Szász and Phillips Operators Based on q-Integers

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ABSTRACT

In this thesis, q-Szász-Durrmeyer ($0 < q < 1$) and q-Phillips ($q > 0$) operators are defined and some properties of these operators are studied. More precisely, local approximation results for continuous functions in terms of modulus of continuity are proved and Voronovskaja type asymptotic results are investigated.

Keywords: q-Szász-Durrmeyer operators, k-functional, modulus of continuity, q-calculus, q-Phillips operators, q-integers, q-gamma functions, rate of convergence, q-integral.

ÖZ

Bu tezde, q-Szász-Durrmeyer $(0<q<1)$ ve q-Phillips $(q>0)$ operatörleri tanımlanmış ve bu operatörlerin bazı özellikleri incelenmiştir. Daha açık olarak, süreklilik modülü cinsinden, sürekli fonksiyonlar için yerel yaklaşım sonuçları ispatlanmış ve Voronovskaja tipli asimtotik sonuçlar incelenmiştir.

Anahtar kelimeler: q-Szazs-Durrmeyer operatörleri, k-fonksiyonel, süreklilik modülü, q-hesap, q-Phillips operatörleri, q-tamsayılar, q-gamma fonksiyonları, yakınsama hızı, q-integral.

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NOTATIONS AND SYMBOLS

In this thesis we shall often make use of the following symbols:

$:=$	is the sign indicating equal by definition . "$a := b$" indicates that a is the quantity to be defined or explained, and b provides the definition or explanation. "$b := a$" has the same meaning,
$\mathbb{N}$	the set of natural numbers,
$\mathbb{N}_0$	the set of natural numbers including zero,
$\mathbb{Z}$	the set of all integers numbers,
$\mathbb{R}$	the set of real numbers,
$\mathbb{R}_+$	the set of positive real numbers,
$\mathbb{R}^+$	the set of positive real numbers,
(a, b)	an open interval,
$[a, b]$	a closed interval,
$\overline{E}$	closure of E,
$\mathbb{R}^{[0,\infty)}$	all real functions on the interval $[0,\infty)$,
$\mathbb{R}^{[0,1]}$	all real functions on the interval $[0, 1]$.
	Let X □be an interval of the real axis.
$F(X)$	the set of all real-valued functions defined on X.
$B(X)$	the set of all real-valued and bounded functions defined on X.
$C(X)$	the set of all real-valued and continuous functions defined on X.
$C[a, b]$	the set of all real-valued and continuous functions defined on the compact interval $[a, b]$.

For $f \in B(X)$ or $f \in C(X)$

$\|f\|_\infty$ is the sup-norm, namely $\|f\|_\infty := \sup\{|f(x)| : x \in X\}$.

$\Delta f(x_i)$ is the forward difference defined as

$$\Delta f(x_i) = f(x_{i+1}) - f(x_i) = f(x_i + h) - f(x_i), \text{ with step size } h,$$

$$\Delta^0 f(x_i) = f(x_i), \quad \Delta^r f(x_i) = \Delta(\Delta^{r-1} f(x_i)),$$

$\Delta_h^k f(x)$ is the finite diference of order $k \in \mathbb{N}$, with step size $h \in \mathbb{N} \backslash \{0\}$ and

Starting point $x \in X$. It's formula is given by

$$\Delta_h^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + jh).$$

$C^r[a, b]$ the set of all real-valued, r-times continuously differentiable function,

where $r \in \mathbb{N}$.

$C_B[0, \infty)$ the space of all real-valued continuous bounded functions f on $[0, \infty)$

endowed with the norm $\|f\| = \sup_{x \in [0,\infty)} |f(x)|$.

$\omega(f;\delta)$ the first modulus of continuity . It is defined as

$$\omega(f;\delta) = \sup_{0 < h \le \delta} \sup_{x \in [0,\infty)} |f(x+h) - f(x)|.$$

$\omega_2(f;\sqrt{\delta})$ the second modulus of continuity. It is defined as

$$\omega_2(f;\sqrt{\delta}) = \sup_{0 < h \le \delta} \sup_{x \in [0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|$$

where $f \in C_B[0, \infty)$ and $\delta > 0$,

$d(x, E)$ the distance between x and E . It is defined as

$$d(x, E) = \inf\{|t - x| : t \in E\}.$$

$K_2(f;\delta)$ the Peetre's K-functional. It is defined as

$$K_2(f;\delta) = \inf_{g \in C_B^2[0,\infty)} \left\{ \|f-g\| + \delta \|g''\| \right\}.$$

where $C_B^2[0,\infty) := \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$.

e_n denotes the n-th monomial with, $e_n : [a,b] \to \mathbb{R}$
$x \to x^n$, $n \in \mathbb{N}_0$.

Let $m \in \mathbb{N}$,

$C_m[0,\infty) := \{ f \in C[0,\infty) : \exists M_f > 0 \text{ s.t. } |f(x)| < M_f (1 + x^m)$

and $\|f\|_m := \sup_{x \in [0,\infty)} \dfrac{|f(x)|}{1+x^m} \}$

$$C_m^*[0,\infty) := \left\{ f \in C_m[0,\infty) : \lim_{x \to \infty} \frac{|f(x)|}{1+x^m} < \infty \right\}.$$

$\Omega_m(f;\delta)$; the weighted modulus of continuity. It is defined as

$$\Omega_m(f;\delta) = \sup_{x \ge 0,\ 0 < h \le \delta} \frac{|f(x+h) - f(x)|}{1+(x+h)^m}, \text{ where } f \in C_m^*[0,\infty).$$

$$(x-a)_q^n = \begin{cases} 1 & \textit{if } n = 0 \\ \prod_{j=0}^{n-1} (x - q^j a) & \textit{if } n \ge 1. \end{cases},$$

$$(x-a)_q^\infty = \prod_{j=0}^{\infty} (x - q^j a).$$

Chapter 1

INTRODUCTION

Positive linear operators play a fundamental role in Approximation Theory. In the last few decades, the theory of positive linear operators has been an intensively investigated area of research. Especially, Computer-aided geometric design is effected by the theory of linear operators. In 1885, the first proof of Karl Weierstrass's Theorem on approximation by algebraic or trigonometric polynomials was presented as the key moment in the development of Approximation Theory. Since its' proof was complicated , many famous mathematicians was attemted to find simpler proofs. Sergej N. Bernstein constructed well-known Bernstein polynomials as follows:

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

for any $f \in C\left[0,1\right], x \in [0,1]$ and $n \in \mathbb{N}$.

Due to the importance of the Bernstein polynomials, a variety of their generalizations and related topics have been studied (see, e.g [2]-[7], [9], [11], [12], [15], [19], [23], [27]-[29], [33], [35], [37], [38], [40], [44]-[49]). Recently, an intensive research has been conducted on operators based on q-integers, see [2]-[4], [9], [27], [33], [35], [37], [38], [40], [45]- [49]. In [29], N. I. Mahmudov introduced q-Szász-Mirakjan operators as follows:

$$S_{n,q}(f)(x) := S_{n,q}(f;x) = \frac{1}{\prod\limits_{j=0}^{\infty}\left(1+(1-q)q^j\left[n\right]x\right)} \sum_{k=0}^{\infty} f\left(\frac{[k]}{q^{k-2}\left[n\right]}\right) q^{\frac{k(k-1)}{2}} \frac{\left[n\right]^k x^k}{[k]!}, \tag{1.0.1}$$

where $x \in [0,\infty)\,, 0 < q < 1, n \in \mathbb{N}, f \in C\left[0,\infty\right)$ and investigated approximation properties of q-Szász-Mirakjan operators. Moreover, in [30], N. I. Mahmudov introduced Szász operators based on the q-integers

$$M_{n,q}\left(f;x\right) := \sum_{k=0}^{\infty} f\left(\frac{[k]}{[n]}\right) \frac{1}{q^{\frac{k(k-1)}{2}}} \frac{[n]^k x^k}{[k]!} e_q\left(-[n]\, q^{-k}x\right). \tag{1.0.2}$$

where $q > 1, n \in \mathbb{N}, f : [0, \infty) \to \mathbb{R}$ and $e_q\left(-[n]\, q^{-k}x\right) := e_q^{-[n]q^{-k}x} = \sum_{j=0}^{\infty} \frac{\left(-[n]q^{-k}x\right)^j}{[j]_q!}$.

In this thesis, we use these two operators to obtain new operators. This thesis consist of five chapters and is organized as follows:

In Chapter 2, we give some basic definitions and elementary properties about linear and positive operators. We give the definition of Szász operator and Phillips operator. Moreover, we give some basic definitions and some elementary properties related to q-integers. At the end of this chapter, definitions of two q-Szász operators and their some basic properties are given.

In Chapter 3, we introduce the following q-Szász-Durrmeyer operator

$$D_{n,q}(f;x) = [n] \sum_{k=0}^{\infty} q^k s_{n,k}(q;x) \int_0^{\infty/(1-q)} s_{n,k}(q;t) f(t) d_q t. \tag{1.0.3}$$

where $x \in [0, \infty)\,, f \in \mathbb{R}^{[0,\infty)}, 0 < q < 1, n \in \mathbb{N}$ and $s_{n,k}(q;x) = \frac{1}{E_q([n]x)} q^{\frac{k(k-1)}{2}} \frac{[n]^k x^k}{[k]!} = e_q(-[n]\, x) q^{\frac{k(k-1)}{2}} \frac{[n]^k x^k}{[k]!}$ by using q-Szász-Mirakjan operators which was introduced by N. I. Mahmudov in (1.0.1). Here, we can say that operator (1.0.3) generalize the sequence of classical Szász-Durrmeyer operators. As we mention before, the approximation of functions by using linear positive operators introduced via q-Calculus is currently under intensive research. The q-Bernstein polynomials $B_{n,q}\left(f;x\right)$, $n = 1, 2, ..$, $0 < q < \infty$, were introduced by G. M. Phillips in [40]. While for $q = 1$ these polynomials coincide with the classical ones, for $q \neq 1$ we obtain new polynomials possessing interesting properties, see [9], [35], [36]. In [45], T. Trif introduced the q-Meyer-König and Zeller operators for each positive integer n, and $f \in C[0, 1]$. Like the classical operators, the q-Bernstein operators and the q-Meyer-König and Zeller operators share some good properties such as the shape-preserving properties and monotonicity for convex function. In [23], H. Karslı and V. Gupta introduced and studied approximation properties of q-Chlodowsky operators. In [38], M. A. Özarslan and H. Aktuğlu studied Local approximation properties of certain class of linear positive operators via I-convergence. Very recently, V. Gupta [12] introduced and studied approximation properties of q-Durrmeyer operators. V. Gupta and H. Wang [15] introduce the q-Durrmeyer type operators and studied estimation of the rate of convergence for continuous functions in terms of modulus of continuity. In [11] authors studied some direct local and global approximation theorems for the q-Durrmeyer operators $M_{n,q}$ for $0 < q < 1$. Some other analogues of the Bernstein-Durrmeyer operators related to the Bernstein basis functions $p_{n,k}(q;x)$ have been studied by M. M. Derriennic [6]. In [2], [3] q-Szász-Mirakjan operators were defined and their approximation properties were investigated. In [2], q-Szász-Mirakjan operator were defined as follows

$$\mathbb{S}_{n,q}\left(f\right)\left(x\right) := E_q\left(-\frac{[n]\,x}{b_n}\right)\sum_{k=0}^{\infty} f\left(\frac{[k]\,b_n}{[n]}\right)\frac{[n]^k\,x^k}{[k]!b_n^k}, \tag{1.0.4}$$

where $0 \leq x < \dfrac{b_n}{(1-q)\,[n]}$, $E_q\left(-\frac{[n]x}{b_n}\right) := E_q^{\left(-\frac{[n]x}{b_n}\right)} = \sum\limits_{j=0}^{\infty} q^{j(j-1)/2}\frac{\left(-\frac{[n]x}{b_n}\right)^j}{[j]!}$, $f \in C\left[0,\infty\right)$, and $\{b_n\}$ is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$. Although, from the structural point of view the q-Szász-Mirakjan operators have some similarity to the classical Szász-Mirakjan operators, they have convergence properties similar to the Bernstein-Chlodowsky operators. That is, the interval of convergence of the series grows as $n \to \infty$ as in Bernstein–Chlodowsky operators. In this context, N. I. Mahmudov in [29] introduced q-Szász-Mirakjan operator $S_{n,q}(f)(x)$ and investigated their approximation properties.

In this chapter,

- we introduce the q-Szász-Durrmeyer operators $D_{n,q}$ and evaluate the moments of $D_{n,q}$,

- we prove local approximation result for continuous functions in terms of modulus of continuity,
- we study Voronovskaja type result for the q-Szász-Durrmeyer operators.

In Chapter 4, we introduce the following q-Phillips operators

$$P_{n,q}(f;x) = [n] \sum_{k=1}^{\infty} q^{k-1} s_{n,k}(q;qx) \int_{0}^{\infty/(1-q)} s_{n,k-1}(q;t) f(t) d_q t + e_q\left(-[n]\,qx\right) f(0), \tag{1.0.5}$$

where $x \in [0,\infty)$, $f \in \mathbb{R}^{[0,\infty)}$, $0 < q < 1, n \in \mathbb{N}$ and $s_{n,k}(q;x) = \frac{1}{E_q([n]x)} q^{\frac{k(k-1)}{2}} \frac{[n]^k x^k}{[k]!} = e_q(-[n]\,x) q^{\frac{k(k-1)}{2}} \frac{[n]^k x^k}{[k]!}$ by using q-Szász-Mirakjan operators which was introduced by N. I. Mahmudov in (1.0.1). Here we can say that the operator (1.0.5) generalizes the sequence of classical Phillips operators.

R. S. Phillips [39] defined the well-known positive linear operators

$$P_n(f;x) = n \sum_{k=1}^{\infty} e(-nx) \frac{n^k x^k}{k!} \int_{0}^{\infty} e(-nt) \frac{n^{k-1} t^{k-1}}{(k-1)!} f(t) dt + e\left(-nx\right) f(0),$$

where $x \in [0,\infty)$. Some approximation properties of these operators were studied by Gupta and Srivastava [17] and by May [34]. Bézier variant of these Phillips operators were proposed and studied by Gupta [16], where the rate of convergence for the Bézier variant of the Phillips operators for bounded variation functions was discussed. As we mention before, N. I. Mahmudov in [29] introduced the q-Szász-Mirakjan operators and investigated their approximation properties. Very recently Gupta [13] proposed another sequence of q-Phillips operators based on q-Szász basis functions considered in [2] as

$$P_n^q\left(f\left(t\right);x\right) = \frac{[n]_q}{b_n} \sum_{k=1}^{\infty} s_{n,k}^q(x) \int_{0}^{\frac{qb_n}{1-q^n}} q^{\frac{1}{2}} s_{n,k-1}^q(t) f(q^{-1}t) d_q t + E_q\left(-[n]_q \frac{x}{b_n}\right) f(0),$$

where

$$s_{n,k}^q(x) = \frac{\left([n]_q x\right)^k}{q^{\frac{k+1}{2}} [k]_q! (b_n)^k} E_q\left(-[n]_q \frac{x}{b_n}\right).$$

In this chapter,

- we introduce q-parametric Phillips operators,
- we study the approximation properties of the q-Phillips operators,
- we establish some local approximation results for continuous functions in terms of modulus of continuity ,
- we obtain inequalities for the weighted approximation error of q-Phillips operators,
- we study Voronovskaja type asymptotic formula for the q-Phillips operators.

We have recently considered the q-analogue of well known Phillips operators [39] for the case when $0 < q < 1$. In chapter 5, we consider the other case i.e. $q > 1$ and here we discuss the approximation properties of q-Phillips operators, for this case. Therefore, we introduce the following q-Phillips operators

$$P_{n,q}(f;x) = [n] \sum_{k=1}^{\infty} q^{2-k} s_{n,k}(q;qx) \int_{0}^{\infty/1-\frac{1}{q}} s_{n,k-1}(q;t)f(t)d_{\frac{1}{q}}t + E_{\frac{1}{q}}\left(-[n]\,x\right) f(0), \tag{1.0.6}$$

where $x \in [0,\infty)$, $f \in \mathbb{R}^{[0,\infty)}$, $q > 1$, $n \in \mathbb{N}$ and

$$s_{n,k}(q;x) = \frac{1}{q^{\frac{k(k-1)}{2}}} \frac{[n]^k x^k}{[k]!} e_q(-[n]\,q^{-k}x) = \frac{1}{q^{\frac{k(k-1)}{2}}} \frac{[n]^k x^k}{[k]!} E_{\frac{1}{q}}(-[n]\,q^{-k}x)$$

by using q-parametric Szász operators which was introduced by N. I. Mahmudov in (1.0.2). As we mention above, in [30], N. I. Mahmudov introduced a q-generalization of the Szász operators in the case $q > 1$. Notice that different q-generalizations of Szász-Mirakjan operators were introduced and studied by A. Aral and V. Gupta [2], [3], by C. Radu [43] and by N. I. Mahmudov [29] in the case $0 < q < 1$. Notice that the rate of approximation by the q-Szász operators for $q > 1$ is of order $q^{-\frac{n}{2}}$, which is better than $\sqrt{\frac{1}{n}}$ (rate of approximation for the classical Szász-Mirakjan operators).

In this chapter,

- we construct q-parametric Phillips operators in the case $q > 1$ and evaluate the moments of $P_{n,q}$,
- we establish the local approximation result for continuous functions in terms of modulus of continuity,
- we obtain a Voronovskaja type asymptotic result for the q-Phillips operators.

Chapter 2

PRELIMINARY AND AUXILIARY RESULTS

2.1 Positive Linear Operators

In this part, we mention about some basic definitions and some elementary properties including positive and linear operators. For more detail on this topic see [8].

Definition 2.1.1. *([8]) Suppose that X and Y be two linear spaces of real functions. We say that, $L : X \to Y$ is linear operator if*

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g),$$

for all $f, g \in X$ and for all $\alpha, \beta \in \mathbb{R}$. Furthermore, if $\forall\ f \geq 0,\ f \in X$ implies that $Lf \geq 0$, then L is a positive operator.

Throughout this section, we employ the notation $L(f; x)$ but in some case $(Lf)(x)$, to highlight the argument of the function $Lf \in Y$.

Proposition 2.1.2. *[8]Suppose that $L : X \to Y$ be a linear positive operator. Then*

(i) L is said to be monotonic, İf $f, g \in X$ with $f \leq g$ then $Lf \leq Lg$.

(ii) for all $f \in X$ we have $|Lf| \leq L|f|$.

Definition 2.1.3. *([8]) Let X, Y be two linear normed spaces of real functions such that $X \subseteq Y$ and let $L : X \to Y$. To each operator L we can assign a non-negative number $\|L\|$ defined by*

$$\|L\| := \sup_{f \in X,\ \|f\|=1} \|Lf\| = \sup_{f \in X,\ \|f\| \leq 1} \|Lf\|.$$

It is clear that, $\|.\|$ satisfies all the properties of a norm and therefore is called the operator norm.

If we consider $X = Y = C[a,b]$ the following can be stated regarding the continuity and the operator norm:

Corollary 2.1.4. *([8]) If $L : C[a,b] \to C[a,b]$ is positive linear operator then L is also continuous and $\|L\| = \|Le_0\|$ where $e_0 = t^0$.*

The following result provides a neccesary and sufficient condition for the convergence of a positive linear operator towards the identity operator. This classical result of approximation theory is mostly known as Bohman-Korovkin Theorem.

Theorem 2.1.5. *([8]) Assume that $L_n : C[a,b] \to C[a,b]$ be a sequence of positive linear operators and let $e_i = t^i$. If $\lim_{n\to\infty} L_n(e_i) = e_i, i = 0,1,2,$ uniformly on $[a,b]$, then $\lim_{n\to\infty} L_n(f) = f$ uniformly on $[a,b]$ for every $f \in C[a,b]$.*

Because of the above theorem the monomials $e_i = t^i$, $i = 0,1,2,$ play an important role in the approximation theory of linear and positive operators on spaces of continuous function. They are often called Korovkin test-functions.

Many mathematicians have been inspired from this elegant and simple result to extend the last theorem in different directions, generalizing the notion of sequence and considering different spaces. In this direction Korovkin-type approximation theory arose such as a special branch of approximation theory. The complete and comprehensive exposure on this topic can be found in [1].

The following inequality which is called Cauchy-Schwarz inequality is used in many estimates

$$(L(fg))^2 \leq L\left(f^2\right) L\left(g^2\right), \quad f,g \in C[a,b].$$

The following theorem gives the Hölder-type inequality for positive linear operators which reduces to the inequality of Cauchy-Schwarz in case $p = q = 2$.

Theorem 2.1.6. *([8]) Assume that* $L : C[a,b] \to C[a,b]$ *be a positive linear operator and* $L(e_0) = e_0$. *For* $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in C[a,b]$ *one has*

$$L(|fg|;x) \leq L(|f|^p;x)^{\frac{1}{p}} L(|g|^q;x)^{\frac{1}{q}}.$$

The following quantities play an important role for positive linear operators $L : C[a,b] \to C[a,b]$. For $n \geq 0$, the moments of order n is denoted by

$$L((e_1 - x)^n;x) := L((e_1 - x)^n)(x), \;\; x \in [a,b]$$

and for $n \geq 1$, the absolute moments of odd order n is denoted by

$$L(|e_1 - x|^n;x) := L(|e_1 - x|^n)(x), \;\; x \in [a,b].$$

The first absolute moments $L(|e_1 - x|;x)$ and the second order moments $L\left((e_1 - x)^2;x\right)$ are very important moments. In general, it is difficult to compute the first absolute moments, hence the Cauchy-Schwarz inequality is used to estimate as follows:

$$L(|e_1 - x|;x) \leq \sqrt{L((e_0^2);x)}\sqrt{L\left((e_1 - x)^2;x\right)}. \tag{2.1.1}$$

But sometimes this approximation is too harsh. Therefore, we give some alternative ways as follows:

Proposition 2.1.7. *[8] Suppose that* L, p, q, f *and* x *are given as in Theorem 2.1.6, and let* $0 \leq n = n_1 + n_2$ *be a decomposition of the non-negative number* n *with* $n_1, n_2 \geq 0$. *Then*

$$L(|e_1 - x|^n;x) \leq L(|e_1 - x|^{n_1.p};x)^{\frac{1}{p}} L(|e_1 - x|^{n_2.q};x)^{\frac{1}{q}}.$$

For the case $n = 1$, $n = n_1 + n_2 = 0 + 1$, $p = q = 2$, *this reduces to (2.1.1).*

Proposition 2.1.8. *[8]If* $L : C[a,b] \to C[a,b]$ *is a positive linear operator such that* $Le_0 = e_0$ *and* $1 \leq s < r$. *Then*

$$L(|e_1 - x|^s;x)^{\frac{1}{s}} \leq L(|e_1 - x|^r;x)^{\frac{1}{r}}, \;\; x \in [a,b].$$

Example 2.1.9. *([8]) (i) Let* $L : C[a,b] \to C[a,b]$ *be a positive linear operator with* $Le_0 = e_0$*. Then we obtain*

$$L(|e_1 - x|; x) \leq L\left((e_1 - x)^2; x\right)^{\frac{1}{2}} \leq L\left(|e_1 - x|^3; x\right)^{\frac{1}{3}} \leq L\left((e_1 - x)^4; x\right)^{\frac{1}{4}} ...$$

(ii) An alternative way to bound the third term via Cauchy-Schwarz is

$$L\left(|e_1 - x|^3; x\right)^{\frac{1}{3}} \leq L\left((e_1 - x)^2; x\right)^{\frac{1}{6}} L\left((e_1 - x)^4; x\right)^{\frac{1}{6}}.$$

A recurrence formula for moments of higher order is given .

Proposition 2.1.10. *[8]If* L *is a linear operator and* $k \in \mathbb{N}_0$*, then*

$$L\left((e_1 - x)^k; x\right) = L(e_k; x) - \sum_{l=0}^{k-1} \binom{k}{l} x^{k-l} L\left((e_1 - x)^l; x\right). \qquad (2.1.2)$$

Proof. Write

$$\begin{aligned}
L(e_k; x) &= L((e_1 - x + x)^k; x) \\
&= L\left(\sum_{l=0}^{k} \binom{k}{l} x^{k-l} (e_1 - x)^l; x\right) \\
&= \sum_{l=0}^{k} \binom{k}{l} x^{k-l} L\left((e_1 - x)^l; x\right) \\
&= L\left((e_1 - x)^k; x\right) + \sum_{l=0}^{k-1} \binom{k}{l} x^{k-l} L\left((e_1 - x)^l; x\right),
\end{aligned}$$

which implises the representation of the k-th moment. □

Remark 2.1.11. *([8])*

(i) The equality (2.1.2) holds without the assumption $Le_i = e_i$, for $i = 0, 1$.

(ii) To compute $L\left((e_1 - x)^k; x\right)$ we can use Proposition 2.1.10 if we know $L(e_k; x)$ and $L\left((e_1 - x)^l; x\right)$, $0 \leq l \leq k - 1$.

