Martingale Theory

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ABSTRACT

From ages to ages there had been expectation of individuals on a specific predictions and future occurrences. So also in a game, different participant that involves in those specified game have their various expectations of the results or the output of the game they are involved in. That is why we need a mathematical theory that helps in prediction of the future expectations in our day to day activities. Therefore the Martingale Theory is a very good theory that explains and dissects the expectation of a gamer in a given game of chance. So in this thesis, we shall talk about the Martingale Theory expressing the expectations of a gamer in a game of chance, and also discuss the gaming strategies so as to enlighten everyone involved in a specific game their required expectation after proper understanding of the Martingale Theory.

Keywords: Martingale, Game of chance, Random walk, Stopping time.

ÖZ

Eski zamanlardan günümüze kadar insanların gelecekteki oluşumlar ile ilgili belirli öngörüleri ve beklentileri olmuştur. Hatta farklı katılımcıların dahil olduğu belirli bir oyunda, oyuncunun dahil olduğu oyunun sonucuna yönelik çeşitli beklentileri vardır. Bu sebebledir ki, günlük hayatımızda gelecekle ilgili beklentiler hakkında öngörüde bulunabilmek için matematiksel Teoriye ihtiyaç duyulmaktadır. Bir şans oyununda oyuncunun beklentisini açıklamak ve incelemek için Martingale Teorisi kullanılmaktadır. Bu tezde oyuncunun beklentisi ifade etmek için Martingale Teorisi hakkında konuşacağız ve ayrıca oyun stratejilerini tartışacağız.

Anahtar kelimeler: Martingale, şans oyunu, rasgele adım, durma zamanı

To God Almighty and my Dearest family

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Chapter 1

INTRODUCTION

Martingale is a betting strategy that was traced back as at 18 centuries in France. This strategy was introduced for a game in which a specific gambler wins his stake if a coin comes up heads and loses it if same coin comes up tails. The gambler needs to double his bet after every loss since he/she is not ready to loose nor give up and his/her aim is to recover all previous losses plus win and gain a profit that is equivalent to the original stake. This same Martingale strategy has been applied to some other games like the roulette, as the probability of hitting either red or black is close to 0.5.

We can also describe a Martingale as a model of a fair game in which the knowledge of the past events or the knowledge of the already known result of the game can never help to predict the result or the mean of the expected winnings. Consequently, a Martingale is a sequence of random variables or rather a stochastic process for which, at a given time in the realized sequence, the expectation of the next value in the sequence is equal to the present observed value inconsequential of the knowledge of all previously observed values.

On the other hand, in a non-martingales process, we may still have a situation where the expected value of the process at one time is equal to the expected value of the process at the next time. However, knowledge of the previous outcomes, for instance, “the previous cards drawn out from a set of cards” may be able to reduce the uncertainty of upcoming outcomes. Thus, the expected or resulting value of the next outcome given a definite knowledge of the present and all previous outcomes may definitely be higher than the current outcome provided we use the said winning strategy. Martingale does not include the possibility of the winning strategies based on already known game history, and thus making the system a model of fair games.

Since a gambler with inexhaustible measure of wealth will almost surely flip head, with this said reason, the Martingale betting strategy was concluded to be as a sure system of gaming by those who recommended it. Even though none of the gamblers possesses an inexhaustible wealth, and the exponential movement of the bets placed would eventually bankrupt the gamer and the gamblers who chose to use the Martingale system often wins a minute net reward, thus appearing to have a faultless and accurate strategy. However, the gambler's expected results and values mostly ends up being zero (or even less than zero) because the small probability that he will suffer an unimaginable loss exactly measures up and balance up with his gain. (In a casino, the expected result of a gambler is negative, simply because to the house's edge.) The possibility of catastrophic loss may not really be small since the bet size always rises in an exponential rate. The fact that strings of consecutive losses definitely occur more often than just an ordinary intuitional suggestions, can make the gambler go bankrupt quickly.

1.1 A brief example on how the Martingale System works?

The most effective and basic system of betting outside even money bets is the Martingale system. In which a player has to double his bet after every loss in other to regain the previous loses. And immediately the player wins, the system back to its original state. This implies that just one win by the player will definitely recover all the accumulated losses and result in a net profit of the initial stake. Although it's one of the most aggressive systems because of the risk it carries, since its progression is geometrical. The smaller the wins you need to recover your previous losses, the more catastrophic and dangerous the system becomes.

Below is a brief description of how this system works, lets imagine we want to place a bet on horses in a given horse race competition. And if we want to constantly bet on a single horse in the given horse race, and let's assume the horses are numbered $1\ to\ 10$, and we want to constantly place 1 chip bet on the horse 3 of 10 always. If the sequence of the results of the winning horse in five consecutive races were to be $5, 7, 4, 6, 3,$ respectively, the following would occur:

1. The first bet is 1 chip on the 3^{rd} horse. But since the winning horse was 5, thus, we lose and having a net profit of -1.
2. Since we lost, and so we must double our bet in other to regain fully. Therefore we bet 2 chips instead of 1 (on horse 3 again). But since the winner again is the 7^{th} horse, we lose. And the Net profit now becomes: -3 i.e. $-1-2$
3. Another loss means doubling the bet again which means we place 4 chips to regain fully and with the outcome result, which is the 4^{th} horse winning, We lose again. Therefore our net profit is: -7 i.e. $-1-2-4$
4. We have lost again, so need to double the bet again. So we put bet 8 chips on the bet for full recovery of the lost chips. With the result, the 6^{th} horse won the race and making us loose the chip and giving a net profit of : $-15,$ i.e. $-1-2-4-8$
5. After this loss, we still have to double the bet and therefore we bet 16 chips. This time, we win with our 3^{rd} horse winning the race. Thus we have a net profit of : $-1-2-4-8+16 = +1.$

With just one single win, we have been able to turn our previous losses i.e. fifteen times the original stake into a win, although very small profit but no loss. Of course, we could have arrived at the same sets of results if we had used a different initial bet (and thus we create more risk in other to win more) or by betting a different thing entirely other than even said money bet.

If you examine and study the just concluded example very well, we can deduce that doubling the bets simply implies betting only 1 chip extra more than our present net loss on each occasion. This is the main reason why we can easily conclude that the system works: Since the sum of the combined previous losses will definitely be one lesser than our next bet, then this implies that we always stand a chance of making a profit on every spin. But with what we deduce in the above example, just a little short and relatively small loss streak increases the required bankroll totally and significantly. We have to bet 16 good chips just to be able to win only one because of the accumulated losses. This is a very dangerous decision which is encountered in a system as aggressive as this so called Martingale system. And also we can’t forget the fact that it takes just a single win to recover from all the losses one might have accumulated. Therefore this fact makes the system more interesting and attractive, in addition to the danger of the system.

1.2 Advantages and Disadvantages

As far as the Martingales system is advantageous so also it has some disadvantages. One of the major disadvantages to the Martingales system with a specified bankroll that increases at a geometric rate is that it is definitely easy for a player using this system to end up in a disastrous and a catastrophic situation where he will definitely be faced with an unrecoverable losses, in which the chances are: is either what the gambler has at hand might not be able to meet up with the table limit, or he totally runs out of chips. And, even mathematically, if neither of these never occurs in any circumstances, the player would always hope that he will definitely gains back the money he had invested in the bet and with this, making him ending up needing infinitely numerous time (in which no person has) to ensure that he eventually recovers all what he has lost and if possible add some additional gain. Since every roulette table in all the various casino has a betting maximum/limit or a certain amount of chip or cash to put in the bet, and also there is a limit to every player's bankroll, it is mathematically certain that at some point during the commencement of the game that the Martingale system will result in the player either losing all of his chips or coming short of the table limit and so he/she won't be able to continue in the bet or game. Other betting system tries to reduce the odds of this happening by decreasing the rate and the speed at which losses have to be recovered.

And also since almost all gambler are so eager get back their losses after accumulating a reasonable number of loses. Then the situation becomes more difficult since you have to bet more than what you've already loosed and it requires more risk since it's not certain that one is definitely going to win in the next bet. Then it puts the gambler in a very difficult position of choice. Because the ending result might be catastrophic since it's a game of chance. The winner can easily say the system is very nice and productive one while the looser mostly says the system sucks.

Chapter 2

REVIEW OF PROBABILITY THEORY

2.1 Probability Terminology

For a proper understanding of the theory of martingale, there are necessary probability terms that are needed to be explained.

