u+a-1)^2} \\
&= \frac{a(a-1)}{(u+a-1)^2}-1<0 \qquad \text{on } (1-a,+\infty).
\end{aligned}
$$

iii.

$$
\begin{aligned}
h_1''(u) &= \frac{-2a(a-1)}{(u+a-1)^3} \\
&= \frac{2a(a-1)}{(1-a-u)^3}>0 \qquad \text{on } (1-a,+\infty).
\end{aligned}
$$

iv. $\lim_{u\to(1-a)^+} h_1(u) = \lim_{u\to(1-a)^+} \frac{u(u-1)}{u+a-1} = +\infty$ and $h_1(1)=0$.

So the prey nullcline is monotone decreasing for the interval $(1-a,1]$.

Case II: $(a=1)$

In this case,

$$h_1(u)=1-u.$$

The graph of $h_1(u)$ cuts the axes at $u=1$ and $v=1$ so it is monotone decreasing.

Case III: $(a>1)$

i.

$$h_1(0)=0.$$

ii.

$$h_1'(0)=\frac{1}{a-1}>0.$$

iii.

$$\begin{aligned} h_1'(u) &= \frac{a(a-1)}{(u+a-1)^2} - 1 = 0 \Rightarrow \\ a(a-1) &= (u+a-1)^2 \Rightarrow \\ a^2 - a &= u^2 + 2au - 2u + a^2 - 2a + 1 \Rightarrow \end{aligned}$$

$$u^2 + (2a-2)u + (-a+1) = 0.$$

$$\begin{aligned} u_{1,2} &= \frac{-2a+2 \pm \sqrt{(2a-2)^2 - 4(-a+1)}}{2} \\ &= \frac{-2a+2 \pm 2\sqrt{a(a-1)}}{2} \\ &= \pm\sqrt{a(a-1)} + 1 - a. \end{aligned}$$

The positive one is

$$u = \sqrt{a(a-1)} + 1 - a \quad \in \quad (0,1).$$

iv.

$$h_1''(u) = \frac{-2a(a-1)}{(u+a-1)^3} < 0 \qquad \text{when} \qquad u \geq 0.$$

We conclude that the prey nullcline starts from the origin and increases. It is concave down with maximum at $\sqrt{a(a-1)} + 1 - a$.

On the other hand, the predator nullcline is independent from the sign of $a-1$. It starts from the origin and is concave down since

i.

$$h_2(0) = 0.$$

ii.

$$\begin{aligned} h_2'(u) &= \frac{1}{2c}\left[(de-c) + \frac{2\,(b+(c-de)u)\,(c-de) - 4c(b-d)}{2\sqrt{(b+(c-de)u)^2 - 4c(b-d)u}}\right] \\ &= \frac{1}{2c}\left[(de-c) + \frac{(b+(c-de)u)\,(c-de) - 2c(b-d)}{\sqrt{(b+(c-de)u)^2 - 4c(b-d)u}}\right]. \end{aligned}$$

Consider the following limit:

$$
\begin{aligned}
\lim_{u\to+\infty} h_2'(u) &= \frac{de-c}{2c} + \frac{1}{2c}.\frac{(c-de)^2}{\sqrt{(c-de)^2}} \\
&= \frac{de-c+\sqrt{(c-de)^2}}{2c} \\
&= \frac{de-c+|c-de|}{2c} \\
&= \frac{de-c+c-de}{2c} \\
&= 0
\end{aligned}
$$

since

$$
c-de = \frac{\delta K}{\alpha\varepsilon} - \frac{\beta}{\alpha}.\frac{K}{\varepsilon} = \frac{K}{\varepsilon\alpha}(\delta-\beta) > 0 \tag{5.1.2}
$$

by virtue of (5.1.1). Similarly, it follows from (5.1.1) that

$$
\begin{aligned}
\lim_{u\to 0^+} h_2'(u) &= \frac{de-c}{2c} + \frac{b(c-de)}{2c\,|b|} + \frac{(c-de)^2u}{\sqrt{(b+(c-de)u)^2-4c(b-d)u}} - \frac{2c(b-d)}{2c\,|b|} \\
&= \frac{1}{2c}\left[de-c+\frac{b(c-de)-2c(b-d)}{b}\right] \\
&= \frac{d-b}{b} > 0.
\end{aligned}
$$

Hence,

$$
h_2'(u) \to \begin{cases} 0^+, & \text{as } u\to+\infty, \\ \text{constant} > 0, & \text{as } u \to 0^+, \end{cases}
$$

that is,

$$
h_2'(u) > 0 \quad \text{for all} \quad u > 0.
$$

Thus, $h_2(u)$ is an increasing function.