Corollary 2.1.12. *([8]) If L is a linear operator with $Le_i = e_i$, $i = 0, 1$, then we obtain the following moments*

$$L\left((e_1 - x)^3; x\right) = L(e_3; x) - x^3 - 3xL\left((e_1 - x)^2; x\right),$$
$$L\left((e_1 - x)^4; x\right) = L(e_4; x) - x^4 - \left(4xL\left((e_1 - x)^3; x\right) + 6x^2L\left((e_1 - x)^2; x\right)\right).$$

The degree of convergence of positive linear operators towards the identity operator are measured by using the main tools the first modulus of smoothness and the second modulus of smoothness. For $f \in C[a, b]$ and $\delta \geq 0$, first modulus of smoothness is defined as

$$\begin{aligned} \omega(f; \delta) &:= \omega_1(f; \delta) \\ &:= \sup\{|f(x + h) - f(x)| : x,\ x + h \in [a, b],\ 0 \leq h \leq \delta\}; \end{aligned} \tag{2.1.3}$$

and the second modulus of smoothness is defined as

$$\begin{aligned} \omega_2(f; \delta) := \sup\{&|f(x + h) - 2f(x) + f(x - h)| : \\ &x, x \pm h \in [a, b],\ 0 \leq h \leq \delta\}. \end{aligned} \tag{2.1.4}$$

Definition 2.1.13. *([8]) For $k \in \mathbb{N}$, $\delta \in \mathbb{R}_+$ and $f \in C[a, b]$ the modulus of smoothness of order k is defined by*

$$\omega_k(f; \delta) := \sup\left\{\left|\Delta_h^k f(x)\right| 0 \leq h \leq \delta, x,\ x + kh \in [a, b]\ \right\}. \tag{2.1.5}$$

Proposition 2.1.14. *(see [8])*

1) $\omega_k(f; 0) = 0.$

2) $\omega_k(f; .)$ is a positive, continuous and non-decreasing function on $\mathbb{R}_+$.

3) $\omega_k(f; .)$ is sub-additive, i.e., $\omega_1(f; \delta_1 + \delta_2) \leq \omega_1(f; \delta_1) + \omega_1(f; \delta_2), \delta_i \geq 0,\ i = 1, 2.$

4) $\forall\, \delta \geq 0$, $\omega_{k+1}(f;\delta) \leq 2\omega_k(f;\delta)$.

5) If $f \in C^1[a,b]$ then $\omega_{k+1}(f;\delta) \leq \delta\omega_k(f';\delta)$, $\delta \geq 0$.

6) If $f \in C^r[a,b]$ then $\omega_r(f;\delta) \leq \delta^r \sup\limits_{\delta\in[a,b]} \left|f^{(r)}(\delta)\right|$.

7) $\forall\, \delta > 0$ and $n \in \mathbb{N}$, $\omega_k(f;n\delta) \leq n^k\omega_k(f;\delta)$.

8) $\forall\, \delta > 0$ and $r > 0$, $\omega_k(f;r\delta) \leq (1+[r])^k\omega_k(f;\delta)$, where $[r]$ is the integer part of r.

9) If $\delta \geq 0$ is fixed, then $\omega_k(f;.)$ is a seminorm on $C[a,b]$.

2.2 Szász Operator

In this section, we give the definition of Szász operators which was defined by Otto Szász in 1950.

Definition 2.2.1. *The positive linear operators*

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{(nx)^k}{k!}$$

where $x \in [0,\infty) \subset \mathbb{R}$ and $n \in \mathbb{N}$ are called Szász operators.

These operators are a generalization of the Bernstain polinomials to infinite intervals.

2.3 Phillips Operator

In this part, we give the definition of Phillips operators which was defined by R. S. Phillips [39].

Definition 2.3.1. *Let $C_B[0,\infty)$ be the space of real valued continuous bounded func-*

tions f *on* $[0,\infty)$ *endowed with the norm* $\|f\| = \sup_{x\geq 0} |f(x)|$ *. The positive linear operators*

$$P_n(f;x) = n\sum_{k=1}^{\infty} e(-nx)\frac{n^k x^k}{k!}\int_0^{\infty} e(-nt)\frac{n^{k-1}t^{k-1}}{(k-1)!}f(t)dt + e(-nx)f(0),$$

where $x \in [0,\infty)$ *. These operators are called Phillips operators.*

2.4 q-Integers

Definition 2.4.1. *Consider an arbitrary function* $f(x)$ *and* $q \in \mathbb{R}^+ \setminus \{1\}$ *. The following expression*

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x} \tag{2.4.1}$$

is called q*-derivative of the function* $f(x)$.

For any constants a and b, D_q has the following property

$$D_q(af(x) + bg(x)) = aD_q f(x) + bD_q g(x).$$

Therefore, we can say that D_q is a linear operator on the space of polynomials.

Definition 2.4.2. *For any* $n \in \mathbb{N}$ *and* $q \in \mathbb{R}^+$, *the* q*-analogue of* n *(*q*-integer) is defined by*

$$[n] = [n]_q := \begin{cases} \frac{1-q^n}{1-q} = 1 + q + q^2 + ... + q^{n-1} & \text{if } q \neq 1 \\ n & \text{if } q = 1 \end{cases} \quad \text{and} \quad [0] := 0 \ . \tag{2.4.2}$$

By using Definition 2.4.2, we define

$$\mathbb{N}_q = \{[n], \text{ with } n \in \mathbb{N}\}. \tag{2.4.3}$$

It is clear that, the set of q-integers $\mathbb{N}_q$ generalizes the set of nonnegative integers $\mathbb{N}$, which we recover by putting $q = 1$.

Definition 2.4.3. *For any $n \in \mathbb{N}$ and $q \in \mathbb{R}^+$, the q-analogue of $n!$ (q-factorial) is defined by*

$$[n]! = [n]_q! := \begin{cases} [1]\,[2]\,...\,[n] & if \;\; n = 1, 2, ... \\ 1 & if \;\; n = 0 \end{cases}. \tag{2.4.4}$$

Definition 2.4.4. *We define q-analogue of $(x-a)^n$ as*

$$(x-a)_q^n := \begin{cases} (x-a)(x-qa)...(x-q^{n-1}a) & if \;\; n \geq 1 \\ 1 & if \;\; n = 0. \end{cases} \tag{2.4.5}$$

Definition 2.4.5. *For integers $0 \leq k \leq n$, q-binomial coefficient is defined by*

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]\,[n-1]\,...\,[n-k+1]}{[k]!} = \frac{[n]!}{[k]!\,[n-k]!}. \tag{2.4.6}$$

Here, we can say that the q-binomial coefficients reduce to the ordinary binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ as $q \to 1$.

Lemma 2.4.6. *Let n be a nonnegative integer and a be a number. Then we have*

$$(x+a)_q^n := \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-1)/2} a^j x^{n-j} \tag{2.4.7}$$

which is called the Gauss's binomial formula.

Lemma 2.4.7. *For a given nonnegative integer n, we have the following formula*

$$\frac{1}{(1-x)_q^n} = 1 + \sum_{j=1}^{n} \frac{[n]\,[n+1]\,...\,[n+j-1]}{[j]!} x^j \tag{2.4.8}$$

which is called Heine's binomial formula.

Now, we consider Lemma 2.4.6 with x and a replaced by 1 and x respectively to obtain the following Gauss's binomial formula

$$(1+x)_q^n := \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-1)/2} x^j. \tag{2.4.9}$$

At this position, someone may wonder, what happen if the limit of formula (2.4.9) and (2.4.8) are taken as $n \to \infty$. In the ordinary calculus, i.e., $q = 1$, the answer is either infinitely large or infinitely small, it depends on the value of x. But, in quantum calculus it is different. For example, when $|q| < 1$, the infinite product

$$(1+x)_q^{\infty} = (1+x)(1+qx)(1+q^2x)...$$

converges to finite limit. Another one, if we suppose $|q| < 1$, we have

$$\lim_{n\to\infty}[n] = \lim_{n\to\infty}\frac{1-q^n}{1-q} = \frac{1}{1-q} \tag{2.4.10}$$

and

$$\begin{aligned}\lim_{n\to\infty}\begin{bmatrix} n \\ j \end{bmatrix} &= \lim_{n\to\infty}\frac{[n]!}{[j]!\,[n-j]!} \\ &= \lim_{n\to\infty}\frac{[n]\,[n-1]\,...\,[n-j+1]}{[j]!} \\ &= \lim_{n\to\infty}\frac{(1-q^n)(1-q^{n-1})...(1-q^{n-j+1})}{(1-q)(1-q^2)...(1-q^j)} \\ &= \frac{1}{(1-q)(1-q^2)...(1-q^j)}. \end{aligned} \tag{2.4.11}$$

Therfore, the q-integers and q-binomial coefficients behave in a very different way when n is large as compared to their ordinary counterparts. Assume that $|q| < 1$. We obtain two important identities of formal power series in x by using (2.4.10) and (2.4.11) to Gauss's and Heine's binomial formulas as $n \to \infty$:

$$(1+x)_q^{\infty} = \sum_{j=0}^{\infty} q^{j(j-1)/2}\frac{x^j}{(1-q)(1-q^2)...(1-q^j)} \tag{2.4.12}$$

$$\frac{1}{(1-x)_q^{\infty}} = \sum_{j=0}^{\infty}\frac{x^j}{(1-q)(1-q^2)...(1-q^j)}. \tag{2.4.13}$$

Moreover, we have

$$\begin{aligned}\frac{1}{(1-x)_q^{\infty}} &= \sum_{j=0}^{\infty}\frac{\left(\frac{x}{1-q}\right)^j}{1\left(\frac{1-q^2}{1-q}\right)\cdots\left(\frac{1-q^j}{1-q}\right)} \\ &= \sum_{j=0}^{\infty}\frac{\left(\frac{x}{1-q}\right)^j}{[j]!}. \end{aligned} \tag{2.4.14}$$

Since each term in the sums of (2.4.12) and (2.4.13) has no meaning when $q = 1$, they have no classical analogues. Interestingly, the two identities were discovered by Euler, who lived before Gauss and Heine. In this respect, the formula (2.4.12) is called

Euler's first identities or E_1 and the formula (2.4.13) is called Euler's second identities or E_2.

There are two q-analogues of the exponential function e^x.These are given from the following definitions:

Definition 2.4.8. *([19]) A q-analogue of the classical exponential function e^x is*

$$e_q^x := e_q(x) = \sum_{j=0}^{\infty} \frac{x^j}{[j]!}. \tag{2.4.15}$$

Since we have (2.4.14), we can say that

$$e_q^{x/(1-q)} = \frac{1}{(1-x)_q^{\infty}} \tag{2.4.16}$$

or

$$e_q^x = \frac{1}{(1-(1-q)x)_q^{\infty}}, \quad |x| < \frac{1}{1-q}, \quad |q| < 1. \tag{2.4.17}$$

Definition 2.4.9. *([19]) Another q-analogue of the classical exponential function is*

$$E_q^x := E_q(x) = \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{x^j}{[j]!} = (1+(1-q)x)_q^{\infty}, \quad |q| < 1. \tag{2.4.18}$$

According to (2.4.17) and (2.4.18), we can say that two exponential functions are closely related. These relations are given as follows

$$e_q^x E_q^{-x} = 1 \tag{2.4.19}$$

and by using (2.4.12) and (2.4.13), we obtain

$$\begin{aligned} e_{1/q}^x &= \sum_{j=0}^{\infty} \frac{(1-1/q)^j x^j}{(1-1/q)(1-1/q^2)...(1-1/q^j)} \\ &= \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{(1-q)^j x^j}{(1-q)(1-q^2)...(1-q^j)} \end{aligned}$$

and so

$$e_{1/q}^x = E_q^x. \tag{2.4.20}$$

Definition 2.4.10. *The q-Jackson integral of f(x) is defined as*

$$\int f(x) d_q x = (1-q)x \sum_{j=0}^{\infty} q^j f(q^j x). \tag{2.4.21}$$

Here we use Jackson formula (2.4.21) to define the definite q-integral.

Definition 2.4.11. *Let $a > 0$. The definite q-integral is defined as*

$$\int_0^a f(x) d_q x = a(1-q) \sum_{j=0}^{\infty} q^j f(q^j a). \tag{2.4.22}$$

Definition 2.4.12. *([19]) The q-improper integral of $f(x)$ on $[0, \infty)$ is defined to be*

$$\int_0^\infty f(x) d_q x = \sum_{j=-\infty}^{\infty} \int_{q^{j+1}}^{q^j} f(x) d_q x = (1-q) \sum_{j=-\infty}^{\infty} q^j f(q^j) \tag{2.4.23}$$

if $0 < q < 1$, *or*

$$\int_0^\infty f(x) d_q x = \sum_{j=-\infty}^{\infty} \int_{q^j}^{q^j+1} f(x) d_q x = (q-1) \sum_{j=-\infty}^{\infty} q^j f(q^j) \tag{2.4.24}$$

if $q > 1$.

Definition 2.4.13. *([18], [25]) For $A > 0$, the q-improper integral is defined as*

$$\int_0^{\infty/A} f(x) d_q x = (1-q) \sum_{j=-\infty}^{\infty} \frac{q^j}{A} f\left(\frac{q^j}{A}\right). \tag{2.4.25}$$

In this thesis, the following two q-gamma function are used.

Definition 2.4.14. *([7]) If $x > 0$, the q-Gamma function is defined as follows*

$$\Gamma_q(x) = \int_0^{\frac{1}{1-q}} t^{x-1} E_q(-qt) d_q t. \tag{2.4.26}$$

Definition 2.4.15. *([7]) For every A, $x > 0$, the q-gamma function is defined to be*

$$\gamma_q^A(x) = \int_0^{\infty/A(1-q)} t^{x-1} e_q(-t) d_q t. \tag{2.4.27}$$

Theorem 2.4.16. *([7]) For every A, $x > 0$ one has*

$$\Gamma_q(x) = K(A; x) \gamma_q^A(x), \tag{2.4.28}$$

where $K(A; x) = \frac{1}{1+A} A^x \left(1 + \frac{1}{A}\right)_q^x (1+A)_q^{1-x}$.

In particular for any positive integer n

$$K(A;n) = q^{\frac{n(n-1)}{2}} \tag{2.4.29}$$

and

$$\Gamma_q(n) = q^{\frac{n(n-1)}{2}} \gamma_q^A(n) \text{ see [7].} \tag{2.4.30}$$

2.5 q-Parametric Szász Operators

In this section, we would like to draw attention to the q-parametric Szász operators.

In [2], A. Aral and V. Gupta introduced q-Szász-Mirakjan operators as follows

$$\mathbb{S}_{n,q}(f)(x) := E_q\left(-\frac{[n]\,x}{b_n}\right)\sum_{k=0}^{\infty} f\left(\frac{[k]\,b_n}{[n]}\right)\frac{[n]^k\,x^k}{[k]!b_n^k},$$

where $0 \le x < \frac{b_n}{(1-q)[n]}$, $0 < q < 1$, $n \in \mathbb{N}$, $f \in C\,[0,\infty)$ and $\{b_n\}$ is a sequence of positive numbers such that $\lim\limits_{n\to\infty} b_n = \infty$. Moreover, they investigated the approximation properties of the defined operator.

In [29], N. I. Mahmudov introduced q-Szász-Mirakjan operators as follows

$$S_{n,q}(f)(x) := S_{n,q}(f;x) = \frac{1}{\prod\limits_{j=0}^{\infty}(1+(1-q)q^j\,[n]\,x)}\sum_{k=0}^{\infty} f\left(\frac{[k]}{q^{k-2}\,[n]}\right) q^{\frac{k(k-1)}{2}}\frac{[n]^k\,x^k}{[k]!},$$

where $x \in [0,\infty)\,,\ 0 < q < 1$, $n \in \mathbb{N}$, $f \in C\,[0,\infty)$ and investigated approximation properties of q-Szász-Mirakjan operators. Here $S_{n,q}$ is linear and positive operator as a classical Szász-Mirakjan operator $S_{n.}$ In the theory of approximation by positive operators, moments $S_{n,q}(t^m;x)$ have very important role. In this respect, we will only mention the following recurrence formula and explicit formulas for moments $S_{n,q}(t^m;x)$, $m = 0,1,2,3,4$.

Lemma 2.5.1. *([29]) Let $0 < q < 1$. The following recurrence formula holds*

$$S_{n,q}(t^{m+1};x) = \sum_{j=0}^{m}\binom{m}{j}\frac{x}{q^{2j-m-1}\,[n]^{m-j}} S_{n,q}(t^j;x). \tag{2.5.1}$$

Lemma 2.5.2. *[29] Let $0 < q < 1$. We have*

$$S_{n,q}(1;x) = 1, \;\; S_{n,q}(t;x) = qx, \; S_{n,q}(t^2;x) = qx^2 + \frac{q^2}{[n]}x,$$
$$S_{n,q}(t^3;x) = \frac{q^3 x}{[n]^2} + (2q^2+q)\frac{x^2}{[n]} + x^3,$$
$$S_{n,q}(t^4;x) = \frac{q^4 x}{[n]^3} + (3q^3+3q^2+q)\frac{x^2}{[n]^2} + \left(3q+2+\frac{1}{q}\right)\frac{x^3}{[n]} + \frac{x^4}{q^2}.$$

In [30], N. I. Mahmudov introduced a q-generalization of the Szász operators in the case $q > 1$ as follows:

Definition 2.5.3. *([30]) Let $q > 1$ and $n \in \mathbb{N}$. For $f : [0,\infty) \to \mathbb{R}$ we define the Szász operators based on the q-integers*

$$M_{n,q}(f;x) := \sum_{k=0}^{\infty} f\left(\frac{[k]}{[n]}\right) \frac{1}{q^{\frac{k(k-1)}{2}}} \frac{[n]^k x^k}{[k]!} e_q\left(-[n]\, q^{-k} x\right).$$

Before as we mentioned that different q-generalizations of Szász-Mirakjan operators were introduced and studied by A. Aral and V. Gupta [2], [3], by C. Radu [43] and by N. I. Mahmudov [29] in the case $0 < q < 1$. When N. I. Mahmudov defined q-Szász operators for $q > 1$, he noticed that the rate of approximation by the q-Szász operators for $q > 1$ is of order $q^{-\frac{n}{2}}$, which is better than $\sqrt{\frac{1}{n}}$ (rate of approximation for the classical Szász-Mirakjan operators). Therefore he found that the approximation properties of his q-Szász operators are better than the classical Szász-Mirakjan operators and the other q-Szász-Mirakjan operators.

The operator $M_{n,q}$ is linear and positive operator as classical Szász operator $S_{n.}$ Also in the theory of approximation by positive operators, moments $M_{n,q}(t^m;x)$ have very important role. In this respect, we will only mention the following recurrence formula and explicit formulas for moments $M_{n,q}(t^m;x)$, $m = 0,1,2,3,4.$

Lemma 2.5.4. *([30]) Let $q > 1$. The following recurrence formula holds*

$$M_{n,q}(t^{m+1};x) = \sum_{j=0}^{m} \binom{m}{j} \frac{xq^j}{[n]^{m-j}} M_{n,q}(t^j;q^{-1}x)$$

Lemma 2.5.5. *[30] Let $q > 1$. We have*

$$\begin{aligned}
&M_{n,q}(1;x) = 1, \;\; M_{n,q}(t;x) = x, \; M_{n,q}(t^2;x) = x^2 + \frac{1}{[n]}x, \\
&M_{n,q}(t^3;x) = x^3 + \frac{(2+q)}{[n]}x^2 + \frac{1}{[n]^2}x \\
&M_{n,q}(t^4;x) = x^4 + \left(3+2q+q^2\right)\frac{x^3}{[n]} + (3+3q+q^2)\frac{x^2}{[n]^2} + \frac{1}{[n]^3}x.
\end{aligned}$$

Chapter 3

ON q-SZÁSZ-DURRMEYER OPERATORS

In this chapter, we introduce the q-Szász-Durrmeyer operators $D_{n,q}$ and evaluate the moments of $D_{n,q}$. We prove local approximation result for continuous functions in terms of modulus of continuity. Furthermore, we study Voronovskaja type result for the q-Szász-Durrmeyer operators. (see [32])

3.1 Moments

In this section firstly, we introduce the following so called q-Szász-Durrmeyer operators which generalize the sequence of classical Szász-Durrmeyer operators.

Definition 3.1.1. *For $f \in \mathbb{R}^{[0,\infty)}$, $0 < q < 1$ and $n \in \mathbb{N}$, we define the following q-Szász-Durrmeyer operator*

$$D_{n,q}(f;x) = [n] \sum_{k=0}^{\infty} q^k s_{n,k}(q;x) \int_0^{\infty/(1-q)} s_{n,k}(q;t) f(t) d_q t. \tag{3.1.1}$$

where $x \in [0,\infty)$ and $s_{n,k}(q;x) = \frac{1}{E_q([n]x)} q^{\frac{k(k-1)}{2}} \frac{[n]^k x^k}{[k]!} = e_q(-[n]\, x) q^{\frac{k(k-1)}{2}} \frac{[n]^k x^k}{[k]!}$.

It is clear that $s_{n,k}(q;x) \geq 0$ for all $q \in (0,1)$ and $x \in [0,\infty)$. Moreover $\sum\limits_{k=0}^{\infty} s_{n,k}(q;x) = \frac{1}{E_q([n]x)} \sum\limits_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{[n]^k x^k}{[k]!} \overset{\text{(by 2.4.18)}}{=} \frac{1}{E_q([n]x)} E_q([n]\, x) = 1.$

Secondly, we calculate $D_{n,q}(t^i;x)$ for $i = 0,1,2$. By the definition (2.4.15) of q-

Gamma function γ_q, we have

$$\begin{aligned}
\int_0^{\infty/(1-q)} t^s s_{n,k}(q;t)d_qt &= \int_0^{\infty/(1-q)} t^s \frac{1}{E_q([n]\,t)} q^{\frac{k(k-1)}{2}} \frac{[n]^k\, t^k}{[k]!} d_qt \\
&= \int_0^{\infty/(1-q)} t^s e_q(-[n]\,t) q^{\frac{k(k-1)}{2}} \frac{[n]^k\, t^k}{[k]!} d_qt \\
&= \frac{1}{[n]^{s+1}} \frac{1}{[k]!} q^{\frac{k(k-1)}{2}} \int_0^{\infty/(1-q)} ([n]\,t)^{k+s}\, e_q(-[n]\,t)\,[n]\, d_qt \\
&= \frac{1}{[n]^{s+1}} \frac{1}{[k]!} q^{\frac{k(k-1)}{2}} \int_0^{\infty/(1-q)[n]^{-1}} (u)^{k+s}\, e_q(-u) d_qu \\
&= \frac{1}{[n]^{s+1}} \frac{1}{[k]!} q^{\frac{k(k-1)}{2}} \gamma_q^{[n]^{-1}}(k+s+1) \\
&= \frac{1}{[n]^{s+1}} \frac{1}{[k]!} q^{\frac{k(k-1)}{2}} \frac{\Gamma_q(k+s+1)}{q^{(k+s+1)(k+s)/2}} \\
&= \frac{1}{[n]^{s+1}} \frac{1}{[k]!} q^{\frac{k(k-1)}{2}} \frac{[k+s]!}{q^{(k+s+1)(k+s)/2}},\ s \in \mathbb{N} \cup \{0\}\,.
\end{aligned}$$

Lemma 3.1.2. *We have*

$$D_{n,q}(1;x) = 1, \qquad D_{n,q}(t;x) = \frac{1}{q^2}x + \frac{1}{[n]\,q},$$
$$D_{n,q}(t^2;x) = \frac{1}{q^6}x^2 + \frac{(1+q)^2}{q^5\,[n]}x + \frac{1+q}{q^3}\frac{1}{[n]^2}.$$

Proof. We know that, (see [29])

$$S_{n,q}(1;x) = 1, \quad S_{n,q}(t;x) = qx, \quad S_{n,q}(t^2;x) = qx^2 + \frac{q^2}{[n]}x.$$

