Tunneling of Massive Vector Particles From Rotating Charged Black Strings

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We study the quantum tunneling of charged massive vector bosons from a charged static and a rotating black string. We apply the standard methods, first we use the WKB approximation and the Hamilton-Jacobi equation, and then we end up with a set of four linear equations. Finally, solving for the radial part by using the determinant of the metric equals zero, the corresponding tunneling rate and the Hawking temperature is recovered in both cases. The tunneling rate deviates from pure thermality and is consistent with an underlying unitary theory.

I. INTRODUCTION

In his seminal paper [1], Steven Hawking showed that black holes radiate thermally due to the quantum effects and this radiation is known as Hawking radiation. Thus, for the first time, it has been established a relation between thermodynamics and space-time geometry. Furthermore, the entropy of the black hole is shown to be proportional to the surface area of the black hole.

Besides the Hawking's original method, today there exists a number of different approaches deriving the Hawking temperature [2–4]. The tunneling method [5–13], has been studied in details and shown to be very successful for calculating the Hawking temperature for different types of particles emitted from static as well as stationary space-time metrics [14–18]. Hawking temperature depends on the black hole mass M, charge Q and angular momentum J, using the tunneling approach, it is also shown that, the Hawking temperature for a particular black hole configuration remains unaltered and unaffected by the nature of particles emitted from the black hole. Moreover, the radiation spectrum is shown to deviate from pure thermality due to the conservation of energy, and hence the theory is consistent with an underlying unitary theory.

Due to the non-linearity of the Einstein's field equations it is very difficult to find exact solutions. However, apart from the standard solutions characterized with spherical symmetry, solutions with cylindrical symmetry have also been found, such solutions are known as cylindrical black holes or black strings [30, 31]. The tunneling of scalar and Dirac particles from charged static/rotating black string has been also investigated [26–29]. Recently, the tunneling of massive spin-1 particles has attracted interest [19–25]. Therefore, in this paper, we aim to study the tunneling of massive vector bosons $W^{\pm}$(spin-1 particles) from the space-time of a charged static and a rotating black string. First, we derive the field equations by using the Lagrangian given by the Glasgow-Weinberg-Salam model. We then use the WKB approximation and the separation of variables which results with a set of four linear equations, solving for the radial part by using the determinant of the metric equals zero, we found the tunneling rate and the corresponding Hawking temperature in both cases.

The paper is organized as follows. In Sec. II, we investigate the tunneling of massive vector particles from the static charged black strings and calculate the corresponding tunneling rate and the Hawking temperature. In Sec. III, we extend our calculations for the case of tunneling of massive vector particles from a rotating charged black string. In Sec. IV, we comment on our results.

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II.   TUNNELING FROM STATIC CHARGED BLACK STRINGS

We can begin by writing the Einstein-Hilbert action with a negative cosmological constant in the presence of an electromagnetic field given by

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\,(R - 2\Lambda) - \frac{1}{16\pi} \int d^4x \sqrt{-g} F^{\mu\nu} F_{\mu\nu}, \tag{1}$$

where the Maxwell electromagnetic tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2}$$

If one takes into account the cylindrical symmetries of the space-time, then the line element for a static charged black string with negative cosmological constant in the presence of electromagnetic fields is shown to be [30, 31]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\theta^2 + \alpha^2 r^2 dz^2, \tag{3}$$

where

$$f(r) = \alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2}, \tag{4}$$

and

$$\alpha^2 = -\frac{1}{3}\Lambda, \quad b = 4GM, \quad c^2 = 4GQ^2. \tag{5}$$

Solving for $\alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2} = 0$, one can easily find the outer horizon given by [26]

$$r_+ = \frac{b^{\frac{1}{3}}\sqrt{s} + \sqrt{2\sqrt{s^2 - 4p^2} - s}}{2\alpha}, \tag{6}$$