1. **Outcome**: It can be easily explained as the result of an experiment. For instance if a coin is tossed, the point at which it comes to rest gives either a head or a tail. That's the outcome of the result of the experiment of tossing the coin.
2. **Trial**: Each time we roll a die or toss a coin is called a trial.
3. **Experiment**: It consist of one or more trails which may give different results
4. **Favourable Outcome**: When an experiment is performed and the desired outcome becomes the real outcome of the experiment then we say it is favourable. For example if we want a 4 to come up in the experiment of tossing a die, and after tossing the die, we get a 4 as an outcome. Then we say the outcome is favourable.
5. **Equal likely Outcome**: While performing a probability experiment with either a die or a coin. The instrument used i.e. a die or a coin has an equal likely outcome if it is not bent or the die is not unequally customized to land with a desired figure.
6. **Event**: An event is a subset of the sample space in which a probability can be assigned. i.e.

7. **Sample space**: The sample space is the collection or sets of all the sample points or all the possible outcome of an experiment.

2.2 Definition of Some Terms

2.2.1 Sigma-Algebra (σ-Algebra)

Definition 2.1: A Sigma-Algebra (σ-Algebra) or Sigma-Field (σ-field) on a given non empty set X is a collection of the subsets of X which is closed under Compliments, Union of countably many sets and Intersection of countably many sets.

Consider a set X. A σ–algebra $\mathcal{F}$ of subsets of X is a collection $\mathcal{F}$of subsets of X satisfying the following conditions:

(a) $\emptyset \in \mathcal{F}$

(b) if $\mathcal{B} \in \mathcal{F}$ then its complement B^c is also in $\mathcal{F}$

(c) if $\mathcal{B}_1, \mathcal{B}_2, \ldots$ is a countable collection of sets in $\mathcal{F}$ then their union $\bigcup_{n=1}^{\infty} \mathcal{B}_n$

Sometimes we will just write "Sigma-Algebra" instead of "Sigma-Algebra of subsets of X." There are two extreme examples of sigma-algebras: the collection $\{\emptyset$, X$\}$ is a sigma-algebra of subsets of X and also the set P(X) of all subsets of X is a sigma-algebra. Any sigma-algebra $\mathcal{F}$ of subsets of X lies between these two extremes: $\{\emptyset, X\} \subset \mathcal{F} \subset \mathrm{P}(X)$.

Algebra is required to be closed under finitely many set operations. That is to say, a σ-algebra is an algebra of sets, completed to include countably infinite operations. The pair $(X, \mathcal{F})$ is also a field of sets, called a **Measurable space**. For example,

If $X = \{p, q, r, s\}$, one possible σ-algebra on X is

$F = \{\emptyset, \{p,\ q\}, \{r,\ s\}, \{p,\ q,\ r,\ s\}\}$, where $\emptyset$ is the empty set. However, a finite algebra is always a σ-algebra. If $\{A_1,\ A_2,\ A_3, \dots\}$ is a countable partition of X then the collection of all unions of sets in the partition with the empty set is a σ-algebra.

2.2.2 Probability Space

Definition 2.2: A probability space $(\Omega\ \mathcal{F}\ \mathrm{P})$ consist of

1. The sample space Ω which is the set of all possible outcomes $\omega \in \Omega$ of some random experiment.
2. The event $\mathcal{F}$ which is the collection of all possible events under consideration and each subset of $\mathcal{F}$ is an event.
3. And P which is the assignment of the probabilities to the events, in other words it implies the function P from events to probability.

The set of all possible subsets of Ω is denoted by 2^{Ω} thus $\mathcal{F} \subset 2^{\Omega}$, consisting of all allowed events i.e. those events to which one can assign probabilities.

2.2.3 Probability Measure

Definition 2.3: Measurable space is a pair $(\Omega\ \mathcal{F})$ with $\mathcal{F}$ a $\sigma-$field of subsets of Ω is called a measurable space.

2.2.4 Filtration

Definition 2.4: A sequence of σ-algebra's $\mathcal{F}_1, \mathcal{F}_2 \dots$ on a Ω such that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \dots \mathcal{F}_n \subset \mathcal{F}$ is called a filtration. (Since $\mathcal{F}_n$ is a σ-algebra for each n). As $\mathcal{F}_n$ increases, our knowledge at time n increases. It contains all events A such that at time n it is possible to conclude whether A occurs or not.

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space, A filtration $(\Omega, \mathcal{F}, \mathrm{P})$ is an increasing family $(\mathcal{F}_t)_{t\geq 0}$ of sub- σ-algebra of $\mathcal{F}$. That is to say for each t, $\mathcal{F}_t$ is a σ-algebra including in $\mathcal{F}$ and if $s \leq t$, $\mathcal{F}_s \subset \mathcal{F}_t$. A probability space $(\Omega, \mathcal{F}, \mathrm{P})$ endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$ is called a filtered probability space.

2.2.5 Adapted Process

Definition 2.5: If $\mathcal{F}$ is a $\sigma - field$ on Ω, then a function $\xi: \Omega \to \mathbb{R}$ is said to be $\mathcal{F} - measurable$ if $\{\xi \in B\} \in \mathcal{F}$ for every Borel set $\mathfrak{B} \in B$ if $(\Omega, \mathcal{F}, \mathrm{P})$ is a probability space, then such a function ξ is called a random variable.

Definition 2.6: A process or a sequence of random variables $(X_n)_{n\geq 0}$ is called Adapted (to the Filtration $\mathcal{F}_n$) namely $\mathcal{F}_1, \mathcal{F}_2, \ . \ . \ .$ if X_n is $\mathcal{F}_n - measurable$ for each $n = 1, 2, \ . \ . \ .$

Example 2.2: If $\mathcal{F}_n = \sigma(X_1, . \ . \ ., X_n)$ is the $\sigma - algebra$ generated by $X_1, . \ . \ ., X_n$ then $X_1, X_2, \ . \ . \ .$ is adapted to $\mathcal{F}_1, \mathcal{F}_2, \ . \ . \ .$

2.2.6 Indicator Function

For any event A$\in \mathcal{F}$, the function $I_A(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}$ is a random variable. We call such random variable an Indicator Function.

2.2.6.1 Properties of Indicator Random Variables

1. $I_\emptyset(\omega) = 0$ and $I_\Omega(\omega) = 1$
2. $I_{A^c}(\omega) = 1 - I_A(\omega)$
3. $I_A(\omega) \leq I_B(\omega)$ if and only if $A \subseteq B$
4. $I_{\cap_i A_i}(\omega) = \prod_i I_{A_i}(\omega)$
5. If A_i are disjoint then $I_{\cup_i A_i}(\omega) = \sum_i I_{A_i}(\omega)$

2.2.7 Measurable space

Definition 2.7: A pair $(\Omega, \mathcal{F})$ with $\mathcal{F}$ being a σ-field of subsets of Ω is called a measurable space. Given a measurable space, a probability measure P is a function $P: \mathcal{F} \to [0,1]$, with the following properties:

a) $0 \leq P(A) \leq 1$, for all $A \in \mathcal{F}$ (nonnegative)

b) $P\,(\Omega) = 1$

c) $P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$ disjoint sets $A_n \cap A_m = \emptyset$ for all $n = m$

A probability space is a triplet $(\Omega, \mathcal{F}, \mathrm{P})$ with P, a probability measure on $(\Omega, \mathcal{F})$

2.2.8 Random Variable

Definition 2.8: If $\mathcal{F}$ is a $\sigma - field$ on Ω, then a function $\xi: \Omega \to \mathbb{R}$ is said to be $\mathcal{F}$ $-$measurable if $\{\xi \in B\} \in \mathcal{F}$ for every Borel set $B \in \mathfrak{B}$, if $(\Omega, \mathcal{F}, \mathrm{P})$ is a probability space, then such a function ξ is called a random variable. A random variable can be furthermore explained as the collection or set of values obtained from a probability experiment which are subject to variation simply because of chance.

2.2.9 Stochastic Process

Definition 2.9: Given a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and a measurable space $(\mathcal{S}, \mathcal{F})$, an S-valued stochastic process is a collection of S-valued random variables on Ω, indexed by a set T (time). In other words a stochastic process $\mathbb{P}$ (also known as random process) is a map $\mathbb{P}: T \longrightarrow f_0(\Omega, \mathcal{F}, \mathrm{P})$ where $f_0(\Omega, \mathcal{F}, \mathrm{P})$ is the space of (equivalence classes of) bounded measurable functions for a probability space $(\Omega, \mathcal{F}, \mathrm{P})\ to\ \mathbb{R}$.