Using the above formulas we get

$$\begin{aligned}
D_{n,q}(1;x) &= [n] \sum_{k=0}^{\infty} q^k s_{n,k}(q;x) \int_0^{\infty/(1-q)} s_{n,k}(q;t) d_qt \\
&= [n] \sum_{k=0}^{\infty} q^k s_{n,k}(q;x) \frac{1}{[n]} \frac{q^{k(k-1)/2}}{q^{(k+1)k/2}} \\
&= \sum_{k=0}^{\infty} s_{n,k}(q;x) \frac{q^{k(k+1)/2}}{q^{(k+1)k/2}} \\
&= S_{n,q}(1;x) \\
&= 1,
\end{aligned}$$

$$
\begin{aligned}
D_{n,q}(t;x) &= [n]\sum_{k=0}^{\infty} q^k s_{n,k}(q;x) \int\limits_{0}^{\infty/(1-q)} t s_{n,k}(q;t) d_q t \\
&= [n]\sum_{k=0}^{\infty} q^k s_{n,k}(q;x) \frac{[k+1]}{[n]^2} \frac{q^{k(k-1)/2}}{q^{(k+2)(k+1)/2}} \\
&= \sum_{k=0}^{\infty} s_{n,k}(q;x) \frac{[k+1]}{[n]} \frac{1}{q^{k+1}} \\
&= \sum_{k=0}^{\infty} s_{n,k}(q;x) \frac{([k]+q^k)}{[n]} \frac{1}{q^{k+1}} \\
&= \sum_{k=0}^{\infty} \frac{[k]}{q^{k+1}[n]} s_{n,k}(q;x) + \sum_{k=0}^{\infty} \frac{1}{q[n]} s_{n,k}(q;x) \\
&= \frac{1}{q^3}\, S_{n,q}(t;x) + \frac{1}{q[n]} S_{n,q}(1;x) \\
&= \frac{1}{q^3}\, qx + \frac{1}{[n]\,q} \\
&= \frac{x}{q^2} + \frac{1}{[n]\,q},
\end{aligned}
$$

and

$$
\begin{aligned}
D_{n,q}(t^2;x) &= [n]\sum_{k=0}^{\infty} q^k s_{n,k}(q;x)\int_0^{\infty/(1-q)} t^2 s_{n,k}(q;t)d_qt \\
&= [n]\sum_{k=0}^{\infty} q^k s_{n,k}(q;x)\frac{[k+2]\,[k+1]}{[n]^3}\frac{q^{k(k-1)/2}}{q^{(k+3)(k+2)/2}} \\
&= \sum_{k=0}^{\infty} s_{n,k}(q;x)\frac{\left([k]^2+q^k(2+q)\,[k]+q^{2k}(1+q)\right)}{[n]^2}\frac{1}{q^{2k+3}} \\
&= \sum_{k=0}^{\infty} s_{n,k}(q;x)\frac{[k]^2}{[n]^2}\frac{1}{q^{2k+3}}+\sum_{k=0}^{\infty} s_{n,k}(q;x)\frac{q^k(2+q)\,[k]}{[n]^2}\frac{1}{q^{2k+3}} \\
&+ \sum_{k=0}^{\infty} s_{n,k}(q;x)\frac{q^{2k}(1+q)}{[n]^2}\frac{1}{q^{2k+3}} \\
&= \sum_{k=0}^{\infty} s_{n,k}(q;x)\frac{[k]^2}{[n]^2}\frac{1}{q^{2k+3}}+\sum_{k=0}^{\infty} s_{n,k}(q;x)\frac{(2+q)\,[k]}{[n]^2}\frac{1}{q^{k+3}} \\
&+ \sum_{k=0}^{\infty} s_{n,k}(q;x)\frac{(1+q)}{[n]^2}\frac{1}{q^3} \\
&= \frac{1}{q^7}\sum_{k=0}^{\infty}\frac{[k]^2}{q^{2k-4}\,[n]^2}s_{n,k}(q;x)+\frac{(2+q)}{q^5\,[n]}\sum_{k=0}^{\infty}\frac{[k]}{[n]\,q^{k-2}}s_{n,k}(q;x) \\
&+ \frac{(1+q)}{[n]^2\,q^3}\sum_{k=0}^{\infty} s_{n,k}(q;x) \\
&= \frac{1}{q^7}S_{n,q}(t^2;x)+\frac{(2+q)}{q^5\,[n]}S_{n,q}(t;x)+\frac{(1+q)}{[n]^2\,q^3}S_{n,q}(1;x) \\
&= \frac{1}{q^7}\left(qx^2+\frac{q^2}{[n]}x\right)+\frac{(2+q)}{q^5\,[n]}qx+\frac{(1+q)}{[n]^2\,q^3} \\
&= \frac{1}{q^6}x^2+\frac{1}{q^5\,[n]}x+\frac{2q}{q^5\,[n]}x+\frac{1}{q^3\,[n]}x+\frac{(1+q)}{[n]^2\,q^3} \\
&= \frac{1}{q^6}x^2+\frac{(1+q)^2}{q^5\,[n]}x+\left(\frac{1}{q^3}+\frac{1}{q^2}\right)\frac{1}{[n]^2}.
\end{aligned}
$$

□

Lemma 3.1.3. *For all $0<q<1$ the following identity holds:*

$$
D_{n,q}(t^m;x)=\frac{1}{[n]^m\,q^{(m^2+m)/2}}\sum_{s=0}^{m}C_{s,m}(q)\,[n]^s\sum_{k=0}^{\infty}\frac{[k]^s}{[n]^s}\frac{1}{q^{mk}}s_{n,k}(q;x).
$$

Proof. Indeed, we have

$$\begin{aligned}
D_{n,q}(t^m;x) &= [n]\sum_{k=0}^{\infty} q^k s_{n,k}(q;x)\int_0^{\infty/(1-q)} t^m s_{n,k}(q;t)d_qt \\
&= [n]\sum_{k=0}^{\infty} q^k s_{n,k}(q;x)\frac{1}{[n]^{m+1}}\frac{1}{[k]!}q^{\frac{k(k-1)}{2}}\frac{[k+m]!}{q^{(k+m+1)(k+m)/2}} \\
&= \sum_{k=0}^{\infty}\frac{[k+m]\ldots[k+1]}{[n]^m}\frac{1}{q^{(m^2+2mk+m)/2}}s_{n,k}(q;x) \\
&= \frac{1}{[n]^m\, q^{(m^2+m)/2}}\sum_{k=0}^{\infty}\frac{[k+m]\ldots[k+1]}{q^{mk}}s_{n,k}(q;x) \\
&= \frac{1}{[n]^m\, q^{(m^2+m)/2}}\sum_{k=0}^{\infty}\frac{1}{q^{mk}}\sum_{s=0}^{m}C_{s,m}(q)\,[k]^s\, s_{n,k}(q;x) \\
&= \frac{1}{[n]^m\, q^{(m^2+m)/2}}\sum_{k=0}^{\infty}\sum_{s=0}^{m}C_{s,m}(q)\,[k]^s\,\frac{1}{q^{mk}}s_{n,k}(q;x) \\
&= \frac{1}{[n]^m\, q^{(m^2+m)/2}}\sum_{s=0}^{m}C_{s,m}(q)\,[n]^s\sum_{k=0}^{\infty}\frac{[k]^s}{[n]^s}\frac{1}{q^{mk}}s_{n,k}(q;x)
\end{aligned}$$

where $C_{s,m}(q)$ is defined by the following formula

$$[k+1]\,[k+2]\ldots[k+m] = \prod_{s=1}^{m}\left([s]+q^s\,[k]\right) = \sum_{s=0}^{m}C_{s,m}(q)\,[k]^s\,.$$

Here $C_{s,m}(q) > 0$, $s = 0, ..., m$ are constants independent of k. □

3.2 Local Approximation

Let $C_B\,[0,\infty)$ be the space of all real-valued continuous bounded functions f on $[0,\infty)$, endowed with the norm $\|f\| = \sup_{x\in[0,\infty)}|f(x)|$. The Peetre's K-functional is introduced by

$$K_2(f;\delta) = \inf_{g\in C_B^2[0,\infty)}\left\{\|f-g\| + \delta\,\|g''\|\right\},$$

where $C_B^2\,[0,\infty) := \{g \in C_B\,[0,\infty) :\ g', g'' \in C_B\,[0,\infty)\}$.

By $[[8], p.177,$ Theorem 2.4] there exists an absolute constant $M > 0$ such that

$$K_2(f,\delta) \le M\omega_2(f;\sqrt{\delta}), \tag{3.2.1}$$

where $\delta > 0$ and the second order modulus of smoothness is defined as

$$\omega_2(f;\sqrt{\delta}) = \sup_{0<h\leq\delta} \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|,$$

where $f \in C_B[0,\infty)$ and $\delta > 0$. Also we let

$$\omega(f;\delta) = \sup_{0<h\leq\delta} \sup_{x\in[0,\infty)} |f(x+h) - f(x)| \tag{3.2.2}$$

Lemma 3.2.1. *Let $f \in C_B[0,\infty)$. Consider the operators*

$$D^*_{n,q}(f;x) = D_{n,q}(f;x) + f(x) - f\left(\frac{1}{q^2}x + \frac{1}{[n]\,q}\right). \tag{3.2.3}$$

Then, for all $g \in C^2_B[0,\infty)$, we have

$$\left|D^*_{n,q}(g;x) - g(x)\right| \leq \left\{\frac{(1-q^2)(1+2q^2+2q^4)}{q^6}x^2 + \left(\frac{1+2q+3q^2}{q^5\,[n]}\right)x + \frac{1+2q}{q^3\,[n]^2}\right\}\|g''\|. \tag{3.2.4}$$

Proof. From (3.2.3), we have

$$\begin{aligned} D^*_{n,q}(t-x;x) &= D_{n,q}(t-x;x) - \left(\frac{1}{q^2}x + \frac{1}{[n]\,q} - x\right) \qquad (3.2.5)\\ &= D_{n,q}(t;x) - xD_{n,q}(1;x) - \left(\frac{1}{q^2}x + \frac{1}{[n]\,q}\right) + x = 0. \end{aligned}$$

Let $x \in [0,\infty)$ and $g \in C^2_B[0,\infty)$. Using the Taylor's formula

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du,$$

we can write by (3.2.5) that

$$\begin{aligned} &D^*_{n,q}(g;x) - g(x)\\ &= D^*_{n,q}((t-x)g'(x);x) + D^*_{n,q}\Big(\int_x^t (t-u)g''(u)du;x\Big)\\ &= g'(x)D^*_{n,q}((t-x);x) + D_{n,q}\Big(\int_x^t (t-u)g''(u)du;x\Big)\\ &- \int_x^{\frac{x}{q^2}+\frac{1}{[n]q}} \left(\frac{x}{q^2} + \frac{1}{[n]\,q} - u\right)g''(u)du\\ &= D_{n,q}\Big(\int_x^t (t-u)g''(u)du;x\Big) - \int_x^{\frac{x}{q^2}+\frac{1}{[n]q}} \left(\frac{x}{q^2} + \frac{1}{[n]\,q} - u\right)g''(u)du. \end{aligned}$$

On the other hand, since

$$\begin{aligned}\left|\int_x^t (t-u)g''(u)du\right| &\leq \left|\int_x^t |t-u|\,|g''(u)|\,du\right| \\ &\leq \|g''\| \left|\int_x^t |t-u|\,du\right| \\ &\leq (t-x)^2\,\|g''\|\end{aligned}$$

and

$$\begin{aligned}&\left|\int_x^{\frac{x}{q^2}+\frac{1}{[n]q}} \left(\frac{x}{q^2}+\frac{1}{[n]\,q}-u\right) g''(u)du\right| \\ &\leq \left(\frac{x}{q^2}+\frac{1}{[n]\,q}-x\right)^2 \|g''\| \\ &= \left(\left(\frac{1}{q^2}-1\right)x+\frac{1}{[n]\,q}\right)^2 \|g''\| \\ &= \left[\left(\frac{1}{q^2}-1\right)^2 x^2+2\left(\frac{1}{q^2}-1\right)\frac{1}{[n]\,q}x+\frac{1}{[n]^2\,q^2}\right]\|g''\|\,,\end{aligned}$$

we conclude that

$$\begin{aligned}\left|D_{n,q}^{*}(g;x)-g(x)\right| &= \Bigg|D_{n,q}\left(\int_x^t (t-u)g''(u)du;x\right) \\ &\quad - \int_x^{\frac{x}{q^2}+\frac{1}{[n]q}} \left(\frac{x}{q^2}+\frac{1}{[n]\,q}-u\right) g''(u)du\Bigg| \\ &\leq D_{n,q}((t-x)^2\,\|g''\|\,;x) \\ &\quad + \left[x^2\left(\frac{1}{q^2}-1\right)^2+2\left(\frac{1}{q^2}-1\right)\frac{1}{[n]\,q}x+\frac{1}{[n]^2\,q^2}\right]\|g''\| \\ &\leq D_{n,q}((t-x)^2;x)\,\|g''\| \\ &\quad + \left[\left(\frac{1}{q^2}-1\right)^2 x^2+2\left(\frac{1}{q^2}-1\right)\frac{1}{[n]\,q}x+\frac{1}{[n]^2\,q^2}\right]\|g''\|\,.\end{aligned}$$

Thus, from the fact that

$$
\begin{aligned}
& D_{n,q}((t-x)^2;x) \\
& = D_{n,q}(t^2-2tx+x^2;x) \\
& = D_{n,q}(t^2;x) - 2xD_{n,q}(t;x) + x^2 D_{n,q}(1;x) + x^2 - x^2 \\
& \leq \left| D_{n,q}(t^2;x) - x^2 \right| + 2x \left| D_{n,q}(t;x) - x \right| \\
& \leq \left(\frac{1}{q^6} - 1 \right) x^2 + \frac{(1+q)^2}{q^5 [n]} x + \left(\frac{1}{q^3} + \frac{1}{q^2} \right) \frac{1}{[n]^2} + 2 \left(\frac{1}{q^2} - 1 \right) x^2 + \frac{2}{[n]\, q} x \\
& = \left[\left(\frac{1}{q^6} - 1 \right) + 2 \left(\frac{1}{q^2} - 1 \right) \right] x^2 + \left(\frac{(1+q)^2}{q^5 [n]} + \frac{2}{[n]\, q} \right) x + (\frac{1}{q^3} + \frac{1}{q^2}) \frac{1}{[n]^2} \\
& = \left(\frac{1}{q^6} + \frac{2}{q^2} - 3 \right) x^2 + \left(\frac{(1+q)^2}{q^5 [n]} + \frac{2}{[n]\, q} \right) x + \left(\frac{1}{q^3} + \frac{1}{q^2} \right) \frac{1}{[n]^2},
\end{aligned}
$$

we get

$$
\begin{aligned}
& \left| D^*_{n,q}(g;x) - g(x) \right| \\
& \leq \left\{ \left(\frac{1}{q^6} + \frac{2}{q^2} - 3 \right) x^2 + \left(\frac{(1+q)^2}{q^5 [n]} + \frac{2}{[n]\, q} \right) x + \left(\frac{1}{q^3} + \frac{1}{q^2} \right) \frac{1}{[n]^2} \right\} \|g''\| \\
& + \left[\left(\frac{1}{q^2} - 1 \right)^2 x^2 + 2 \left(\frac{1}{q^2} - 1 \right) \frac{1}{[n]\, q} x + \frac{1}{[n]^2 q^2} \right] \|g''\| \\
& = \left\{ \left[\left(\frac{1}{q^6} - 1 \right) + 2 \left(\frac{1}{q^2} - 1 \right) + \left(\frac{1}{q^2} - 1 \right)^2 \right] x^2 \right. \\
& + \left(\frac{1+(2+q)q}{q^5 [n]} + \frac{2}{[n]\, q} + 2 \left(\frac{1}{q^2} - 1 \right) \frac{1}{[n]\, q} \right) x \\
& \left. + (\frac{1}{q^3} + \frac{1}{q^2}) \frac{1}{[n]^2} + \frac{1}{[n]^2 q^2} \right\} \|g''\| \\
& = \left\{ \left(\frac{(1+q^2-2q^6}{q^6} \right) x^2 + \left(\frac{1+2q+3q^2}{q^5 [n]} \right) x + \frac{1+2q}{q^3 [n]^2} \right\} \|g''\| \\
& = \left\{ \left(\frac{(1-q^2)(1+2q^2+2q^4)}{q^6} \right) x^2 + \left(\frac{1+2q+3q^2}{q^5 [n]} \right) x + \frac{1+2q}{q^3 [n]^2} \right\} \|g''\| .
\end{aligned}
$$

□

Lemma 3.2.2. *For $f \in C_B[0,\infty)$, we have*

$$\|D_{n,q} f\| \leq \|f\| .$$

Proof. Since $s_{n,k}(q;x) \geq 0$ for all $q \in (0,1)$ and $x \in [0,\infty)$, we get the following

$$\begin{aligned}
|D_{n,q}(f;x)| &= \left| [n] \sum_{k=0}^{\infty} q^k s_{n,k}(q;x) \int_0^{\infty/(1-q)} s_{n,k}(q;t) f(t) d_q t \right| \\
&\leq [n] \sum_{k=0}^{\infty} q^k \left| s_{n,k}(q;x) \right| \int_0^{\infty/(1-q)} \left| s_{n,k}(q;t) \right| \left| f(t) \right| d_q t \\
&= [n] \sum_{k=0}^{\infty} q^k s_{n,k}(q;x) \int_0^{\infty/(1-q)} s_{n,k}(q;t) \left| f(t) \right| d_q t \\
&\leq \|f\| \, [n] \sum_{k=0}^{\infty} q^k s_{n,k}(q;x) \int_0^{\infty/(1-q)} s_{n,k}(q;t) d_q t \\
&= \|f\| \, D_{n,q}(1;x) \\
&= \|f\| .
\end{aligned}$$

□

Lemma 3.2.3. *Let the operators $D^*_{n,q}$ be defined by (3.2.3) and let $f \in C_B[0,\infty)$. Then*

$$\left\| D^*_{n,q}(f;.) \right\| \leq 3 \|f\| .$$

Proof. By Lemma 3.2.2 and (3.2.3)

$$\left\| D^*_{n,q}(f;.) \right\| \leq \|D_{n,q}(f;.)\| + 2\|f\| \leq 3\|f\| .$$

□

Theorem 3.2.4. *Let $f \in C_B[0,\infty)$. Then, for every $x \in [0,\infty)$, there exists a constant $L > 0$ such that*

$$|D_{n,q}(f;x) - f(x)| \leq L\omega_2(f; \sqrt{\delta_n(q;x)}) + \omega(f; \alpha_n(q;x)),$$

where

$$\delta_n(q;x) = \frac{(1-q^2)(1+2q^2+2q^4)}{q^6} x^2 + \left(\frac{1+2q+3q^2}{q^5[n]} \right) x + \frac{1+2q}{q^3[n]^2}$$

and

$$\alpha_n(q;x) = \left(\frac{1}{q^2} - 1\right)x + \frac{1}{[n]\,q}.$$

Proof. From (3.2.3) and for all $g \in C_B^2\,[0,\infty)$, we can write that

$$\begin{aligned}
&|D_{n,q}(f;x) - f(x)| \\
&\leq \left|D^*_{n,q}(f;x) - f(x)\right| + \left|f(x) - f\left(\frac{x}{q^2} + \frac{1}{[n]\,q}\right)\right| \\
&= \left|D^*_{n,q}(f;x) - f(x) + D^*_{n,q}(g;x) - D^*_{n,q}(g;x) + g(x) - g(x)\right| \\
&+ \left|f(x) - f\left(\frac{x}{q^2} + \frac{1}{[n]\,q}\right)\right| \\
&\leq \left|D^*_{n,q}(f-g;x) - (f-g)(x)\right| + \left|f(x) - f\left(\frac{x}{q^2} + \frac{1}{[n]\,q}\right)\right| \\
&+ \left|D^*_{n,q}(g;x) - g(x)\right| \\
&\leq \left|D^*_{n,q}(f-g;x)\right| + |(f-g)(x)| + \left|f(x) - f\left(\frac{x}{q^2} + \frac{1}{[n]\,q}\right)\right| \\
&+ \left|D^*_{n,q}(g;x) - g(x)\right|.
\end{aligned}$$

Now, taking into account boundedness of $D^*_{n,q}$ and the inequality (3.2.4), we get

$$\begin{aligned}
&|D_{n,q}(f;x) - f(x)| \\
&\leq 4\,\|f-g\| + \left|f(x) - f\left(\frac{x}{q^2} + \frac{1}{[n]\,q}\right)\right| \\
&+ \left\{\frac{(1-q^2)(1+2q^2+2q^4)}{q^6}x^2 + \frac{1+2q+3q^2}{q^5\,[n]}x + \frac{1+2q}{q^3\,[n]^2}\right\}\|g''\| \\
&\leq 4\,\|f-g\| + \omega\left(f;\left(\frac{1}{q^2}-1\right)x + \frac{1}{[n]\,q}\right) + \delta_n(q;x)\,\|g''\|.
\end{aligned}$$

Now, taking infimum on the right-hand side over all $g \in C_B^2\,[0,\infty)$ and using (3.2.1), we get the following result

$$\begin{aligned}
|D_{n,q}(f;x) - f(x)| &\leq 4K_2(f;\delta_n(q;x)) + \omega(f;\alpha_n(q;x)) \\
&\leq 4A\omega_2(f;\sqrt{\delta_n(q;x)}) + \omega(f;\alpha_n(q;x)) \\
&= L\omega_2(f;\sqrt{\delta_n(q;x)}) + \omega(f;\alpha_n(q;x))
\end{aligned}$$

where $L = 4A > 0$. □

Theorem 3.2.5. *Let $0 < \alpha \leq 1$ and E be any bounded subset of the interval $[0, \infty)$. Then, if $f \in C_B[0, \infty)$ is locally $Lip(\alpha)$, i.e. the condition*

$$|f(y) - f(x)| \leq M |y - x|^{\alpha}, \; y \in E \text{ and } x \in [0, \infty) \tag{3.2.6}$$

holds, then, for each $x \in [0, \infty)$, we have

$$|D_{n,q}(f; x) - f(x)| \leq M \left\{ \delta_n^{\frac{\alpha}{2}}(q; x) + 2 (d(x, E))^{\alpha} \right\},$$

where $\delta_n(q; x)$ is the same as in Theorem 3.2.4, M is a constant depending on α and f; and $d(x, E)$ is the distance between x and E defined as

$$d(x, E) = \inf \{|y - x| : y \in E\}.$$

Proof. Let $\overline{E}$ denote the closure of E in $[0, \infty)$. Then, there exists a point $x_0 \in \overline{E}$ such that $|x - x_0| = d(x, E)$. Using the triangle inequality

$$|f(y) - f(x)| \leq |f(y) - f(x_0)| + |f(x) - f(x_0)|$$

we get, by (3.2.6)

$$\begin{aligned}
|D_{n,q}(f; x) - f(x)| &= |D_{n,q}(f; x) - D_{n,q}(f(x); x)| \\
&\leq D_{n,q}(|f(y) - f(x)|; x) \\
&\leq D_{n,q}(|f(y) - f(x_0)|; x) + D_{n,q}(|f(x) - f(x_0)|; x) \\
&= D_{n,q}(|f(y) - f(x_0)|; x) + |f(x) - f(x_0)| \\
&\leq M \{D_{n,q}(|y - x_0|^{\alpha}; x) + |x - x_0|^{\alpha}\} \\
&\leq M \{D_{n,q}(|y - x|^{\alpha} + |x - x_0|^{\alpha}; x) + |x - x_0|^{\alpha}\} \\
&= M \{D_{n,q}(|y - x|^{\alpha}; x) + 2 |x - x_0|^{\alpha}\}.
\end{aligned}$$

Using the Hölder inequality with $p = \frac{2}{\alpha}$, $q = \frac{2}{2-\alpha}$ we find that

$$
\begin{aligned}
&|D_{n,q}(f;x) - f(x)| \\
&\leq M\left\{[D_{n,q}(|y-x|^{\alpha p};x)]^{\frac{1}{p}}\,[D_{n,q}(1^q;x)]^{\frac{1}{q}} + 2\left(d\left(x,E\right)\right)^{\alpha}\right\} \\
&= M\left\{\left[D_{n,q}(|y-x|^2;x)\right]^{\frac{\alpha}{2}} + 2\left(d\left(x,E\right)\right)^{\alpha}\right\} \\
&= M\left\{\left[\left[\left(\frac{1}{q^6}-1\right)+2\left(\frac{1}{q^2}-1\right)\right]x^2\right.\right. \\
&\left.+\left(\frac{1+(2+q)q}{q^5\,[n]}+\frac{2}{[n]\,q}\right)x+(\frac{1}{q^3}+\frac{1}{q^2})\frac{1}{[n]^2}\right]^{\frac{\alpha}{2}} \\
&\left.+2\left(d\left(x,E\right)\right)^{\alpha}\right\} \\
&\leq M\left\{\left[\left[\left(\frac{1}{q^6}-1\right)+2\left(\frac{1}{q^2}-1\right)\right]x^2+\left(\frac{1+(2+q)q}{q^5\,[n]}+\frac{2}{[n]\,q}\right)x\right.\right. \\
&\left.+(\frac{1}{q^3}+\frac{1}{q^2})\frac{1}{[n]^2}+\left[\left(\frac{1}{q^2}-1\right)^2x^2+2\left(\frac{1}{q^2}-1\right)\frac{1}{[n]\,q}x+\frac{1}{[n]^2\,q^2}\right]\right]^{\frac{\alpha}{2}} \\
&\left.+2\left(d\left(x,E\right)\right)^{\alpha}\right\} \\
&= M\left\{\left[\left(\frac{(1-q^2)(1+2q^2+2q^4)}{q^6}\right)x^2+\left(\frac{1+2q+3q^2}{q^5\,[n]}\right)x+\frac{1+2q}{q^3\,[n]^2}\right]^{\frac{\alpha}{2}}\right. \\
&\left.+2\left(d\left(x,E\right)\right)^{\alpha}\right\} \\
&= M\left\{\delta_n^{\frac{\alpha}{2}}(q;x)+2\left(d\left(x,E\right)\right)^{\alpha}\right\}.
\end{aligned}
$$

□

3.3 Voronovskaja Type Theorem

In this section, we prove Voronovskaja type result for the q-Szász-Durrmeyer operators.