where

$$s = \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\left(\frac{4p^2}{3}\right)^3}\right)^{\frac{1}{3}} + \left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\left(\frac{4p^2}{3}\right)^3}\right)^{\frac{1}{3}}, \tag{7}$$

$$p^2 = \frac{c^2}{b^{\frac{4}{3}}}. \tag{8}$$

Let us now write the Lagrangian density which describes the $W^\pm$-bosons in a background electromagnetic field given by [19]

$$\mathcal{L} = -\frac{1}{2}\left(D_\mu^+ W_\nu^+ - D_\nu^+ W_\mu^+\right)\left(D^{-\mu}W^{-\nu} - D^{-\nu}W^{-\mu}\right) + \frac{m_W^2}{\hbar^2}W_\mu^+ W^{-\mu} - \frac{i}{\hbar}e F^{\mu\nu}W_\mu^+ W_\nu^-, \tag{9}$$

where $D_{\pm\mu} = \nabla_\mu \pm \frac{i}{\hbar}eA_\mu$ and ∇_μ is the covariant geometric derivative. Also, e gives the charge of the W^+ boson, A_μ is the electromagnetic vector potential of the black string given by $A_\mu = (-h(r), 0, 0, 0)$, here $h(r) = 2Q/\alpha r$, where Q is the charge of the black string. Using the above Lagrangian the equation of motion for the W-boson field reads

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\left(D^{\pm\nu}W^{\pm\mu} - D^{\pm\mu}W^{\pm\nu}\right)\right] \pm \frac{ieA_\mu}{\hbar}\left(D^{\pm\nu}W^{\pm\mu} - D^{\pm\mu}W^{\pm\nu}\right) + \frac{m_W^2}{\hbar^2}W^{\pm\nu} \pm \frac{i}{\hbar}eF^{\mu\nu}W_\mu^\pm = 0 \tag{10}$$

where $F^{\mu\nu} = \nabla^\mu A^\nu - \nabla^\nu A^\mu$. In this work, we will investigate the tunneling of W^+ boson, therefore one needs to solve the following equation

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\alpha}g^{\nu\beta}\left(\partial_\beta W_\alpha^+ - \partial_\alpha W_\beta^+ + \frac{i}{\hbar}eA_\beta W_\alpha^+ - \frac{i}{\hbar}eA_\alpha W_\beta^+\right)\right] \tag{11}$$

$$+ \frac{ieA_\mu g^{\mu\alpha}g^{\nu\beta}}{\hbar}\left(\partial_\beta W_\alpha^+ - \partial_\alpha W_\beta^+ + \frac{i}{\hbar}eA_\beta W_\alpha^+ - \frac{i}{\hbar}eA_\alpha W_\beta^+\right) + \frac{m_W^2 g^{\nu\beta}}{\hbar^2}W_\beta^+ + \frac{i}{\hbar}eF^{\nu\alpha}W_\alpha^+ = 0,$$

for $\nu = 0, 1, 2, 3$. Using the WKB approximation

$$W_\mu^+(t, r, \theta, z) = C_\mu(t, r, \theta, z) \exp\left(\frac{i}{\hbar} S(t, r, \theta, z)\right), \tag{12}$$

where the action is given by

$$S(t, r, \theta, z) = S_0(t, r, \theta, z) + \hbar\, S_1(t, r, \theta, z) + \hbar^2 S_2(t, r, \theta, z) + \dots \tag{13}$$