Now, let us check the concavity of $h_2(u)$.

$$
h_2''(u) = \frac{2de(d-b)(b-k)}{\sqrt{\left(4c(d-b)u+(b+(c-de)u)^2\right)^3}} < 0 \quad \text{for} \quad u > 0,
$$

i.e., it is concave down. The limit of the function which defines the predator nullcline is

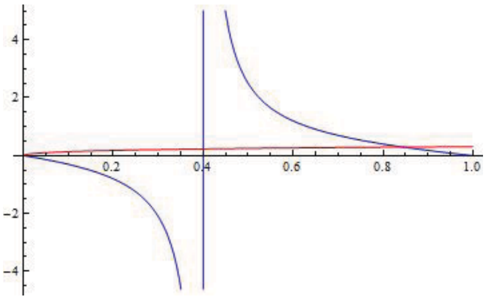

Figure 5.1: The prey and predator nullclines of the system (5.0.2) with $a < 1$.

evaluated as follows:

$$\begin{aligned}\lim_{u\to+\infty} h_2(u) &= \lim_{u\to+\infty} \frac{(de-c)u - b + \sqrt{(b+(c-de)u)^2 - 4c(b-d)u}}{2c} \\ &= \lim_{u\to+\infty} \frac{\sqrt{(b+(c-de)u)^2 - 4c(b-d)u} - (b-(de-c)u)}{2c}.\end{aligned}$$

Multiplying the expression in the numerator by the conjugate gives

$$\begin{aligned}\lim_{u\to+\infty} h_2(u) &= \lim_{u\to+\infty} \frac{(b+(c-de)u)^2 - 4c(b-d)u - (b-(de-c)u)^2}{2c\left[\sqrt{(b+(c-de)u)^2 - 4c(b-d)u} + (b-(de-c)u)\right]} \\ &= \lim_{u\to+\infty} \frac{-4c(b-d)u}{2c\left[\sqrt{(b+(c-de)u)^2 - 4c(b-d)u} + (b-(de-c)u)\right]} \\ &= \frac{-2(b-d)}{\sqrt{(c-de)^2} - (de-c)} = \frac{-2(b-d)}{|c-de| - (de-c)} \\ &= \frac{-2(b-d)}{c-de-de+c} = \frac{-2(b-d)}{2c-2de} = \frac{d-b}{c-de},\end{aligned}$$

by virtue of (5.1.2).

Summarizing the above discussion, we conclude the following.

i. If $a < 1$ as in Case I, the system (5.0.2) has a unique equilibrium point (u_*, v_*) in the population quadrant with $u_* \in (1-a, 1)$, as shown in Figure 5.1, where the blue curve represents the prey nullcline and the red one is the predator nullcline.

ii. When $a = 1$, as in Case II stated above, the system (5.0.2) has a unique equilibrium point with positive coordinates (u_*, v_*) such that

$$u_* = \frac{c + d(e+1) - \sqrt{(c+d(e+1))^2 - 4de(b+c)}}{2de}$$

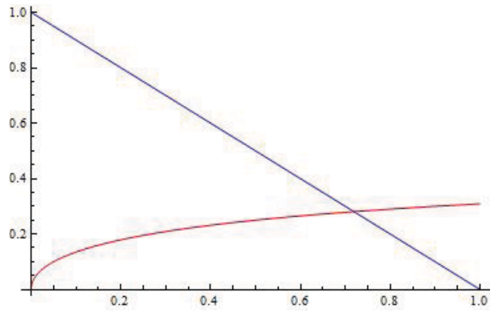

Figure 5.2: The prey and predator nullclines of the system (5.0.2) with $a = 1$.

and

$$v_* = 1 - u_*.$$

The result is illustrated in Figure 5.2. The blue curve is the prey nullcline and the red curve is the predator nullcline.