Lemma 3.3.1. *Let* $0 < q < 1$. *We have*

$$D_{n,q}(t^3;x) = \frac{(q+1)(q+q^2+1)}{[n]^3 q^6} + \frac{(q+1)(q+q^2+1)^2}{[n]^2 q^9}x + \frac{(q+q^2+1)^2}{[n] q^{11}}x^2 + \frac{1}{q^{12}}x^3,$$

$$D_{n,q}(t^4;x) = \frac{(q^2+1)(q+q^2+1)(q+1)^2}{[n]^4 q^{10}} + \frac{(q+q^2+1)(q^2+1)^2(q+1)^3}{[n]^3 q^{14}}x + \frac{(q+1)(q^2+1)^2(q+q^2+1)^2}{[n]^2 q^{17}}x^2 + \frac{(q+1)^2(q^2+1)^2}{[n] q^{19}}x^3 + \frac{1}{q^{20}}x^4.$$

Proof. In order to prove Lemma 3.3.1, we have to use Lemma 2.5.2. For $f(t) = t^3$, we get

$$\begin{aligned}
&D_{n,q}(t^3;x)\\
&= \frac{1}{[n]^3 q^6}\sum_{k=0}^{\infty}\frac{[k+3][k+2][k+1]}{q^{3k}} s_{n,k}(q;x)\\
&= \frac{1}{[n]^3 q^6}\sum_{k=0}^{\infty}\left\{\frac{[k]^3 + q^k(3+2q+q^2)[k]^2 + q^{2k}(3+4q+3q^2+q^3)[k]}{q^{3k}}\right.\\
&\left. + \frac{q^{3k}(1+q)(1+q+q^2)}{q^{3k}}\right\} s_{n,k}(q;x)\\
&= \frac{1}{[n]^3 q^6}\left\{\sum_{k=0}^{\infty}\frac{[k]^3}{q^{3k}} s_{n,k}(q;x) + \sum_{k=0}^{\infty}\frac{(3+2q+q^2)[k]^2}{q^{2k}} s_{n,k}(q;x)\right.\\
&\left. + \sum_{k=0}^{\infty}\frac{(3+4q+3q^2+q^3)[k]}{q^k} s_{n,k}(q;x) + \sum_{k=0}^{\infty}(1+q)\left(1+q+q^2\right) s_{n,k}(q;x)\right\}\\
&= \frac{1}{q^6}\sum_{k=0}^{\infty}\frac{[k]^3}{[n]^3 q^{3k}} s_{n,k}(q;x) + \frac{(3+2q+q^2)}{[n] q^6}\sum_{k=0}^{\infty}\frac{[k]^2}{[n]^2 q^{2k}} s_{n,k}(q;x)\\
&+ \frac{(3+4q+3q^2+q^3)}{[n]^2 q^6}\sum_{k=0}^{\infty}\frac{[k]}{[n] q^k} s_{n,k}(q;x) + \frac{(1+2q+2q^2+q^3)}{[n]^3 q^6}\sum_{k=0}^{\infty} s_{n,k}(q;x)\\
&= \frac{1}{q^{12}}\sum_{k=0}^{\infty}\frac{[k]^3}{[n]^3 q^{3k-6}} s_{n,k}(q;x) + \frac{(3+2q+q^2)}{[n] q^{10}}\sum_{k=0}^{\infty}\frac{[k]^2}{[n]^2 q^{2k-4}} s_{n,k}(q;x)\\
&+ \frac{(3+4q+3q^2+q^3)}{[n]^2 q^8}\sum_{k=0}^{\infty}\frac{[k]}{[n] q^{k-2}} s_{n,k}(q;x) + \frac{(1+2q+2q^2+q^3)}{[n]^3 q^6}\sum_{k=0}^{\infty} s_{n,k}(q;x)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{q^{12}}\sum_{k=0}^{\infty}\left(\frac{[k]}{[n]\,q^{k-2}}\right)^3 s_{n,k}(q;x) + \frac{(3+2q+q^2)}{[n]\,q^{10}}\sum_{k=0}^{\infty}\left(\frac{[k]}{[n]\,q^{k-2}}\right)^2 s_{n,k}(q;x) \\
&+ \frac{(3+4q+3q^2+q^3)}{[n]^2\,q^8}\sum_{k=0}^{\infty}\frac{[k]}{[n]\,q^{k-2}}s_{n,k}(q;x) + \frac{(1+2q+2q^2+q^3)}{[n]^3\,q^6}\sum_{k=0}^{\infty}s_{n,k}(q;x) \\
&= \frac{1}{q^{12}}\left(\frac{q^3}{[n]^2}x + \left(2q^2+q\right)\frac{x^2}{[n]} + x^3\right) + \frac{(3+2q+q^2)}{[n]\,q^{10}}\left(qx^2 + \frac{q^2}{[n]}x\right) \\
&+ \frac{(3+4q+3q^2+q^3)\,q}{[n]^2\,q^8}x + \frac{(1+2q+2q^2+q^3)}{[n]^3\,q^6} \\
&= \frac{1}{q^9\,[n]^2}x + \frac{(2q^2+q)}{q^{12}\,[n]}x^2 + \frac{1}{q^{12}}x^3 + \frac{(3+2q+q^2)}{[n]\,q^9}x^2 + \frac{(3+2q+q^2)}{[n]^2\,q^8}x \\
&+ \frac{(3+4q+3q^2+q^3)}{[n]^2\,q^7}x + \frac{(1+2q+2q^2+q^3)}{[n]^3\,q^6} \\
&= \frac{(1+2q+2q^2+q^3)}{[n]^3\,q^6} + \left(\frac{1}{q^9\,[n]^2} + \frac{(3+2q+q^2)}{[n]^2\,q^8} + \frac{(3+4q+3q^2+q^3)}{[n]^2\,q^7}\right)x \\
&+ \left(\frac{(2q^2+q)}{q^{12}\,[n]} + \frac{(3+2q+q^2)}{[n]\,q^9}\right)x^2 + \frac{1}{q^{12}}x^3 \\
&= \frac{(1+2q+2q^2+q^3)}{[n]^3\,q^6} + \left(\frac{1+3q+5q^2+5q^3+3q^4+q^5}{[n]^2\,q^9}\right)x \\
&+ \left(\frac{1+2q+3q^2+2q^3+q^4}{[n]\,q^{11}}\right)x^2 + \frac{1}{q^{12}}x^3 \\
&= \frac{(q+1)\,(q+q^2+1)}{[n]^3\,q^6} + \frac{(q+1)\,(q+q^2+1)^2}{[n]^2\,q^9}x + \frac{(q+q^2+1)^2}{[n]\,q^{11}}x^2 + \frac{1}{q^{12}}x^3
\end{aligned}
$$

and for $f(t) = t^4$, we have

$$
\begin{aligned}
D_{n,q}(t^4;x) &= \frac{1}{[n]^4\,q^{10}}\sum_{k=0}^{\infty}\frac{[k+4]\,[k+3]\,[k+2]\,[k+1]}{q^{4k}}s_{n,k}(q;x) \\
&= \frac{1}{[n]^4\,q^{10}}\sum_{k=0}^{\infty}\left\{\frac{[k]^4 + q^k\,(4+3q+2q^2+q^3)\,[k]^3}{q^{4k}}\right. \\
&+ \frac{q^{2k}\,(6+9q+9q^2+7q^3+3q^4+q^5)\,[k]^2}{q^{4k}} \\
&+ \frac{q^{3k}\,(4+9q+12q^2+12q^3+8q^4+4q^5+q^6)\,[k]}{q^{4k}} \\
&\left.+\frac{q^{4k}\,(1+3q+5q^2+6q^3+5q^4+3q^5+q^6)}{q^{4k}}\right\} s_{n,k}(q;x)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{[n]^4 q^{10}}\sum_{k=0}^{\infty}\frac{[k]^4}{q^{4k}}s_{n,k}(q;x) + \frac{1}{[n]^4 q^{10}}\sum_{k=0}^{\infty}\frac{q^k\left(4+3q+2q^2+q^3\right)[k]^3}{q^{4k}}s_{n,k}(q;x)\\
&+ \frac{1}{[n]^4 q^{10}}\sum_{k=0}^{\infty}\frac{q^{2k}\left(6+9q+9q^2+7q^3+3q^4+q^5\right)[k]^2}{q^{4k}}s_{n,k}(q;x)\\
&+ \frac{1}{[n]^4 q^{10}}\sum_{k=0}^{\infty}\frac{q^{3k}\left(4+9q+12q^2+12q^3+8q^4+4q^5+q^6\right)[k]}{q^{4k}}s_{n,k}(q;x)\\
&+ \frac{1}{[n]^4 q^{10}}\sum_{k=0}^{\infty}\frac{q^{4k}\left(1+3q+5q^2+6q^3+5q^4+3q^5+q^6\right)}{q^{4k}}s_{n,k}(q;x)\\
&= \frac{1}{q^{18}}\sum_{k=0}^{\infty}\frac{[k]^4}{[n]^4 q^{4k-8}}s_{n,k}(q;x) + \frac{\left(4+3q+2q^2+q^3\right)}{[n]\, q^{16}}\sum_{k=0}^{\infty}\frac{[k]^3}{[n]^3 q^{3k-6}}s_{n,k}(q;x)\\
&+ \frac{\left(6+9q+9q^2+7q^3+3q^4+q^5\right)}{[n]^2 q^{14}}\sum_{k=0}^{\infty}\frac{[k]^2}{[n]^2 q^{2k-4}}s_{n,k}(q;x)\\
&+ \frac{\left(4+9q+12q^2+12q^3+8q^4+4q^5+q^6\right)}{[n]^3 q^{12}}\sum_{k=0}^{\infty}\frac{[k]}{[n]\, q^{k-2}}s_{n,k}(q;x)\\
&+ \frac{\left(1+3q+5q^2+6q^3+5q^4+3q^5+q^6\right)}{[n]^4 q^{10}}\sum_{k=0}^{\infty}s_{n,k}(q;x)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{q^{18}}\sum_{k=0}^{\infty}\left(\frac{[k]}{[n]\, q^{k-2}}\right)^4 s_{n,k}(q;x)\\
&+ \frac{\left(4+3q+2q^2+q^3\right)}{[n]\, q^{16}}\sum_{k=0}^{\infty}\left(\frac{[k]}{[n]\, q^{k-2}}\right)^3 s_{n,k}(q;x)\\
&+ \frac{\left(6+9q+9q^2+7q^3+3q^4+q^5\right)}{[n]^2 q^{14}}\sum_{k=0}^{\infty}\left(\frac{[k]}{[n]\, q^{k-2}}\right)^2 s_{n,k}(q;x)\\
&+ \frac{\left(4+9q+12q^2+12q^3+8q^4+4q^5+q^6\right)}{[n]^3 q^{12}}\sum_{k=0}^{\infty}\frac{[k]}{[n]\, q^{k-2}}s_{n,k}(q;x)\\
&+ \frac{\left(1+3q+5q^2+6q^3+5q^4+3q^5+q^6\right)}{[n]^4 q^{10}}\sum_{k=0}^{\infty}s_{n,k}(q;x)\\
&= \frac{1}{q^{18}}\left(\frac{q^4}{[n]^3}x + \left(3q^3+3q^2+q\right)\frac{x^2}{[n]^2} + \left(3q+2+\frac{1}{q}\right)\frac{x^3}{[n]} + \frac{x^4}{q^2}\right)\\
&+ \frac{\left(4+3q+2q^2+q^3\right)}{[n]\, q^{16}}\left(\frac{q^3}{[n]^2}x + \left(2q^2+q\right)\frac{x^2}{[n]} + x^3\right)\\
&+ \frac{\left(6+9q+9q^2+7q^3+3q^4+q^5\right)}{[n]^2 q^{14}}\left(qx^2 + \frac{q^2}{[n]}x\right)\\
&+ \frac{\left(4+9q+12q^2+12q^3+8q^4+4q^5+q^6\right)}{[n]^3 q^{11}}x\\
&+ \frac{\left(1+3q+5q^2+6q^3+5q^4+3q^5+q^6\right)}{[n]^4 q^{10}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{q^{14}[n]^3}x + \frac{(3q^2+3q+1)}{q^{17}[n]^2}x^2 + \frac{(3q^2+2q+1)}{q^{19}[n]}x^3 \\
&+ \frac{1}{q^{20}}x^4 + \frac{(4+3q+2q^2+q^3)}{[n]^3 q^{13}}x \\
&+ \frac{(4+3q+2q^2+q^3)(2q^2+q)}{[n]^2 q^{16}}x^2 \\
&+ \frac{(4+3q+2q^2+q^3)}{[n] q^{16}}x^3 \\
&+ \frac{(6+9q+9q^2+7q^3+3q^4+q^5)}{[n]^2 q^{13}}x^2 \\
&+ \frac{(6+9q+9q^2+7q^3+3q^4+q^5)}{[n]^3 q^{12}}x \\
&+ \frac{(4+9q+12q^2+12q^3+8q^4+4q^5+q^6)}{[n]^3 q^{11}}x \\
&+ \frac{(1+3q+5q^2+6q^3+5q^4+3q^5+q^6)}{[n]^4 q^{10}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(1+3q+5q^2+6q^3+5q^4+3q^5+q^6)}{[n]^4 q^{10}} + \Bigg(\frac{1}{q^{14}[n]^3} \\
&+ \frac{(4+3q+2q^2+q^3)}{[n]^3 q^{13}} + \frac{(6+9q+9q^2+7q^3+3q^4+q^5)}{[n]^3 q^{12}} \\
&+ \frac{(4+9q+12q^2+12q^3+8q^4+4q^5+q^6)}{[n]^3 q^{11}}\Bigg)x \\
&+ \Bigg(\frac{(3q^2+3q+1)}{q^{17}[n]^2} + \frac{(4+3q+2q^2+q^3)(2q^2+q)}{[n]^2 q^{16}} \\
&+ \frac{(6+9q+9q^2+7q^3+3q^4+q^5)}{[n]^2 q^{13}}\Bigg)x^2 \\
&+ \left(\frac{(3q^2+2q+1)}{q^{19}[n]} + \frac{(4+3q+2q^2+q^3)}{[n] q^{16}}\right)x^3 + \frac{1}{q^{20}}x^4
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(1+3q+5q^2+6q^3+5q^4+3q^5+q^6)}{[n]^4 q^{10}} \\
&+ \Bigg(\frac{1+4q+3q^2+2q^3+q^4+6q^2+9q^3+9q^4+7q^5+3q^6}{q^{14}[n]^3} \\
&+ \frac{q^7+4q^3+9q^4+12q^5+12q^6+8q^7+4q^8+q^9}{q^{14}[n]^3}\Bigg)x \\
&+ \Bigg(\frac{3q^2+3q+1+4q^2+11q^3+8q^4+5q^5+2q^6}{q^{17}[n]^2} \\
&+ \frac{6q^4+9q^5+9q^6+7q^7+3q^8+q^9}{q^{17}[n]^2}\Bigg)x^2 \\
&+ \left(\frac{3q^2+2q+1+4q^3+3q^4+2q^5+q^6}{q^{19}[n]}\right)x^3 + \frac{1}{q^{20}}x^4
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(1+3q+5q^2+6q^3+5q^4+3q^5+q^6)}{[n]^4 q^{10}} \\
&+ \frac{(1+4q+9q^2+15q^3+19q^4+19q^5+15q^6+9q^7+4q^8+q^9)}{[n]^3 q^{14}} x \\
&+ \frac{(1+3q+7q^2+11q^3+14q^4+14q^5+11q^6+7q^7+3q^8+q^9)}{[n]^2 q^{17}} x^2 \\
&+ \frac{(1+2q+3q^2+4q^3+3q^4+2q^5+q^6)}{[n]\, q^{19}} x^3 + \frac{1}{q^{20}} x^4 \\
&= \frac{(q^2+1)(q+q^2+1)(q+1)^2}{[n]^4 q^{10}} + \frac{(q+q^2+1)(q^2+1)^2(q+1)^3}{[n]^3 q^{14}} x \\
&+ \frac{(q+1)(q^2+1)^2(q+q^2+1)^2}{[n]^2 q^{17}} x^2 + \frac{(q+1)^2(q^2+1)^2}{[n]\, q^{19}} x^3 + \frac{1}{q^{20}} x^4.
\end{aligned}
$$

□

Lemma 3.3.2. *Assume that* $q_n \in (0,1)$, $q_n \to 1$ *and* $q_n^n \to a$ *as* $n \to \infty$. *For every* $x \in [0,\infty)$ *there hold*

$$\lim_{n\to\infty} [n]_{q_n} D_{n,q_n}(t-x;x) = (1-a)2x+1, \tag{3.3.1}$$

$$\lim_{n\to\infty} [n]_{q_n} D_{n,q_n}((t-x)^2;x) = 2(1-a)x^2+2x, \tag{3.3.2}$$

$$\lim_{n\to\infty} [n]_{q_n}^2 D_{n,q_n}((t-x)^4;x) = 12x^2+24(1-a)x^3+12(1-a)^2x^4. \tag{3.3.3}$$

Proof. First of all we write explicit formula for $D_{n,q_n}(t-x;x)$

$$
\begin{aligned}
D_{n,q_n}(t-x;x) &= D_{n,q_n}(t;x) - xD_{n,q_n}(1;x) \\
&= \frac{1}{q_n^2}x + \frac{1}{[n]_{q_n} q_n} - x \\
&= x\left(\frac{1}{q_n^2} - 1\right) + \frac{1}{[n]_{q_n} q_n} \\
&= x\frac{(1-q_n)(1+q_n)}{q_n^2} + \frac{1}{[n]_{q_n} q_n}.
\end{aligned}
$$

Then we get

$$
\begin{aligned}
\lim_{n\to\infty} [n]_{q_n} D_{n,q_n}(t-x;x) &= \lim_{n\to\infty} [n]_{q_n} \left(x\frac{(1-q_n)(1+q_n)}{q_n^2} + \frac{1}{[n]_{q_n} q_n} \right) \\
&= \lim_{n\to\infty} \left(\frac{(1-q_n^n)(1+q_n)}{q_n^2} x + \frac{1}{q_n} \right) \\
&= (1-a)2x + 1.
\end{aligned}
$$

Next, we calculate $D_{n,q_n}((t-x)^2;x)$, as follows

$$
\begin{aligned}
D_{n,q_n}((t-x)^2;x) &= D_{n,q_n}(t^2;x) - 2xD_{n,q_n}(t;x) + x^2 D_{n,q_n}(1;x) + x^2 - x^2 \\
&= \left(\frac{1}{q_n^6} - 1 \right) x^2 + \left(\frac{1+(2+q_n)q_n}{q_n^5 [n]_{q_n}} \right) x + (\frac{1}{q_n^3} + \frac{1}{q_n^2}) \frac{1}{[n]_{q_n}^2} \\
&- 2\left(\frac{1}{q_n^2} - 1 \right) x^2 - \frac{2}{[n]_{q_n} q_n} x \\
&= \frac{(1-q_n)(1+q_n+q_n^2)(1+q_n^3)}{q_n^6} x^2 + \frac{1+(2+q_n)q_n}{q_n^5 [n]_{q_n}} x \\
&+ (\frac{1}{q_n^3} + \frac{1}{q_n^2}) \frac{1}{[n]_{q_n}^2} - 2\frac{(1-q_n)(1+q_n)}{q_n^2} x^2 - \frac{2}{[n]_{q_n} q_n} x
\end{aligned}
$$

and we obtain

$$
\begin{aligned}
&\lim_{n\to\infty} [n]_{q_n} D_{n,q_n}((t-x)^2;x) \\
&= \lim_{n\to\infty} [n]_{q_n} \left(\frac{(1-q_n)(1+q_n+q_n^2)(1+q_n^3)}{q_n^6} x^2 + \frac{1+(2+q_n)q_n}{q_n^5 [n]_{q_n}} x \right. \\
&\left. +(\frac{1}{q_n^3} + \frac{1}{q_n^2}) \frac{1}{[n]_{q_n}^2} - 2\frac{(1-q_n)(1+q_n)}{q_n^2} x^2 - \frac{2}{[n]_{q_n} q_n} x \right) \\
&= \lim_{n\to\infty} \left(\frac{(1-q_n^n)(1+q_n+q_n^2)(1+q_n^3)}{q_n^6} x^2 + \frac{1+(2+q_n)q_n}{q_n^5} x \right. \\
&\left. +(\frac{1}{q_n^3} + \frac{1}{q_n^2}) \frac{1}{[n]_{q_n}} - 2\frac{(1-q_n^n)(1+q_n)}{q_n^2} x^2 - \frac{2}{q_n} x \right) \\
&= (1-a)\, 6x^2 + 4x - (1-a)\, 4x^2 - 2x \\
&= 2\,(1-a)\, x^2 + 2x.
\end{aligned}
$$

Finally, we give an explicit formula for $D_{n,q_n}((t-x)^4\,;x)$

$$
\begin{aligned}
&D_{n,q_n}((t-x)^4\,;x)\\
&= D_{n,q_n}(t^4;x) - 4xD_{n,q_n}(t^3;x) + 6x^2 D_{n,q_n}(t^2;x) - 4x^3 D_{n,q_n}(t;x) + x^4\\
&= \frac{(1+3q_n+5q_n^2+6q_n^3+5q_n^4+3q_n^5+q_n^6)}{[n]_{q_n}^4\, q_n^{10}}\\
&+ \frac{(1+4q_n+9q_n^2+15q_n^3+19q_n^4+19q_n^5+15q_n^6+9q_n^7+4q_n^8+q_n^9)}{[n]_n^3\, q_n^{14}}x\\
&+ \frac{(1+3q_n+7q_n^2+11q_n^3+14q_n^4+14q_n^5+11q_n^6+7q_n^7+3q_n^8+q_n^9)}{[n]_{q_n}^2\, q_n^{17}}x^2\\
&+ \frac{(1+2q_n+3q_n^2+4q_n^3+3q_n^4+2q_n^5+q_n^6)}{[n]_{q_n}\, q_n^{19}}x^3 + \frac{1}{q_n^{20}}x^4\\
&- 4x\left(\frac{(1+2q_n+2q_n^2+q_n^3)}{[n]_{q_n}^3\, q_n^6} + \frac{(1+3q_n+5q_n^2+5q_n^3+3q_n^4+q_n^5)}{[n]_n^2\, q_n^9}x\right.\\
&\left.+\frac{(1+2q_n+3q_n^2+2q_n^3+q_n^4)}{[n]_{q_n}\, q_n^{11}}x^2 + \frac{1}{q_n^{12}}x^3\right)\\
&+ 6x^2\left(\frac{x^2}{q_n^6} + \left(\frac{1+(2+q_n)q_n}{q_n^5\,[n]_{q_n}}\right)x + \left(\frac{1}{q_n^3}+\frac{1}{q_n^2}\right)\frac{1}{[n]_{q_n}^2}\right)\\
&- 4x^3\left(\frac{x}{q_n^2} + \frac{1}{[n]_{q_n}\, q_n}\right) + x^4
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{n\to\infty} [n]_{q_n}^2\, D_{n,q_n}((t-x)^4\,;x)\\
&= \lim_{n\to\infty}\left\{\frac{(1+3q_n+5q_n^2+6q_n^3+5q_n^4+3q_n^5+q_n^6)}{[n]_{q_n}^2\, q_n^{10}}\right.\\
&+ \frac{(1+4q_n+9q_n^2+15q_n^3+19q_n^4+19q_n^5+15q_n^6+9q_n^7+4q_n^8+q_n^9)}{[n]_{q_n}\, q_n^{14}}x\\
&+ \frac{(1+3q_n+7q_n^2+11q_n^3+14q_n^4+14q_n^5+11q_n^6+7q_n^7+3q_n^8+q_n^9)}{q_n^{17}}x^2\\
&+ [n]_{q_n}\frac{(1+2q_n+3q_n^2+4q_n^3+3q_n^4+2q_n^5+q_n^6)}{q_n^{19}}x^3 + \frac{[n]_{q_n}^2}{q_n^{20}}x^4\\
&- 4x\left(\frac{(1+2q_n+2q_n^2+q_n^3)}{[n]_{q_n}\, q_n^6} + \frac{(1+3q_n+5q_n^2+5q_n^3+3q_n^4+q_n^5)}{q_n^9}x\right.\\
&\left.+ [n]_{q_n}\frac{(1+2q_n+3q_n^2+2q_n^3+q_n^4)}{q_n^{11}}x^2 + [n]_{q_n}^2\frac{1}{q_n^{12}}x^3\right)\\
&+ 6x^2\left([n]_{q_n}^2\frac{x^2}{q_n^6} + [n]_{q_n}\left(\frac{1+(2+q_n)q_n}{q_n^5}\right)x + \left(\frac{1}{q_n^3}+\frac{1}{q_n^2}\right)\right)\\
&\left.-4x^3\left([n]_{q_n}^2\frac{x}{q_n^2} + [n]_{q_n}\frac{1}{q_n}\right) + [n]_{q_n}^2\, x^4\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= 72x^2 - 72x^2 + 12x^2 \\
&+ \lim_{n\to\infty}\left\{[n]_{q_n}\frac{(1+2q_n+3q_n^2+4q_n^3+3q_n^4+2q_n^5+q_n^6)}{q_n^{19}}x^3\right. \\
&+ \frac{[n]_{q_n}^2}{q_n^{20}}x^4 - 4[n]_{q_n}\frac{(1+2q_n+3q_n^2+2q_n^3+q_n^4)}{q_n^{11}}x^3 - 4[n]_{q_n}^2\frac{1}{q_n^{12}}x^4 \\
&+ 6[n]_{q_n}^2\frac{1}{q_n^6}x^4 + 6[n]_{q_n}\left(\frac{1+(2+q_n)q_n}{q_n^5}\right)x^3 \\
&\left.-4[n]_{q_n}^2\frac{1}{q_n^2}x^4 - 4[n]_{q_n}\frac{1}{q_n}x^3 + [n]_{q_n}^2 x^4\right\} \\
&= 12x^2 + \lim_{n\to\infty}\left\{x^4\left(\frac{[n]_{q_n}^2}{q_n^{20}} - 4[n]_{q_n}^2\frac{1}{q_n^{12}} + 6[n]_{q_n}^2\frac{1}{q_n^6} - 4[n]_{q_n}^2\frac{1}{q_n^2} + [n]_{q_n}^2\right)\right. \\
&+ x^3\left([n]_{q_n}\frac{(1+2q_n+3q_n^2+4q_n^3+3q_n^4+2q_n^5+q_n^6)}{q_n^{19}}\right. \\
&- 4[n]_{q_n}\frac{(1+2q_n+3q_n^2+2q_n^3+q_n^4)}{q_n^{11}} \\
&\left.\left.+6[n]_{q_n}\left(\frac{1+(2+q_n)q_n}{q_n^5}\right) - 4[n]_{q_n}\frac{1}{q_n}\right)\right\} \\
&= 12x^2 + \lim_{n\to\infty}\left\{x^4[n]_{q_n}^2\left(\frac{1}{q_n^{20}} - \frac{4}{q_n^{12}} + \frac{6}{q_n^6} - \frac{4}{q_n^2} + 1\right)\right. \\
&+ x^3[n]_{q_n}\left(\frac{(1+2q_n+3q_n^2+4q_n^3+3q_n^4+2q_n^5+q_n^6)}{q_n^{19}}\right. \\
&\left.\left.-\frac{4(1+2q_n+3q_n^2+2q_n^3+q_n^4)}{q_n^{11}} + \frac{6(1+(2+q_n)q_n)}{q_n^5} - \frac{4}{q_n}\right)\right\} \\
&= 12x^2 + \lim_{n\to\infty}\left\{x^4[n]_{q_n}^2 q_n^{-20}(q_n+1)^2(q_n-1)^2\left(1+2q_n^2+3q_n^4\right.\right. \\
&\left.+4q_n^6+q_n^8-2q_n^{10}-5q_n^{12}-2q_n^{14}+q_n^{16}\right) + x^3[n]_{q_n}q_n^{-19}(1-q_n)\left(1+3q_n\right. \\
&+ 6q_n^2+10q_n^3+13q_n^4+15q_n^5+16q_n^6+16q_n^7+12q_n^8+4q_n^9 \\
&\left.\left.-8q_n^{10}-16q_n^{11}-20q_n^{12}-20q_n^{13}-14q_n^{14}-2q_n^{15}+4q_n^{16}+4q_n^{17}\right)\right\} \\
&= 12x^2 + \lim_{n\to\infty}\left\{x^4 q_n^{-20}(1-q_n^n)^2(q_n+1)^2\left(1+2q_n^2+3q_n^4\right.\right. \\
&\left.+4q_n^6+q_n^8-2q_n^{10}-5q_n^{12}-2q_n^{14}+q_n^{16}\right) \\
&+ x^3 q_n^{-19}(1-q_n^n)\left(1+3q_n+6q_n^2+10q_n^3+13q_n^4\right. \\
&+ 15q_n^5+16q_n^6+16q_n^7+12q_n^8+4q_n^9-8q_n^{10}-16q_n^{11} \\
&\left.\left.-20q_n^{12}-20q_n^{13}-14q_n^{14}-2q_n^{15}+4q_n^{16}+4q_n^{17}\right)\right\} \\
&= 12x^2 + 24(1-a)x^3 + 12(1-a)^2x^4
\end{aligned}
$$

and we prove (3.3.3). □

Theorem 3.3.3. *Let $q_n \in (0,1)$. Then the sequence $\{D_{n,q_n}(f)\}$ converges to f uniformly on $[0,A]$ for each $f \in C_2^*[0,\infty)$ if and only if $\lim\limits_{n\to\infty} q_n = 1$.*