We can now use the last three equations and neglect the terms of higher order of $\hbar$, then one can find the following set of four equations:

$$
\begin{aligned}
0 =\ & C_0 \left(-(\partial_1 S_0)^2 - \frac{(\partial_2 S_0)^2}{r^2 f} - \frac{(\partial_3 S_0)^2}{\alpha^2 r^2 f} - \frac{m^2}{f}\right) + C_1 \left((\partial_1 S_0)(eA_0 + \partial_0 S_0)\right) + C_2 \left(\frac{(\partial_2 S_0)}{r^2 f}(\partial_0 S_0 + eA_0)\right) \\
& + C_3 \left(\frac{(\partial_3 S_0)}{\alpha^2 r^2 f}(\partial_0 S_0 + eA_0)\right),
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
0 =\ & C_0 \left(-(\partial_1 S_0)(eA_0 + \partial_0 S_0)\right) + C_1 \left(-f\frac{(\partial_2 S_0)^2}{r^2} - f\frac{(\partial_3 S_0)^2}{\alpha^2 r^2} + (\partial_0 S_0 + eA_0)^2 - m^2 f\right) + C_2 \left(f\frac{(\partial_1 S_0)(\partial_2 S_0)}{r^2}\right) \\
& + C_3 \left(f\frac{(\partial_1 S_0)(\partial_3 S_0)}{\alpha^2 r^2}\right),
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
0 =\ & C_0 \left(-\partial_2 S_0 \left(\frac{\partial_0 S_0 + eA_0}{f}\right)\right) + C_1 \left(f(\partial_2 S_0)(\partial_1 S_0)\right) + C_2 \left(-f(\partial_1 S_0)^2 - \frac{(\partial_3 S_0)^2}{\alpha^2 r^2} + \frac{(\partial_0 S_0 + eA_0)^2}{f} - m^2\right) \\
& + C_3 \left(\frac{(\partial_2 S_0)(\partial_3 S_0)}{\alpha^2 r^2}\right),
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
0 =\ & C_0 \left(-\partial_3 S_0 \left(\frac{\partial_0 S_0 + eA_0}{f}\right)\right) + C_1 \left(f(\partial_3 S_0)(\partial_1 S_0)\right) + C_3 \left(-f(\partial_1 S_0)^2 - \frac{(\partial_2 S_0)^2}{r^2} + \frac{(\partial_0 S_0 + eA_0)^2}{f} - m^2\right) \\
& + C_2 \left(\frac{(\partial_2 S_0)(\partial_3 S_0)}{r^2}\right).
\end{aligned}
\tag{17}
$$

From the metric (3), it is clear that due to the space-time symmetries we can use the following ansatz for the action

$$S_0(t, r, \theta, z) = -Et + W(r) + J_1\theta + J_2 z + C, \tag{18}$$

where E, J_1, J_2 and C are constants. Therefore, the non-zero elements of the coefficient matrix Ξ are given by

$$
\begin{aligned}
\Xi_{11} &= -(W')^2 - \frac{J_1^2}{r^2 f} - \frac{J_2^2}{\alpha^2 r^2 f} - \frac{m^2}{f} \\
\Xi_{12} &= -\Xi_{21} = W'(eA_0 - E) \\
\Xi_{13} &= \frac{J_1}{r^2 f}(eA_0 - E) \\
\Xi_{14} &= \frac{J_2}{\alpha^2 r^2 f}(eA_0 - E) \\
\Xi_{22} &= \left(-f\frac{J_1^2}{r^2} - f\frac{J_2^2}{\alpha^2 r^2} + (eA_0 - E)^2 - m^2 f\right) \\
\Xi_{23} &= f\frac{W' J_1}{r^2} \\
\Xi_{24} &= f\frac{W' J_2}{\alpha^2 r^2} \\
\Xi_{31} &= -J_1\frac{(eA_0 - E)}{f}
\end{aligned}
$$

$$\Xi_{32} = fJ_1W'$$

$$\Xi_{33} = \left(-f(W')^2 - \frac{J_2^2}{\alpha^2 r^2} + \frac{(eA_0 - E)^2}{f} - m^2\right)$$

$$\Xi_{34} = \frac{J_1 J_2}{\alpha^2 r^2}$$

$$\Xi_{41} = -\frac{J_2(eA_0 - E)}{f}$$

$$\Xi_{42} = fJ_2W'$$

$$\Xi_{43} = \frac{J_1 J_2}{r^2}$$

$$\Xi_{44} = \left(-f(W')^2 - \frac{J_1^2}{r^2} + \frac{(eA_0 - E)^2}{f} - m^2\right). \tag{19}$$