The notion of a stochastic process is very important both in mathematical theory and its applications in science, engineering, economics, etc. It is used to model a large number of various phenomena where the quantity of interest varies discretely or continuously through time in a non-predictable fashion.

Every stochastic process can be viewed as a function of two variables t and ω. For each fixed $(t, \omega) \to X_t(\omega)$ is a random variable, as postulated in the definition.

However, if we change our point of view and keep ω fixed, we see that the stochastic process is a function mapping ω to the real-valued function $(t, \omega) \rightarrow X_t(\omega)$ These functions are called the trajectories of the stochastic process X.

2.2.10 Mathematical Expectation

Definition 2.10: Giving that X as a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ then the expected value of X denoted by $\mathbb{E}(X)$ is defined as the Lebesque Integral $\int_\Omega XdP = \int_\Omega X(\omega)Pd(\omega)$ provided the Integral exist, then it's called Expected value of X .

In probability theory, mathematical expectation, also known as the expected value of a random variable can be described as the average value of a long-run experiment. For example the mathematical expectation of the experiment of rolling a die is 3.5 because the average value of large numbered rolled die in its extreme state is 3.5 which is the expected value.

2.2.10.1 Properties of Expectation

1. $\mathbb{E}I_A = P(A)$ for any $A \in \mathcal{F}$.
2. If $X(\omega) = \sum_{n=1}^{N} C_n I_{A_n}$ is a sample function, then $\mathbb{E}X = \sum_{n=1}^{N} C_n P(A_n)$
3. If X and Y are integrable random variable then for any constants $\alpha, \beta \in \mathbb{R}$, the random variable $\alpha X + \beta Y$ is integrable and $\mathbb{E}(\alpha X + \beta Y) = \alpha\mathbb{E}X + \beta\mathbb{E}Y$ (linearity)
4. $\mathbb{E}X = c$ if $X(\omega) = c$ with probability 1 almost surely.
5. Monotonicity: if $X \geq Y$ almost surely, then $\mathbb{E}X \geq \mathbb{E}Y$ and also if $X \geq Y$ almost surely and $\mathbb{E}X = \mathbb{E}Y$ then $X = Y$

2.2.11 Conditional Expectation

We can recall that the conditional probability of A given B

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \qquad P(B) > 0$$

Clearly, $P(A|B) = P(A)$ if A and B are independent.

Given that $P(B) > 0,$ the conditional distribution function of a random variable X given B is given as

$$\mathbb{F}_x(x|B) = \frac{P(X_{\leq x}, B)}{P(B)}, \qquad x \in \mathbb{R}$$

and therefore

$$\mathbb{E}(X|B) = \frac{\mathbb{E}(XI_B)}{P(B)}$$

is the conditional expectation of X given B.

2.2.11.1 Conditioning on an Event.

Definition 2.11: For any integrable random variable ξ and any event $B \in \mathcal{F}$ such that $P(B) \neq 0,$ the conditional expectation of ξ given B is defined by

$$\mathbb{E}(\xi|B) = \frac{1}{P(B)} \int_B \xi dP.$$

2.2.11.2 Conditioning on a Discrete Random Variable

The conditional expectation of a discrete random variable η with possible value $y_1, y_2, \ . \ . \ .$ such that $P(\eta = y_n) \neq 0$ for each n. Founding out the value of η amounts to finding out which of the events $\{\eta = y_n\}$ has occurred or not conditioning by η should therefore be the same as conditioning by the events $\{\eta = y_n\}$.

Definition 2.12: Let ξ be an integrable random variable and let $y_1, y_2, \ . \ . \ .$ such that $P(\eta = y_n) \neq 0$ for each n, then the conditional expectation of ξ given η is defined to be a random variable $\mathbb{E}(\xi|\eta)$ such that

$$\mathbb{E}(\xi|\eta)(\omega) = \mathbb{E}(\xi|\{\eta = y_n\}) \text{ if } \eta(\omega) = y_n.$$

2.2.11.3 Conditioning on an Arbitrary Random Variable

Definition 2.13: Let ξ be an integrable random variable and let η be an arbitrary random variable, then the conditional expectation of ξ given η is defined to be a random variable $\mathbb{E}(\xi|\eta)$ such that

1. $\mathbb{E}(\xi|\eta)$ is $\sigma(\eta) - measurable$
2. For any $A \in \sigma(\eta)$, $\int_A \mathbb{E}(\xi|\eta)dP = \int_A \xi dP$.

The conditional probability of an event $A \in \mathcal{F}$ given η is $P(A|\eta) = \mathbb{E}(1_A|\eta)$.

2.2.11.4 Conditioning on a σ −Algebra

Definition 2.14: Let ξ be an integrable random variable on a probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{G}$ be a σ −Algebra contain in $\mathcal{F}$. Then the conditional expectation of ξ given $\mathcal{G}$ is defined be a random variable $\mathbb{E}(\xi|\mathcal{G})$ such that.

1. $\mathbb{E}(\xi|\mathcal{G})$ is $\mathcal{G}$ −measurable
2. For any $A \in \mathcal{G}$

$$\int_A \mathbb{E}(\xi|\mathcal{G})\, dP = \int_A \xi dP$$

The condition probability of an event $A \in \mathcal{F}$ given a σ −Algebra $\mathcal{G}$ can be defined by $P(A|\mathcal{G}) = \mathbb{E}(1_A|\mathcal{G})$. The notion of condition expectation with respect to σ −Algebra extends conditional on a random variable η in the sense that

$\mathbb{E}(\xi|\sigma(\eta)) = \mathbb{E}(\xi|\eta)$, when $\sigma(\eta)$ is the σ −Algebra generated by η

2.2.12 General Properties of Conditional Expectation

Given that $a, b \in \mathbb{R}$, $\xi, \eta \in \mathbb{L}_1$, $and\ \mathcal{G}, \mathcal{H}$ are sub- σ −Algebra on Ω

1. (Linearity) Let $\xi, \eta \in \mathbb{L}_1(\Omega, \mathcal{F}, P)$ then

$$\mathbb{E}(a\xi + b\eta|\mathcal{G}) = a\mathbb{E}(\xi|\mathcal{G}) + b\mathbb{E}(\eta|\mathcal{G})$$

Proof: For any $B \in \mathcal{G}$,

$$\int_B (a\mathbb{E}(\xi|\mathcal{G}) + b\mathbb{E}(\eta|\mathcal{G}))dP$$

$$a\int_B \mathbb{E}(\xi|\mathcal{G})dP + b\int_B \mathbb{E}(\eta|\mathcal{G}))dP$$

$$a\int_B \xi dP + b\int_B \eta dP$$

Since ξ and η are $\mathcal{G}$ $-$measurable, and also by uniqueness we have

$$\int_B \big(a\mathbb{E}(\xi|\mathcal{G}) + b\mathbb{E}(\eta|\mathcal{G})\big)dP = \mathbb{E}(a\xi + b\eta|\mathcal{G})$$

2. $\mathbb{E}\big(\mathbb{E}(\xi|\mathcal{G})\big) = \mathbb{E}(\xi)$ (By replacing $B = \Omega$ in 1)
3. (Positivity) If $\xi \geq 0$, then $\mathbb{E}(\xi|\mathcal{G}) \geq 0$ almost surely.
4. (Monotonic) If $0 \leq \xi_n \uparrow \xi$, then $\mathbb{E}(\xi_n|\mathcal{G}) \uparrow \mathbb{E}(\xi|\mathcal{G})$ almost surely.
5. (Fatou's Lemma) if $\xi_n \geq 0$ then $\mathbb{E}(\liminf \xi_n|\mathcal{G}) \leq \liminf \mathbb{E}(\xi_n|\mathcal{G})$ almost surely
6. (Dominated Convergence Theorem) $|\xi_n(\omega)| \leq \eta(\omega), \forall n < \infty$ and $\xi_n \to \xi$ almost surely, then $\mathbb{E}(\xi_n|\mathcal{G}) \to \mathbb{E}(\xi|\mathcal{G})$ almost surely.
7. (Jensen's Inequality) If $g: \mathbb{R} \to \mathbb{R}$ is convex, and $\mathbb{E}|g: (\xi)| < \infty$, then $\mathbb{E}(g: (\xi|\mathcal{G})) \geq g: (\mathbb{E}(\xi|\mathcal{G})$, almost surely.

