Multipoint Nonlocal Problem for Ordinary Differential Equations

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ABSTRACT

Boundary value problems with nonlocal boundary conditions have been considered in numerous investigations.

In this thesis different approaches are analyzed by considering nonlocal boundary conditions of the solution of second order ordinary differential equations. For the existences and uniqueness of the solution the method of contraction mapping of the multipoint nonlocal problems is applied. The finite-difference analogue of the method is also discussed.

Keywords: Boundary Value Problems , Second order ordinary differential equations, multipoint nonlocal condition.

ÖZ

Yerel olmayan sınır koşulları ile Sınır değer problemleri sayısız araştırmalarda dikkate alınmıştır.

Bu tez çalışmasında farklı yaklaşımlar ikinci mertebeden adi diferansiyel denklemlerin çözümünün yerel olmayan sınır koşullarını dikkate alınarak analiz edilmektedir. Çözümün varoluşunun ve teklik çoklu yerel olmayan problemlerin daralma haritalama yöntemi uygulanır. Yöntemin sonlu farklar analogu da tartışılmıştır.

Anahtar Kelimeler: Sınır Değer Problemleri, İkinci mertebeden adi diferansiyel denklemler, çoklu yerel olmayan Sınır Problemleri.

DEDICATION

This study is respectfully dedicated to my parents, my beloved wife PAIMAN and my granule daughter SARA.

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INTRODUCTION

Linear second-order ordinary differential equations, with multipoint boundary value problems were establish by V. A. II'in and E. A. Moiseev in [1], induced by the proceedings of (Bitsadze and Samarskii) on nonlocal linear elliptic boundary problems [2]. Some of the latest results for nonlinear multipoint boundary value problems have been considered [3 − 6].

Furthermore, various partial differential equations involving boundary value problems, in which the boundary conditions are represented as the ratio between the values of the desired functions calculated at different points on the boundary or within the area under consideration, are investigated by many authors. A considerable amount of work has been devoted to the issue of finding effective features for the boundary value problem uniquely solvable with nonlocal conditions, for different classes of differential operator equations and partial differential equations .

In this thesis investigations of nonlocal problems of second-order ordinary differential equations, with boundary conditions of the first kind are reviewed.

In Chapter 2, a nonlocal boundary value problem was studied for unique the solution belonging to the class $C^4[0,1]$ (That is $u(x)$ has a continuous fourth derivative on the interval [0, 1]).

In Chapter 3, we describe the difference approximation of the differential operator and its boundary value condition in a uniform grid with a mesh -step h, where only one solution of the difference problem exists for each h, when $h \to 0$. The solution approaches to the solution of differential problems with second-order accuracy.

In Chapter 4, we demonstrate some numerical examples as an application of the theoretical results given above.

Chapter 2

ON SOME BOUNDARY VALUE PROBLEMS WITH NONLOCAL CONDITION

2.1 Introduction

Boundary value problems of differential equations (ordinary and partial) with nonlocal conditions arise in many applications. In the paper [2] Boundary value problem was formulated and investigated in which nonlocal condition is connected to the values of a solution on some part of the boundary and on some interior curve (Bitsadze - Samarskii problem). Numerical solution of such problems is considered in [3]. Problems with the integral nonlocal condition in the heat problems are investigated, and its generalizations were studied for various equations of mathematical physics.

In this Chapter an extensive class of nonlocal problems is considered, the necessary and sufficient conditions about the existence and uniqueness of solution of the continuous problem and difference problem are provided.

2.2 Formulation of the boundary value problems

We suppose that the problem:

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+q(x)u(x)=f(x) \quad (2.1)$$

$$p_1 \geq p(x) \geq p_o > 0\,, \quad q_1 \geq q(x) \geq q_o \geq 0,$$

$$u(0)=\mu \quad (2.2)$$

$$u(1)=cu(a)+d\,, \qquad 0 \leq a \leq 1 \quad (2.3)$$

where c, d are real numbers and condition (2.3) is a simplest nonlocal condition of type Bitsadze-Samarskii.

Assume that the functions $p(x), q(x), f(x),$ are such that solution of the corresponding first boundary value problem exists and belongs to some class of function, for example W_2^k(Sobolev Space).

Therefore, we define the unknown boundary condition by

$$u(1)=\lambda. \quad (2.4)$$

For arbitrary constant real number λ, the solution of equations $(2.1), (2.2), (2.4)$, exists and is unique.

We denote this solution by $u(x,\lambda)$. Consequently, by the maximum principle for a boundary value problem it follows that $u(x,\lambda)$, is a continuously differentiable function of the argument with respect to λ.

Let us denote

$$\phi(\lambda)=\lambda-cu(a,\lambda)-d. \quad (2.5)$$

It's clear that $u(x,\lambda)$, will be a solution of problem $(2.1) - (2.3)$, if and only if there exists λ_o, such that $\phi(\lambda_o) = 0$. The main element of solvability is to establish equality $\phi'_\lambda(\lambda) \neq 0$, that is:

$$\phi'(\lambda) = 1 - cu'_\lambda(a,\lambda) \neq 0. \tag{2.6}$$

Let us consider in addition, on ordinary differential equation with Dirichlet boundary condition on the interval $[0,1]$

$$-\frac{d}{dx}\left[p(x)\frac{d\omega}{dx}\right] + q(x)\omega = 0, \tag{2.7}$$

$$\omega(0) = 0\,, \quad \omega(1) = 1. \tag{2.8}$$

Theorem 2.1: In order to solve problem $(2.1)-(2.3)$ uniquely for any value of $f(x), u, \mu$ it is necessary and sufficient condition that

$$c\omega(a) \neq 1, \tag{2.9}$$

where $\omega(x)$, is a solution of problem $(2.7), (2.8)$.

Proof: We explore differential equation (2.1), with boundary value conditions $(2.2), (2.4)$. The function $v(x,\lambda) = u(x,\lambda) - \lambda x - \mu(1-x)$, satisfies a boundary value problem

$$-\frac{d}{dx}\left[p(x)\frac{dv}{dx}\right] + q(x)v = f_1(x) + \lambda f_2(x), \tag{2.10}$$

$$v(0) = 0, \quad v(1) = 0,$$

where

$$f_1(x) = f(x) - \mu\left(\frac{dp}{dx} - q(x)x + q(x)\right),$$

and

$$f_2(x) = \frac{dp}{dx} - q(x)x.$$

By using Green's function $G(x,s)$, of problem (2.10), we will write a solution

$$v(x,\lambda) = \int_0^1 G(x,s)[f_1(s) + \lambda f_2(s)]ds. \tag{2.11}$$

Since $v(x,\lambda)$, is the solution of equation (2.10) and satisfies the boundary value conditions

$$v(0,\lambda) = u(0,\lambda) - \lambda(0) - \mu(1-0) = u(0) - \mu = 0,$$

where from condition (2.2) $u(0) = \mu$

$$v(1,\lambda) = u(1,\lambda) - \lambda(1) - \mu(1-1) = u(1) - \lambda = 0,$$

and also from condition (2.4), $u(1) = \lambda$.

Consequently, $u(x,\lambda) = v(x,\lambda) + \lambda x + \mu(1-x)$, substituting equation for $u(x,\lambda)$, in equation (2.5), and considering equation (2.11), we obtain

$$\phi(\lambda) = \lambda - c\left(\int_0^1 G(a,s)[f_1(s) + \lambda f_2(s)]ds + \lambda a + \mu(1-a)\right) - d.$$

Taking the derivative of $\phi(\lambda)$, with respect to λ we obtain

$$\phi'(\lambda) = 1 - c\left(\int_0^1 G(a,s)[f_2(s)]ds + a\right).$$

From this equality, we derive the condition

$$1 - c\left(\int_0^1 G(a,s)f_2(s)ds + a\right) \neq 0. \qquad (2.12)$$

It's clear that condition $\phi'(\lambda) \neq 0$ is equivalent to the condition $1 - c\omega(a) \neq 0$. this provides the existence of the unique root of the equation $\phi(\lambda) = 0$, which does not depend on values of μ *and* $f(x)$.