The nontrivial solution of this equation [20]

$$\Xi(C_0, C_1, C_2, C_3)^T = 0, \tag{20}$$

is obtained by using the determinant of the matrix equals zero, $\det \Xi = 0$, it follows

$$m^2\left(-r^2(E - eA_0)^2\alpha^2 + f^2r^2\alpha^2(W')^2 + \left((m^2r^2 + J_1^2)\alpha^2 + J_2^2\right)f\right)^3 = 0. \tag{21}$$

Solving this equation for the radial part leads to the following integral

$$W_{\pm}(r) = \pm \int \frac{\sqrt{(E - eA_0)^2 - f(r)\left(m^2 + \frac{J_1^2}{r^2} + \frac{J_2^2}{\alpha^2 r^2}\right)}}{f(r)} dr. \tag{22}$$

Expanding the function $f(r)$ in Taylor's series near the horizon

$$f(r_+) \approx f'(r_+)(r - r_+), \tag{23}$$

and by integrating around the pole at the outer horizon r_+, gives

$$W_{\pm}(r) = \pm \frac{i\pi(E - eA_0)}{f'(r_+)}. \tag{24}$$

Now we can set the probability of the ingoing particle to 100% (since every outside particle falls into the black hole), it follows

$$P_- \simeq e^{-2ImW_-} = 1,$$

which implies $ImC = -ImW_-$. For the outgoing particle we have $ImS_+ = ImW_+ + ImC$, and also we make use of $W_+ = -W_-$, which leads to the probability for the outgoing particle given by

$$P_+ = e^{-2ImS} \simeq e^{-4ImW_+}. \tag{25}$$

In this way the tunneling rate of particles tunneling from inside to outside the horizon is given by

$$\Gamma = \frac{P_+}{P_-} \simeq e^{(-4ImW_+)}. \tag{26}$$

We can find the Hawking temperature simply by compering the last result with the Boltzmann factor $\Gamma = e^{-\beta E_{net}}$, where $E_{net} = (E - eA_0)$ and $\beta = 1/T_H$, yielding

$$T_H = \frac{f'(r_+)}{4\pi}. \tag{27}$$

Using Eqn.(4), one can recover the Hawking temperature for a static charged black string [26]

$$T_H = \frac{1}{4\pi}\left(2\alpha^2 r_+ + \frac{b}{\alpha r_+^2} - \frac{2c^2}{\alpha^2 r_+^3}\right). \tag{28}$$

III.   TUNNELING FROM ROTATING CHARGED BLACK STRINGS (RCBSS)

Lemos derived a rotating charged cylindrically symmetric exact solution of Einstein equations for a black string [30]. The line element for a RCBSs is given by [26]

$$ds^2 = -F(r)\,dt^2 + R^2(r)\,(N dt + d\theta)^2 + \frac{dr^2}{G(r)} + \alpha^2 r^2 dz^2, \tag{29}$$

where the lapse function F and the shift function N are given as

$$G = \left(\alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2} \right), \tag{30}$$

$$F = fG, \tag{31}$$

$$f = \left(\gamma^2 - \frac{\omega^2}{\alpha^2} \right)^2 \frac{r^2}{R^2}, \tag{32}$$

$$N = -\frac{\gamma\omega}{\alpha^2 R^2} \left(\frac{b}{\alpha r} - \frac{c^2}{\alpha^2 r^2} \right), \tag{33}$$

and

$$R^2 = \gamma^2 r^2 - \frac{\omega^2}{\alpha^4} \left(\alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2} \right). \tag{34}$$