Chapter 3

THE MARTINGALE THEORY

3.1 Basic Definition

Definition 3.1: A sequence of random variables or random process $(X_n)_{n\geq 0}$ is a Martingale with respect to a filtration $(\mathcal{F}_n)_{n\geq 0}$ if for all $n \geq 0$

1. If X_n is integrable for all n that is $\mathbb{E}[|X_n|] < \infty$

2. $(X_n)_{n\geq 0}$ is adapted to $(\mathcal{F}_n)_{n\geq 0}$

3. $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$ (or generally) $\mathbb{E}[X_{n+k}|\mathcal{F}_n] = X_n \; with \; k > 0$

If (1) and (2) holds but instead of (3) we have

4. $\mathbb{E}[|X_{n+1}|\mathcal{F}_n] \leq X_n$, then we say X_n is a **Supermartingale** with respect to the filtration $\mathcal{F}_n$ and on the other hand if (1) and (2) holds but in place of (3) we have

5. $\mathbb{E}[|X_{n+1}|\mathcal{F}_n] \geq X_n$, then we say X_n is a **Submartingale** with respect to filtration. X_n will only be a submartingale or supermatingale if and only if X_n itself is a Martingale.

Since Martingales describe a fair game of chance, a submartingale can also be termed as a favourable game in which the expectation of the gambler is greater than the previous or foreknowledge of the games result. Also a supermartingale is also called an unfavorable game where the expectation of the gambler is less than the previous result of a known game.

The random sequence X_n models the outcomes of particular random phenomena that take place through time, whereas the filtration $\mathcal{F}_n$ tells us precisely what is known in each period. If X_n is a supermartingale, then at time n we have partial information about the outcome of this random phenomena at least inasmuch as the values of $X_1, \ldots, X_n$ are concerned, and conditioning on this information, we expect the value of X_{n+1} to be less than or equal to the observed value at date n For instance, if the daily value of a given stock is modeled as a Martingale, then we expect the value of the stock on Friday to be equal to its value on Thursday, conditional on all the information available to us on Thursday. It is important to note that the expected values of the terms of a Martingale remains constant through time.

3.2 Some Examples of Martingales

Example 3.1: Let $(\xi_n)_{n\geq 1}$be an independent random variable with $\mathbb{E}(\xi_n) = 0$ for all $n \geq 1$. The process $(X_n)_{n\geq 0}$ defined by $X_n = X_0 + \xi_1 + \ldots + \xi_n$ is a Martingale as long as the random variable X_0 is independent of $(\xi_n)_{n\geq 1}$ and $\mathbb{E}(X_0) < \infty$.

Solution: Since $\mathbb{E}|X_n| \leq \mathbb{E}|X_0| + \sum_i^n \mathbb{E}|\xi_i| < \infty$ for all $n \geq 0$ and since the independent properties implies

$$\mathbb{E}(X_{n+1} - X_n | X_n, \ldots X_0) \equiv \mathbb{E}(\xi_{n+1} | X_n, \ldots X_0) = \mathbb{E}(\xi_{n+1}) = 0.$$

Remark: It is obvious that $\mathbb{E}(\xi_n) \geq 0$ for all $n \geq 0$ then $(X_n)_{n\geq 0}$ is a submartingale, although if $\mathbb{E}|\xi_n| \leq 0$ for all $n \geq 1$, then $(X_n)_{n\geq 0}$ is a supermartingale. Generally, if $(\xi_n)_{n\geq 1}$ are independent random variables with $\mathbb{E}|\xi_n| \leq \infty$ for all $n \geq 1$, then the process $X_n = X_0 + (\xi_1 - \mathbb{E}(\xi_n)) + \ldots + (\xi_n - \mathbb{E}(\xi_n))$ $n \geq 0$ is a Martingale.■

Example 3.2: Sum of independent zero-mean random variables. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent random variables with $\mathbb{E}|\xi_n| < \infty, \ for\ all\ k$ $\mathbb{E}|\xi_n| = 0, \ for\ all\ k.$

Define: $S_n = \xi_1 + \xi_2 + \ . \ . \ . + \xi_n$ $\qquad S_n = 0$

$\mathcal{F}_n = \sigma(\xi_1, \xi_2, . \ . \ . \xi_n)$ $\qquad \mathcal{F}_n = \{\emptyset\ , \Omega\}$

then for any $n \geq 1$, we have (almost surely)

$\mathbb{E}(S_n|\mathcal{F}_{n-1}) = \ \mathbb{E}(S_{n-1}|\mathcal{F}_{n-1}) + \mathbb{E}(\xi_n|\mathcal{F}_{n-1}) = S_{n-1} + \mathbb{E}(\xi_n) = S_{n-1}.$

Hence

$\mathbb{E}(S_n|\mathcal{F}_{n-1}) = S_{n-1}$ (almost surely)■

Example 3.3: Products of non-negative independent random variables of mean 1.

Let $\xi_1, \xi_2, \ . \ . \ .$ be a sequence of independent non-negative random variables with

$\mathbb{E}(\xi_k) = 1\ for\ all\ k$

Define $P_n = \xi_1, \xi_2\ , \ . \ . \ ., \ \xi_n$ $\qquad P_n = 1$

$\mathcal{F}_n = \sigma(\xi_1, \xi_2, . \ . \ ., \xi_n)$ $\qquad \mathcal{F}_n = \{\emptyset\ , \Omega\}$

For $n \geq 1,$ we have (almost surely)

$\mathbb{E}(P_n|\mathcal{F}_{n-1}) = \ \mathbb{E}(P_{n-1}\xi_n|\mathcal{F}_{n-1})$

$P_{n-1}(\mathcal{F}_n - measurable) = \ P_{n-1}\ \mathbb{E}(P_n|\mathcal{F}_{n-1})$

$(\xi_n independent\ of\ \mathcal{F}_{n-1}) = \ P_{n-1}\ \mathbb{E}(\xi_n)$

$\mathbb{E}(\xi_n) = 1 = P_{n-1}$ therefore P is a Martingale.■

Example 3.4: Show that if ξ_n is a Martingale with respect to filtration $\mathcal{F}_n$ then

$\mathbb{E}(\xi_1) = \mathbb{E}(\xi_2) = \ . \ . \ . = \mathbb{E}(\xi_n)$

Solution: taking expectation on both sides of the equality

$\xi_n = \ \mathbb{E}(\xi_{n+1}|\mathcal{F}_n),$ Since is a Martingale, we get

$\mathbb{E}(\xi_n) = \mathbb{E}(\mathbb{E}(\xi_{n+1}|\mathcal{F}_n))$ Since ξ_n is independent of $\mathcal{F}_n$ then $\mathbb{E}(\xi_{n+1})$ for each $n.$ ■

Example 3.5: Given that ξ is an integrable random variable defined on a probability space (Ω $\mathcal{F}$ P), and $\mathcal{F}_n$ a filtration in $\mathcal{F}$, we define $\xi_n = \mathbb{E}(\xi|\mathcal{F}_n)$ for all positive integer n, in other form of $\mathbb{E}(\xi|\mathcal{F}_n)$, we want to show that ξ_n is a martingale with respect to $\mathcal{F}_n$.

Solution: Since by definition of the conditional expectation, ξ_n is $\mathcal{F}_n$ −measurable for each n and also $|\xi_n| \leq \mathbb{E}(|\xi||\mathcal{F}_n)$ hence we have $\mathbb{E}(\xi_n) \leq \mathbb{E}\big(\mathbb{E}(|\xi||\mathcal{F}_n)\big) = \mathbb{E}(|\xi|) \leq \infty$ which definitely means that ξ_n is integrable for each n. thus have $\mathbb{E}\big(\mathbb{E}(\xi_{n+1}|\mathcal{F}_n)\big) = \mathbb{E}(\mathbb{E}(\xi|\mathcal{F}_{n+1})|\mathcal{F}_n)$ almost-surely

$$= \mathbb{E}(\xi|\mathcal{F}_n) = \xi_n$$

3.3 Random Walk

Definition 3.2: Random Walk (Drunkard's Walk) Let $\xi_1, \xi_2, \ . \ . \ .$ be a integer valued random variable having common density f. Let X_0 be n integer-valued random variable that is independent of the $\xi_i's$ and the set $X_n = X_0 + \xi_1 + \xi_2 + . \ . \ . + \xi_n$. The sequence $(X_n)_{n\geq 0}$ is called a **Random Walk**.