Hence $\omega(x) = \int_0^1 G(x,s)\left(\frac{dp(s)}{ds} - q(s)s\right)ds + x$ is a solution of problem (2.7), (2.8). From equation (2.12), $\phi(\lambda)$ has a zero λ_o, which implies that $u(1,\lambda_o) = cu(a,\lambda_o) + d$.

Consequently, $u(1,\lambda_o)$, is a solution of problem (2.1) – (2.3), and we obtain condition (2.9) .

Therefore,the theorem is proved.

Corollary 2.1: Sufficient solvability condition of problem $(2.1) - (2.3)$ is the inequality

$$-\infty < c < \frac{1}{\omega(a)} \quad , \qquad (2.13)$$

which follows from condition (2.9).

Since for the solution of problem $(2.7), (2.8)$, and $0 < \omega(a) < 1$, where $0 < a < 1$, we may establish the following assessment $0 \leq \omega(x) \leq 1$, in particular, follows another more limiting sufficient solvability condition $(2.1) - (2.3)$, from inequality (2.13).

That is $\max \omega(x) = 1$

$$-\infty < c \leq 1, \qquad (2.14)$$

with which coincides with the results of [2].

We note that condition (2.9) provides existence to the unique solution of a boundary value problem $(2.1) - (2.3)$, with a nonlocal condition (2.3).

Remark 2.1: Condition (2.13) might be obtained from the Theorem 2.1, but it also might be obtained from inequality (2.6), directly. Indeed, by using a representation of the solution through a Green's function, we can show that $u'_\lambda(x,\lambda)$(which is the derivative of a function $u(x,\lambda)$), is a solution of problem $(2.7), (2.8)$, that is $\omega(x) = u'_\lambda(x,\lambda)$.

Consequently, from condition (2.9), we obtain $-\infty < c\omega(a) < 1$, from which condition (2.13), follows.

Let us generalize the result of Theorem 2.1 for different differential equations, boundary value and nonlocal conditions.

Let us consider more general conditions instead of a nonlocal condition (2.3)

$$u(1) = \sum_{i=1}^{n} c_i u(a_i) + d, \qquad 0 \leq a_i < 1. \tag{2.15}$$

As we proved Theorem 2.1 we analogically obtain the following condition: In order to obtain the unique solution of problem $(2.1), (2.2), (2.15)$, for any $f(x), \mu$ it is a necessary and sufficient condition so that it satisfies

$$\sum_{i=1}^{n} c_i \omega(a_i) \neq 1, \tag{2.16}$$

where $\omega(x)$, is a solution of problem $(2.7), (2.8)$. Sufficient condition, that as analogical as inequality (2.14), might be formulated by the following way:

$$-\infty < \sum_{i=1}^{k} c_i \omega(a_i) < 1, \tag{2.17}$$

where the summation performs to particular indexes *I*, for which $c_i > 0$.

Remark 2.2: If in differential problem $(2.1)-(2.3)$, instead of boundary value condition (2.2), we substitute a condition

$$u'(0)=\mu, \tag{2.18}$$

then in Theorem 2.1, in which $\omega(x)$ is a solution of equation (2.7), we consider value conditions

$$\omega'(0)=0,\ \omega(1)=1. \tag{2.19}$$

We analogically consider a boundary value problem $(2.1),(2.15),(2.18)$.

Let us consider a nonlinear differential equation.

$$-\frac{d}{dx}\left[\mu\left(x,\frac{du}{dx}\right)\frac{du}{dx}\right]+\mathrm{g}(x,u)=f(x), \tag{2.20}$$

with conditions $(2.2),(2.3)$, suppose that for each finite value x,u,u' already satisfies

$$\frac{\partial \mathrm{g}}{\partial u}\geq 0\,,\quad \frac{\partial \mu(p)p}{\partial p}\geq c_1>0. \tag{2.21}$$

We shall consider an equivalent problem $(2.20),(2.2),(2.4)$, with unknown value λ of the problem $(2.20),(2.2),(2.3)$, as well as in a linear case.

Let us denote $\bar{u}(x)=u(x,\lambda_1)-u(x,\lambda_2)$, where $u(x,\lambda_i), i=1,2$, is a solution of problem $(2.20),(2.2),(2.4)$, where $\lambda=\lambda_i$. We obtain a boundary value problem for a function $\bar{u}(x)$

$$-\frac{d}{dx}\left(p(x)\frac{d\bar{u}}{dx}\right)+q(x)\bar{u}=0, \tag{2.22}$$

$$\bar{u}(0)=0,\quad \bar{u}(1)=\lambda_1-\lambda_2, \tag{2.23}$$

where

$$p(x) = \int_0^1 \frac{\partial \tilde{k}}{\partial p} dt\,, \qquad q(x) = \int_0^1 \frac{\partial \tilde{g}}{\partial u} dt, \tag{2.24}$$

$$k = k(x, p) = \mu(x, p)p,$$

$$\tilde{k} = k\left(x, \frac{du_\theta}{dx}\right),$$

$$\tilde{g} = g(x, u_\theta)\,,$$

$$u_\theta = \theta u_1 + (1 - \theta)u_2\,, \qquad 0 \le \theta \le 1.$$

Considering inequality (2.21), we have

$$p(x) \ge p_o > 0, q(x) \ge 0.$$

Further researches of a problem coincide with the research in the linear case.

In conclustion, we obtain that Theorem 2.1 might be applied to a nonlinear problem (2.20), (2.2), (2.3), where $\omega(x)$ is a solution of problem (2.7), (2.8), of which functions $p(x), q(x)$, are defined in equation (2.24).

In this case we have the sufficient solvability conditions (2.13), (2.14), as well as in the linear case.

Remark 2.3: Let us consider the more general nonlinear equation instead of the equation (2.20)

$$-\frac{d}{dx}k_1\left(x,u,\frac{du}{dp}\right)+k_o\left(x,u,\frac{du}{dx}\right)=f(x). \qquad (2.25)$$

We assume that inequalities (2.26) satisfy for every finite ξ_1,ξ_2, and for any real values x,p_o,p_1

$$c_1\sum_{i=0}^{1}\xi_i^2\le\sum_{i,j=0}^{1}\frac{\partial k_i(x,p_o,p_1)}{\partial p_j}\xi_i\,\xi_j\le c_2\sum_{i=0}^{1}\xi_i^2\ , \qquad (2.26)$$

where $c_1>0$.

Theorem 2.1 might be applied for the solution of the differential equation $(2.25),(2.2),\ (2.3)$, as well as for the case above, however $\omega(x)$ is a solution of the following equation:

$$-\frac{d}{dx}\left(p(x)\frac{d\omega}{dx}+r_1(x)\omega\right)+r_o(x)\frac{d\omega}{dx}+q(x)\omega=0, \qquad (2.27)$$

with boundary value condition (2.8), where functions $p(x),q(x)$, are defined as in equation (2.24), and

$$r_1(x)=\int_0^1\frac{\partial\tilde{k}_1(x,u)}{\partial u}dt,\ \ r_o(x)=\int_0^1\frac{\partial\tilde{k}_o(x,u)}{\partial p}dt.$$

Since we have no maximum principle in the general case of an equation (2.27) , then sufficient condition (2.14) is incorrect (condition (2.13), is applicable).

In conclusion of the section (2.2) , let us consider a linear equation (2.1), with boundary condition (2.2), and nonlocal condition

$$cu(1) = \int_{x_1}^{x_2} \rho(x)\, u(x)dx + d, \quad 0 \le x_1 < x_2 \le 1. \tag{2.28}$$

By applying the same method of research as problem $(2.1) - (2.3)$, we obtain that a necessary and sufficient condition for solvability is a condition $(2.1), (2.2), (2.28)$

$$\int_{x_1}^{x_2} \rho(x)\, \omega(x)dx \neq c, \tag{2.29}$$

where $\omega(x)$ is a solution of problem $(2.7), (2.8)$.

In particular, we obtain that the problem is still solvable when $c = 0$, and $\rho(x) > 0$, a special case of a nonlinear differential equation.