Noted that the rotation parameter $a = J/M$, constant $\alpha^2 = -\Lambda/3$, where Λ is the cosmological constant, M is the ADM mass, Q is the charge of the black string, and J is the angular momentum. In addition, b and c are defined as

$$b = 4M \left(1 - \frac{3a^2\alpha^2}{2} \right), \tag{35}$$

$$c^2 = 4Q^2 \left(\frac{1 - 3a^2\alpha^2/2}{1 - a^2\alpha^2/2} \right). \tag{36}$$

Furthermore, γ^2 and ω^2/α^2 are defined as

$$\gamma^2 = \frac{2GM}{b} \pm \frac{2G}{b} \sqrt{M^2 - \frac{8J\alpha^2}{9}}, \tag{37}$$

$$\frac{\omega^2}{\alpha^2} = \frac{4GM}{b} \mp \frac{4G}{b} \sqrt{M^2 - \frac{8J\alpha^2}{9}}, \tag{38}$$

or

$$\gamma = \sqrt{\frac{1 - \frac{a^2\alpha^2}{2}}{1 - \frac{3a^2\alpha^2}{2}}}, \tag{39}$$

$$\omega = \frac{a\alpha^2}{\sqrt{1 - \frac{3a^2\alpha^2}{2}}}. \tag{39}$$

Let us now introduce the electromagnetic field associated with the vector potential of the RCBSs

$$A_\mu = (A_0, 0, A_2, 0) \tag{40}$$

where $A_0 = -\gamma h(r)$, $A_2 = \frac{\omega}{\alpha^2} h(r)$, and $h(r)$ is an arbitrary function of r for the line charge density along the z-line given by $Q = \frac{Q_z}{\Delta z} = \gamma \lambda$. To exactly reveal the massive vector particle's tunneling radiation, we should solve the Proca equation in Eqn.(11). Following the standard procedure, we use the WKB approximation Eqn.(12) with the action Eqn(13) in the background of the RCBSs spacetime and neglect the factors of higher orders of $\hbar$. Then using the following ansatz for the action

$$S_0 = -Et + W(r) + J_1\theta + J_2 z + k, \tag{41}$$

where E, J_1, J_2 and k are constants, we get four decoupled equations such as:

$$\frac{C_0}{fG^2R^2r^2\alpha^2}\Big[fG^3R^2r^2\alpha^2W'^2 + G\Big[\big((m^2r^2\alpha^2 + J_2^2)\,fG - r^2\,(eA_2 + J_1)\,N\alpha^2\,((eA_2 + J_1)\,N - eA_0 + E)\big)\,R^2$$

$$+r^2\alpha^2 fG\,(eA_2 + J_1)^2\Big]\Big] - ((eA_2 + J_1)\,N - eA_0 + E)\,\frac{W'}{fG}C_1 + \frac{(-erA_2 - J_1 r)}{fGr}\,((eA_2 + J_1)\,N - eA_0 + E)\,C_2$$

$$-\frac{C_3 J_2}{fG}\,((eA_2 + J_1)\,N - eA_0 + E) = 0, \tag{42}$$

$$\frac{C_0}{fGR^2\alpha^2r^2}\left(-\alpha^2 fG^2R^2 eA_0 r^2 W' + \alpha^2 R^2 fG^2W'r^2 E\right) + 2\Big[\alpha^2\big(((-N^2A_2 + NA_0)\,R^2 + fGA_2)\,e - R^2NE\big)\,r^2 J_1$$

$$+\frac{1}{2}\alpha^2 r^2\left(-R^2N^2 + fG\right)J_1^2 - \alpha^2 R^2 e\,(NA_2 - A_0)\,r^2 E$$

$$-\frac{1}{2}R^2\alpha^2r^2E^2 + \alpha^2r^2\left(\frac{1}{2}\left(m^2fG - e^2\,(NA_2 - A_0)^2\right)R^2 + \frac{1}{2}fGe^2A_2^2\right) + \frac{1}{2}R^2 fGJ_2^2\Big]\frac{C_1}{fGR^2\alpha^2r^2}$$