A one dimension random walk is a Markov chain whose state space is given by the integers $i = 0, \ \pm 1, \pm 2, . \ . \ .$ for some probability P, $P_{i,i+1} = P$ and $P_{i,i-1} = 1 - P$.

Example 3.5: Let ξ_n be a symmetric random walk that is $\xi_n = \zeta_1 + . \ . \ . + \zeta_n$, where $\zeta_1, \zeta_2 \ . \ . \ .$ is a sequence of independent random variables such that $P(\zeta_n = 0) = P(\zeta_n = 1) = {}^1/_2$ (a sequence of coin tosses). Show that ${\xi_n}^2 - n$ is a Martingale with respect to the filtration $\mathcal{F}_n = \sigma(\zeta_1, . \ . \ . , \zeta_n)$

Solution: ${\xi_n}^2 - n$ =$(\zeta_1 + . \ . \ . + \zeta_n)^2 - n$ is a function of $\zeta_1, \zeta_2, . \ . \ . , \zeta_n$, hence it is measurable with respect to the:

1. $\sigma - algebra \ \mathcal{F}_n$ generated by $\zeta_1, \ \zeta_2, . \ . \ . , \zeta_n$. That is to say ${\xi_n}^2 - n$ is adapted to $\mathcal{F}_n$ since $|\xi_n| = |\zeta_1 + . \ . \ . + \zeta_n| \leq |\zeta_1| + . \ . \ . + |\zeta_n| = n$ it follows that
2. $\mathbb{E}\left(\left|{\xi_n}^2 - n\right|\right) \leq \mathbb{E}\left({\xi_n}^2\right) + n \leq n^2 + n \leq \infty$, Therefore ${\xi_n}^2 - n$ is integrable for each… n

3. $\mathbb{E}(\xi_{n+1}^2 | \mathcal{F}_n) = \mathbb{E}(\xi_n^2 | \mathcal{F}_n) + 2\mathbb{E}(\xi_n\zeta_{n+1} | \mathcal{F}_n) + \mathbb{E}(\zeta_{n+1}^2 | \mathcal{F}_n)$

$$= \xi_n^2 + 2\xi_n\mathbb{E}(\zeta_{n+1} | \mathcal{F}_n) + \mathbb{E}(\zeta_{n+1}^2)$$

$$= \xi_n^2 + 2\mathbb{E}(\zeta_{n+1})\,\xi_n + \mathbb{E}(\zeta_{n+1}^2) = \xi_n^2 + 1.$$

This simply means that $\mathbb{E}(\xi_{n+1}^2 - (n+1) | \mathcal{F}_n) = \xi_n^2 - n - 1 + 1 = \xi_n^2 - n$ therefore $\xi_n^2 - n$ is a Martingale.

3.4 Game of Chance: Fair and Unfair Games

For instance, suppose you take part in a game roulette, Let $\xi_1, \xi_2, . . .$ be a sequence of integrable random variables where ζ_n are your winnings or loses per unit stake in a game n. If your stake in each game is one, then your total winning after n games will be $\xi_n = \zeta_1 + \zeta_2 + . . . + \zeta_n$. We take filtration $\mathcal{F}_n = \sigma(\zeta_1, \zeta_2, . . ., \zeta_n)$, also we take $\xi_0 = 0$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

If $n - 1$ rounds of the game have been played so far your accumulated knowledge will be represented by the $\sigma - algebra\ \ \mathcal{F}_{n-1}$.

The game is called a fair game if

$$\mathbb{E}(\xi_n | \mathcal{F}_{n-1}) = \xi_{n-1}.$$

That is to say you expect that your fortune at step $\boldsymbol{n}$ will on averagely be the same as at step $n - 1$.

The game will be favourable to you if

$\mathbb{E}(\xi_n | \mathcal{F}_{n-1}) \geq \xi_{n-\mathbf{1}}$. Thus ξ_n is a **Sub-Martingale** and the game is unfavorable to you if ξ_n is **Super-Martingale**, $\mathbb{E}(\xi_n | \mathcal{F}_{n-\mathbf{1}}) \leq \xi_{n-\mathbf{1}}$ for $n = 1, 2, . . .$ with respect to filtration $\mathcal{F}_n$.

Definition 3.3: A gambling strategy $\alpha_1, \alpha_2, . . .$ with respect to filtration $\mathcal{F}_1, \mathcal{F}_2 . . .$ is a sequence of random variables such that α_n is $\mathcal{F}_{n-\mathbf{1}} - measurable$ for each $n = 1, 2, \ldots$ where $\mathcal{F}_0 = \{\emptyset, \Omega\}$. If you follow a strategy $\alpha_1, \alpha_2, . . .$ then your winning after n games will be

$\eta_n = \alpha_1\zeta_1 + \alpha_2\zeta_2 + . . . + \alpha_n\zeta_n$

$= \alpha_1(\xi_1 - \xi_0) + . . . + \alpha_n(\xi_n - \xi_{n-1})$

3.5 A Fundamental Principle: You can't beat the system

Let $\alpha_1, \alpha_2, . . .$ be a gambling strategy.

1. If $\alpha_1, \alpha_2, . . .$ is a bounded sequence and $\xi_0, \xi_1, \xi_2, . . .$ is a Martingale, then $\eta_0, \eta_1, \eta_2, . . .$ is a Martingale (a fair game will always result to a fair one no matter what you do).
2. If $\alpha_1, \alpha_2, . . .$ is a non-negative bounded sequence and $\xi_0, \xi_1, \xi_2, . . .$ is a supermartingale, then $\eta_0, \eta_1, \eta_2, . . .$ is a supermartingale. That is, an unfavourable game will always result to unfavorable game irrelevant of what you do.
3. If $\alpha_1, \alpha_2, . . .$ is a non-negative bounded sequence and $\xi_0, \xi_1, \xi_2, . . .$ is a sub-martingale, then $\eta_0, \eta_1, \eta_2, . . .$is a submartingale (a favourable game turns to a favourable one always).

3.6 Stopping Times

Naturally every gambler or someone engaging in a certain betting should have a plan that helps him to regulate himself on when to stop which could be based on a specific condition, since the possibility of him to continue for a very long period of time might be very small because he may definitely run out of chips at some time. Therefore he should know the time to stop which is his stopping time.

Now the stopping decision on this wise can be modeled as a random variable which is measured with respect to the $\sigma - algebra$ which gives the necessary available information at each time t and this idea brought up the need for a stopping time.

Definition 3.4: A map $\tau: \Omega \rightarrow \{0, 1, 2, . . . ; \infty\}$ is called a Stopping Time if

a) $\{ \tau \leq n\} = \{\omega: \tau(\omega) \leq n\} \in \mathcal{F}_n, \quad \forall n \leq \infty.$

Equivalently

b) $\{ \tau = n\} = \{\omega: \tau(\omega) = n\} \in \mathcal{F}_n, \quad \forall n \leq \infty.$

Proof: Let us prove the implication

$\boldsymbol{a}) \to \boldsymbol{b})$

If τ has property a), then $\{ \tau \leq n\} \in \mathcal{F}_n$ and $\{ \tau \leq n-1\} \in \mathcal{F}_{n-1} \subset \mathcal{F}_n$,

so$\{ \tau = n\} = \{ \tau \leq n\} \backslash \{ \tau \leq n-1\} \in \mathcal{F}_n$.

$(\boldsymbol{b}) \to (\boldsymbol{a})$ If τ has property (b), then $\{ \tau = k\} \in \mathcal{F}_k \subset \mathcal{F}_n$,

Therefore $\{ \tau \leq n\} = \{ \tau = 1\} \cup \ldots \cup \{\tau = n\}\} \in \mathcal{F}_n$

Example: Every constant function from Ω into $\{1,2,\ldots\infty\}$ is a stopping time with respect to any filtration $\mathcal{F}_t$ in any σ-algebra on Ω: For instance, $\tau = 1$ is a Stopping time which gives us information to immediately terminate all. At the other hand is when $\tau = \infty$: This is another stopping time which implies that one could never terminate or stop. Generally, if τ is a stopping time with respect to $\mathcal{F}_t$ and t is any positive integer, then $\omega \to \min(\tau(\omega), t)$ is a stopping time with respect to $\mathcal{F}_t$. Clearly, this stopping time explains that the stopping decision must be taken not later than the appointed period t.