iii. In the Case III, where $a > 1$, the system (5.0.2) has either no equilibrium points with positive coordinates or two equilibrium points in the first quadrant. This depends on whether the maximum point of h_1 is smaller or larger than the value of h_2 at the stationary point of h_1. If the maximum point of h_1 is greater than the limit value of h_2 at infinity, we have

$$\begin{aligned}
\max h_1 &= h_1\left(\sqrt{a(a-1)} + 1 - a\right) \\
&= \frac{\left(\sqrt{a(a-1)} + 1 - a\right)\left(a - \sqrt{a(a-1)}\right)}{\sqrt{a(a-1)} + 1 - a + a - 1} \\
&= \frac{a\sqrt{a(a-1)} - a(a-1) + a - \sqrt{a(a-1)} - a^2 + a\sqrt{a(a-1)}}{\sqrt{a(a-1)}} \\
&= 2a - 1 - 2\sqrt{a(a-1)} > \frac{d-e}{c-de} = \lim_{u\to+\infty} h_2(u).
\end{aligned}$$

In this case, there exist two equilibrium points with positive coordinates illustrated in Figure 5.3. In the figure, the blue curve is h_1, the prey nullcline and the red curve is h_2, the predator nullcline. On the other hand, if

$$\max h_1 < \lim_{u\to+\infty} h_2(u),$$

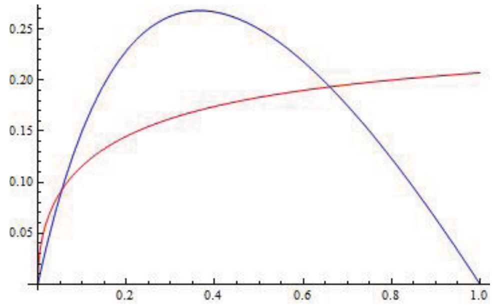

Figure 5.3: The prey and predator nullclines of the system (5.0.2) with $a > 1$.

the system has no equilibrium points with positive coordinates.

Consider now the Jacobian of the system (5.0.2) at the equilibrium point $E_* = (u_*, v_*)$,

$$J(u_*, v_*) = \begin{bmatrix} \theta_3 - a\theta_2^2 & -a\theta_1^2 \\ d\theta_2^2 & -\theta_4 + d\theta_1^2 \end{bmatrix},$$

where

$$\theta_1 = \frac{u_*}{v_* + u_*}, \qquad \theta_2 = \frac{v_*}{v_* + u_*},$$
$$\theta_3 = 1 - 2u_*, \qquad \theta_4 = \frac{b + cv_*(2 + ev_*)}{(1 + ev_*)^2}.$$

The characteristic equation of $J(u_*, v_*)$ is

$$\lambda^2 - (tr(J(u_*, v_*)))\lambda + \det(J(u_*, v_*)) = 0.$$

Thus, E_* is asymptotically stable if

$$tr(J(u_*, v_*)) = \theta_3 - a\theta_2^2 - \theta_4 + d\theta_1^2 < 0, \tag{5.1.3}$$

and

$$\det(J(u_*, v_*)) = (\theta_3 - a\theta_2^2)(-\theta_4 + d\theta_1^2) + ad\theta_1^2\theta_2^2 > 0. \tag{5.1.4}$$

Theorem 5.1.1. *If $d > a$, $b < d < k$ and $a \leq 1$, then the system (5.0.2) has a unique positive equilibrium point that is asymptotically stable when (5.1.3) and (5.1.4) are satisfied [6].*

Chapter 6

A NUTRIENT-PREY-PREDATOR MODEL WITH INTRATROPIC PREDATION

Intratrophic predation is a kind of predation where predator feeds on both prey and predator. In this chapter, we investigate the influence of intratrophic predation on a three trophic leveled nutrient-prey-predator model.

Let $x(t)$ denote the nutrient concentration at time t. It is assumed that, in the absence of prey, the nutrient has a logistic growth, $rx\left(1-\frac{x}{K}\right)$, where r is the growth rate of x and K is the carrying capacity. Let $y(t)$ and $z(t)$ denote the prey and predator populations at time t, respectively. The system is modelled as follows:

$$\begin{aligned} \dot{x} &= rx\left(1-\frac{x}{K}\right)-\frac{m_1xy}{a_1+x}, \\ \dot{y} &= \frac{\alpha m_1xy}{a_1+x}-\frac{m_2yz}{a_2+y+bz}-\gamma y, \\ \dot{z} &= \frac{\beta m_2(y+bz)z}{a_2+y+bz}-\frac{m_2bz^2}{a_2+y+bz}-\delta z, \end{aligned} \tag{6.0.1}$$