Proof. The proof is similar to that of Theorem 2 [12]. □

Theorem 3.3.4. *Assume that $q_n \in (0,1)$, $q_n \to 1$ and $q_n^n \to a$ as $n \to \infty$. For any $f \in C_2^*[0,\infty)$ such that $f', f'' \in C_2^*[0,\infty)$ the following equality holds*

$$\lim_{n\to\infty} [n]_{q_n} \left(D_{n,q_n}(f;x) - f(x)\right) = \left((1-a)\,2x+1\right) f'(x) + f''(x)\left((1-a)x^2 + x\right)$$

for every $x \geq 0$.

Proof. Let $f, f', f'' \in C_2^*[0,\infty)$ and $x \in [0,\infty)$ be fixed. By the Taylor formula we may write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + r(t;x)(t-x)^2, \tag{3.3.4}$$

where $r(t;x)$ is the Peano form of the remainder, $r(.;x) \in C_2^*[0,\infty)$ and $\lim\limits_{t\to x} r(t;x) = 0$. Applying D_{n,q_n} to (3.3.4) we obtain

$$\begin{aligned}
&[n]_{q_n} \left(D_{n,q_n}(f;x) - f(x)\right) \\
&= f'(x)\,[n]_{q_n} D_{n,q_n}(t-x;x) \\
&+ \frac{1}{2}f''(x)\,[n]_{q_n} D_{n,q_n}\left((t-x)^2;x\right) + [n]_{q_n} D_{n,q_n}\left(r(t;x)(t-x)^2;x\right).
\end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$D_{n,q_n}\left(r(t;x)(t-x)^2;x\right) \leq \sqrt{D_{n,q_n}\left(r^2(t;x);x\right)}\sqrt{D_{n,q_n}\left((t-x)^4;x\right)}. \tag{3.3.5}$$

Observe that $r^2(x;x) = 0$ and $r^2(.;x) \in C_2^*[0,\infty)$. Then it follows from Theorem 3.3.3 that

$$\lim_{n\to\infty} D_{n,q_n}\left(r^2(t;x);x\right) = r^2(x;x) = 0 \tag{3.3.6}$$

uniformly with respect to $x \in [0,A]$. Now from (3.3.5), (3.3.6) and Lemma 3.3.2 we get immediately

$$\lim_{n\to\infty} [n]_{q_n} D_{n,q_n}\left(r(t;x)(t-x)^2;x\right) = 0.$$

Then we get the following

$$
\begin{aligned}
&\lim_{n\to\infty} [n]_{q_n} \left(D_{n,q_n}(f;x) - f(x)\right) \\
&= \lim_{n\to\infty} \Bigg(f'(x)\, [n]_{q_n} D_{n,q_n}(t-x;x) + \frac{1}{2} f''(x)\, [n]_{q_n} D_{n,q_n}\left((t-x)^2;x\right) \\
&+ [n]_{q_n} D_{n,q_n}\left(r(t;x)(t-x)^2;x\right)\Bigg) \\
&= \left((1-a)\,2x+1\right) f'(x) + \frac{1}{2} f''(x) \left(2(1-a)x^2+2x\right) \\
&= \left((1-a)\,2x+1\right) f'(x) + f''(x) \left((1-a)x^2+x\right).
\end{aligned}
$$

□

Chapter 4

ON CERTAIN q-PHILLIPS OPERATORS

In this chapter, we introduce q-parametric Phillips operators $P_{n,q}$ and evaluate the moments of $P_{n,q}$. We study the approximation properties of the q-Phillips operators, establish some local approximation result for continuous functions in terms of modulus of continuity and obtain inequalities for the weighted approximation error of q-Phillips operators. Furthermore, we study Voronovskaja type asymptotic formula for the q-Phillips operators.(see [31])

4.1 Moments

In this section firstly, we introduce the following so called q-Phillips operators which generalize the sequence of classical Phillips operators.

Definition 4.1.1. *For $f \in \mathbb{R}^{[0,\infty)}$, $0 < q < 1$ and $n \in \mathbb{N}$ we define the following q-parametric Phillips operators*

$$P_{n,q}(f;x) = [n] \sum_{k=1}^{\infty} q^{k-1} s_{n,k}(q;qx) \int_{0}^{\infty/(1-q)} s_{n,k-1}(q;t) f(t) d_q t + e_q\left(-[n]\,qx\right) f(0), \tag{4.1.1}$$

where $x \in [0,\infty)$ and $s_{n,k}(q;x) = \frac{1}{E_q([n]x)} q^{\frac{k(k-1)}{2}} \frac{[n]^k x^k}{[k]!} = e_q(-[n]\,x) q^{\frac{k(k-1)}{2}} \frac{[n]^k x^k}{[k]!}$.

It is clear that $s_{n,k}(q;x) \geq 0$ for all $q \in (0,1)$ and $x \in [0,\infty)$. Moreover $\sum\limits_{k=0}^{\infty} s_{n,k}(q;x) = \frac{1}{E_q([n]x)} \sum\limits_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{[n]^k x^k}{[k]!} \overset{\text{(by 2.4.18)}}{=} \frac{1}{E_q([n]x)} E_q([n]\,x) = 1.$

Secondly, we calculate $P_{n,q}(t^i;x)$ for $i = 0,1,2$. By the definition (2.4.15) of q-Gamma function γ_q, we have

$$\begin{aligned}
\int_0^{\infty/(1-q)} t^s s_{n,k}(q;t)d_qt &= \int_0^{\infty/(1-q)} t^s \frac{1}{E_q([n]\,t)} q^{\frac{k(k-1)}{2}} \frac{[n]^k\, t^k}{[k]!} d_qt \\
&= \int_0^{\infty/(1-q)} t^s e_q(-[n]\,t) q^{\frac{k(k-1)}{2}} \frac{[n]^k\, t^k}{[k]!} d_qt \\
&= \frac{1}{[n]^{s+1}} \frac{1}{[k]!} q^{\frac{k(k-1)}{2}} \int_0^{\infty/(1-q)} ([n]\,t)^{k+s}\, e_q(-[n]\,t)\,[n]\,d_qt \\
&= \frac{1}{[n]^{s+1}} \frac{1}{[k]!} q^{\frac{k(k-1)}{2}} \int_0^{\infty/(1-q)[n]^{-1}} (u)^{k+s}\, e_q(-u) d_qu \\
&= \frac{1}{[n]^{s+1}} \frac{1}{[k]!} q^{\frac{k(k-1)}{2}} \gamma_q^{[n]^{-1}}(k+s+1) \\
&= \frac{1}{[n]^{s+1}} \frac{1}{[k]!} q^{\frac{k(k-1)}{2}} \frac{\Gamma_q(k+s+1)}{q^{(k+s+1)(k+s)/2}} \\
&= \frac{1}{[n]^{s+1}} \frac{1}{[k]!} q^{\frac{k(k-1)}{2}} \frac{[k+s]!}{q^{(k+s+1)(k+s)/2}} \\
&= \frac{1}{[n]^{s+1}} \frac{[k+s]!}{[k]!} \frac{1}{q^{(2k+s)(s+1)/2}},\ s \in \mathbb{N}\cup\{0\}\,.
\end{aligned}$$

Lemma 4.1.2. *We have*

$$\begin{aligned}
P_{n,q}(1;x) = 1, \quad P_{n,q}(t;x) &= x, \\
P_{n,q}(t^2;x) &= \frac{1}{q^2}x^2 + \frac{(1+q)}{q^2\,[n]}x, \\
P_{n,q}((t-x)^2\,;x) &= \left(\frac{1}{q^2}-1\right)x^2 + \frac{(1+q)}{q^2\,[n]}x.
\end{aligned}$$

Proof. We know that, see [29],

$$S_{n,q}(1;x) = 1, \quad S_{n,q}(t;x) = qx, \quad S_{n,q}(t^2;x) = qx^2 + \frac{q^2}{[n]}x.$$

Using the above formulas we obtain the following for $f(t)=1$, $f(t)=t$, $f(t)=t^2$, respectively

$$
\begin{aligned}
P_{n,q}(1;x) &= [n]\sum_{k=1}^{\infty} q^{k-1}s_{n,k}(q;qx)\int_{0}^{\infty/(1-q)} s_{n,k-1}(q;t)d_qt + e_q\left(-[n]\,qx\right) \\
&= [n]\sum_{k=1}^{\infty} q^{k-1}s_{n,k}(q;qx)\frac{1}{[n]}\frac{1}{q^{k-1}} + e_q\left(-[n]\,qx\right) \\
&= \sum_{k=1}^{\infty} s_{n,k}(q;qx) + e_q\left(-[n]\,qx\right) = \sum_{k=0}^{\infty} s_{n,k}(q;qx) = 1,
\end{aligned}
$$

$$
\begin{aligned}
P_{n,q}(t;x) &= [n]\sum_{k=1}^{\infty} q^{k-1}s_{n,k}(q;qx)\int_{0}^{\infty/(1-q)} ts_{n,k-1}(q;t)d_qt \\
&= [n]\sum_{k=1}^{\infty} q^{k}s_{n,k}(q;qx)\frac{[k]}{[n]^2}\frac{1}{q^{2k-1}} \\
&= \sum_{k=0}^{\infty} s_{n,k}(q;qx)\frac{[k]}{[n]}\frac{1}{q^{k}} \\
&= \frac{1}{q^2}\sum_{k=0}^{\infty} s_{n,k}(q;qx)\frac{[k]}{[n]}\frac{1}{q^{k-2}} = \frac{1}{q^2}q^2x = x
\end{aligned}
$$

and

$$
\begin{aligned}
P_{n,q}(t^2;x) &= [n]\sum_{k=1}^{\infty} q^{k-1}s_{n,k}(q;qx)\int_{0}^{\infty/(1-q)} t^2s_{n,k-1}(q;t)d_qt \\
&= \sum_{k=1}^{\infty} s_{n,k}(q;qx)\frac{[k+1]\,[k]}{[n]^2}\frac{1}{q^{2k+1}} = \sum_{k=0}^{\infty} s_{n,k}(q;qx)\frac{[k+1]\,[k]}{[n]^2}\frac{1}{q^{2k+1}} \\
&= \sum_{k=0}^{\infty} s_{n,k}(q;qx)\frac{\left([k]+q^k\right)[k]}{[n]^2}\frac{1}{q^{2k+1}} = \sum_{k=0}^{\infty} s_{n,k}(q;qx)\frac{[k]^2}{[n]^2}\frac{1}{q^{2k+1}} \\
&+ \sum_{k=0}^{\infty} s_{n,k}(q;qx)\frac{q^k\,[k]}{[n]^2}\frac{1}{q^{2k+1}} \\
&= \frac{1}{q^5}\sum_{k=0}^{\infty} s_{n,k}(q;qx)\frac{[k]^2}{[n]^2}\frac{1}{q^{2k-4}} + \frac{1}{[n]\,q^3}\sum_{k=0}^{\infty} s_{n,k}(q;qx)\frac{[k]}{[n]}\frac{1}{q^{k-2}} \\
&= \frac{1}{q^5}\left(q^3x^2 + \frac{q^3}{[n]}x\right) + \frac{1}{[n]\,q}x = \frac{1}{q^2}x^2 + \frac{1}{q^2\,[n]}x + \frac{1}{[n]\,q}x \\
&= \frac{1}{q^2}x^2 + \frac{(1+q)}{q^2\,[n]}x.
\end{aligned}
$$

Since $P_{n,q}$ are the positive linear operators

$$\begin{aligned}
P_{n,q}((t-x)^2;x) &= P_{n,q}(t^2-2tx+x^2;x) \\
&= P_{n,q}(t^2;x)-2xP_{n,q}(t;x)+x^2P_{n,q}(1;x) \\
&= \frac{1}{q^2}x^2+\frac{(1+q)}{q^2[n]}x-2x^2+x^2 \\
&= \left(\frac{1}{q^2}-1\right)x^2+\frac{(1+q)}{q^2[n]}x
\end{aligned}$$

which complete the proof. □

Lemma 4.1.3. *For all* $0<q<1$ *the following identity holds:*

$$P_{n,q}(t^m;x)=\frac{1}{[n]^m q^{(m^2-m)/2}}\sum_{s=1}^{m}C_{s,m}(q)\,[n]^s\sum_{k=0}^{\infty}\frac{[k]^s}{[n]^s}\frac{1}{q^{km}}s_{n,k}(q;qx).$$

Proof. Indeed, we have

$$\begin{aligned}
P_{n,q}(t^m;x) &= [n]\sum_{k=1}^{\infty}q^{k-1}s_{n,k}(q;qx)\int_0^{\infty/(1-q)}t^m s_{n,k-1}(q;t)d_qt \\
&= [n]\sum_{k=1}^{\infty}q^{k-1}s_{n,k}(q;qx)\frac{1}{[n]^{m+1}}\frac{1}{[k-1]!}q^{\frac{(k-1)(k-2)}{2}}\frac{[k-1+m]!}{q^{(k+m)(k-1+m)/2}} \\
&= \sum_{k=1}^{\infty}\frac{[k-1+m]\ldots[k]}{[n]^m}\frac{1}{q^{(m^2+2mk-m)/2}}s_{n,k}(q;qx) \\
&= \sum_{k=0}^{\infty}\frac{[k-1+m]\ldots[k]}{[n]^m q^{(m^2+2mk-m)/2}}s_{n,k}(q;qx).
\end{aligned}$$

Using $[k+s]=[s]+q^s[k]$, we obtain

$$[k]\,[k+1]\ldots[k+m-1]=\prod_{s=0}^{m-1}([s]+q^s[k])=\sum_{s=1}^{m}C_{s,m}(q)\,[k]^s$$

where $C_{s,m}(q)>0$, $s=1,2,...,m$ are the constants independent of k. Hence

$$\begin{aligned}
P_{n,q}(t^m;x) &= \frac{1}{[n]^m q^{(m^2-m)/2}}\sum_{k=0}^{\infty}\frac{1}{q^{km}}\sum_{s=1}^{m}C_{s,m}(q)\,[k]^s\,s_{n,k}(q;qx) \\
&= \frac{1}{[n]^m q^{(m^2-m)/2}}\sum_{k=0}^{\infty}\sum_{s=1}^{m}C_{s,m}(q)\,[k]^s\,\frac{1}{q^{km}}s_{n,k}(q;qx) \\
&= \frac{1}{[n]^m q^{(m^2-m)/2}}\sum_{s=1}^{m}C_{s,m}(q)\,[n]^s\sum_{k=0}^{\infty}\frac{[k]^s}{[n]^s}\frac{1}{q^{km}}s_{n,k}(q;qx).
\end{aligned}$$

□

4.2 Approximation Properties

Lemma 4.2.1. *Let* $f \in C_B^2[0,\infty)$*. Then, for all* $f \in C_B^2[0,\infty)$ *, we have*

$$|P_{n,q}(f;x) - f(x)| \leq \left\{ \left(\frac{1-q^2}{q^2} \right) x^2 + \frac{(1+q)}{q^2[n]} x \right\} \|f''\| . \tag{4.2.1}$$

Proof. Let $x \in [0,\infty)$ and $f \in C_B^2[0,\infty)$. Using the Taylor's formula

$$f(t) - f(x) = (t-x)f'(x) + \int_x^t (t-u)f''(u)du,$$

we can write

$$\begin{aligned} P_{n,q}(f;x) - f(x) &= P_{n,q}((t-x)f'(x);x) + P_{n,q}\left(\int_x^t (t-u)f''(u)du; x \right) \\ &= f'(x)P_{n,q}((t-x);x) + P_{n,q}\left(\int_x^t (t-u)f''(u)du; x \right) \\ &= P_{n,q}\left(\int_x^t (t-u)f''(u)du; x \right) . \end{aligned}$$

On the other hand, since

$$\left| \int_x^t (t-u)f''(u)du \right| \leq \int_x^t |t-u| \, |f''(u)| \, du \leq \|f''\| \int_x^t |t-u| \, du \leq (t-x)^2 \|f''\| ,$$

we conclude that

$$\begin{aligned} |P_{n,q}(f;x) - f(x)| &= \left| P_{n,q}(\int_x^t (t-u)f''(u)du; x) \right| \\ &\leq P_{n,q}((t-x)^2 \|f''\| ; x) \\ &= \left\{ \left(\frac{1-q^2}{q^2} \right) x^2 + \frac{(1+q)}{q^2[n]} x \right\} \|f''\| . \end{aligned}$$

□

Lemma 4.2.2. *For* $f \in C_B[0,\infty)$ *, we have*

$$\|P_{n,q}f\| \leq \|f\| .$$

Proof. Let $f \in C_B[0,\infty)$. By Definition 4.1.1, we get the following result

$$
\begin{aligned}
|P_{n,q}(f;x)| &= \left| [n] \sum_{k=0}^{\infty} q^{k-1} s_{n,k}(q;qx) \int_0^{\infty/(1-q)} s_{n,k-1}(q;t) f(t) d_q t \right| \\
&\leq [n] \sum_{k=0}^{\infty} q^{k-1} |s_{n,k}(q;qx)| \int_0^{\infty/(1-q)} |s_{n,k-1}(q;t)| \, |f(t)| \, d_q t \\
&= [n] \sum_{k=0}^{\infty} q^{k-1} s_{n,k}(q;qx) \int_0^{\infty/(1-q)} s_{n,k-1}(q;t) \, |f(t)| \, d_q t \\
&\leq \|f\| \, [n] \sum_{k=0}^{\infty} q^{k-1} s_{n,k}(q;qx) \int_0^{\infty/(1-q)} s_{n,k-1}(q;t) d_q t \\
&= \|f\| \, P_{n,q}(1;x) \\
&= \|f\| .
\end{aligned}
$$

□

Theorem 4.2.3. *Let $f \in C_B[0,\infty)$. Then, for every $x \in [0,\infty)$, there exists a constant $M > 0$ such that*

$$|P_{n,q}(f;x) - f(x)| \leq M\omega_2(f; \sqrt{\delta_n(x)})$$

where

$$\delta_n(x) = \left(\frac{1-q^2}{q^2} \right) x^2 + \frac{(1+q)}{q^2 [n]} x.$$

Proof. Now, taking into account boundedness of $P_{n,q}$, we get

$$
\begin{aligned}
|P_{n,q}(f;x) - f(x)| &= |P_{n,q}(f;x) - P_{n,q}(g,x) - f(x) + g(x) + P_{n,q}(g,x) - g(x)| \\
&\leq |P_{n,q}(f-g;x) - (f-g)(x)| + |P_{n,q}(g;x) - g(x)| \\
&\leq . \, |P_{n,q}(f-g;x) + (f-g)(x)| + |P_{n,q}(g;x) - g(x)| \\
&\leq 2\|f-g\| + \left\{ \left(\frac{1-q^2}{q^2} \right) x^2 + \frac{(1+q)}{q^2 [n]} x \right\} \|g''\| \\
&\leq 2\left(\|f-g\| + \delta_n(x) \|g''\| \right)
\end{aligned}
$$

where $g \in C_B^2[0,\infty)$. Now, taking infimum on the right-hand side over all $g \in C_B^2[0,\infty)$ and using (3.2.1), we get the following result

$$\begin{aligned} |P_{n,q}(f;x) - f(x)| &\leq 2K_2(f;\delta_n(x)) \\ &\leq 2A\omega_2(f;\sqrt{\delta_n(x)}) \\ &= M\omega_2(f;\sqrt{\delta_n(x)}) \end{aligned}$$

where $M = 2A > 0$. □

Theorem 4.2.4. *Let $0 < \alpha \leq 1$ and E be any bounded subset of the interval $[0,\infty)$. Then, if $f \in C_B[0,\infty)$ is locally $Lip(\alpha)$, i.e. the condition*

$$|f(y) - f(x)| \leq L|y-x|^{\alpha},\ y \in E \text{ and } x \in [0,\infty) \tag{4.2.2}$$

holds, then, for each $x \in [0,\infty)$, we have

$$|P_{n,q}(f;x) - f(x)| \leq L\left\{\delta_n^{\frac{\alpha}{2}}(q;x) + 2\left(d(x,E)\right)^{\alpha}\right\},$$

where $\delta_n(q;x)$ is the same as in Theorem 4.2.3, L is a constant depending on α and f; and $d(x,E)$ is the distance between x and E defined as

$$d(x,E) = \inf\{|t-x| : t \in E\}.$$

Proof. Let $\overline{E}$ denote the closure of E in $[0,\infty)$. Then, there exists a point $x_0 \in \overline{E}$ such that $|x - x_0| = d(x,E)$. Using the triangle inequality

$$|f(t) - f(x)| \leq |f(t) - f(x_0)| + |f(x) - f(x_0)|$$

we get, by (4.2.2)

$$
\begin{aligned}
|P_{n,q}(f;x)-f(x)| &= |P_{n,q}(f;x)-P_{n,q}(f(x);x)| \\
&\le P_{n,q}(|f(t)-f(x)|;x) \\
&\le P_{n,q}(|f(t)-f(x_0)|;x)+P_{n,q}(|f(x)-f(x_0)|;x) \\
&= P_{n,q}(|f(t)-f(x_0)|;x)+|f(x)-f(x_0)| \\
&\le L\left\{P_{n,q}(|t-x_0|^\alpha;x)+|x-x_0|^\alpha\right\} \\
&\le L\left\{P_{n,q}(|t-x|^\alpha+|x-x_0|^\alpha;x)+|x-x_0|^\alpha\right\} \\
&= L\left\{P_{n,q}(|t-x|^\alpha;x)+2\,|x-x_0|^\alpha\right\}.
\end{aligned}
$$

Using the Hölder inequality with $p=\frac{2}{\alpha}$, $q=\frac{2}{2-\alpha}$ we find that

$$
\begin{aligned}
|P_{n,q}(f;x)-f(x)| &\le L\left\{[P_{n,q}(|t-x|^{\alpha p};x)]^{\frac{1}{p}}\,[P_{n,q}(1^q;x)]^{\frac{1}{q}}+2\,(d\,(x,E))^\alpha\right\} \\
&= L\left\{\left[P_{n,q}(|t-x|^2;x)\right]^{\frac{\alpha}{2}}+2\,(d\,(x,E))^\alpha\right\} \\
&\le L\left\{\left[\left(\frac{1-q^2}{q^2}\right)x^2+\frac{(1+q)}{q^2\,[n]}x\right]^{\frac{\alpha}{2}}+2\,(d\,(x,E))^\alpha\right\} \\
&= L\left\{\delta_n^{\frac{\alpha}{2}}(q;x)+2\,(d\,(x,E))^\alpha\right\}.
\end{aligned}
$$

□

We consider the following classes of functions:

$$
\begin{aligned}
C_m\,[0,\infty) &:= \{f\in C\,[0,\infty): \exists\, M_f>0 \text{ s.t } |f(x)|<M_f(1+x^m) \text{ and} \\
&\qquad \|f\|_m := \sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^m}\Bigg\}, \\
C_m^*\,[0,\infty) &:= \left\{f\in C_m\,[0,\infty): \lim_{x\to\infty}\frac{|f(x)|}{1+x^m}<\infty\right\}, \quad m\in\mathbb{N}.
\end{aligned}
$$

Next, we obtain a direct approximation theorem in $C_1^*\,[0,\infty)$ and an estimation in terms of the weighted modulus of continuity. It is known that, if f is not uniformly continuous on the interval $[0, \infty)$, then the usual first modulus of continuity $\omega(f, \delta)$ does not tend to zero, as $\delta \to 0$. For every $f \in C_m^*[0, \infty)$, the weighted modulus of continuity is defined as follows

$$\Omega_m(f, \delta) = \sup_{x \geq 0,\ 0 < h \leq \delta} \frac{|f(x+h) - f(x)|}{1 + (x+h)^m}.$$

(see [26])

Lemma 4.2.5. *([26]) Let $f \in C_m^*[0, \infty)$, $m \in \mathbb{N}$. Then*

(1) $\Omega_m(f, \delta)$ is a monotone increasing function of δ,

(2) $\lim\limits_{\delta \to 0^+} \Omega_m(f, \delta) = 0$,

(3) for any $\alpha \in [0, \infty)$, $\Omega_m(f, \alpha\delta) \leq (1 + \alpha)\Omega_m(f, \delta)$.