3.7 First Hitting time

Suppose that a coin is tossed repeatedly and you win or lose $\mathcal{F}_1$, depending on which way it lands. Suppose that you start the game with for instance say $\mathcal{F}_5$in your pocket and decide to play until you have $\mathcal{F}_{10}$ or until you lose everything. If ξ_n is the amount you have at step n, then the time when you stop the game is $\tau = \min\{n: \xi_n = 10 \; or \; 0\}$ and it's called the **FIRST HITTING TIME** (of 10 or 0 by the random sequence ξ_n). τ is a stopping time with respect to the filtration $\mathcal{F}_n = \sigma(\xi_1, \xi_2, \ldots, \xi_n)$ because $\{ \tau = n\} = \{0 < \xi_1 < 10\} \cap \ldots \cap \{0 < \xi_{n-1} < 10\} \cap \{\xi = 10\}$ or Each of the sets on the right hand side belongs to $\mathcal{F}_n$, so their intersection. This proves that $\{ \tau = n\} \in \mathcal{F}_n$ for each $n \,\&\, \tau$ is a stopping time.

Definition 3.5: We call $\xi_{\tau\wedge n}$ the sequence stopped at τ, which is often denoted by ξ_n^τ. Thus, for each $\omega \in \Omega, \xi_n^\tau(\omega) = \xi_{\tau(\omega)\wedge n}(\omega)$, where $\tau \wedge n = \min(\tau, n)$ and $\xi_{\tau\wedge n}$ is a stopped process at τ.

Proposition 3.1: Let τ be a stopping time.

1. If ξ_n is a Martingale, then $\xi_{\tau\wedge n}(\mathbb{E}(\xi_{\tau\wedge n}) = \mathbb{E}(\xi_0)$ is also a Martingale.
2. If ξ_n is a Supermartingale, then so is $\xi_{\tau\wedge n}$, in particular, $(\mathbb{E}(\xi_{\tau\wedge n}) \leq \mathbb{E}(\xi_0)\ \forall n$.
3. If ξ_n is a Submartingale, then $(\mathbb{E}(\xi_{\tau\wedge n}) \geq \mathbb{E}(\xi_0))$ is a submartingale also.

3.8 Optional Stopping Theorem

If ξ_n is a Martingale, then particularly $\mathbb{E}(\xi_n) = \mathbb{E}(\xi_1)\ for\ each\ n.$

Example 3.5: Show that $\mathbb{E}(\xi_\tau)$ is not necessarily equal to $\mathbb{E}(\xi_1)$ for a stopping time τ. However, if the equality $\mathbb{E}(\xi_\tau) = \mathbb{E}(\xi_1)$ does hold, it can be very useful. The optional stopping theorem provides sufficient conditions for this to happen.

Theorem3.1: **(Optional Stopping Theorem)**

Let ξ_n be a Martingale and τ a stopping time with respect to a filtration $\mathcal{F}_n$ such that the following conditions hold:

1. $\tau < \infty$ almost surely
2. ξ_τ is integrable
3. $\mathbb{E}\left(\xi_n 1_{\{\tau > n\}}\right) \to 0$ as $n \to \infty$

   Then $\mathbb{E}(\xi_\tau) = \mathbb{E}(\xi_1)$.

Proof: Because $\mathbb{E}(\xi_\tau) = \mathbb{E}(\xi_{\tau\wedge n}) + (\xi_\tau - \xi_n)1_{\{\tau>n\}}$, it follows that

$$\mathbb{E}(\xi_\tau) = \mathbb{E}(\xi_{\tau\wedge n}) + \mathbb{E}\left(\xi_\tau 1_{\{\tau>n\}}\right) - \mathbb{E}\left(\xi_n 1_{\{\tau>n\}}\right).$$

Since $\xi_{\tau\wedge n}$ is a martingale by proposition, the first term on the right hand side is equal to $\mathbb{E}(\xi_{\tau\wedge n}) = \mathbb{E}(\xi_1)$ the last term tends to zero by assumption (3).

The term (middle term) $\mathbb{E}\left(\xi_n 1_{\{\tau>n\}}\right) = \sum_{k=n+1}^{\infty} \mathbb{E}\left(\xi_k 1_{\{\tau>k\}}\right)$ tends to zero as $n \to \infty$ since the series $\mathbb{E}(\xi_\tau) = \sum_{k=1}^{\infty} \mathbb{E}\left(\xi_k 1_{\{\tau=k\}}\right)$ is convergent and by assumption (2), it follows that $\mathbb{E}(\xi_\tau) = \mathbb{E}(\xi_1),$ as required.

Example 3.6: (Expectation of the first Hitting time for a random walk)

Let ξ_n be a symmetric random walk and let K be a positive integer. We define the first hitting time of $\pm K\ by\ \xi$ to be $\tau = \min\{n: |\xi_n| = K\}$.

It is obvious that τ is a stopping time and it has been showed that $\xi_n^2 - n$ is a martingale.

If the Optional Stopping Theorem can be applied, Then $\mathbb{E}(\xi_\tau^2 - \tau) = \mathbb{E}(\xi_1^2 - 1) = 0$ since $\mathbb{E}(\xi_1^2) = 1$, Hence, $\mathbb{E}(\tau) = \mathbb{E}(\xi_\tau^2) = K^2$, since $|\xi_\tau| = K$.

Now, verifying the conditions of the Optional Stopping Theorem.

1. Show that $P(\boldsymbol{\tau} = \infty) = 0$, we may estimate $P(\tau > 2Kn)$. We can think of $2Kn$ tosses of a coin as n sequences of $2k$ tosses. A necessary condition for $\tau > 2Kn$ is that no one of these n contains heads only therefore $P(\tau > 2Kn) \leq (1 - \frac{1}{2^{2k}})^n \to 0$ as $n \to \infty$. Because $(\tau > 2Kn)$ for $n = 1, 2,. . .$ is a contracting sequence of sets and it follows $P(\boldsymbol{\tau} = \infty) = P \cup (\tau > 2Kn) \leq \sum_{n=1}^{\infty} P(\tau > 2Kn)$ contracting sequence. $\mathbf{lim}_{n\to\infty} P(\tau > 2Kn) = 0$ (by Borel-Cantelli).
2. We need to show that $\mathbb{E}(\xi_\tau^2 - \tau) < \infty$ Indeed

$\mathbb{E}(\tau) =$

$\sum_{n=1}^{\infty} nP(\tau = n) \sum_{n=1}^{\infty} \sum_{k=1}^{2k} (2kn + k)\, P(\tau = 2kn + k) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{2k} 2k(n + 1)\, P(\tau > 2kn) \leq \sum_{k=1}^{\infty} \left(\sum_{k=1}^{2k} 2k\right)(n + 1) P(\tau = 2kn) = 4k^2 \sum_{n=1}^{\infty} (n + 1)\, P(\tau >$

$2kn) \leq 4k^2 \sum_{n=1}^{\infty}(n+1)\,(1-\frac{1}{2^{2k}})^n < \infty.$

Since $1-\frac{1}{2^{2k}} < 1$. The above series is therefore convergent moreover, $(\xi_\tau^2) = K^2$ so $\mathbb{E}(|\xi_\tau^2 - \tau|) \leq \mathbb{E}(\xi_\tau^2) + \mathbb{E}(\tau) = K^2 + \mathbb{E}(\tau) < \infty.$

3. Since $\xi_n^2 \leq K^2$ on $(\tau > n)$, $\mathbb{E}\big(\xi_n^2 1_{\{\tau>n\}}\big) \leq K^2 P\{\tau > n\} \to 0$ as $n \to \infty$ Moreover, $\mathbb{E}\big(n1_{\{\tau>n\}}\big) \leq$, $\mathbb{E}\big(\tau 1_{\{\tau>n\}}\big) = \mathbb{E}(\tau)P(\tau > n) < \infty$ as $n \to \infty$ it then follows that $\mathbb{E}\left((\xi_n^2 - n)1_{\{\tau>n\}}\right) \to \infty.$

3.9 Martingale Inequality and Convergence

We begin with classical inequalities for martingales, called the DOOB INEQUALITIES, which are useful in studying convergence of Martingales and later the properties of stochastic integrals, then we present a classical result known as Doob's Martingale Convergence Theorem, which provides the limit $\lim_n \xi_n$ of a Martingale.