$$+\left[-\alpha^2 fG^2R^2A_2er^2W' - \alpha^2 G^2 fR^2W'r^2 J_1\right]\frac{C_2}{fGR^2\alpha^2r^2} - GJ_2W'C_3 = 0, \tag{43}$$

$$\Big[-\alpha^2\left(-fG^2R\left(-R^2N^2 + fG\right)rE - fG^2\left(N^2A_0 erR^2 - fGA_0er\right)R\right)rJ_1 - \alpha^2 RfG^2\left((reA_2N^2 - 2A_0erN)\,R^2 - fGA_2er\right)rE$$

$$-\alpha^2r^2 fG^2R^3NE^2 + \alpha^2 fG^2\left(e^2\,(NA_2 - A_0)\,A_0 rNR^2 - fGA_2A_0 e^2r\right)Rr\Big]\frac{C_0}{f^2G^3R^3r^2\alpha^2} + \Big[-\alpha^2r^2 fG^2R\left(-R^2N^2 + fG\right)W'J_1$$

$$-2\alpha^2 fG^2R\left(-\frac{1}{2}erN\,(NA_2 - A_0)\,R^2 + \frac{1}{2}fGA_2er\right)rW' + \alpha^2 R^3 fG^2Nr^2W'E\Big]\frac{C_1}{f^2G^3R^3r^2\alpha^2}$$

$$+\Big[-\alpha^2\left(fG^2R^3NrE - fG^2NA_0erR^3\right)rJ_1 + f^2G^3R^3J_2^2 + \alpha^2 fG^2\left(fGrm^2 + e^2\,(NA_2 - A_0)\,A_0 r\right)R^3r$$

$$-\alpha^2 R^3 fG^2\,(NA_2er - 2A_0er)\,rE - \alpha^2r^2 fG^2R^3E^2 + r^2\alpha^2 R^3 f^2G^4W'^2\Big]\frac{C_2}{f^2G^3R^3r^2\alpha^2}$$

$$+\left[-\alpha^2 r^2 fG^2 R\left(-R^2 N^2+fG\right)J_2 J_1+\alpha^2 R^3 r^2 fG^2 NJ_2 E-\alpha^2 fG^2 R\left(\left(-N^2 A_2+NA_0\right)R^2+fGA_2\right)er^2 J_2\right]C_3=0,$$
(44)

$$\left(-fG^2 A_0 erR+RrG^2 fE\right)\frac{J_2}{R\alpha^2 r^3 fG^2}C_0-\frac{W'J_2 C_1}{r^2\alpha^2}+\left(-RfG^2 A_3 er-J_1 RfG^2 r\right)\frac{J_2}{R\alpha^2 r^3 fG^2}C_2$$
(45)

$$-\Big[r\left(-R^2 N^2+fG\right)J_1^2-2\left(\left(\left(-N^2 A_2+NA_0\right)R^2+fGA_2\right)e-R^2 NE\right)rJ_1-rR^2 fG^2 W'^2+rR^2 E^2$$

$$+2erR^2\left(NA_2-A_0\right)E-\left(\left(m^2 fG-e^2\left(NA_2-A_0\right)^2\right)R^2+fGe^2 A_2^2\right)r\Big]\frac{C_3}{GrfR^2}=0.$$

Then the non-zero elements of the coefficient matrix Θ are calculated as following

$$\Theta_{11}=\Big[fG^3 R^2 r^2\alpha^2 W'^2+G\Big[\left(\left(m^2 r^2\alpha^2+J_2^2\right)fG-r^2\left(eA_2+J_1\right)N\alpha^2\left(\left(eA_2+J_1\right)N-eA_0+E\right)\right)R^2$$
(46)

$$+r^2\alpha^2 fG\left(eA_2+J_1\right)^2\Big]\Big],$$

$\Theta_{12}=-\left(\left(eA_2+J_1\right)N-eA_0+E\right)W',$

$\Theta_{13}=\left(-erA_2-J_1 r\right)\left(\left(eA_2+J_1\right)N-eA_0+E\right),$