In the next theorem, we give an expression of the approximation error with the operators $P_{n,q}$, by means of Ω_1.

Theorem 4.2.6. *If $f \in C_1^*[0, \infty)$, then the inequality*

$$\|P_{n,q}(f) - f\|_2 \leq k(q)\Omega_1 \left(f; \frac{1}{\sqrt{[n]}} \right)$$

holds, where $k(q)$ is a constant independent of f and n.

Proof. In order to prove this theorem , first of all we calculate $|f(t) - f(x)|$ by using the definition of $\Omega_1(f, \delta)$ and Lemma 4.2.5 as follows

$$\begin{aligned} |f(t) - f(x)| &\leq (1 + x + |t - x|) \left(\frac{|t - x|}{\delta} + 1 \right) \Omega_1(f, \delta) \\ &\leq (1 + 2x + t) \left(\frac{|t - x|}{\delta} + 1 \right) \Omega_1(f, \delta). \end{aligned}$$

Then

$$
\begin{aligned}
&|P_{n,q}(f;x) - f(x)| \\
&\leq P_{n,q}(|f(t) - f(x)|\,; x) \\
&\leq P_{n,q}\left((1+2x+t)\left(\frac{|t-x|}{\delta}+1\right)\Omega_1(f,\delta); x\right) \\
&= \Omega_1(f,\delta)\left(P_{n,q}\left((1+2x+t); x\right) + P_{n,q}\left((1+2x+t)\frac{|t-x|}{\delta}; x\right)\right).
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality to the second term, we get

$$
P_{n,q}\left((1+2x+t)\frac{|t-x|}{\delta}; x\right) \leq \left(P_{n,q}\left((1+2x+t)^2; x\right)\right)^{\frac{1}{2}}\left(P_{n,q}\left(\frac{|t-x|^2}{\delta^2}; x\right)\right)^{\frac{1}{2}}.
$$

Consequently

$$
\begin{aligned}
|P_{n,q}(f;x) - f(x)| \leq \Omega_1(f,\delta)\,&(P_{n,q}\left((1+2x+t); x\right) \\
&+ \left(P_{n,q}\left((1+2x+t)^2; x\right)\right)^{\frac{1}{2}}\left(P_{n,q}\left(\frac{|t-x|^2}{\delta^2}; x\right)\right)^{\frac{1}{2}}\Bigg).
\end{aligned}
\tag{4.2.3}
$$

On the other hand, there is a positive constant $K(q)$ such that

$$
P_{n,q}\left((1+2x+t); x\right) = 1 + 3x \leq 3(1+x), \tag{4.2.4}
$$

$$
\begin{aligned}
&\left(P_{n,q}\left((1+2x+t)^2; x\right)\right)^{\frac{1}{2}} \\
&= \left((1+2x)^2 P_{n,q}(1; x) + 2(1+2x)P_{n,q}(t; x) + P_{n,q}\left(t^2; x\right)\right)^{\frac{1}{2}} \\
&= \left((1+2x)^2 + 2(1+2x)x + \frac{1}{q^2}x^2 + \frac{(1+q)}{q^2\,[n]}x\right)^{\frac{1}{2}} \\
&\leq \left(4(1+x)^2 + 4(1+x)(1+x) + \frac{1}{q^2}x^2 + 2\frac{1}{q^2}x + 1\right)^{\frac{1}{2}} \\
&\leq K(q)(1+x)
\end{aligned}
$$

and

$$
\begin{aligned}
\left(P_{n,q}\left(\frac{|t-x|^2}{\delta^2}; x\right)\right)^{\frac{1}{2}} &= \frac{1}{\delta q}\sqrt{(1-q^2)x^2 + \frac{(1+q)}{[n]}x} \\
&= \frac{1}{\delta q}\sqrt{\frac{(1+q)(1-q^n)}{[n]}x^2 + \frac{(1+q)}{[n]}x} \\
&\leq \frac{2}{\delta q\sqrt{[n]}}\sqrt{x^2+x} \;\leq \frac{2}{\delta q\sqrt{[n]}}(1+x).
\end{aligned}
\tag{4.2.5}
$$

Now from (4.2.3), (4.2.4) and (4.2.5) we have

$$
\begin{aligned}
&|P_{n,q}(f;x)-f(x)| \\
&\le \Omega_1(f,\delta)\left\{P_{n,q}\left((1+2x+t);x\right)+P_{n,q}\left((1+2x+t)\frac{|t-x|}{\delta};x\right)\right\} \\
&\le \Omega_1(f,\delta)\left\{P_{n,q}\left((1+2x+t);x\right)\right. \\
&+\left(P_{n,q}\left((1+2x+t)^2;x\right)\right)^{\frac{1}{2}}\left(P_{n,q}\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{\frac{1}{2}}\Bigg\} \\
&\le \Omega_1(f,\delta)\left\{3(1+x)+K(q)(1+x)\frac{2}{\delta q\sqrt{[n]}}(1+x)\right\} \\
&= \Omega_1(f,\delta)\left\{3(1+x)+K(q)\frac{2(1+x)^2}{\delta q\sqrt{[n]}}\right\} \\
&\le (1+x^2)\Omega_1(f,\delta)\left\{3K_1+K(q)\frac{4}{\delta q\sqrt{[n]}}\right\},
\end{aligned}
$$

where

$$K_1=\sup_{x\ge 0}\frac{1+x}{1+x^2}.$$

If we take $\delta=\frac{1}{\sqrt{[n]}}$, then from the above inequality we obtain the following

$$\frac{|P_{n,q}(f;x)-f(x)|}{1+x^2}\le \Omega_1(f,\delta)\left(3K_1+K(q)\frac{4}{q}\right)$$

$$\|P_{n,q}(f)-f\|_2\le k(q)\Omega_1\left(f;\delta\right).$$

□

4.3 Voronovskaja Type Theorem

In this section, we proceed to state and prove a Voronovkaja type theorem for the q-Phillips operators. We first prove the following lemma:

Lemma 4.3.1. *Let* $0<q<1$. *We have*

$$
\begin{aligned}
P_{n,q}(t^3;x) &= \frac{1}{q^6}x^3+\frac{[2]\,[3]}{[n]\,q^6}x^2+\frac{[2]\,[3]}{[n]^2\,q^5}x, \\
P_{n,q}(t^4;x) &= \frac{1}{q^{12}}x^4+\frac{[2]\,[3]\,(1+q^2)}{[n]\,q^{12}}x^3+\frac{[2]\,[3]^2\,(1+q^2)}{[n]^2\,q^{11}}x^2+\frac{[2]^2\,[3]\,(1+q^2)}{[n]^3\,q^9}x.
\end{aligned}
$$

Proof. Simple calculations show that

$$\begin{aligned}
P_{n,q}(t^3;x) &= \frac{1}{[n]^3 q^3}\sum_{k=0}^{\infty}\frac{[k+2][k+1][k]}{q^{3k}}s_{n,k}(q;qx)\\
&= \frac{1}{[n]^3 q^3}\sum_{k=0}^{\infty}\frac{[k]^3+q^k(2+q)[k]^2+q^{2k}(1+q)[k]}{q^{3k}}s_{n,k}(q;qx)\\
&= \frac{1}{[n]^3 q^3}\left\{\sum_{k=0}^{\infty}\frac{[k]^3}{q^{3k}}s_{n,k}(q;qx)+\sum_{k=0}^{\infty}\frac{(2+q)[k]^2}{q^{2k}}s_{n,k}(q;qx)\right.\\
&\left.+\sum_{k=0}^{\infty}\frac{(1+q)[k]}{q^k}s_{n,k}(q;qx)\right\}\\
&= \frac{1}{q^3}\sum_{k=0}^{\infty}\frac{[k]^3}{[n]^3 q^{3k}}s_{n,k}(q;qx)+\frac{(2+q)}{[n]q^3}\sum_{k=0}^{\infty}\frac{[k]^2}{[n]^2q^{2k}}s_{n,k}(q;qx)\\
&+\frac{(1+q)}{[n]^2q^3}\sum_{k=0}^{\infty}\frac{[k]}{[n]q^k}s_{n,k}(q;qx)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{q^9}\sum_{k=0}^{\infty}\frac{[k]^3}{[n]^3 q^{3k-6}}s_{n,k}(q;qx)+\frac{(2+q)}{[n]q^7}\sum_{k=0}^{\infty}\frac{[k]^2}{[n]^2q^{2k-4}}s_{n,k}(q;qx)\\
&+\frac{(1+q)}{[n]^2q^5}\sum_{k=0}^{\infty}\frac{[k]}{[n]q^{k-2}}s_{n,k}(q;qx)\\
&= \frac{1}{q^9}\sum_{k=0}^{\infty}\left(\frac{[k]}{[n]q^{k-2}}\right)^3 s_{n,k}(q;qx)+\frac{(2+q)}{[n]q^7}\sum_{k=0}^{\infty}\left(\frac{[k]}{[n]q^{k-2}}\right)^2 s_{n,k}(q;qx)\\
&+\frac{(1+q)}{[n]^2q^5}\sum_{k=0}^{\infty}\frac{[k]}{[n]q^{k-2}}s_{n,k}(q;qx)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{q^9}\left(\frac{q^4}{[n]^2}x+\left(2q^4+q^3\right)\frac{x^2}{[n]}+q^3x^3\right)+\frac{(2+q)}{[n]q^7}\left(q^3x^2+\frac{q^3}{[n]}x\right)\\
&+\frac{(1+q)q^2}{[n]^2q^5}x\\
&= \frac{1}{q^5[n]^2}x+\frac{(2q+1)}{q^6[n]}x^2+\frac{1}{q^6}x^3+\frac{(2+q)}{[n]q^4}x^2+\frac{(2+q)}{[n]^2q^4}x\\
&+\frac{(1+q)}{[n]^2q^3}x\\
&= \frac{1}{q^6}x^3+\frac{(1+2q+2q^2+q^3)}{q^6[n]}x^2+\frac{(1+2q+2q^2+q^3)}{q^5[n]^2}x\\
&= \frac{1}{q^6}x^3+\frac{(1+q)(1+q+q^2)}{[n]q^6}x^2+\frac{(1+q)(1+q+q^2)}{[n]^2q^5}x\\
&= \frac{1}{q^6}x^3+\frac{[2][3]}{[n]q^6}x^2+\frac{[2][3]}{[n]^2q^5}x
\end{aligned}$$

and

$$
\begin{aligned}
&P_{n,q}(t^4;x)\\
&=\frac{1}{[n]^4q^6}\sum_{k=0}^{\infty}\frac{[k+3]\,[k+2]\,[k+1]\,[k]}{q^{4k}}s_{n,k}(q;qx)\\
&=\frac{1}{[n]^4q^6}\sum_{k=0}^{\infty}\left\{\frac{[k]^4+q^k\,(3+2q+q^2)\,[k]^3}{q^{4k}}\right.\\
&\left.+\frac{q^{2k}\,(3+4q+3q^2+q^3)\,[k]^2+q^{3k}\,(1+2q+2q^2+q^3)\,[k]}{q^{4k}}\right\}s_{n,k}(q;qx)\\
&=\frac{1}{q^{14}}\sum_{k=0}^{\infty}\left(\frac{[k]}{[n]\,q^{k-2}}\right)^4s_{n,k}(q;qx)\\
&+\frac{(3+2q+q^2)}{[n]\,q^{12}}\sum_{k=0}^{\infty}\left(\frac{[k]}{[n]\,q^{k-2}}\right)^3s_{n,k}(q;qx)\\
&+\frac{(3+4q+3q^2+q^3)}{[n]^2q^{10}}\sum_{k=0}^{\infty}\left(\frac{[k]}{[n]\,q^{k-2}}\right)^2s_{n,k}(q;qx)\\
&+\frac{(1+2q+2q^2+q^3)}{[n]^3q^8}\sum_{k=0}^{\infty}\frac{[k]}{[n]\,q^{k-2}}s_{n,k}(q;qx)\\
&=\frac{1}{q^{14}}\left(\frac{q^5}{[n]^3}x+\left(3q^3+3q^2+q\right)\frac{q^2}{[n]^2}x^2+\left(3q+2+\frac{1}{q}\right)\frac{q^3}{[n]}x^3+q^2x^4\right)\\
&+\frac{(3+2q+q^2)}{[n]\,q^{12}}\left(\frac{q^4}{[n]^2}x+\left(2q^2+q\right)\frac{q^2}{[n]}x^2+q^3x^3\right)\\
&+\frac{(3+4q+3q^2+q^3)}{[n]^2q^{10}}\left(q^3x^2+\frac{q^3}{[n]}x\right)+\frac{(1+2q+2q^2+q^3)\,q^2}{[n]^3q^8}x\\
&=\frac{1}{q^{12}}x^4+\frac{1+2q+3q^2+(3+2q+q^2)q^3}{[n]\,q^{12}}x^3\\
&+\frac{1+3q+3q^2+(3+2q+q^2)(2q+1)q^2+(3+4q+3q^2+q^3)q^4}{[n]^2q^{11}}x^2\\
&+\frac{1+(3+2q+q^2)q+(3+4q+3q^2+q^3)q^2+(1+2q+2q^2+q^3)q^3}{[n]^3q^9}x\\
&=\frac{1}{q^{12}}x^4+\frac{(1+q)\,(1+q^2)(1+q+q^2)}{[n]\,q^{12}}x^3+\frac{(1+q)\,(1+q^2)(1+q+q^2)^2}{[n]^2q^{11}}x^2\\
&+\frac{(1+q)^2\,(1+q^2)(1+q+q^2)}{[n]^3q^9}x\\
&=\frac{1}{q^{12}}x^4+\frac{[2]\,[3]\,(1+q^2)}{[n]\,q^{12}}x^3+\frac{[2]\,[3]^2\,(1+q^2)}{[n]^2q^{11}}x^2+\frac{[2]^2\,[3]\,(1+q^2)}{[n]^3q^9}x.
\end{aligned}
$$

□

Theorem 4.3.2. *Let $q_n \in (0,1)$. Then the sequence $\{P_{n,q_n}(f)\}$ converges to f uniformly on $[0,A]$ for each $f \in C_2^*[0,\infty)$ if and only if $\lim\limits_{n\to\infty} q_n = 1$.*

Proof. The proof is similar to that of Theorem 2 [12]. □

Lemma 4.3.3. *Assume that $q_n \in (0,1)$, $q_n \to 1$ and $q_n^n \to a$ as $n \to \infty$. For every $x \in [0,\infty)$ there hold*

$$\lim_{n\to\infty} [n]_{q_n} P_{n,q_n}((t-x)^2;x) = 2(1-a)x^2 + 2x,$$
$$\lim_{n\to\infty} [n]_{q_n}^2 P_{n,q_n}((t-x)^4;x) = 12x^2 + 24(1-a)x^3 + 12(1-a)^2x^4.$$

Proof. First, we have

$$\begin{aligned}
\lim_{n\to\infty} [n]_{q_n} P_{n,q_n}((t-x)^2;x) &= \lim_{n\to\infty} [n]_{q_n} \left\{ \left(\frac{1}{q_n^2} - 1 \right) x^2 + \frac{(1+q_n)}{q_n^2 [n]_{q_n}} x \right\} \\
&= \lim_{n\to\infty} \left(\frac{(1-q_n^n)(1+q_n)}{q_n^2} x^2 + \frac{(1+q_n)}{q_n^2} x \right) \\
&= 2(1-a)x^2 + 2x.
\end{aligned}$$

In order to calculate the second limit we need expression for $P_{n,q_n}((t-x)^4;x)$:

$$\begin{aligned}
&P_{n,q_n}((t-x)^4;x) \\
&= P_{n,q_n}(t^4;x) - 4xP_{n,q_n}(t^3;x) + 6x^2P_{n,q_n}(t^2;x) - 4x^3P_{n,q_n}(t;x) + x^4 \\
&= \frac{1}{q_n^{12}}x^4 + \frac{[2]_{q_n}[3]_{q_n}(1+q_n^2)}{[n]_{q_n} q_n^{12}}x^3 + \frac{[2]_{q_n}[3]_{q_n}^2(1+q_n^2)}{[n]_{q_n}^2 q_n^{11}}x^2 + \frac{[2]_{q_n}^2[3]_{q_n}(1+q_n^2)}{[n]_{q_n}^3 q_n^9}x \\
&- 4x\left\{ \frac{1}{q_n^6}x^3 + \frac{[2]_{q_n}[3]_{q_n}}{[n]_{q_n} q_n^6}x^2 + \frac{[2]_{q_n}[3]_{q_n}}{[n]_{q_n}^2 q_n^5}x \right\} + 6x^2\left\{ \frac{1}{q_n^2}x^2 + \frac{[2]_{q_n}}{q_n^2[n]_{q_n}}x \right\} - 3x^4 \\
&= \frac{(1-4q_n^6+6q_n^{10}-3q_n^{12})}{q_n^{12}}x^4 \\
&+ \left\{ \frac{[2]_{q_n}[3]_{q_n}(1+q_n^2) - 4[2]_{q_n}[3]_{q_n}q_n^6 + 6q_n^{10}[2]_{q_n}}{q_n^{12}[n]_{q_n}} \right\} x^3 \\
&+ \left\{ \frac{[2]_{q_n}[3]_{q_n}^2(1+q_n^2) - 4q_n^6[2]_{q_n}[3]_{q_n}}{q_n^{11}[n]_{q_n}^2} \right\} x^2 + \left\{ \frac{[2]_{q_n}^2[3]_{q_n}(1+q_n^2)}{q_n^9[n]_{q_n}^3} \right\} x
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(1+2q_n^2+3q_n^4-3q_n^8)\,(1-q_n^n)^2\,(q_n+1)^2}{q_n^{12}\,[n]_{q_n}^2}x^4 \;+\left\{\frac{(q_n^n-1)(q_n+1)}{1}\right.\\
&\times\left.\frac{(2q_n^7-4q_n^2-5q_n^3-6q_n^4-6q_n^5-2q_n^6-2q_n+6q_n^8+6q_n^9-1)}{q_n^{12}\,[n]_{q_n}^2}\right\}x^3\\
&+\left\{\frac{[2]_{q_n}[3]_{q_n}(1+q_n^2)-4q_n^6[2]_{q_n}[3]_{q_n}}{q_n^{11}\,[n]_{q_n}^2}\right\}x^2+\left\{\frac{[2]_{q_n}^2[3]_{q_n}(1+q_n^2)}{q_n^9\,[n]_{q_n}^3}\right\}x
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{n\to\infty}[n]_{q_n}^2\,P_{n,q_n}((t-x)^4;x)\\
&=\lim_{n\to\infty}\frac{(1-q_n^n)^2}{(1-q_n)^2}\left\{\frac{(1+2q_n^2+3q_n^4-3q_n^8)\,(1-q_n^n)^2\,(q_n+1)^2}{q_n^{12}\,[n]_{q_n}^2}x^4\right.\\
&+\frac{(q_n^n-1)(q_n+1)}{1}\\
&\times\frac{(2q_n^7-4q_n^2-5q_n^3-6q_n^4-6q_n^5-2q_n^6-2q_n+6q_n^8+6q_n^9-1)}{q_n^{12}\,[n]_{q_n}^2}x^3\\
&+\left.\frac{[2]_{q_n}[3]_{q_n}(1+q_n^2)-4q_n^6[2]_{q_n}[3]_{q_n}}{q_n^{11}\,[n]_{q_n}^2}x^2+\frac{[2]_{q_n}^2[3]_{q_n}(1+q_n^2)}{q_n^9\,[n]_{q_n}^3}\right\}\\
&=\lim_{n\to\infty}\frac{(1-q_n^n)^2}{(1-q_n)^2}\left\{\frac{(1+2q_n^2+3q_n^4-3q_n^8)\,(1-q_n^n)^2\,(q_n+1)^2}{q_n^{12}\,[n]_{q_n}^2}x^4\right.\\
&+\frac{(q_n^n-1)(q_n+1)}{1}\\
&\times\frac{(2q_n^7-4q_n^2-5q_n^3-6q_n^4-6q_n^5-2q_n^6-2q_n+6q_n^8+6q_n^9-1)}{q_n^{12}\,[n]_{q_n}^2}x^3\\
&+\frac{(q_n+1)\,(q_n+2q_n^2+q_n^3+q_n^4-4q_n^6+1)\,(q_n+q_n^2+1)}{q_n^{11}\,[n]_{q_n}^2}x^2\\
&+\left.\left\{\frac{(1+q_n)^2(1+q_n^2)(1+q_n+q_n^2)}{q_n^9\,[n]_{q_n}^3}\right\}x\right\}\\
&=12\,(1-a)^2\,x^4+24\,(1-a)\,x^3+12x^2.
\end{aligned}
$$

□

Theorem 4.3.4. *Assume that $q_n\in(0,1)$, $q_n\to 1$ and $q_n^n\to a$ as $n\to\infty$. For any $f\in C_2^*[0,\infty)$ such that $f',f''\in C_2^*[0,\infty)$ the following equality holds*

$$\lim_{n\to\infty}[n]_{q_n}\left(P_{n,q_n}(f;x)-f(x)\right)=\left((1-a)x^2+x\right)f''(x)$$

uniformly on any $[0, A]$, $A > 0$.

Proof. Let $f, f', f'' \in C_2^*[0,\infty)$ and $x \in [0,\infty)$ be fixed. By the Taylor formula we may write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + r(t;x)(t-x)^2, \qquad (4.3.1)$$

where $r(t;x)$ is the Peano form of the remainder, $r(.;x) \in C_2^*[0,\infty)$ and $\lim_{t\to x} r(t;x) = 0$. Applying P_{n,q_n} to (4.3.1) we obtain

$$\begin{aligned}
&[n]_{q_n}\left(P_{n,q_n}(f;x) - f(x)\right)\\
&= f'(x)\,[n]_{q_n} P_{n,q_n}(t-x;x) + \frac{1}{2}f''(x)\,[n]_{q_n} P_{n,q_n}\left((t-x)^2;x\right)\\
&+ [n]_{q_n} P_{n,q_n}\left(r(t;x)(t-x)^2;x\right)\\
.&= \frac{1}{2}f''(x)\,[n]_{q_n} P_{n,q_n}\left((t-x)^2;x\right)\\
&+ [n]_{q_n} P_{n,q_n}\left(r(t;x)(t-x)^2;x\right).
\end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$P_{n,q_n}\left(r(t;x)(t-x)^2;x\right) \le \sqrt{P_{n,q_n}\left(r^2(t;x);x\right)}\sqrt{P_{n,q_n}\left((t-x)^4;x\right)}. \qquad (4.3.2)$$

Observe that $r^2(x;x) = 0$ and $r^2(.;x) \in C_2^*[0,\infty)$. Then it follows from Theorem 4.3.2 that

$$\lim_{n\to\infty} P_{n,q_n}\left(r^2(t;x);x\right) = r^2(x;x) = 0 \qquad (4.3.3)$$

uniformly with respect to $x \in [0, A]$. Now from (4.3.2), (4.3.3) and Lemma 4.3.3 we get immediately

$$\lim_{n\to\infty} [n]_{q_n} P_{n,q_n}\left(r(t;x)(t-x)^2;x\right) = 0.$$

Then we get the following

$$\begin{aligned}
\lim_{n\to\infty}[n]_{q_n}\left(P_{n,q_n}(f;x) - f(x)\right) &= \lim_{n\to\infty}\Bigg(\frac{1}{2}f''(x)\,[n]_{q_n} P_{n,q_n}\left((t-x)^2;x\right)\\
&+ [n]_{q_n} P_{n,q_n}\left(r(t;x)(t-x)^2;x\right)\Bigg)\\
&= \left((1-a)x^2 + x\right) f''(x).
\end{aligned}$$

□

Chapter 5

APPROXIMATION BY q-PHILLIPS OPERATORS FOR $Q > 1$

In this chapter, we construct q-parametric Phillips operators in the case $q > 1$ and evaluate the moments of $P_{n,q}$. We establish the local approximation result for continuous functions in terms of modulus of continuity. Furthermore, we obtain a Voronovskaja type asymptotic result for the q-Phillips operators.