3.9.1 Doob's Decomposition Theorem

Let $(\Omega, \mathcal{F})$ be a measurable space and (Ω_t) a sub-martingale with respect to a filtration $(\mathcal{F}_t)$ in $\mathcal{F}$, then there exist two sequences (Y_t) and (Z_t) in $\mathbb{R}$ such that $(\Omega_t) = (Y_t) + (Z_t)$ where

1. (Y_t) is a martingale with respect to $(\mathcal{F}_t)$
2. $Z_1 \leq Z_2 \leq. . .$
3. Z_{t+1} is $\mathcal{F}_t$ $-$measurable for each m

Every submartingale can be split into a martingale and increasing predictable process.

Proposition 3.2: Doob's Maximal Inequality

Suppose that $\xi_n, n \in N$, is a non-negative submartingale with respect to filtration $\mathcal{F}_n$. then for any $\lambda > 0$, $\lambda P(max\xi_k \geq \lambda) \leq \mathbb{E}\left(\xi_n 1_{\{\max_{k\leq n} \xi_k \geq \lambda\}}\right)$, where 1_A is the characteristic function of a set A.

Proof: Let $\xi_n^* = \max_{k\leq n} \xi_k$for $\lambda > 0$, define $\tau = \min\{k \leq n: \xi_k \geq \lambda\}$, if there is a $k \leq n$ such that $\xi_k \geq \lambda$ and $\tau = n$ otherwise. Then τ is a stopping time that $\tau \leq n$ almost surely. Since ξ_n is a submartingale, ($\mathbb{E}(\xi_{n+1}|\mathcal{F}_n) \geq \xi_n$). So $\mathbb{E}(\xi_n) \geq \mathbb{E}(\xi_\tau)$

But $\mathbb{E}(\xi_\tau) = \mathbb{E}\left(\xi_\tau 1_{\{\xi_n^* \geq \lambda\}}\right) + \mathbb{E}\left(\xi_\tau 1_{\{\xi_n^* < \lambda\}}\right)$. Observe that if $\xi_n^* \geq \lambda$, then $\xi_\tau \geq \lambda$. (Since $\xi_n^* = \max_{k\leq n} \xi_k$). Moreover, if $\xi_n^* < \lambda$, then $\tau = n$ and so $\xi_\tau = \xi_n$ therefore $\mathbb{E}(\xi_n) \geq \mathbb{E}(\xi_\tau) \geq \lambda P(\xi_n^* \geq \lambda) + \mathbb{E}\left(\xi_n 1_{\{\xi_n^* < \lambda\}}\right)$ it follows that $\lambda P(\xi_n^* \geq \lambda) \leq$ $\mathbb{E}(\xi_n) - \mathbb{E}\left(\xi_n 1_{\{\xi_n^* \geq \lambda\}}\right)$.

Theorem 3.2: (Doob's Maximal L^2 inequality)

If $\xi_n, n \in N$, is a non-negative square integrable submartingale with respect to filtration $\mathcal{F}_n$. Then $\mathbb{E}\left|\max_{k\leq n} \xi_k\right|^2 \leq 4\mathbb{E}\left|\xi_n\right|^2$

Proof: Put $\xi_n^* = \max_{k\leq n} \xi_k$. $\mathbb{E}|\xi_r^*|^2 = 2\int_0^\infty tP(\xi_n^* > t)dt$ and by proposition 1, Doob's Maximal Inequality $\leq 2\int_0^\infty \mathbb{E}\left(\xi_n 1_{\{\xi_n^* \geq t\}}\right)dt = 2\int_{\{\xi_n^* > t\}}^\infty (\xi_n dP)dt$ by Fubini

Theorem,

$= 2\int_\Omega \xi_n (\int_0^{\xi_n^*} dt)dP = 2\int_\Omega \xi_n\, \xi_n^* dP = 2\,\mathbb{E}(\int_\Omega \xi_n\, \xi_n^*)$ by Cauchy-Schwartz Inequality

$\leq 2(\mathbb{E}|\xi_n|^2)^{1/2}(\mathbb{E}|\xi_n^*|^2)^{1/2}$, that is $\mathbb{E}\,|\xi_n^*|^2 \leq 2(\mathbb{E}|\xi_n|^2)^{1/2}(\mathbb{E}\,|\xi_n^*|^2)^{1/2}$

$\mathbb{E}\,(|\xi_n^*|^2)^{1/2} \leq 2(\mathbb{E}|\xi_n|^2)^{1/2} \Longrightarrow \mathbb{E}\,(|\xi_n^*|^2) \leq 4\,\mathbb{E}|\xi_n|^2$.

Definition 3.7: Given an adapted sequence of random variables $\xi_1, \xi_2, \ldots$ and two real numbers a$< b$, we define a gambling strategy $\varphi_1, \varphi_2, \ldots$ by putting $\varphi_1 = 0$ for $n = 1,2, \ldots$

$$\varphi_{n+1} = \begin{cases} 1, & if\, \varphi_n = 0 \;\; and\; \xi_n < a \\ 1, & if\; \varphi_n = 1 \;\; and\; \xi_n \le b \\ 0, & otherwise \end{cases}$$

It will be called the Upcrossings strategy. Each $k = 1,2, \ldots$ such that $\varphi_k = 1$ and $\varphi_{k+1} = 0$ will be called an Upcrossings of the interval $[a, b]$

The Upcrossings form a (finite or infinite) increasing sequence $U_1 < U_2 < \cdots$

The number of Upcrossings made up to time n, that is the largest k such that $U_k \le n$ will be denoted by $U_n[a, b]$, (we put $U_n[a, b] = 0$ if no such k exists.

We refrain from playing the game and wait until ξ_n becomes less than a. As soon as, this happens, we start playing unit stakes at each round of the game and continue until ξ_n becomes greater than b.

The strategy φ_n is defined in such a way that $\varphi_n = 0$ whenever we refrain from playing the n^{th} games and $\varphi_n = 1$ otherwise.

Proposition 3.3: The Upcrossings strategy φ_n is a gambling strategy.

Proof: Prove that φ_n is $\mathcal{F}_{n-1} - measure$ for each n, using induction

Lemma. (Upcrossings Inequality)

If $\xi_1, \xi_2, . . .$ is a supermartingale and $a < b$, then

$$(b - a)\mathbb{E}(U_n[a, b] \le \mathbb{E}((\xi_n - a)^{-})$$

Where $(\xi_n - a)^{-} = \max\{0, -(\xi_n - a)\}$

Proof: Let $\zeta_n = \alpha_1(\xi_1 - \xi_0) + \alpha_2(\xi_2 - \xi_1) + . . . + \alpha_n(\xi_n - \xi_{n-1})$ be the total winnings at step $n = 1,2, \ldots$ if the Upcrossings strategy is followed.

We put $\zeta_0 = 0$. Since $\xi_1, \xi_2, . . .$ is a super-martingale, by proposition, the (Fundamental Principle), ζ_n is a supermartingale.

Let us fix an n and put $k = U_n[a, b]$ so that $0 < U_1 < U_2 < \ldots < U_k \le n$.

Each Upcrossings increases the total winnings by at least $b - a$,

$\zeta_{U_i} - \zeta_{U_i - 1} \ge b - a$ for $i = 1, \ldots, k$

(We put $U_0 = 0$ for simplicity)

Moreover, $\zeta_n - \zeta_{U_k} \geq -(\xi_n - a)^-$,

it follows that $\zeta_n \geq (b-a)U_n[a,b] - (\xi_n - a)^-$ (fundamental inequality)

Taking expectation on both sides,

$$\mathbb{E}(\zeta_n) \geq (b-a)\mathbb{E}(U_n[a,b]) - \mathbb{E}(\xi_n - a)^-$$

Since ζ_n is a supermartingale $0 = \mathbb{E}(\zeta_1) \geq \mathbb{E}(\zeta_n)$,

hence $(b-a)\mathbb{E}(U_n[a,b]) \leq \mathbb{E}((\xi_n - a)^-)$

3.9.2 Doob's Martingale Convergence Theorem.

Theorem 3.3: (Doob's MCT)

Suppose that $\xi_1, \xi_2, . . .$ is a supermartingale with respect to a filtration $\mathcal{F}_1, \mathcal{F}_2, . . .$ such that $\sup_n \mathbb{E}(|\xi_n|) < \infty$ Then there is an integrable random variable ξ such that $\lim_{n\to\infty} \xi_n = \xi$ almost surely.