$\Theta_{14}=-J_2\left(\left(eA_2+J_1\right)N-eA_0+E\right),$

$\Theta_{21}=\left(-\alpha^2 fG^2 R^2 eA_0 r^2 W'+\alpha^2 R^2 fG^2 W'r^2 E\right),$

$$\Theta_{22}=2\Big[\alpha^2\left(\left(\left(-N^2 A_2+NA_0\right)R^2+fGA_2\right)e-R^2 NE\right)r^2 J_1+\frac{1}{2}\alpha^2 r^2\left(-R^2 N^2+fG\right)J_1^2-\alpha^2 R^2 e\left(NA_2-A_0\right)r^2 E$$

$$-\alpha^2 r^2 fG^2 R^3 NE^2+\alpha^2 r^2\left(\frac{1}{2}\left(m^2 fG-e^2\left(NA_2-A_0\right)^2\right)R^2+\frac{1}{2}fGe^2 A_2^2\right)+\frac{1}{2}R^2 fGJ_2^2\Big],$$

$\Theta_{23}=\left[-\alpha^2 fG^2 R^2 A_2 er^2 W'-\alpha^2 G^2 fR^2 W'r^2 J_1\right],$

$\Theta_{24}=-GJ_2 W',$

$$\Theta_{31}=\Big[-\alpha^2\left(-fG^2 R\left(-R^2 N^2+fG\right)rE-fG^2\left(N^2 A_0 erR^2-fGA_0 er\right)R\right)rJ_1$$

$$-\alpha^2 RfG^2\left(\left(reA_2 N^2-2A_0 erN\right)R^2-fGA_2 er\right)rE-\alpha^2 r^2 fG^2 R^3 NE^2+\alpha^2 fG^2$$

$$+\left(e^2\left(NA_2-A_0\right)A_0 rNR^2-fGA_2 A_0 e^2 r\right)Rr\Big],$$

$$\Theta_{32}=\Big[-\alpha^2 r^2 fG^2 R\left(-R^2 N^2+fG\right)W'J_1-2\alpha^2 fG^2 R\left(-\frac{1}{2}erN\left(NA_2-A_0\right)R^2+\frac{1}{2}fGA_2 er\right)rW'$$

$$+\alpha^2 R^3 fG^2 Nr^2 W'E\Big],$$

$$\Theta_{33}=\Big[-\alpha^2\left(fG^2 R^3 NrE-fG^2 NA_0 erR^3\right)rJ_1+f^2 G^3 R^3 J_2^2+\alpha^2 fG^2\left(fGrm^2+e^2\left(NA_2-A_0\right)A_0 r\right)R^3 r$$

$$-\alpha^2 R^3 fG^2\left(NA_2 er-2A_0 er\right)rE-\alpha^2 r^2 fG^2 R^3 E^2+r^2\alpha^2 R^3 f^2 G^4 W'^2\Big],$$

$$\Theta_{34}=\Big[-\alpha^2 r^2 fG^2 R\left(-R^2 N^2+fG\right)J_2 J_1+\alpha^2 R^3 r^2 fG^2 NJ_2 E-\alpha^2 fG^2 R\left(\left(-N^2 A_2+NA_0\right)R^2+fGA_2\right)er^2 J_2\Big],$$