Chapter 3

A NONLOCAL BOUNDARY VALUE PROBLEM IN DIFFERENCE INTERPRETATIONS

In this Chapter we want to study a nonlocal boundary value problem in a difference interpretation.

Assume that $u(x)$ is the exact solution of problem

$$[k(x)u']' - q(x)u = -f(x) \quad \text{when } 0 < x < 1 \qquad (3.1)$$

$$u(0) = 0\,, \quad u(1) = \sum_{i=1}^{m} \alpha_i u(\,\xi_i\,) \qquad (3.2)$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$ and constants $\alpha_1,\ \alpha_2, \cdots,\ \alpha_m$, and real numbers satisfying the inequalities $-\infty < \ \alpha_1 + \ \alpha_2 + \ \alpha_3 + \cdots + \alpha_m \ \leq 1$. The purpose of this research is to prove that the conditions $k(x) \in C^3[\,0{,}1\,]\,, q(x)$ *and* $f(x) \in C^2[0{,}1]$, $k(x) \geq m_o > 0$, $q(x) \geq 0$ are satisfied everywhere on the interval [0,1]. (These conditions we will call conditions A).

Theorem 3. 1: If $k(x), q(x)$ and $f(x)$ satisfy conditions A, then classical solution of problem $(3.1), (3.2)$, belongs to $C^4[0{,}1]$. (That is $u(x)$ has a continuous fourth derivative on the interval [0, 1]).

Let us consider the difference interpretations of problem $(3.1), (3.2)$, where $y(x)$ is the solution of finite difference, for any equidistant grid function $y(x)$, defined on a uniform grid

$$\omega_h = \{x_i = ih, i = 1{,}2, \cdots, N-1,\ x_o = 0, x_N = 1\}.$$

By following A.A.Samarskii ([8]page 148), we approximate the differential operator by the difference operator in equation (3.1).

$$\Lambda y_i = (a\, y_{\bar{x}})_x - d_i y_i = -\varphi_i$$

$$\frac{1}{h}\left(a_{i+1}\frac{y_{i+1} - y_i}{h} - a_i\,\frac{y_i - y_{i-1}}{h}\right) - d_i y_i = -\varphi_i\,. \qquad (3.3)$$

$$a_i = k_{i-1/2} = k(x_i - h/2).$$

$$d_i = q_i = q(x_i),\ \varphi_i = f_i = f(x_i).$$

We will consider h as a mesh-step, where h is less than a half of the smallest segments $[\,0,\ \xi_1\,],\ \ [\,\xi_1,\ \xi_2\,], \cdots,\ [\,\xi_m, 1\,]$, let i_l be defined by the condition such as $(i_l)h \le \xi_l \le (i_l + 1)h$.

We will suppose that normally ξ_l is not a grid point and the uniform grid with increment $h = 1/N$ is as shown in Figure 3.1

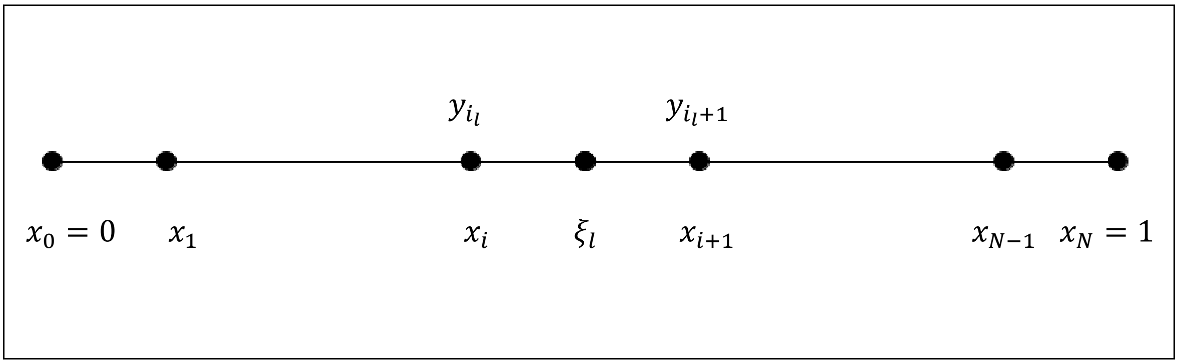

Figure 3.1: One-dimensional desctete grid with mesh-step h on the interval [0,1].

where

$$x_i = (i_l)h\,,\quad x_{i+1} = (i_l + 1)h$$

Since $u(1) = \sum_{l=1}^{m} \alpha_l u(\xi_l)$, we apply Lagrange (Linear) interpolation of the points $(i_l)h,\ (i_l+1)h$, as shown in Table 3.1 then

Table 3.1: Table of two grid points on grid set ω_h

	$\boldsymbol{x}$	$\boldsymbol{f(x)}$
1	$(i_l)\,h$	y_{i_l}
2	$(i_l+1)h$	y_{i_l+1}

$$u(x) \approx L_l = y_{i_l}\left[\frac{x-(i_l+1)h}{(i_l)h-(i_l+1)h}\right] + y_{i_l+1}\left[\frac{x-(i_l)h}{(i_l+1)h-(i_l)h}\right]$$

$$= y_{i_l}\frac{[x-(i_l+1)h]}{-h} + y_{i_l+1}\frac{[x-(i_l)h]}{h}$$

$$= y_{i_l}\frac{[(i_l+1)h-x]}{h} + y_{i_l+1}\frac{[x-(i_l)h]}{h}$$

where $x = \xi_l$

$$u(\xi_l) \approx y_{i_l}\frac{[(i_l+1)h-\xi_l]}{h} + y_{i_l+1}\frac{[\xi_l-(i_l)h]}{h}.$$

In order to find the solution of the difference equation, we will approximate the boundary condition (3.2), by the following:

$$y_o = 0, \qquad \mathcal{L}(h)y = \sum_{l=1}^{m} \alpha_l u(\xi_l) - y_N = 0$$

$$y_o = 0,\ \mathcal{L}(h)y = \sum_{l=1}^{m} \alpha_l \left\{y_{i_l}\frac{[(i_l+1)\,h-\xi_l]}{h} + y_{i_l+1}\frac{[\xi_l-(i_l)\,h]}{h}\right\} - y_N = 0. \qquad (3.4)$$

Equivalently

$$y_o = 0, \qquad y_N = \sum_{l=1}^{m} \alpha_l \left\{y_{i_l}\frac{[(i_l+1)h-\xi_l]}{h} + y_{i_l+1}\frac{[\xi_l-(i_l)h]}{h}\right\}$$

Theorem 3.2: If conditions A are satisfied, then only one solution $y(x)$ of difference problem (3.3), (3.4), exists for each $h > 0$, $h \to 0$, and approaches to the solution $u(x)$, of the differential problem (3.1), (3.2), with the second order accuracy in h.

Proof: In order to prove Theorem 3.2 by following A.A. Samarskii ([8] Chapter 3), we assume that $z_i = y_i - u_i$, besides, we note that a grid function $z(x)$, is the solution of the difference problem (3.5):

From equation (3.3)

$\Lambda z_i = -(\varphi + \Lambda u_i)$, is the error approximation, u(x) ∈ $C^4[0,1]$

$$\Lambda z_i = -\psi_i, \tag{3.5}$$

$z_o = 0$, $\mathcal{L}(\text{h})(z_i + u_i) = 0$, so we can get $\mathcal{L}(h)(z_i) = -\mathcal{L}(h)(u_i)$, consequently by using boundary value condition (3.4), we obtain

$$z_o = 0\,, \mathcal{L}(h)z = -\sum_{l=1}^{m} \alpha_l \left\{ u_{i_l} \frac{[(i_l+1)\,h-\xi_l]}{h} + u_{i_l+1} \frac{[\xi_l - i_l h]}{h} \right\} + u_N. \tag{3.6}$$

We have the condition that (for example [8] page 151) when $u(x) \in C^4[0,1]$, the differential equation operator approximates to second order of accuracy in step h, that is:

$$|\psi| = \max_{0 \le i \le N} |\psi_i| = O(h^2).$$

which may be easily verified by Taylor's formula.

$$|\mathcal{L}(\text{h})\text{z}| = O(h^2)$$

Before our substitution of $z_i = \tilde{z}_i - Ai$, where $i = \frac{x_i}{h}$, and we note that

$$\left(1 - \sum_{i=1}^{m} \alpha_i \xi_i\right) \geq 1 - (\alpha_1 \xi_{1+} \alpha_2 \xi_2 + \cdots + \alpha_m \xi_m) \geq$$

$$1 - (\alpha_1 \xi_{m+} \alpha_2 \xi_m + \cdots + \alpha_m \xi_m) \geq$$

$$1 - (\alpha_1 + \alpha_2 + \cdots + \alpha_m)\, \xi_m \geq$$

$$1 - \xi_m > 0,$$

where $(\alpha_1 + \alpha_2 + \cdots + \alpha_m) \leq 1$.