5.1 Moments for $P_{n,q}(f; x)$

In this section firstly, we introduce the following so called q-Phillips operators.

Definition 5.1.1. *Let $q > 1$ and $n \in \mathbb{N}$. For $f : [0, \infty) \to \mathbb{R}$ we define the Phillips operator based on the q-integers*

$$P_{n,q}(f; x) = [n] \sum_{k=1}^{\infty} q^{2-k} s_{n,k}(q; qx) \int_{0}^{\infty/1-\frac{1}{q}} s_{n,k-1}(q; t) f(t) d_{\frac{1}{q}} t + E_{\frac{1}{q}} \left(- [n] \, x\right) f(0), \tag{5.1.1}$$

where $x \in [0, \infty)$ and

$$s_{n,k}(q; x) = \frac{1}{q^{\frac{k(k-1)}{2}}} \frac{[n]^k \, x^k}{[k]!} e_q(- [n] \, q^{-k} x) = \frac{1}{q^{\frac{k(k-1)}{2}}} \frac{[n]^k \, x^k}{[k]!} E_{\frac{1}{q}}(- [n] \, q^{-k} x).$$

It is obvious that $s_{n,k}(q; x) \geq 0$ for all $q > 1$ and $x \in [0, \infty)$. Moreover $\sum\limits_{k=0}^{\infty} s_{n,k}(q; x) \overset{\text{(see [30])}}{=} M_{n,q}(1; x) = 1.$

Secondly, we calculate $P_{n,q}(t^i;x)$ for $i = 0,1,2$ by using the Definition 2.4.14 of q-Gamma function Γ_q. Now, we write explicitly

$$\begin{aligned}
&\int_0^{\infty/1-\frac{1}{q}} t^s s_{n,k}(q;t) d_{\frac{1}{q}}t \\
&= \int_0^{\infty/1-\frac{1}{q}} t^s \frac{1}{q^{\frac{k(k-1)}{2}}} \frac{[n]^k t^k}{[k]!} E_{\frac{1}{q}}(-[n]\, q^{-k}t) d_{\frac{1}{q}}t \\
&= \int_0^{\infty/1-\frac{1}{q}} \frac{t^{s+k} [n]^k [n]^s (q^{-k+1})^{k+s}}{q^{\frac{k(k-1)}{2}} [k]!\, [n]^{s+1} q^{-k+1} (q^{-k+1})^{k+s}} E_{\frac{1}{q}}(-q^{-1} [n]\, q^{-k+1}t)\, [n]\, q^{-k+1} d_{\frac{1}{q}}t \\
&= \frac{q^{k-1} q^{(k-1)(k+s)}}{[n]^{s+1} [k]! q^{\frac{k(k-1)}{2}}} \int_0^{\infty/1-\frac{1}{q}} \left([n]\, q^{-k+1}t\right)^{(k+s+1)-1} E_{\frac{1}{q}}(-q^{-1} [n]\, q^{-k+1}t)\, [n]\, q^{-k+1} d_{\frac{1}{q}}t.
\end{aligned} \tag{5.1.2}$$

Here, we substitute $u = [n]_q\, q^{-k+1}t$ and $d_{\frac{1}{q}}u = [n]_q\, q^{-k+1} d_{\frac{1}{q}}t$ in (5.1.2), we get the following

$$\begin{aligned}
&\int_0^{\infty/1-\frac{1}{q}} t^s s_{n,k}(q;t) d_{\frac{1}{q}}t \\
&= \frac{q^{k-1} q^{(k-1)(k+s)}}{[n]^{s+1} [k]! q^{\frac{k(k-1)}{2}}} \int_0^{\infty/1-\frac{1}{q}} (u)^{(k+s+1)-1} E_{\frac{1}{q}}(-q^{-1}u)\, [n]\, d_{\frac{1}{q}}u \\
&= \frac{q^{(k-1)(k+s+1)}}{[n]^{s+1} [k]! q^{\frac{k(k-1)}{2}}} \Gamma_{\frac{1}{q}}(k+s+1) = \frac{q^{(k-1)(k+s+1)}}{[n]^{s+1} [k]! q^{\frac{k(k-1)}{2}}} [k+s]_{\frac{1}{q}}!.
\end{aligned} \tag{5.1.3}$$

From q-calculus (see [19]), we know that

$$\begin{aligned}
[k+s]_{\frac{1}{q}} &= \frac{1-\left(\frac{1}{q}\right)^{k+s}}{1-\frac{1}{q}} \\
&= \frac{q^{k+s}-1}{q^{k+s}} \frac{q}{q-1} \\
&= \frac{q^{k+s}-1}{q-1} \frac{q}{q^{k+s}} \\
&= [k+s] \frac{1}{q^{k+s-1}}
\end{aligned}$$

and because of this result, we have

$$\begin{aligned}[k+s]_{\frac{1}{q}}! &= [k+s]_{\frac{1}{q}}\,[k+s-1]_{\frac{1}{q}}\,...\,[2]_{\frac{1}{q}}\,[1]_{\frac{1}{q}} \\ &= [k+s]\,\frac{1}{q^{k+s-1}}\,[k+s-1]\,\frac{1}{q^{k+s-2}}...\,[2]\,\frac{1}{q}\,[1] \\ &= [k+s]!\frac{1}{q^{(k+s-1)(k+s)/2}}. \end{aligned} \tag{5.1.4}$$

Finally, if we substitute (5.1.4) in (5.1.3), we get

$$\begin{aligned}\int_0^{\infty/1-\frac{1}{q}} t^s s_{n,k}(q;t)d_{\frac{1}{q}}t &= \frac{q^{(k-1)(k+s+1)}}{[n]^{s+1}\,[k]!q^{\frac{k(k-1)}{2}}}\frac{1}{q^{\frac{(k+s-1)(k+s)}{2}}}\,[k+s]! \\ &= \frac{1}{q^{\frac{s(s+1)}{2}}q^{1-k}}\frac{1}{[n]^{s+1}\,[k]!}\,[k+s]!. \end{aligned}$$

Moreover, we have

$$\int_0^{\infty/1-\frac{1}{q}} t^s s_{n,k-1}(q;t)d_{\frac{1}{q}}t = \frac{1}{q^{\frac{s(s+1)}{2}}q^{2-k}}\frac{1}{[n]^{s+1}\,[k-1]!}\,[k+s-1]!.$$

Lemma 5.1.2. *([30]) Let $q>1$. We have*

$$\begin{aligned}&M_{n,q}(1\,;x)=1,\;\; M_{n,q}(t;x)=x,\; M_{n,q}(t^2;x)=x^2+\frac{1}{[n]}x, \\ &M_{n,q}(t^3;x)=x^3+\frac{2+q}{[n]}x^2+\frac{1}{[n]^2}x, \\ &M_{n,q}(t^4;x)=x^4+(3+2q+q^2)\frac{x^3}{[n]}+(3+3q+q^2)\frac{x^2}{[n]^2}+\frac{1}{[n]^3}x.\end{aligned}$$

Lemma 5.1.3. *Suppose that $q>1$. Then, we have*

$$P_{n,q}(1;x)=1,\;\; P_{n,q}(t;x)=x,\; P_{n,q}(t^2;x)=x^2+\frac{(1+q)}{q^2\,[n]}x. \tag{5.1.5}$$

Proof. In order to prove Lemma 5.1.3, we shall use Lemma 5.1.2. First for $f(t)=1$, we have

$$\begin{aligned}
P_{n,q}(1;x) &= [n]\sum_{k=1}^{\infty} q^{2-k} s_{n,k}(q;qx) \int_{0}^{\frac{\infty}{1-\frac{1}{q}}} s_{n,k-1}(q;t)d_{\frac{1}{q}}t + E_{\frac{1}{q}}\left(-[n]\,x\right) f(0) \\
&= [n]\sum_{k=1}^{\infty} q^{2-k} s_{n,k}(q;qx) \frac{1}{qq^{2-k}\,[n]^2\,[k-1]!}\,[k]! + E_{\frac{1}{q}}\left(-[n]\,x\right) \\
&= \sum_{k=1}^{\infty} s_{n,k}(q;qx) + E_{\frac{1}{q}}\left(-[n]\,x\right) \\
&= \sum_{k=0}^{\infty} s_{n,k}(q;qx) \\
&= 1 \quad \text{(by Lemma 5.1.2)}
\end{aligned}$$

Next for $f(t) = t$ and $f(t) = t^2$, we get proceess as follows:

$$\begin{aligned}
P_{n,q}(t;x) &= [n]\sum_{k=1}^{\infty} q^{2-k} s_{n,k}(q;qx) \int_{0}^{\frac{\infty}{1-\frac{1}{q}}} t s_{n,k-1}(q;t)d_{\frac{1}{q}}t + E_{\frac{1}{q}}\left(-[n]\,x\right) f(0) \\
&= [n]\sum_{k=1}^{\infty} q^{2-k} s_{n,k}(q;qx) \frac{1}{qq^{2-k}\,[n]^2\,[k-1]!}\,[k]! \\
&= \frac{1}{q}\sum_{k=0}^{\infty} \frac{[k]}{[n]} s_{n,k}(q;qx) \\
&\overset{\text{(by Lemma 5.1.2)}}{=} \frac{1}{q}qx = x,
\end{aligned}$$

$$
\begin{aligned}
P_{n,q}(t^2;x) &= [n]\sum_{k=1}^{\infty} q^{2-k} s_{n,k}(q;qx) \int_0^{\frac{\infty}{1-\frac{1}{q}}} t^2 s_{n,k-1}(q;t) d_{\frac{1}{q}}t + E_{\frac{1}{q}}\left(-[n]\,x\right) f(0) \\
&= [n]\sum_{k=1}^{\infty} q^{2-k} s_{n,k}(q;qx) \frac{1}{q^3 q^{2-k}\,[n]^3\,[k-1]!}\,[k+1]! \\
&= \frac{1}{q^3}\sum_{k=1}^{\infty} s_{n,k}(q;qx)\frac{[k+1]\,[k]}{[n]^2} \\
&= \frac{1}{q^3}\sum_{k=1}^{\infty} \frac{[k]+q\,[k]^2}{[n]^2} s_{n,k}(q;qx) \\
&= \frac{1}{q^2}\sum_{k=1}^{\infty} \frac{[k]^2}{[n]^2} s_{n,k}(q;qx) + \frac{1}{q^3\,[n]}\sum_{k=1}^{\infty} s_{n,k}(q;qx)\frac{[k]}{[n]} \\
&= \frac{1}{q^2}\sum_{k=0}^{\infty} \frac{[k]^2}{[n]^2} s_{n,k}(q;qx) + \frac{1}{q^3\,[n]}\sum_{k=0}^{\infty} s_{n,k}(q;qx)\frac{[k]}{[n]} \\
&\overset{\text{(by Lemma 5.1.2)}}{=} \frac{1}{q^2}\left((qx)^2 + \frac{1}{[n]}qx\right) + \frac{1}{q^3\,[n]}qx \\
&= x^2 + \frac{(1+q)}{q^2\,[n]}x.
\end{aligned}
$$

□

Our next lemma gives the explicit formula for the moments $P_{n,q}(t^m;x)$.

Lemma 5.1.4. *For all $q>1$ the following identity holds:*

$$
P_{n,q}(t^m;x) = \frac{1}{[n]^m\, q^{m(m+1)/2}}\sum_{s=1}^{m} C_{s,m}(q)\,[n]^s\, M_{n,q}(t^s;qx).
$$

Proof. Firstly, we use definition 5.1.1 for $f(t)=t^m$ where $m\in\mathbb{N}$ to write the following

$$P_{n,q}(t^m;x) = [n]\sum_{k=1}^{\infty} q^{2-k} s_{n,k}(q;qx) \int_0^{\frac{\infty}{1-\frac{1}{q}}} t^m s_{n,k-1}(q;t) d_{\frac{1}{q}} t$$

$$= [n]\sum_{k=1}^{\infty} q^{2-k} s_{n,k}(q;qx) \frac{1}{q^{m(m+1)/2} q^{2-k} [n]^{m+1} [k-1]!} [k+m-1]!$$

$$= \sum_{k=1}^{\infty} \frac{[k+m-1] \dots [k]}{[n]^m} \frac{1}{q^{m(m+1)/2}} s_{n,k}(q;qx)$$

$$= \sum_{k=0}^{\infty} \frac{[k+m-1] \dots [k]}{[n]^m q^{m(m+1)/2}} s_{n,k}(q;qx).$$

From now on, we need to find $[k][k+1] \dots [k+m-1]$. To do this, we employ $[k+s] = [s] + q^s [k]$ and then we get

$$[k][k+1] \dots [k+m-1] = \prod_{s=0}^{m-1} ([s] + q^s [k]) = \sum_{s=1}^{m} C_{s,m}(q) [k]^s$$

where $C_{s,m}(q) > 0$, $s = 1, 2, \dots, m$ are the constants independent of k. Hence

$$P_{n,q}(t^m;x) = \frac{1}{[n]^m q^{m(m+1)/2}} \sum_{k=0}^{\infty}\sum_{s=1}^{m} C_{s,m}(q) [k]^s s_{n,k}(q;qx)$$

$$= \frac{1}{[n]^m q^{m(m+1)/2}} \sum_{s=1}^{m} C_{s,m}(q) [n]^s \sum_{k=0}^{\infty} \frac{[k]^s}{[n]^s} s_{n,k}(q;qx)$$

$$= \frac{1}{[n]^m q^{m(m+1)/2}} \sum_{s=1}^{m} C_{s,m}(q) [n]^s M_{n,q}(t^s;qx)$$

where $M_{n,q}$ is the q-Szasz operator (see [30]) . □

5.2 Local Approximation

Lemma 5.2.1. *Let $f \in C_B^2[0,\infty)$. Then, for all $f \in C_B^2[0,\infty)$, we have*

$$|P_{n,q}(f;x) - f(x)| \le \frac{(1+q)}{q^2 [n]} x \|f''\| . \tag{5.2.1}$$

Proof. Let $x \in [0,\infty)$ and $f \in C_B^2[0,\infty)$. Here, if we use Taylor's formula

$$f(t) - f(x) = (t-x)f'(x) + \int_x^t (t-u) f''(u) du,$$

we obtain the following

$$
\begin{aligned}
P_{n,q}(f;x)-f(x) &= P_{n,q}((t-x)f'(x);x)+P_{n,q}\left(\int\limits_x^t (t-u)f''(u)du;x\right)\\
&= f'(x)P_{n,q}((t-x);x)+P_{n,q}\left(\int\limits_x^t (t-u)f''(u)du;x\right)\\
&= P_{n,q}\left(\int\limits_x^t (t-u)f''(u)du;x\right).
\end{aligned}
$$

On the other hand, since

$$\left|\int\limits_x^t (t-u)f''(u)du\right| \leq \int\limits_x^t |t-u|\,|f''(u)|\,du \leq \|f''\| \int\limits_x^t |t-u|\,du \leq (t-x)^2\,\|f''\|$$

we conclude that

$$
\begin{aligned}
|P_{n,q}(f;x)-f(x)| &= \left|P_{n,q}\left(\int\limits_x^t (t-u)f''(u)du;x\right)\right|\\
&\leq P_{n,q}((t-x)^2\,\|f''\|\,;x).
\end{aligned}
$$

Thus, from the fact that

$$
\begin{aligned}
P_{n,q}((t-x)^2;x) &= P_{n,q}(t^2-2tx+x^2;x)\\
&= P_{n,q}(t^2;x)-2xP_{n,q}(t;x)+x^2P_{n,q}(1;x)\\
&= x^2+\frac{(1+q)}{q^2\,[n]}x-2x^2+x^2=\frac{(1+q)}{q^2\,[n]}x,
\end{aligned}
$$

we get

$$|P_{n,q}(f;x)-f(x)| \leq \frac{(1+q)}{q^2\,[n]}x\,\|f''\|\,.$$

□

Lemma 5.2.2. *For* $f \in C_B\,[0,\infty)$, *we have*

$$\|P_{n,q}f\| \leq \|f\|\,. \tag{5.2.2}$$

Proof. Since

$$|P_{n,q}(f;x)| = \left| [n] \sum_{k=0}^{\infty} q^{2-k} s_{n,k}(q;qx) \int_{0}^{\frac{\infty}{1-\frac{1}{q}}} s_{n,k-1}(q;t) f(t) d_{\frac{1}{q}} t \right|$$

$$\leq [n] \sum_{k=0}^{\infty} q^{2-k} |s_{n,k}(q;qx)| \int_{0}^{\frac{\infty}{1-\frac{1}{q}}} |s_{n,k-1}(q;t)| \, |f(t)| \, d_{\frac{1}{q}} t$$

$$= [n] \sum_{k=0}^{\infty} q^{2-k} s_{n,k}(q;qx) \int_{0}^{\frac{\infty}{1-\frac{1}{q}}} s_{n,k-1}(q;t) \, |f(t)| \, d_{\frac{1}{q}} t$$

$$\leq \|f\| \, [n] \sum_{k=0}^{\infty} q^{2-k} s_{n,k}(q;qx) \int_{0}^{\frac{\infty}{1-\frac{1}{q}}} s_{n,k-1}(q;t) d_{\frac{1}{q}} t$$

$$= \|f\| \, P_{n,q}(1;x) = \|f\| \, .$$

□

Theorem 5.2.3. *Let* $f \in C_B[0,\infty)$ *. Then, for every* $x \in [0,\infty)$ *, there exists a constant* $M > 0$ *such that*

$$|P_{n,q}(f;x) - f(x)| \leq M\omega_2(f; \sqrt{\delta_n(x)})$$

where

$$\delta_n(q;x) = \frac{(1+q)}{q^2 [n]} x.$$

Proof. Since $P_{n,q}$ is a positive linear operator and moreover we have inequalities (5.2.1) and (5.2.2), we can write

$$|P_{n,q}(f;x) - f(x)| = |P_{n,q}(f;x) - P_{n,q}(g,x) - f(x) + g(x) + P_{n,q}(g,x) - g(x)|$$

$$\leq |P_{n,q}(f-g;x) - (f-g)(x)| + |P_{n,q}(g;x) - g(x)|$$

$$\leq |P_{n,q}(f-g;x) + (f-g)(x)| + |P_{n,q}(g;x) - g(x)|$$

$$\leq 2\|f-g\| + \frac{(1+q)}{q^2 [n]} x \, \|g''\|$$

$$= 2\|f-g\| + \delta_n(q;x) \, \|g''\|$$

where $g \in C_B^2[0,\infty)$. Now, taking infimum on the right-hand side over all $g \in C_B^2[0,\infty)$ and using (3.2.1), we get the following result

$$\begin{aligned}|P_{n,q}(f;x)-f(x)| &\leq 2K_2(f;\delta_n(q;x)) \\ &\leq 2A\omega_2(f;\sqrt{\delta_n(q;x)}) \\ &= M\omega_2(f;\sqrt{\delta_n(q;x)})\end{aligned}$$

where $M = 2A > 0$. □

Theorem 5.2.4. *Let $0 < \alpha \leq 1$ and E be any bounded subset of the interval $[0,\infty)$. Then, if $f \in C_B[0,\infty)$ is locally $Lip(\alpha)$, i.e. the condition*

$$|f(y)-f(x)| \leq L\,|y-x|^{\alpha},\ y \in E \text{ and } x \in [0,\infty) \tag{5.2.3}$$

holds, then, for each $x \in [0,\infty)$, we have

$$|P_{n,q}(f;x)-f(x)| \leq L\left\{\delta_n^{\frac{\alpha}{2}}(q;x) + 2\,(d\,(x,E))^{\alpha}\right\},$$

where $\delta_n(q;x)$ is the same as in Theorem 5.2.3, L is a constant depending on α and f; and $d\,(x,E)$ is the distance between x and E defined as

$$d\,(x,E) = \inf\{|t-x| : t \in E\}.$$

Proof. Let $\overline{E}$ denote the closure of E in $[0,\infty)$. Then, there exists a point $x_0 \in \overline{E}$ such that $|x-x_0| = d\,(x,E)$. Using the triangle inequality

$$|f\,(t)-f\,(x)| \leq |f\,(t)-f\,(x_0)| + |f\,(x)-f\,(x_0)|$$

we get, by (5.2.3),

$$\begin{aligned}
|P_{n,q}(f;x)-f(x)| &= |P_{n,q}(f;x)-P_{n,q}(f(x);x)| \\
&\le P_{n,q}(|f(t)-f(x)|;x) \\
&\le P_{n,q}(|f(t)-f(x_0)|;x)+P_{n,q}(|f(x)-f(x_0)|;x) \\
&= P_{n,q}(|f(t)-f(x_0)|;x)+|f(x)-f(x_0)| \\
&\le L\left\{P_{n,q}(|t-x_0|^{\alpha};x)+|x-x_0|^{\alpha}\right\} \\
&\le L\left\{P_{n,q}(|t-x|^{\alpha}+|x-x_0|^{\alpha};x)+|x-x_0|^{\alpha}\right\} \\
&= L\left\{P_{n,q}(|t-x|^{\alpha};x)+2|x-x_0|^{\alpha}\right\}.
\end{aligned}$$

Using the Hölder's inequality with $p=\frac{2}{\alpha}$, $q=\frac{2}{2-\alpha}$, we find that

$$\begin{aligned}
|P_{n,q}(f;x)-f(x)| &\le L\left\{[P_{n,q}(|t-x|^{\alpha p};x)]^{\frac{1}{p}}[P_{n,q}(1^{q};x)]^{\frac{1}{q}}+2(d(x,E))^{\alpha}\right\} \\
&= L\left\{\left[P_{n,q}(|t-x|^{2};x)\right]^{\frac{\alpha}{2}}+2(d(x,E))^{\alpha}\right\} \\
&\le L\left\{\left[\frac{(1+q)}{q^2[n]}x\right]^{\frac{\alpha}{2}}+2(d(x,E))^{\alpha}\right\} \\
&= L\left\{\delta_n^{\frac{\alpha}{2}}(q;x)+2(d(x,E))^{\alpha}\right\}.
\end{aligned}$$

□

We consider the following classes of functions:

$$C_m[0,\infty):=\Big\{f\in C[0,\infty):\exists\, M_f>0 \text{ s.t } |f(x)|<M_f(1+x^m) \text{ and } \|f\|_m:=\sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^m}\Big\},$$

$$C_m^*[0,\infty):=\left\{f\in C_m[0,\infty):\lim_{x\to\infty}\frac{|f(x)|}{1+x^m}<\infty\right\},\quad m\in\mathbb{N}.$$

Next, we obtain a direct approximation theorem in $C_1^*[0,\infty)$ and an estimation in terms of the weighted modulus of continuity. It is known that, if f is not uniformly continuous on the interval $[0,\infty)$, then the usual first modulus of continuity $\omega(f,\delta)$ does not tend to zero, as $\delta \to 0$. For every $f \in C_m^*[0,\infty)$, the weighted modulus of continuity is defined as follows

$$\Omega_m(f,\delta) = \sup_{x\geq 0,\ 0<h\leq\delta} \frac{|f(x+h)-f(x)|}{1+(x+h)^m}.$$

(see [26])

In the next theorem we give an expression of the approximation error with the operators $P_{n,q}$, by means of Ω_1.

Theorem 5.2.5. *If $f \in C_1^*[0,\infty)$, then the inequality*

$$\|P_{n,q}(f)-f\|_2 \leq k\,\Omega_1\left(f;\frac{1}{\sqrt{[n]}}\right)$$

holds, where k is a constant independent of f and n.