$\Theta_{41}=\left(-fG^2 A_0 erR+RrG^2 fE\right)J_2,$

$\Theta_{42}=-W'J_2,$

$\Theta_{43}=\left(-RfG^2 A_2 er-J_1 RfG^2 r\right)J_2,$

$$\Theta_{44}=-\Big[r\left(-R^2 N^2+fG\right)J_1^2-2\left(\left(\left(-N^2 A_2+NA_0\right)R^2+fGA_2\right)e-R^2 NE\right)rJ_1-rR^2 fG^2 W'^2$$

$$+rR^2 E^2+2erR^2\left(NA_2-A_0\right)E-\left(\left(m^2 fG-e^2\left(NA_2-A_0\right)^2\right)R^2+fGe^2 A_2^2\right)r\Big].$$

The nontrivial solution of this equation [20]

$$\Theta\left(C_0, C_1, C_2, C_3\right)^T=0,$$
(47)

is obtained by using the determinant of the matrix equals zero, $\det \Theta = 0$, it follows

$$-m^2 \left[-fG^2R^2r^2\alpha^2W'^2 + \left(-f\left(m^2r^2\alpha^2 + J_2^2\right)G + r^2\alpha^2\left(\left(eA_2 + J_1\right)N - eA_0 + E\right)^2 \right)R^2 - Gf\alpha^2r^2\left(eA_2 + J_1\right)^2 \right]^3 = 0. \tag{48}$$

Solving this equation for the radial part leads to the following integral, as noted that $F(r) = f(r)G(r)$,

$$W_\pm(r) = \pm \int \frac{R(r)\sqrt{\left(E - eA_0 + \left(eA_2 + J_1\right)N\right)^2 - F\left[\left(m^2 + \frac{J_2^2}{r^2\alpha^2}\right) + \frac{\left(eA_2 + J_1\right)^2}{R^2}\right]}}{\left(\gamma^2 - \frac{\omega^2}{\alpha^2}\right)r\,G(r)}dr. \tag{49}$$

Integrating around the pole at the outer horizon r_+, and by using $R(r_+) = \gamma r_+$, gives [10, 11]

$$W_\pm(r) = \pm \frac{i\pi\gamma\left(E - eA_0 + \left(eA_2 + J_1\right)N\right)}{\left(\gamma^2 - \frac{\omega^2}{\alpha^2}\right)G'(r_+)}, \tag{50}$$

where $E_{net} = \left(E - eA_0 + \left(eA_2 + J_1\right)N\right)$. By the same way used in the first part, the tunneling rate of particles tunneling from inside to outside the horizon is given by

$$\Gamma = \frac{P_+}{P_-} \simeq e^{(-4ImW_+)}. \tag{51}$$

On the other hand, using Eqns.(30) and (39), it follows

$$\gamma^2 - \frac{\omega^2}{\alpha^2} = 1, \tag{52}$$

and

$$G'(r_+) = \left(2\alpha^2r_+ + \frac{b}{\alpha r_+^2} - \frac{2c^2}{\alpha^2r_+^2}\right). \tag{53}$$

Again, comparing the Boltzmann factor $\Gamma = e^{-\beta E_{net}}$, with the tunneling rate, gives the Hawking temperature [27, 28]

$$T_H = \frac{G'(r_+)}{4\pi}\frac{\left(\gamma^2 - \frac{\omega^2}{\alpha^2}\right)}{\gamma} = \frac{1}{4\pi\gamma}\left(2\alpha^2r_+ + \frac{b}{\alpha r_+^2} - \frac{2c^2}{\alpha^2r_+^2}\right). \tag{54}$$

IV. CONCLUSION

To summarize, in this paper, we derive the charged black strings temperature using the Hamilton-Jacobi method of the tunneling formalism for the massive vector particles. In the case of a static black string, we start from the field equations, then we use the WKB approximation and the separation of variables which results with a set of four equations. In order to work out the Hawking temperature, we solve the radial part by using the determinant of the metric equals zero. Next, we extend our results to the rotating case and calculate the Hawking temperature. Finally, the results presented in this work extend the tunneling method for massive vector bosons in the case of static/rotating black strings and are consistent with those in the literature[26–29].

V. ACKNOWLEDGMENT

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