In order to find the value of A, firstly we need to calculate

$$-\mathcal{L}(h)\left(\frac{x_i}{h}\right) = -\sum_{l=1}^{m} \alpha_i \left\{ \left(\frac{x_i}{h}\right)_{i_l} \frac{[(i_l+1)h - \xi_l]}{h} + \left(\frac{x_i}{h}\right)_{i_l+1} \frac{[\xi_l - i_l h]}{h} \right\} + \left(\frac{x_i}{h}\right)_N$$

$$= -\sum_{l=1}^{m} \alpha_i \left\{ (i_l)\, \frac{[(i_l+1)h - \xi_l]}{h} + (i_l + 1) \frac{[\xi_l - i_l h]}{h} \right\} + \frac{1}{h}$$

$$= \frac{1}{h}\left(1 - \sum_{i=1}^{m} \alpha_i \xi_i\right).$$

And from the boundary condition (3.6)

$$\mathcal{L}(h)(z_i) = -\mathcal{L}(h)(u_i)$$

$$\mathcal{L}(h)(\tilde{z}_i - Ai) = -\mathcal{L}(h)(u_i)$$

$\mathcal{L}(h)(\tilde{z}_i) = -(\mathcal{L}(h)(u_i) - A\mathcal{L}(h)\left(\frac{x_i}{h}\right))$, where $x_i = ih$

$\mathcal{L}(h)(\tilde{z}_i) = -[\mathcal{L}(h)(u_i) - A\mathcal{L}(h)\left(\frac{x_i}{h}\right)]$,

so when $\mathcal{L}(h)(\tilde{z}_i) = 0$, we obtain

$$A = \left(1 - \sum_{i=1}^{m} \alpha_i \xi_i\right)^{-1} h\mathcal{L}(h)z = O(h^3).$$

We approach problem $(3.5), (3.6)$, by using the next difference problem with the homogeneous nonlocal boundary value problem:

$$\Lambda\tilde{z}_i = -\tilde{\psi}_i\,. \tag{3.7}$$

$$\tilde{z}_o = 0\,,\;\; \mathcal{L}(h)\tilde{z} = 0, \tag{3.8}$$

where $\tilde{\psi}_i = \psi_i - A\Lambda(x_i/h)$, $|\tilde{\psi}| = O(h^2)$.

The central part of the current proof is that we must obtain a priori estimation of $\tilde{z}_N$, value which is a solution $\tilde{z}$, of problem (3.7), (3.8) by using the discrete form for L_2-norm of the right hand side $\tilde{\psi}$ of equation (3.7):

$$|\tilde{z}_N| \le \frac{|\alpha|}{m_o}\|\tilde{\psi}\|, \tag{3.9}$$

where we denote $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_m$, m_o is a constant satisfying the condition $k(x) \ge m_o > 0$, and the norm $\|\tilde{\psi}\|$ is determined by the equality:

$$\|\tilde{\psi}\| = \left[h\,\sum_{i=1}^{N-1}\tilde{\psi}_i^2\right]^{1/2}.$$

In order to prove estimation (3.9), we suppose that $\widetilde{m} = \min\,\{\tilde{z}_1, \tilde{z}_2, \cdots, \tilde{z}_{N-1}\}$, and $\widetilde{M} = \max\,\{\tilde{z}_1, \tilde{z}_2, \cdots, \tilde{z}_{N-1}\}$. Since we chose h for any $l = 1, 2, \ldots, m$, then $\widetilde{m} \le \tilde{z}_{i_l} \le \widetilde{M}$, and $\widetilde{m} \le \tilde{z}_{i_l+1} \le \widetilde{M}$.

Moreover, when $\alpha_l \ge 0$

$$\widetilde{m}\sum_{l=1}^{m}\alpha_l \le \sum_{\mathrm{l}=1}^{\mathrm{m}}\alpha_{\mathrm{l}}\,\tilde{z}(\,\xi_{\mathrm{l}}) \le \widetilde{M}\,\sum_{l=1}^{m}\alpha_l$$

$$\widetilde{m}\sum_{l=1}^{m}\alpha_l \le \sum_{l=1}^{m}\alpha_l\left\{\tilde{z}_{i_l}\frac{[(i_l+l)h-\xi_l]}{h} + \tilde{z}_{i_l+1}\frac{[\xi_l - i_l h]}{h}\right\} \le \widetilde{M}\,\sum_{l=1}^{m}\alpha_l\,,$$

hence, from the second condition of equation (3.8), and the equality $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_m$, we obtain:

$$\widetilde{m}\alpha \le \tilde{z}_N \le \widetilde{M}\alpha. \tag{3.10}$$

If each of $\alpha_l \le 0$ then we similarly obtain:

$$\widetilde{m}\alpha \ge \tilde{z}_N \ge \widetilde{M}\alpha. \qquad (3.11)$$

We exclude the trivial cases $\alpha = 0$ and $\tilde{z}_N = 0$, which leads to a homogeneous boundary value problem of the first kind of estimation (3.9), of which is apparent.

Let us consider the cases:

1. $\tilde{z}_N > 0$,
2. $\tilde{z}_N < 0$.

In the first case from the right inequality (3.10), and from the left inequality (3.11) we derive that $\widetilde{M} > 0$, $\widetilde{m} < 0$ when $\alpha > 0$, $\alpha < 0$ respectively.
From the left inequality (3.10) and from the right inequality (3.11), we derive that $\widetilde{m} < 0$, $\widetilde{M} > 0$ when $\alpha > 0$, $\alpha < 0$, respectively.
So, if any of $\tilde{z}_N \neq 0$, and none of α_l, are equal 0, then $\widetilde{M} > 0$, and $\widetilde{m} < 0$.

Moreover, in case $\alpha_l \ge 0$, from inequalities (3.10), and condition $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_m \le 1$, we derive that $\tilde{z}_N \le \widetilde{M}$, $\widetilde{m} \le \tilde{z}_N$, when $\tilde{z}_N > 0$, $\tilde{z}_N < 0$ respectively, that is , $\widetilde{M} > 0$ is a maximum (for $\tilde{z}_N > 0$) and $\widetilde{m} < 0$ is a minimum (for $\tilde{z}_N < 0$) of value $\tilde{z}_i$, on a whole grid $0 \le i \le N$, besides these maximum and minimum approach in inner nodes, that is when $1 \le i \le N-1$.

In the case $\alpha_l \le 0$. To positive values of $\tilde{z}_N > 0$ correspond negative minimum $\widetilde{m} < 0$, to negative values $\tilde{z}_N < 0$ correspond positive maximum $\widetilde{M} > 0$, that is: in this case when $\tilde{z}_N < 0$, $\widetilde{M} > 0$ is as maximum as we mentioned before; when $\tilde{z}_N > 0$, $\widetilde{m} < 0$ is a minimum of values $\tilde{z}_i$, on a whole grid $0 \leq i \leq N$ similarly.

Besides these maximum and minimum approaches we have the inner nodes, that is when $1 \leq i \leq N - 1$.

In order to obtain the estimation of inequality (3.9), due to inequality (3.10), and (3.11), we should set:

1. An estimation $\widetilde{M} \leq \frac{1}{m_o} \|\tilde{\psi}\|$, in case $\tilde{z}_N > 0$
2. An estimation $-\frac{1}{m_o} \|\tilde{\psi}\| \leq \widetilde{m}$, in case $\tilde{z}_N < 0$

we solely have to obtain the first estimation, as the second estimation is proved similarly.