$$x(0), y(0), z(0) \geq 0,$$

where r, K, m_1, a_1, m_2, a_2, γ, $\delta > 0$, $0 < \alpha, \beta \leq 1$, $0 \leq b \leq 1$. Here a_1 is the half-saturation constant, m_1 is the maximal nutrient uptake rate of prey, and α is the net nutrient conversion rate. Similarly, let a_2 denote the half-saturation constant, where m_2 is the maximal uptake rate of predator. Let γ and δ be the death rates of prey and predator, respectively. The measure of intensity of intratrophic predation is denoted as b. The food for the predator is considered by the equation $y + bz$, where $0 \leq b \leq 1$. Clearly, if $b = 0$, there is no intratrophic predation so the predator preys on only the prey population. If $b = 1$, the predator preys on both prey and predator populations equivalently [4].

Theorem 6.0.2. *All solutions are non-negative and bounded.*

Proof. Since

$$\dot{x}|_{x=0} = 0, \qquad \dot{y}|_{y=0} = 0, \qquad \dot{z}|_{z=0} = 0,$$

all solutions are non-negative for $t \geq 0$. Besides this, it is obvious that

$$\dot{x} \leq rx\left(1 - \frac{x}{K}\right).$$

Therefore,

$$\lim_{t\to\infty} \sup x(t) \leq K,$$

and so,

$$\dot{x} + \dot{y} + \dot{z} \leq rx\left(1 - \frac{x}{K}\right) - \gamma y - \delta z \leq rK\left(1 - \frac{x}{K}\right) - \gamma y - \delta z \leq rK - \hat{k}(x + y + z),$$

where

$$\hat{k} = \min\{r, \gamma, \delta\}.$$

Therefore,

$$\lim_{t\to\infty} \sup x(t) + y(t) + z(t) \leq \frac{rK}{\hat{k}},$$

i.e., all solutions are bounded. □

This theorem guarantees that the system (6.0.1) is biologically meaningful.

6.1 Equilibrium Points and Their Stability

The equilibrium points of the system (6.0.1) can be listed as follows;

$$E_0 = (0, 0, 0), E_1 = (K, 0, 0), E_2 = (x_2, y_2, 0), E_3 = (0, y_3, z_3),$$

$$E_4 = (0, 0, z_4), E_5 = (K, 0, z_5), E_6 = (\bar{x}, \bar{y}, \bar{z}),$$

where

$$x_2 = \frac{a_1\gamma}{\alpha m_1 - \gamma}, \quad y_2 = \frac{r\left(1 - \frac{x_1}{K}\right)(a_1 + x_1)}{m_1} = \frac{ra_1\left(\alpha(\alpha m_1 - \gamma)K - \gamma a_1\right)}{K(\alpha m_1 - \gamma)^2},$$

$$y_3 = \frac{(\beta - 1)a_2\gamma b + a_2\delta}{\beta m_2 + b\gamma - \delta}, \quad z_3 = \frac{\beta a_2 \gamma}{\delta - \beta m_2 - b\gamma}, \quad z_4 = z_5 = \frac{a_2\delta}{b\left(m_2(\beta - 1) - \delta\right)},$$

and

$$\bar{x} > 0, \qquad \bar{y} > 0, \qquad \bar{z} > 0.$$

The equilibrium points E_0 and E_1 always exist, but the positive equilibrium point E_2 exists if

$$\alpha m_1 > \gamma \qquad \text{and} \qquad x_1 < K.$$

For the positive E_3, we must have

$$\delta - \beta m_2 - b\gamma > 0 \Rightarrow \delta - b\gamma > \beta m_2, \tag{6.1.1}$$

and

$$(\beta - 1)a_2\gamma b + a_2\delta < 0 \Rightarrow \delta - b\gamma < -\beta\gamma b. \tag{6.1.2}$$