Proof. (Brzezniak, Zastawniak)

Remark: The theorem is valid for Martingales because every Martingale is a supermartingale, it is also valid for supermartingale, since ξ_n is a submartingale if and only if $-\xi_n$ is a supermartingale.

3.9.3 Uniform Integrability and L^1 Convergence of Martingales

The conditions of the Doob's theorem imply pointwise almost surely convergence of martingales. Here we study convergence in L^1.

Definition 3.6: A sequence $\xi_1, \xi_2, . . .$ of random variables is called uniformly integrable if for every $\varepsilon > 0$ there exist an $\mu > 0$ such that $\int_{\{|\xi|>\mu\}} |\xi| dP < \infty$ for all $n = 1, 2, . . .$

Proposition 3.4: Uniform integrability is a necessary condition for a sequence $\xi_1, \xi_2, . . .$ of integrable random variables to converge in L^1.

Lemma: If ξ is integrable, then for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$P(A) < \delta \Rightarrow \int_A |\xi| dP < \varepsilon$$

Proof: Let $\varepsilon > 0$. Since ξ is integrable, by proposition, there is a $\mu > 0$ such that

$$\int_{\{|\xi|>\mu\}} |\xi| dP < \frac{\varepsilon}{2}$$

Now

$$\int_A |\xi| dP = \int_{A\cap\{|\xi|>\mu\}} |\xi| dP + \int_{A\cap\{|\xi|>\mu\}} |\xi| dP \leq \int_A \mu dP + \int_{\{|\xi|>\mu\}} |\xi| dP < \mu P(A) + \frac{\varepsilon}{2}$$

Let $\delta = \frac{\varepsilon}{2\mu}$, then $P(A) < \delta \Rightarrow \int_A |\xi| dP < \varepsilon$ as required.

Proposition 3.5: A uniform integrable sequence of random variables is bounded in L^1, i.e. $\sup_n \mathbb{E}(|\xi_n|) < \infty$

Proof: Because ξ_n is a uniformly integrable sequence, there is a $\mu > 0$ such that for all n $\int_{\{|\xi|>\mu\}} |\xi| dP < 1$ it follows that

$\mathbb{E}(|\xi_n|) = \int_{\{|\xi_n|>\mu\}} |\xi_n| dP + \int_{\{|\xi_n|\leq\mu\}} |\xi_n| dP < 1 + \mu P\{|\xi_n| \leq \mu\} < 1 + \mu < \infty$ for all n, proving that ξ_n is a bounded sequence in L^1.

Theorem 3.4: Every uniform integrable supermartingale/submartingale ξ_n converges in L^1.

Theorem 3.5: Let ξ_n be a uniform integrable martingale, then $\xi_n = \mathbb{E}(\xi|\mathcal{F}_n)$, where $\xi = \lim_n \xi_n$ is the limit of ξ_n in L^1 and $\mathcal{F}_n = \sigma(\xi_1, \xi_2, . . . , \xi_n)$ is the filtration generated by ξ_n.

Proof: For any $m > n$

$\mathbb{E}(\xi_m|\mathcal{F}_n) = \xi_n$ i.e. for any $A \in \mathcal{F}_n$ $\int_A \xi_m dP = \int_A \xi_n dP$.

Let n be an arbitrary integer and let $A \in \mathcal{F}_n$, for any $m > n$

$\left|\int_A (\xi_n - \xi) dP\right| = \left|\int_A (\xi_m - \xi) dP\right| \le \left|\int_A (\xi_m - \xi) dP\right| \le \mathbb{E}(|\xi_m - \xi|) \to 0$ as $m \to \infty$

it follows that $\int_A |\xi_n| dP = \int_A \xi dP$ for any $A \in \mathcal{F}_n$, so $\xi_n = \mathbb{E}(\xi|\mathcal{F}_n)$.

Theorem 3.6: (Kolmogorov's 0-1 law)

Let $\eta_1, \eta_2, . . . \eta_n$ be a sequence of independent random variables. we define the tail $\sigma -$Algebra

$\mathrm{T} = \mathrm{T}_1 \cap \mathrm{T}_2 \cap . . .$, where $\mathrm{T}_n = \sigma(\eta_n, \eta_{n+1}, . . .)$ then $P(A) = 0$ or 1 for any $A \in \mathrm{T}$

Proof: Take any $A \in \mathrm{T}$ and define $\xi_n = \mathbb{E}(1_A|\mathcal{F}_n)$, where $\mathcal{F}_n = \sigma(\eta_1, \eta_2, . . . \eta_n)$ By Theorem 3.4, ξ_n is a uniform integrable martingale, so $\xi_n \to \xi$ in L^1. By Theorem 3.5,

$\mathbb{E}(\xi|\mathcal{F}_n) = \mathbb{E}(1_A|\mathcal{F}_n)$ for all n. Both $\xi = \lim_n \xi_n$ and 1_A are measurable with respect to the $\sigma -$Algebra $\mathcal{F}_\infty = \sigma(\eta_1, \eta_2, . . .)$ because this $\sigma -$Algebra is generated by the family of sets $\mathcal{F}_1 \cup \mathcal{F}_2 \cup . . .$, it follows that $\xi = 1_A$ almost surely. Since η_n is a sequence of independent random variables, the $\sigma -$Algebra $\mathcal{F}_n$ and T_{n+1} are independent. Because $\mathrm{T} \subset \mathrm{T}_{n+1}$, the $\sigma -$Algebra $\mathcal{F}_n$ and T are independent. Being $\mathrm{T} -$mesurable 1_A is therefore independent of $\mathcal{F}_n$ for any n. This means that $\xi_n = \mathbb{E}(1_A|\mathcal{F}_n) = \mathbb{E}(1_A) = P(A)$ as therefore the $\lim_{n\to\infty} \xi_n = \xi$ is also constant and equal to $P(A)$ almost surely. This means that $P(A) = 1_A$ almost surely, so $P(A) = 0$ or 1. Since as $n \to \infty$ $\lim \xi_n = \mathbb{E}(1_A|\mathcal{F}_n) = P(A)$ thus 1_A is $\mathcal{F}_\infty -$measurable.

Chapter 4

CONCLUSION

The martingale system as a betting strategy is a very interesting system in probability theory as a whole as discuss. With the proper knowledge of the martingale system, it can easily be applied to different aspects and areas of life. Since generally most circumstances warrants us to gamble with whatsoever we are engaging in since we might not be able to predict the outcome of the future occurrences, therefore the need of the proper knowledge and understanding of the martingale system is needed. There are few situations we can apply the martingale system to, so as to enhance the knowledge of applications to real situations and not just a theoretical propaganda. One of the areas of application is the prediction of the market prices which is briefly analyzed below.

Consequently, a sequence $\eta = \eta_0, . . . , \eta_t$ is a martingale with respect to a random sequence $\xi = \xi_0, . . . \xi_t$ if for all $n \geq 0$ $\mathbb{E}(\eta_t | \xi_0, . . . , \xi_t) = \eta_n$ is valid then for the market prediction, we can assume that ξ is a random sequence of the price unit, we can define η such that $\eta_t = \sum_{i=0}^{t} \xi_i$, then the sequence η which is the price is a martingale. This is an implication from the price at a given point representing the consensus probability that the said event will definitely happen or definitely occur. Therefore it is the fair price for the proposed gamble. The expectation of the price in the coming future with the little information obtained is definitely equivalent to the present price.

The most important property of the martingale system that's explains the process is the fact that $\mathbb{E}(\eta_t) = \mathbb{E}(\eta_0)$ for all $n \geq 0$. It can be explained further more by the property of expectation which is $\mathbb{E}(\eta_t) = \mathbb{E}(\mathbb{E}(\eta_t|\xi_0, . . \ ., \xi_{t-1}) = \mathbb{E}(\eta_{t-1})$. If this process is repeatedly iterated, it yields the desire equality, although it is applicable only to a constant time.

The Martingale Theory is a good example of a stochastic process which is applicable to our daily life. By studying and understanding the concept, we have and know more about the expectations and possible outcomes of future predictions. Therefore the knowledge of the Martingale Theory cannot be neglected nor overemphasized.

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