Let $\widetilde{M} = \tilde{z}_j$, on condition $j \leq N - 1$.(see [8] page 99) We should apply difference Green's Formula to a grid function $\tilde{z}_i$, on the grid $0 \leq i \leq j + 1$.

From analog of the First Green Formula for grid function:

$$(z, (ay_{\bar{x}})_x) = -(ay_{\bar{x}}, y_{\bar{x}}] + \ a\, z\, y_{\bar{x}}\, |_N - a_1 y_{x,0}\, z_0 \ ,$$

we obtain

$$((a\tilde{z}_{\bar{x}})_x, \tilde{z}) = -(a\tilde{z}_{\bar{x}}, \tilde{z}_{\bar{x}}\ ) + (a\tilde{z}_{\bar{x}})_{j+1}\tilde{z}_j - (a\tilde{z}_{\bar{x}})_1\, \tilde{z}_o. \qquad (3.12)$$

Since the boundary value

$$\tilde{z}_o = 0,\ \tilde{z}_j = \widetilde{M} > 0\ ,$$

$$(a\tilde{z}_{\bar{x}})_{j+1} = a_{j+1}\, \frac{\tilde{z}_{j+1} - \tilde{z}_j}{h} \leq 0,$$

due to equation (3.12), we derive an inequality

$$((a\tilde{z}_{\bar{x}})_x,\ \tilde{z}) \le -(a\tilde{z}_{\bar{x}},\ \tilde{z}_{\bar{x}})$$

$$m_o(\tilde{z}_{\bar{x}}, \tilde{z}_{\bar{x}}) \le (a\tilde{z}_{\bar{x}}, \tilde{z}_{\bar{x}}) \le -(\,(a\tilde{z}_{\bar{x}})_x, \tilde{z}\,), \tag{3.12'}$$

$$m_o(\tilde{z}_{\bar{x}}, \tilde{z}_{\bar{x}}) \le -(\,(a\tilde{z}_{\bar{x}})_x, \tilde{z}\,) \le (\,(-a\tilde{z}_{\bar{x}})_x, \tilde{z}\,) \le (\tilde{\psi}, \tilde{z}).$$

Using right side of the inequality above, differential equation and Cauchy- Schwarz inequality as well we obtain

$$(\tilde{z}_{\bar{x}}, \tilde{z}_{\bar{x}}) \le \frac{1}{m_o}(\tilde{\psi}, \tilde{z}) \le \frac{1}{m_o}\|\tilde{\psi}\|.\|\tilde{z}\|. \tag{3.13}$$

We take into consideration that the grid function $\tilde{z}_i$ satisfies the condition $\tilde{z}_o = 0$, therefore we may say the difference analogue of Poincare inequality is valid:

$$\|\tilde{z}\| \le \|\tilde{z}_{\bar{x}}\|. \tag{3.14}$$

From inequality (3.13), (3.14), we obtain that

$\|\tilde{z}_{\bar{x}}\|^2 \le \frac{1}{m_o}\|\tilde{\psi}\|\ \|\tilde{z}\| \le \frac{1}{m_o}\|\tilde{\psi}\|\ \|\tilde{z}_{\bar{x}}\|$, therefore we can get $\|\tilde{z}_{\bar{x}}\| \le \frac{1}{m_o}\|\tilde{\psi}\|$: In order to set an estimation of $\widetilde{M} \le \frac{1}{m_o}\|\tilde{\psi}\|$, we have to use the Whitney Embedding Theorem and then:

$$\tilde{z}_j = \max_{1\le i\le j} \tilde{z}_i \le \|\tilde{z}_{\bar{x}}\| = \left[h \sum_{i=1}^{j}\left(\frac{\tilde{z}_i - \tilde{z}_{i-1}}{h}\right)^2\right]^{1/2}.$$

Hence $\widetilde{M} \le \frac{1}{m_o}\|\tilde{\psi}\|$, and similarly $-\frac{1}{m_o}\|\tilde{\psi}\| \le \widetilde{m}$.

After that, we use the following cases for an estimation of inequality (3.9):

Case 1: $\alpha \geq 0$, if $\tilde{z}_N > 0$ and $\widetilde{M} \leq \frac{1}{m_o}\|\tilde{\psi}\|$, then $\widetilde{M}\alpha \leq \frac{\alpha}{m_o}\|\tilde{\psi}\|$

$$\tilde{z}_N \leq \widetilde{M}\alpha \leq \frac{\alpha}{m_o}\|\tilde{\psi}\| \Rightarrow \tilde{z}_N \leq \frac{\alpha}{m_o}\|\tilde{\psi}\|. \quad (3.15)$$

And similarly, if $\tilde{z}_N < 0$ and $-\frac{1}{m_o}\|\tilde{\psi}\| \leq \widetilde{m}$, then $-\frac{\alpha}{m_o}\|\tilde{\psi}\| \leq \widetilde{m}\alpha$

$$-\frac{\alpha}{m_o}\|\tilde{\psi}\| \leq \widetilde{m}\alpha \leq \tilde{z}_N \Rightarrow -\frac{\alpha}{m_o}\|\tilde{\psi}\| \leq \tilde{z}_N . \quad (3.16)$$

From inequality (3.15) and (3.16) , we get $|\tilde{z}_N| \leq \frac{|\alpha|}{m_o}\|\tilde{\psi}\|$.

Case 2: $\alpha \leq 0$, if $\tilde{z}_N > 0$ and $\widetilde{M} \leq \frac{1}{m_o}\|\tilde{\psi}\|$, then $\widetilde{M}\alpha \geq \frac{\alpha}{m_o}\|\tilde{\psi}\|$

$$\tilde{z}_N \geq \widetilde{M}\alpha \geq \frac{\alpha}{m_o}\|\tilde{\psi}\| \Rightarrow \tilde{z}_N \geq \frac{\alpha}{m_o}\|\tilde{\psi}\|. \quad (3.17)$$

And similarly, if $\tilde{z}_N < 0$ and $-\frac{1}{m_o}\|\tilde{\psi}\| \leq \widetilde{m}$, then $-\frac{\alpha}{m_o}\|\tilde{\psi}\| \geq \widetilde{m}\alpha$

$$-\frac{\alpha}{m_o}\|\tilde{\psi}\| \geq \widetilde{m}\alpha \geq \tilde{z}_N \Rightarrow -\frac{\alpha}{m_o}\|\tilde{\psi}\| \geq \tilde{z}_N . \quad (3.18)$$

From equations (3.17) and (3.18), we get $|\tilde{z}_N| \leq \frac{|\alpha|}{m_o}\|\tilde{\psi}\|$.

Therefore setting inequality (3.9), the difference problem, and the boundary value problem of the first kind: and from problem (3.7), (3.8)

$$\Lambda\tilde{z}_i = -\tilde{\psi}_i ,$$

$$\tilde{z}_o = 0 , \;\; \tilde{z}_N = \theta(h),$$

for $\theta(h)$, due to inequality (3.9), we may say $\theta(h) = O(h^2)$.

As we know, for such sort of a problem we can say $\|\tilde{z}_{\bar{x}}\| = O(h^2)$. Due to a Whitney Embedding Theorem, we have an estimation

$$\operatorname*{Max}_{1\le i\le N-1} |\tilde{z}_i| = O(h^2).$$

Since $z_i = \tilde{z}_i - Ai,$ the same estimation may be said about $\|z_{\bar{x}}\|$, and, $\max_{1\le i\le N-1}|z_i|$.

Eventually, the standard technique allows us to obtain an estimation of second order along h and for the second difference derivative

$$\|(z_{\bar{x}})_x\| = \left[h \sum_{i=1}^{N-1} \left(\frac{z_{i+1} - 2z_i + z_{i-1}}{h^2}\right)^2 \right]^{1/2} = O(h^2).$$

Therefor the order of convergence is $O(h^2)$

(It means that when $h \to 0$ approximate solution convergences to the exact solution).