Proof. In order to prove this theorem, first of all we calculate $|f(t)-f(x)|$ by using the definition of $\Omega_1(f,\delta)$ and Lemma 4.2.5 as follows

$$\begin{aligned}|f(t)-f(x)| &\leq (1+x+|t-x|)\left(\frac{|t-x|}{\delta}+1\right)\Omega_1(f,\delta)\\ &\leq (1+2x+t)\left(\frac{|t-x|}{\delta}+1\right)\Omega_1(f,\delta).\end{aligned}$$

Then

$$\begin{aligned}&|P_{n,q}(f;x)-f(x)|\\ &\leq P_{n,q}(|f(t)-f(x)|;x)\\ &\leq P_{n,q}\left((1+2x+t)\left(\frac{|t-x|}{\delta}+1\right)\Omega_1(f,\delta);x\right)\\ &= \Omega_1(f,\delta)\left(P_{n,q}\left((1+2x+t);x\right)+P_{n,q}\left((1+2x+t)\frac{|t-x|}{\delta};x\right)\right).\end{aligned}$$

Applying the Cauchy-Schwarz inequality to the second term, we get

$$P_{n,q}\left((1+2x+t)\frac{|t-x|}{\delta};x\right) \leq \left(P_{n,q}\left((1+2x+t)^2;x\right)\right)^{\frac{1}{2}}\left(P_{n,q}\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{\frac{1}{2}}.$$

Consequently

$$|P_{n,q}(f;x) - f(x)| \leq \Omega_1(f,\delta)\left\{P_{n,q}\left((1+2x+t);x\right) + \left(P_{n,q}\left((1+2x+t)^2;x\right)\right)^{\frac{1}{2}}\left(P_{n,q}\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{\frac{1}{2}}\right\}. \tag{5.2.4}$$

On the other hand, we get the following

$$P_{n,q}\left((1+2x+t);x\right) = 1+3x \leq 3(1+x), \tag{5.2.5}$$

$$\begin{aligned}
&\left(P_{n,q}\left((1+2x+t)^2;x\right)\right)^{\frac{1}{2}} \\
&= \left((1+2x)^2 P_{n,q}\left(1;x\right) + 2(1+2x)P_{n,q}\left(t;x\right) + P_{n,q}\left(t^2;x\right)\right)^{\frac{1}{2}} \\
&= \left((1+2x)^2 + 2(1+2x)x + x^2 + \frac{(1+q)}{q^2\,[n]}x\right)^{\frac{1}{2}} \\
&\leq \left(4(1+x)^2 + 4(1+x)(1+x) + x^2 + 2x + 1\right)^{\frac{1}{2}} \\
&\leq 2(1+x)
\end{aligned}$$

and

$$\begin{aligned}
\left(P_{n,q}\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{\frac{1}{2}} &= \frac{1}{\delta}\sqrt{\frac{(1+q)}{q^2\,[n]}x} \\
&= \frac{1}{\delta\sqrt{[n]}}\sqrt{2x} \\
&\leq \frac{1}{\delta\sqrt{[n]}}\sqrt{1+2x+x^2} = \frac{1+x}{\delta\sqrt{[n]}}.
\end{aligned} \tag{5.2.6}$$

Now from (5.2.4), (5.2.5) and (5.2.6) we have

$$\begin{aligned}|P_{n,q}(f;x)-f(x)| &\leq \Omega_1(f,\delta)\left\{P_{n,q}\left((1+2x+t);x\right)\right.\\ &\left.+P_{n,q}\left((1+2x+t)\frac{|t-x|}{\delta};x\right)\right\}\\ &\leq \Omega_1(f,\delta)\left\{P_{n,q}\left((1+2x+t);x\right)\right.\\ &\left.+\left(P_{n,q}\left((1+2x+t)^2;x\right)\right)^{\frac{1}{2}}\left(P_{n,q}\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{\frac{1}{2}}\right\}\\ &\leq \Omega_1(f,\delta)\left(3(1+x)+2(1+x)\frac{1+x}{\delta\sqrt{[n]}}\right)\\ &= \Omega_1(f,\delta)\left(3(1+x)+2\frac{(1+x)^2}{\delta\sqrt{[n]}}\right)\\ &\leq (1+x^2)\Omega_1(f,\delta)\left(3K_1+\frac{4}{\delta\sqrt{[n]}}\right),\end{aligned}$$

where

$$K_1=\sup_{x\geq 0}\frac{1+x}{1+x^2}.$$

If we take $\delta=\frac{1}{\sqrt{[n]}}$,then from the above inequality we obtain the following

$$\frac{|P_{n,q}(f;x)-f(x)|}{1+x^2}\leq \Omega_1(f,\delta)\left(3K_1+4\right)$$
$$\|P_{n,q}(f)-f\|_2\leq k\,\Omega_1\left(f;\delta\right).$$

□

5.3 Asymptotic Formula

In this section, we prove the Voronovskaja type asymptotic result for the q-Phillips operators.

Lemma 5.3.1. *Let* $q > 1$. *We have*

$$P_{n,q}(t^3;x) = x^3 + \frac{(1+q)(1+q+q^2)}{[n]\,q^3}x^2 + \frac{(1+q)(1+q+q^2)}{[n]^2\,q^5}x,$$

$$P_{n,q}(t^4;x) = x^4 + \frac{(1+q)(1+q^2)(1+q+q^2)}{[n]\,q^4}x^3 + \frac{(1+q)(1+q^2)(1+q+q^2)^2}{[n]^2\,q^7}x^2 + \frac{(1+q)^2(1+q^2)(1+q+q^2)}{[n]^3\,q^9}x.$$

Proof. In order to prove Lemma 5.3.1, we shall use Lemma 5.1.2. For $f(t) = t^3$, we have

$$
\begin{aligned}
&P_{n,q}(t^3;x)\\
&= [n]\sum_{k=1}^{\infty} q^{2-k} s_{n,k}(q;qx) \int_0^{\frac{\infty}{1-\frac{1}{q}}} t^3 s_{n,k-1}(q;t)d_{\frac{1}{q}}t + E_{\frac{1}{q}}\left(-[n]\,x\right)f(0)\\
&= [n]\sum_{k=1}^{\infty} q^{2-k} s_{n,k}(q;qx)\frac{1}{q^6 q^{2-k}[n]^4[k-1]!}[k+2]!\\
&= \frac{1}{q^6}\sum_{k=0}^{\infty}\frac{[k+2]\,[k+1]\,[k]}{[n]^3}s_{n,k}(q;qx)\\
&= \frac{1}{q^6}\sum_{k=0}^{\infty}\frac{(1+q)\,[k]+(q+2q^2)\,[k]^2+q^3\,[k]^3}{[n]^3}s_{n,k}(q;qx)\\
&= \frac{1}{q^6}\left\{\sum_{k=0}^{\infty}\frac{q^3\,[k]^3}{[n]^3}s_{n,k}(q;qx)+\sum_{k=0}^{\infty}\frac{(q+2q^2)\,[k]^2}{[n]^3}s_{n,k}(q;qx)\right.\\
&\left.+\sum_{k=0}^{\infty}\frac{(1+q)\,[k]}{[n]^3}s_{n,k}(q;qx)\right\}\\
&= \frac{1}{q^3}\sum_{k=0}^{\infty}\frac{[k]^3}{[n]^3}s_{n,k}(q;qx)+\frac{(q+2q^2)}{[n]\,q^6}\sum_{k=0}^{\infty}\frac{[k]^2}{[n]^2}s_{n,k}(q;qx)\\
&+\frac{(1+q)}{[n]^2\,q^6}\sum_{k=0}^{\infty}\frac{[k]}{[n]}s_{n,k}(q;qx)\\
&= \frac{1}{q^3}\left((qx)^3+\frac{2+q}{[n]}(qx)^2+\frac{1}{[n]^2}qx\right)+\frac{(q+2q^2)}{[n]\,q^6}\left((qx)^2+\frac{1}{[n]}qx\right)\\
&+\frac{(1+q)}{[n]^2\,q^6}qx\\
&= x^3+\frac{2+q}{q\,[n]}x^2+\frac{1}{q^2\,[n]^2}x+\frac{(q+2q^2)}{[n]\,q^4}x^2+\frac{(q+2q^2)}{[n]^2\,q^5}x+\frac{(1+q)}{[n]^2\,q^5}x\\
&= x^3+\left(\frac{2+q}{q\,[n]}+\frac{(q+2q^2)}{[n]\,q^4}\right)x^2+\left(\frac{1}{q^2\,[n]^2}+\frac{(q+2q^2)}{[n]^2\,q^5}+\frac{(1+q)}{[n]^2\,q^5}\right)x\\
&= x^3+\frac{q^3(2+q)+(q+2q^2)}{[n]\,q^4}x^2+\frac{q^3+(q+2q^2)+1+q}{[n]^2\,q^5}x\\
&= x^3+\frac{(1+q)\,(1+q+q^2)}{[n]\,q^3}x^2+\frac{(1+q)\,(1+q+q^2)}{[n]^2\,q^5}x
\end{aligned}
$$

and for $f(t)=t^4$, we have

$$
\begin{aligned}
&P_{n,q}(t^4;x)\\
&=[n]\sum_{k=1}^{\infty}q^{2-k}s_{n,k}(q;qx)\int_0^{\frac{\infty}{1-\frac{1}{q}}}t^4 s_{n,k-1}(q;t)d_{\frac{1}{q}}t+E_{\frac{1}{q}}\left(-[n]\,x\right)f(0)\\
&=[n]\sum_{k=1}^{\infty}q^{2-k}s_{n,k}(q;qx)\frac{1}{q^{10}q^{2-k}\left[n\right]^5\left[k-1\right]!}\left[k+3\right]!\\
&=\frac{1}{q^{10}}\sum_{k=0}^{\infty}\frac{[k+3]\,[k+2]\,[k+1]\,[k]}{[n]^4}s_{n,k}(q;qx)\\
&=\frac{1}{q^{10}}\sum_{k=0}^{\infty}\left\{\frac{(1+q+q^2)(1+q)\,[k]+(1+q+q^2)(q+2q^2)\,[k]^2}{[n]^4}\right.\\
&\left.+\frac{(1+q+q^2)q^3\,[k]^3+q^3(1+q)\,[k]^2+q^3(q+2q^2)\,[k]^3+q^6\,[k]^4}{[n]^4}\right\}s_{n,k}(q;qx)
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{1}{q^4}\sum_{k=0}^{\infty}\left(\frac{[k]}{[n]}\right)^4 s_{n,k}(q;qx)+\frac{(1+q+q^2)q^3+q^3(q+2q^2)}{[n]\,q^{10}}\sum_{k=0}^{\infty}\left(\frac{[k]}{[n]}\right)^3 s_{n,k}(q;qx)\\
&+\frac{(1+q+q^2)(q+2q^2)+q^3(1+q)}{[n]^2\,q^{10}}\sum_{k=0}^{\infty}\left(\frac{[k]}{[n]}\right)^2 s_{n,k}(q;qx)\\
&+\frac{(1+q+q^2)(1+q)}{[n]^3\,q^{10}}\sum_{k=0}^{\infty}\frac{[k]}{[n]}s_{n,k}(q;qx)\\
&=\frac{1}{q^4}\left((qx)^4+(3+2q+q^2)\frac{(qx)^3}{[n]}+(3+3q+q^2)\frac{(qx)^2}{[n]^2}+\frac{1}{[n]^3}qx\right)\\
&+\frac{(1+q+q^2)+(q+2q^2)}{[n]\,q^7}\left((qx)^3+\frac{2+q}{[n]}(qx)^2+\frac{1}{[n]^2}qx\right)\\
&\frac{(1+q+q^2)(q+2q^2)+q^3(1+q)}{[n]^2\,q^{10}}\left((qx)^2+\frac{1}{[n]}qx\right)+\frac{(1+q+q^2)(1+q)}{[n]^3\,q^{10}}qx
\end{aligned}
$$

$$
\begin{aligned}
&=x^4+\frac{(3+2q+q^2)}{q\,[n]}x^3+\frac{(3+3q+q^2)}{q^2\,[n]^2}x^2\\
&+\frac{1}{q^3\,[n]^3}x+\frac{(1+q+q^2)+(q+2q^2)}{[n]\,q^4}x^3\\
&+\frac{((1+q+q^2)+(q+2q^2))(2+q)}{[n]^2\,q^5}x^2+\frac{(1+q+q^2)+(q+2q^2)}{[n]^3\,q^6}x\\
&+\frac{(1+q+q^2)(q+2q^2)+q^3(1+q)}{[n]^2\,q^8}x^2+\frac{(1+q+q^2)(q+2q^2)+q^3(1+q)}{[n]^3\,q^9}x\\
&+\frac{(1+q+q^2)(1+q)}{[n]^3\,q^9}x
\end{aligned}
$$

$$
\begin{aligned}
&= x^4 + \left[\frac{(3+2q+q^2)}{q\,[n]} + \frac{(1+q+q^2)+(q+2q^2)}{[n]\,q^4}\right]x^3 \\
&+ \left[\frac{(3+3q+q^2)}{q^2\,[n]^2} + \frac{((1+q+q^2)+(q+2q^2))(2+q)}{[n]^2\,q^5}\right. \\
&\left.+\frac{(1+q+q^2)(q+2q^2)+q^3(1+q)}{[n]^2\,q^8}\right]x^2 \\
&+ \left[\frac{1}{q^3\,[n]^3} + \frac{(1+q+q^2)+(q+2q^2)}{[n]^3\,q^6}\right. \\
&\left.+ \frac{(1+q+q^2)(q+2q^2)+q^3(1+q)}{[n]^3\,q^9} + \frac{(1+q+q^2)(1+q)}{[n]^3\,q^9}\right]x
\end{aligned}
$$

$$
\begin{aligned}
&= x^4 + \frac{q^3(3+2q+q^2)+(1+q+q^2)+(q+2q^2)}{[n]\,q^4}x^3 \\
&+ \left[\frac{q^6(3+3q+q^2)+q^3((1+q+q^2)+(q+2q^2))(2+q)}{[n]^2\,q^8}\right. \\
&\left.+\frac{(1+q+q^2)(q+2q^2)+q^3(1+q)}{[n]^2\,q^8}\right]x^2 \\
&+ \left[\frac{q^6+q^3\left((1+q+q^2)+(q+2q^2)\right)+(1+q+q^2)(q+2q^2)}{[n]^3\,q^9}\right. \\
&\left.+\frac{q^3(1+q)+(1+q+q^2)(1+q)}{[n]^3\,q^9}\right]x
\end{aligned}
$$

$$
\begin{aligned}
&= x^4 + \frac{(q+1)\,(q^2+1)\,(q+q^2+1)}{[n]\,q^4}x^3 + \frac{(q+1)\,(q^2+1)\,(q+q^2+1)^2}{[n]^2\,q^7}x^2 \\
&+ \frac{(q^2+1)\,(q+q^2+1)\,(q+1)^2}{[n]^3\,q^9}x \\
&= x^4 + \frac{(1+q)\,(1+q^2)(1+q+q^2)}{[n]\,q^4}x^3 + \frac{(1+q)\,(1+q^2)(1+q+q^2)^2}{[n]^2\,q^7}x^2 \\
&+ \frac{(1+q)^2\,(1+q^2)(1+q+q^2)}{[n]^3\,q^9}x.
\end{aligned}
$$

□

Lemma 5.3.2. *Assume that* $q_n > 1$ *and* $q_n \to 1$ *as* $n \to \infty$ *. Then, for every* $x \in [0, \infty)$ *there holds,*

$$\lim_{n\to\infty} [n]_{q_n}\, P_{n,q_n}(t-x;x) = 0, \tag{5.3.1}$$

$$\lim_{n\to\infty} [n]_{q_n}\, P_{n,q_n}((t-x)^2\,;x) = 2x, \tag{5.3.2}$$

$$\lim_{n\to\infty} [n]_{q_n}^2\, P_{n,q_n}((t-x)^4\,;x) = 12x^2. \tag{5.3.3}$$

Proof. First of all, we write explicit formula for $P_{n,q_n}(t-x;x)$

$$P_{n,q_n}(t-x;x) = P_{n,q_n}(t;x) - xP_{n,q_n}(1;x)$$
$$= x - x = 0.$$

Then we get

$$\lim_{n\to\infty} [n]_{q_n} P_{n,q_n}(t-x;x) = 0.$$

Next, we calculate $P_{n,q_n}((t-x)^2;x)$, as follows

$$\begin{aligned} P_{n,q_n}((t-x)^2;x) &= P_{n,q_n}(t^2;x) - 2xP_{n,q_n}(t;x) + x^2P_{n,q_n}(1;x) + x^2 - x^2 \\ &= x^2 + \frac{(1+q_n)}{q_n^2[n]}x - 2x^2 + x^2 \\ &= \frac{(1+q_n)}{q_n^2[n]}x \end{aligned}$$

and we obtain

$$\begin{aligned} \lim_{n\to\infty} [n]_{q_n} P_{n,q_n}((t-x)^2;x) &= \lim_{n\to\infty} [n]_{q_n} \left\{ \frac{(1+q_n)}{q_n^2 [n]_{q_n}} x \right\} \\ &= \lim_{n\to\infty} \frac{(1+q_n)}{q_n^2} x \\ &= 2x. \end{aligned}$$

Finally, we give an explicit formula for $P_{n,q_n}((t-x)^4;x)$

$$\begin{aligned} &P_{n,q_n}((t-x)^4;x) \\ &= P_{n,q_n}(t^4;x) - 4xP_{n,q_n}(t^3;x) + 6x^2P_{n,q_n}(t^2;x) - 4x^3P_{n,q_n}(t;x) + x^4 \\ &= x^4 + \frac{(1+q_n)(1+q_n^2)(1+q_n+q_n^2)}{[n]_{q_n} q_n^4}x^3 + \frac{(1+q_n)(1+q_n^2)(1+q_n+q_n^2)^2}{[n]_{q_n}^2 q_n^7}x^2 \\ &+ \frac{(1+q_n)^2(1+q_n^2)(1+q_n+q_n^2)}{[n]^3 q^9}x \\ &- 4x\left\{ x^3 + \frac{(1+q_n)(1+q_n+q_n^2)}{[n]_{q_n} q_n^3}x^2 + \frac{(1+q_n)(1+q_n+q_n^2)}{[n]_{q_n}^2 q_n^5}x \right\} \\ &+ 6x^2\left\{ x^2 + \frac{(1+q_n)}{q_n^2[n]_{q_n}}x \right\} - 3x^4 \end{aligned}$$

$$
\begin{aligned}
&= x^4 + \frac{(1+q_n)\,(1+q_n^2)(1+q_n+q_n^2)}{[n]_{q_n}\,q_n^4}x^3 + \frac{(1+q_n)\,(1+q_n^2)(1+q_n+q_n^2)^2}{[n]_{q_n}^2\,q_n^7}x^2 \\
&+ \frac{(1+q_n)^2\,(1+q_n^2)(1+q_n+q_n^2)}{[n]_{q_n}^3\,q_n^9}x - 4x^4 - \frac{4\,(1+q_n)\,(1+q_n+q_n^2)}{[n]_{q_n}\,q_n^3}x^3 \\
&- \frac{4\,(1+q_n)\,(1+q_n+q_n^2)}{[n]_{q_n}^2\,q_n^5}x^2 + 6x^4 + \frac{6\,(1+q_n)}{q_n^2\,[n]_{q_n}}x^3 - 3x^4 \\
&= \left[\frac{(1+q_n)\,(1+q_n^2)(1+q_n+q_n^2)}{[n]_{q_n}\,q_n^4} - \frac{4\,(1+q_n)\,(1+q_n+q_n^2)}{[n]_{q_n}\,q_n^3} + \frac{6\,(1+q_n)}{q_n^2\,[n]_{q_n}}\right]x^3 \\
&+ \left[\frac{(1+q_n)\,(1+q_n^2)(1+q_n+q_n^2)^2}{[n]_{q_n}^2\,q_n^7} - \frac{4\,(1+q_n)\,(1+q_n+q_n^2)}{[n]_{q_n}^2\,q_n^5}\right]x^2 \\
&+ \frac{(1+q_n)^2\,(1+q_n^2)(1+q_n+q_n^2)}{[n]_{q_n}^3\,q_n^9}x
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(1+q_n)\,(1+q_n^2)(1+q_n+q_n^2) - 4q_n\,(1+q_n)\,(1+q_n+q_n^2) + 6q_n^2(1+q_n)}{[n]_{q_n}\,q_n^4}x^3 \\
&+ \frac{(1+q_n)\,(1+q_n^2)(1+q_n+q_n^2)^2 - 4q_n^2\,(1+q_n)\,(1+q_n+q_n^2)}{[n]_{q_n}^2\,q_n^7}x^2 \\
&+ \frac{(1+q_n)^2\,(1+q_n^2)(1+q_n+q_n^2)}{[n]_{q_n}^3\,q_n^9}x \\
&= \frac{(q_n^2-q_n+1)\,(q_n+1)\,(q_n-1)^2}{[n]_{q_n}\,q_n^4}x^3 \\
&+ \frac{(q_n+1)\,(q_n+q_n^2+1)\,(q_n-2q_n^2+q_n^3+q_n^4+1)}{[n]_{q_n}^2\,q_n^7}x^2 \\
&+ \frac{(1+q_n)^2\,(1+q_n^2)(1+q_n+q_n^2)}{[n]_{q_n}^3\,q_n^9}x
\end{aligned}
$$

and we prove (5.3.3)

$$\begin{aligned}
&\lim_{n\to\infty}[n]_{q_n}^2 P_{n,q_n}((t-x)^4;x) \\
&= \lim_{n\to\infty}\frac{(1-q_n^n)^2}{(1-q_n)^2}\left\{\frac{(q_n^2-q_n+1)(q_n+1)(q_n-1)^2}{[n]_{q_n}q_n^4}x^3\right. \\
&+\frac{(q_n+1)(q_n+q_n^2+1)(q_n-2q_n^2+q_n^3+q_n^4+1)}{[n]_{q_n}^2 q_n^7}x^2 \\
&\left.+\frac{(1+q_n)^2(1+q_n^2)(1+q_n+q_n^2)}{[n]_{q_n}^3 q_n^9}x\right\} \\
&= \lim_{n\to\infty}\frac{[n]_{q_n}(q_n^2-q_n+1)(q_n+1)(q_n-1)^2}{q_n^4}x^3 \\
&+\lim_{n\to\infty}\frac{(q_n+1)(q_n+q_n^2+1)(q_n-2q_n^2+q_n^3+q_n^4+1)}{q_n^7}x^2 \\
&+\lim_{n\to\infty}\frac{(1+q_n)^2(1+q_n^2)(1+q_n+q_n^2)}{[n]_{q_n}q_n^9}x. \\
&= \lim_{n\to\infty}\frac{(1-q_n^n)(q_n^2-q_n+1)(1-q_n^2)}{q_n^4}x^3 \\
&+\frac{(q_n+1)(q_n+q_n^2+1)(q_n-2q_n^2+q_n^3+q_n^4+1)}{q_n^7}x^2 \\
&+\lim_{n\to\infty}\frac{(1-q_n)(1+q_n)^2(1+q_n^2)(1+q_n+q_n^2)}{(1-q_n^n)q_n^9}x \\
&= 12x^2.
\end{aligned}$$

□

Theorem 5.3.3. *Let $q_n > 1$. Then the sequence $\{P_{n,q_n}(f)\}$ converges to f uniformly on $[0,A]$ for each $f \in C_2^*[0,\infty)$ if and only if $\lim_{n\to\infty} q_n = 1$.*

Proof. The proof is similar to that of Theorem 2 [12]. □

Theorem 5.3.4. *Assume that $q_n > 1$ and $q_n \to 1$ as $n \to \infty$. For any $f \in C_2^*[0,\infty)$ such that $f', f'' \in C_2^*[0,\infty)$ the following equality holds*

$$\lim_{n\to\infty}[n]_{q_n}(P_{n,q_n}(f;x)-f(x)) = xf''(x),$$

for every $x \in [0,\infty)$.

Proof. Let $f, f', f'' \in C_2^*[0,\infty)$ and $x \in [0,\infty)$ be fixed. By the Taylor's formula, we may write

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + r(t;x)(t-x)^2, \tag{5.3.4}$$

where $r(t;x)$ is the Peano form of the remainder, $r(.;x) \in C_2^*[0,\infty)$ and $\lim\limits_{t\to x} r(t;x) = 0$. Applying $P_{n,q_n}(f,\ x)$ to (5.3.4), we obtain

$$\begin{aligned}
&[n]_{q_n}\left(P_{n,q_n}(f;x) - f(x)\right) \\
&= f'(x)\,[n]_{q_n} P_{n,q_n}(t-x;x) + \frac{1}{2}f''(x)\,[n]_{q_n} P_{n,q_n}\left((t-x)^2;x\right) \\
&+ [n]_{q_n} P_{n,q_n}\left(r(t;x)(t-x)^2;x\right).
\end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$P_{n,q_n}\left(r(t;x)(t-x)^2;x\right) \le \sqrt{P_{n,q_n}\left(r^2(t;x);x\right)}\sqrt{P_{n,q_n}\left((t-x)^4;x\right)}. \tag{5.3.5}$$

Observe that $r^2(x;x) = 0$ and $r^2(.;x) \in C_2^*[0,\infty)$. Then, it follows from Theorem 5.3.3 that

$$\lim_{n\to\infty} P_{n,q_n}\left(r^2(t;x);x\right) = r^2(x;x) = 0 \tag{5.3.6}$$

uniformly with respect to $x \in [0,A]$. Now from (5.3.5), (5.3.6) and Lemma 5.3.2, we get

$$\lim_{n\to\infty} [n]_{q_n} P_{n,q_n}\left(r(t;x)(t-x)^2;x\right) = 0.$$

Then, we have

$$\begin{aligned}
&\lim_{n\to\infty} [n]_{q_n}\left(P_{n,q_n}(f;x) - f(x)\right) \\
&= \lim_{n\to\infty}\left\{f'(x)\,[n]_{q_n} P_{n,q_n}(t-x;x) + \frac{1}{2}f''(x)\,[n]_{q_n} P_{n,q_n}\left((t-x)^2;x\right)\right. \\
&\left. + [n]_{q_n} P_{n,q_n}\left(r(t;x)(t-x)^2;x\right)\right\} \\
&= 0 + \frac{1}{2}f''(x)2x + 0 = xf''(x).
\end{aligned}$$

□

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