Combining inequalities (6.1.1) and (6.1.2) gives

$$\beta m_2 < -\beta\gamma b$$

which is not possible. Therefore, E_3 is not a positive equilibrium point. Similarly, E_4 and E_5 are not positive either since

$$\beta - 1 < 0.$$

In biological systems, negative equilibrium points are not of interest since they are not biologically meaningful. Thus, only the stability of E_0, E_1, E_2 and E_6 is important for us. To be able to discuss the stability of these equilibrium points, we need Jacobian matrix of the system (6.0.1) which is evaluated as follows:

$$J = \begin{bmatrix} r\left(1-\frac{2}{K}x\right) - \frac{m_1 a_1 y}{(a_1+x)^2} & -\frac{m_1 x}{a_1+x} & 0 \\ \frac{\alpha m_1 a_1 y}{(a_1+x)^2} & \frac{\alpha m_1 x}{a_1+x} - \frac{m_2 z(a_2+bz)}{(a_2+y+bz)^2} - \gamma & -\frac{m_2 y(a_2+y)}{(a_2+y+bz)^2} \\ 0 & \frac{\beta m_2 a_2 z + m_2 b z^2}{(a_2+y+bz)^2} & c \end{bmatrix},$$

where

$$c = \beta m_2 \frac{(y+2bz)(a_2+y+bz) - bz(y+bz)}{(a_2+y+bz)^2} - \frac{2m_2 bz(a_2+y+bz) - m_2 b^2 z^2}{(a_2+y+bz)^2} - \delta.$$

The Jacobian matrix at the point $E_0 = (0,0,0)$ is

$$J(E_0) = \begin{bmatrix} r & 0 & 0 \\ 0 & -\gamma & 0 \\ 0 & 0 & -\delta \end{bmatrix}.$$

The eigenvalues are

$$\lambda_1 = r > 0, \qquad \lambda_2 = -\gamma < 0, \qquad \lambda_3 = -\delta < 0.$$

Therefore, E_0 is unstable.

Define

$$R_0 = \frac{\alpha m_1 K}{\gamma(a_1+K)}, \qquad R_1 = \frac{\beta m_2 y_2}{\delta(a_2+y_2)}, \qquad R_2 = \frac{r\left(1-\frac{2}{K}x_2\right)(a_1+x_2)^2}{m_1 a_1 y_2}.$$

The Jacobian matrix at $E_1 = (K,0,0)$ is

$$J(E_1) = \begin{bmatrix} -r & -\frac{m_1 k}{a_1+k} & 0 \\ 0 & \frac{\alpha m_1 K}{a_1+K} - \gamma & 0 \\ 0 & 0 & -\delta \end{bmatrix}$$

Assume that $R_0 < 1$. The eigenvalues are

$$\lambda_1 = -r < 0, \qquad \lambda_2 = \frac{\alpha m_1 K}{a_1+K} - \gamma < 0, \qquad \lambda_3 = -\delta < 0.$$

Thus E_1 is stable if $R_0 < 1$.

The Jacobian matrix at the equilibrium point $E_2 = (x_2, y_2, 0)$ is

$$J(E_2) = \begin{bmatrix} r\left(1-\frac{2}{K}x_2\right) - \frac{m_1 a_1 y_2}{(a_1+x_2)^2} & -\frac{m_1 x_2}{a_1+x_2} & 0 \\ \frac{\alpha m_1 a_1 y_2}{(a_1+x_2)^2} & 0 & -\frac{m_2 y_2}{a_2+y_2} \\ 0 & 0 & \frac{\beta m_2 y_2}{a_2+y_2} - \delta \end{bmatrix}.$$

Assume that $R_0 > 1$, $R_1 < 1$ and $R_2 < 1$. The eigenvalues are

$$\begin{aligned}
\lambda_1 &= \frac{\beta m_2 y_2}{a_2 + y_2} - \delta < 0 \quad \text{since} \quad R_1 < 1, \\
\lambda_{2,3} &= \frac{-p \pm \sqrt{p^2 - 4q}}{2},
\end{aligned}$$

where

$$q > 0 \quad \text{and} \quad p > 0$$

since $R_2 < 1$. If $p^2 - 4q \geq 0$, then both λ_2 and λ_3 are negative. If $p^2 - 4q < 0$, then $Re(\lambda_2)$ and $Re(\lambda_3)$ are negative. That is, in any case E_2 is stable under the given assumptions.

Therefore, we conclude that the stability of E_0, E_1 and E_2 are independent of b, i.e., intratrophic predation has no effect on the stability of the equilibrium points E_0, E_1 and E_2.

The stability of the positive steady state E_6 is not discussed here, but we note that intratrophic predation plays an important role in this case.

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