The reason that difference problem (3.3), (3.4), for each $h > 0$ has only one trivial solution when $\varphi_i = 0$, is that problem (3.3), (3.4) is a linear system and due to set estimation inequality (3.9), as well.

Chapter 4

NUMERICAL EXPRIMENTS

In this Chapter, we will show numerical experiments, and use the central-difference scheme, which provides second-order accuracy, for the approximation of the solution.

We apply a procedure of modified Gauss Seidel methods to a specific problem, and carry out the calculations using the MATLAB programming Language.

The following results of numerical experiments are for the nonlocal boundary value problems of a second-order ordinary differential equation.

Example 1: In this problem

$$u''(x) - (x^2 + 1)u(x) = -f(x) \qquad (4.1)$$

with boundary conditions

$$u(0) = 0\ ,\ \ u(1) = -u\left(\frac{1}{4}\right) - u\left(\frac{3}{4}\right),$$

where $u(x) = e^{\pi x \left(\frac{1}{4} - x\right)\left(\frac{3}{4} - x\right)} . \sin(2\pi x)$, is the exact solution on the interval $[0,1]$. Since $\alpha_1 = \alpha_2 = -1$and $\xi_1 = \frac{1}{4}, \xi_2 = \frac{3}{4}$, both are grid point. Let us defined a uniform grid

$\omega_h = \{x_i = ih, i = 1,2,\cdots, N-1,\ x_o = 0, x_N = 1\}$.

By equation (4.1), for $f(x)$, we obtain

$$f(x) = [\, 4\pi^2 - \pi^2 \left(\frac{3}{16} - 2x + 3x^2\right)^2 + \pi\,(-2 + 6\,x) +$$

$$(x^2 + 1)]\; e^{\pi\, x \left(\frac{1}{4} - x\right)\left(\frac{3}{4} - x\right)} .\sin(2\,\pi\, x) \; -$$

$$2\pi^2 \left(\frac{3}{16} - 2x + 3x^2\right)\; e^{\pi\, x \left(\frac{1}{4} - x\right)\left(\frac{3}{4} - x\right)} . \cos(2\pi\, x).$$

Discretizing equation (4.1), and using the central-difference scheme on an equidistant grid, we obtain the finite-difference equation

$$(y_{\bar{x}})_x - (x^2 + 1)y_i = -f(x_i)\,, \quad i = 1, \dots, n-1, \tag{4.2}$$

$$\frac{(y_{i+1} - 2y_i + y_{i-1})}{h^2} - (x^2 + 1)y_i = -f(x_i)\,, \quad i = 1, \dots, n-1.$$

Hence, rearranging

$$y_i = \frac{(y_{i+1} + y_{i-1}) + h^2 f(x_i)}{2 + (x^2 + 1)h^2}\,,$$

where $h = \frac{1}{n}\,, n \geq 2$

with boundary conditions

$$y_0 = 0\,, \quad y_n = -y_{\left(\frac{n}{4}\right)} - y_{\left(3\frac{n}{4}\right)}.$$

Table 4.1, represents the results obtained, where $\varepsilon_h = \|u - y_h\|$, $h = 2^{-m}$, $m = 3, 4, 5, 6$ is the difference between the exact solution and approximate solution, in the maximum norm, and $R_h = \frac{\|u - y_{2^{-m}}\|}{\|u - y_{2^{-(m+1)}}\|}$, is the order of convergence of the solution.

Table 4.1: Maximum Error between exact and approximate solution and the order of convergence of the solution of Example 1.

h	ε_h	R_h
$\frac{1}{8}$	0.0698	$\simeq$ 4.05813953
$\frac{1}{16}$	0.0172	$\simeq$ 3.90909091
$\frac{1}{32}$	0.0044	$\simeq$ 3.38461538
$\frac{1}{64}$	0.0013	

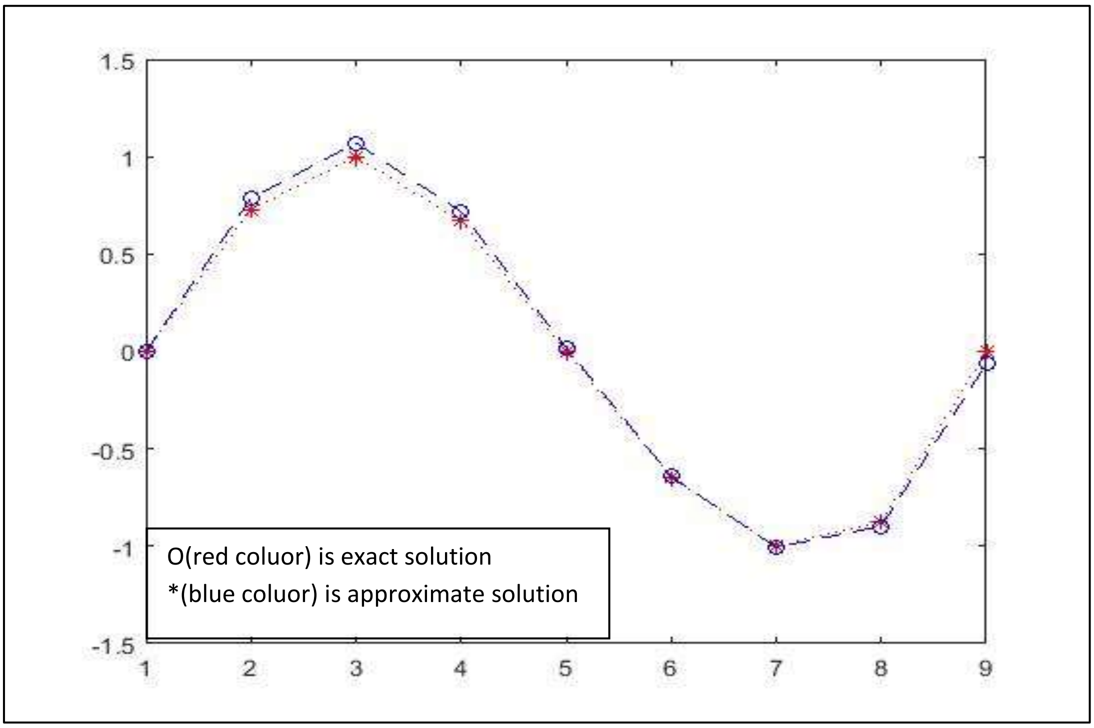

Figure 4.1: Exact solution and approximate solution of Example 1, when $h = \frac{1}{8}$

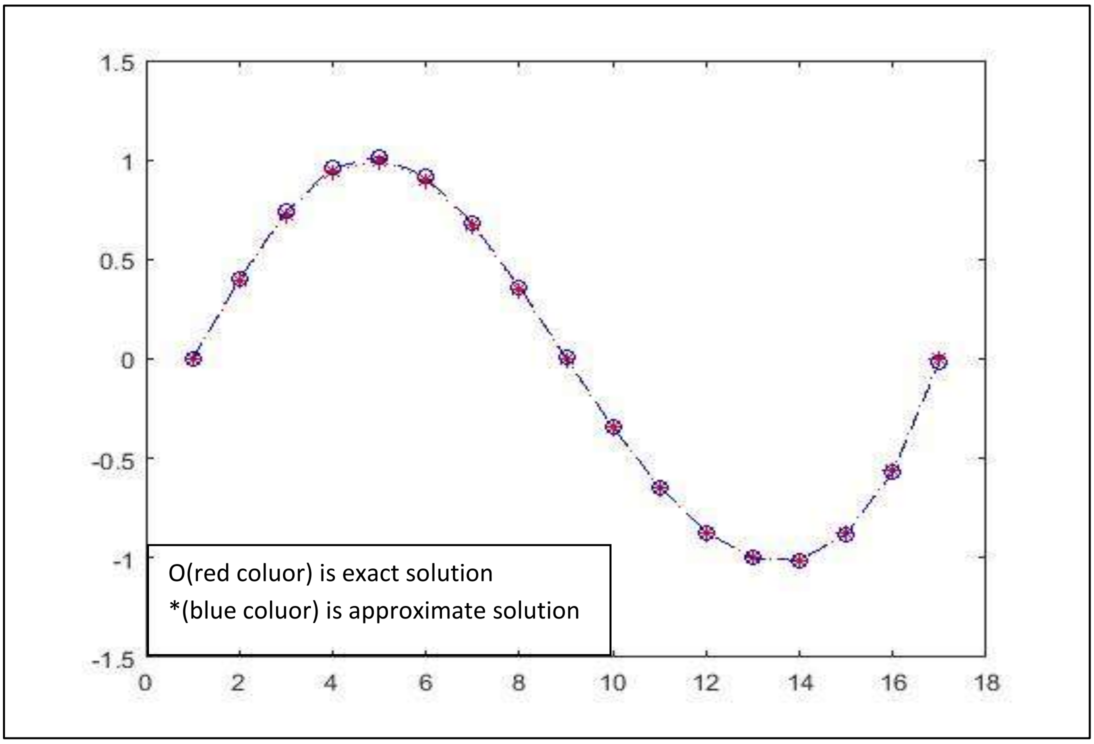

Figure 4.2: Exact solution and approximate solution of Example 1, when $h = \frac{1}{16}$

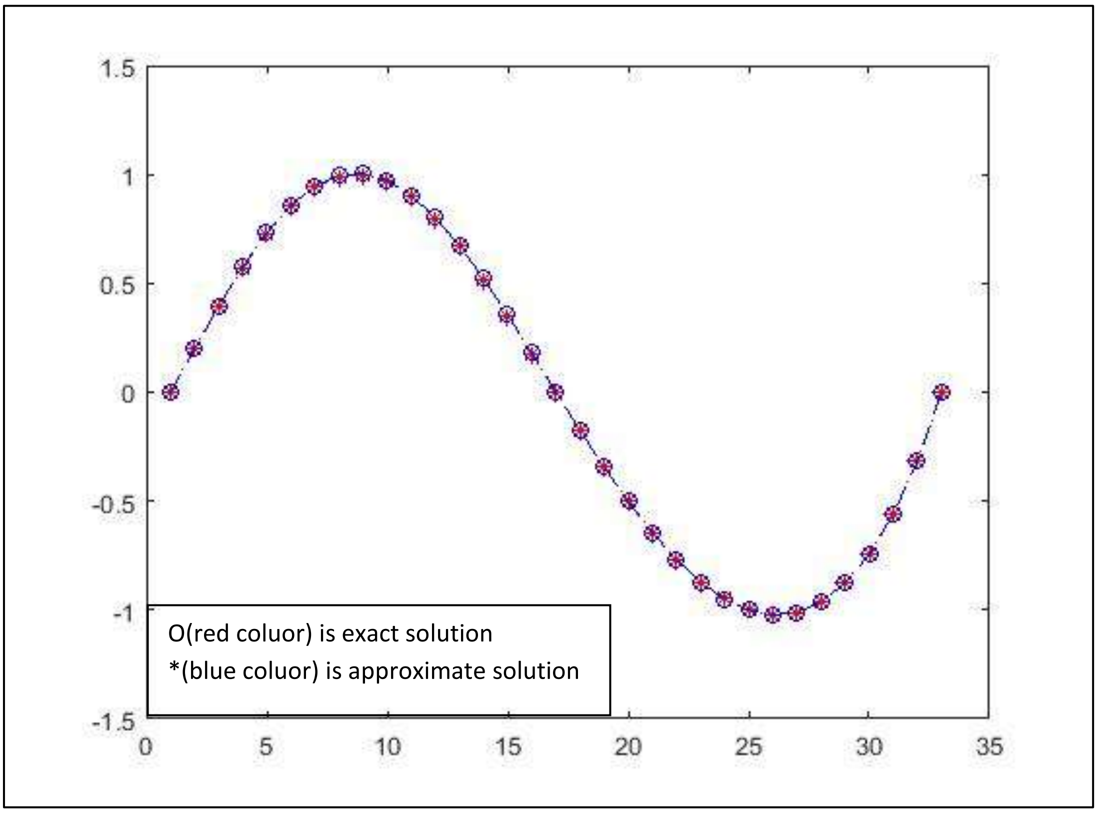

Figure 4.3: Exact solution and approximate solution of Example 1, when $h = \frac{1}{32}$

Example 2: We consider equation (4.1), with $u(x) = e^{\pi x \left(\frac{1}{2}-x\right)\left(\frac{5}{6}-x\right)}.sin(\pi x)$, as the exact solution on the interval [0,1], with the boundary conditions $u(0) = 0$ and $u(1) = u\left(\frac{1}{2}\right) - 2u\left(\frac{5}{6}\right)$, hence $\alpha_1 = 1, \alpha_2 = -2$ and $\xi_1 = \frac{1}{2}, \xi_2 = \frac{5}{6}$, where, ξ_1, is grid point and ξ_2, is not grid point.

By equation (4.1), for $f(x)$, we obtain

$$f(x) = [\,\pi^2 - \pi^2((5/12) - (8/3)x + 3x^2)^2 + \pi \left(\frac{8}{3} - 6x\right) +$$

$$(x^2+1)]\, e^{\pi x \left(\frac{1}{2}-x\right)\left(\frac{5}{6}-x\right)}.sin(\pi x) \; -$$

$$2\pi^2((5/12) - (8/3)x + 3x^2)\, e^{\pi x \left(\frac{1}{2}-x\right)\left(\frac{5}{6}-x\right)}.\cos(\pi x)$$

The following finite difference analogue of problem (4.1), is used for the approximation of the solution.

$$(y_{\bar{x}})_x - (x^2+1)y_i = -f(x_i)\,, \;\; i = 1, \dots, n-1,$$

$$\frac{(y_{i+1} - 2y_i + y_{i-1})}{h^2} - (x^2+1)y_i = -f(x_i)\,, \;\; i = 1, \dots, n-1.$$

Hence, rearranging

$$y_i = \frac{(y_{i+1}+y_{i-1}) + h^2 f(x_i)}{2+(x^2+1)h^2}\,,$$

where $h = \frac{1}{n}, n \geq 2$, with the boundary conditions

$$y_0 = 0\,, \;\; y_n = y_{\left(\frac{n}{2}\right)} - 2\{\, y_i \frac{[(\mathrm{i}-1)\,\mathrm{h} - \xi_2]}{\mathrm{h}} + \mathrm{y}_{i+1} \frac{[\xi_2 - (\mathrm{i})\,\mathrm{h}]}{\mathrm{h}} \}$$

Results are demonstrated in Table (4.2).

Table (4.2), represents the results obtained, where $\varepsilon_h = \|u - y_h\|$ and $h = 2^{-m}$, $m = 3, 4, 5, 6$ is the difference between the exact solution and approximate solution, in the maximum norm, and $R_h = \frac{\|u - y_{2^{-m}}\|}{\|u - y_{2^{-(m+1)}}\|}$, is the order of convergence of the solution.

Table 4.2: Maximum Error between exact solution and approximate solution and the order of convergence of the solution of Example 2.

$\boldsymbol{h}$	$\boldsymbol{\varepsilon_h}$	$\boldsymbol{R_h}$
$\frac{1}{8}$	0.0215	$\simeq$ 3.90909091
$\frac{1}{16}$	0.0055	$\simeq$ 4.23076923
$\frac{1}{32}$	0.0013	$\simeq$ 3.8927980
$\frac{1}{64}$	3.3395e-04	

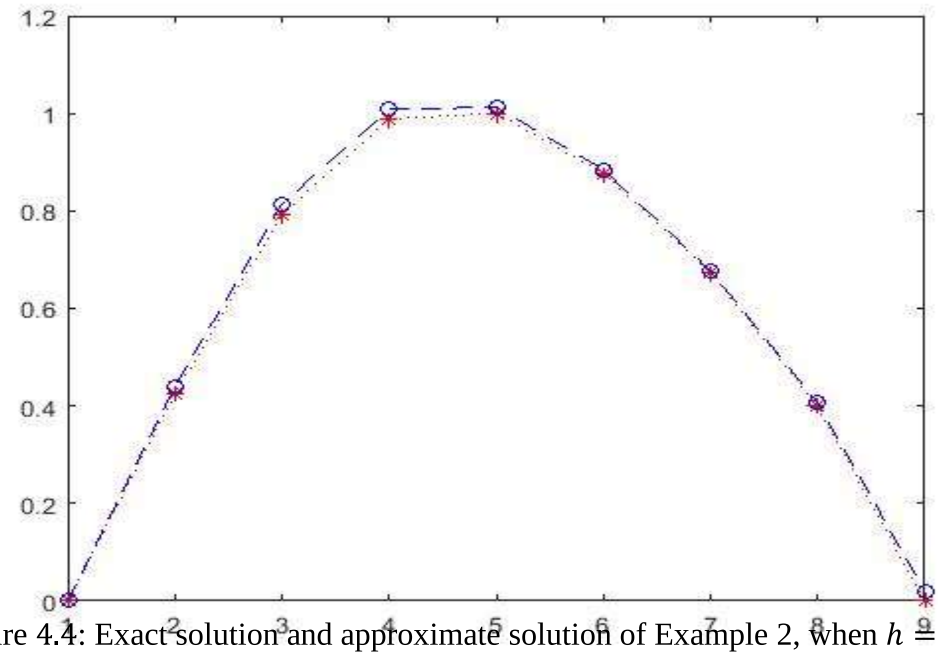

Figure 4.4: Exact solution and approximate solution of Example 2, when $h = \frac{1}{8}$

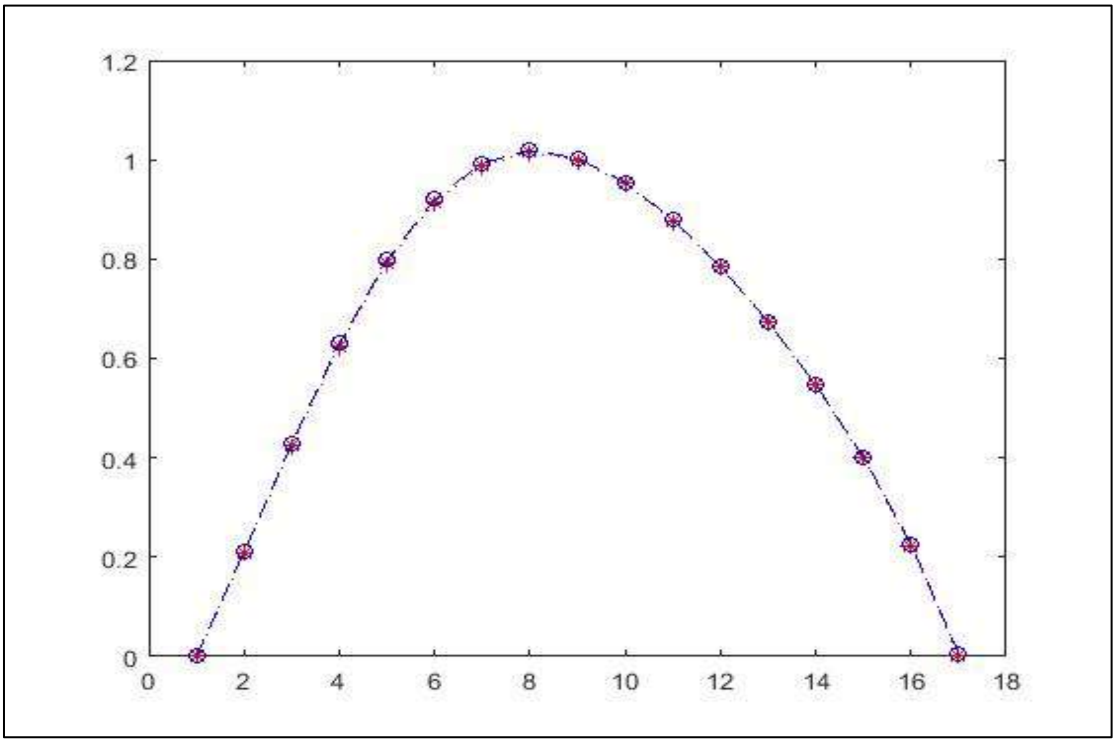

Figure 4.5: Exact solution and approximate solution of Example 2, when $h = \frac{1}{16}$

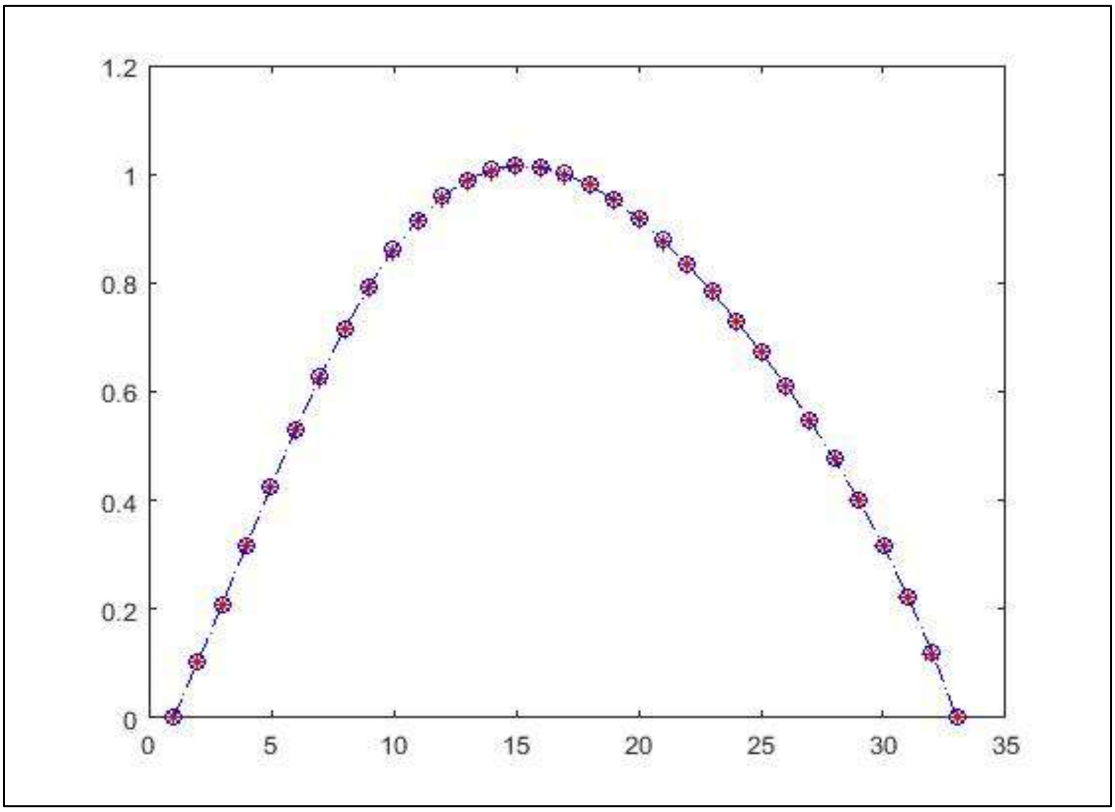

Figure 4.6: Exact solution and approximate solution of Example 2, when $h = \frac{1}{32}$

Chapter 5

CONCLUSION

In this work a wide class of nonlocal problems have been studied, and the necessary and sufficient conditions for the existence and uniqueness of the solution of both the continuous and the difference problems have been provided.

Chapter 2 was concerned with the analysis of a second-order ordinary differential equations with (Bitsadze-Samarskii) type nonlocal boundary conditions. These results were also proved to be true for the difference analogue of the equation.

The investigation of a nonlocal boundary value problem of the first kind was given in detail and specifics in Chapter 3. Again, both the analytical and difference analogue of the equation have been considered.

Finally, numerical experiments have been provided in order to illustrate the theoretical results in Chapter 4. The examples have been calculated using the MATLAB programming Language and the results are consistent with the theoretical results discussed.

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