Lovelock black holes with a power-Yang–Mills source

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ARTICLE INFO

ABSTRACT

We consider the standard Yang–Mills (YM) invariant raised to the power q, i.e., $(F_{\mu\nu}^{(a)}F^{(a)\mu\nu})^q$ as the source of our geometry and investigate the possible black hole solutions. How does this parameter q modify the black holes in Einstein–Yang–Mills (EYM) and its extensions such as Gauss–Bonnet (GB) and the third order Lovelock theories? The advantage of such a power q (or a set of superposed members of the YM hierarchies) if any, may be tested even in a free YM theory in flat spacetime. Our choice of the YM field is purely magnetic in any higher-dimensions so that duality makes no sense. In analogy with the Einstein-power-Maxwell theory, the conformal invariance provides further reduction, albeit in a spacetime for dimensions of multiples of 4.

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1. Introduction

N-dimensional static, spherically symmetric Einstein–Yang–Mills (EYM) black hole solutions in general relativity are well-known by now for which we refer to [1], and references cited therein. YM theory's non-linearity naturally adds further complexity to the already non-linear gravity, thus expectedly the theory and its accompanied solutions become rather complicated. Extension of the Einstein–Hilbert (EH) action with further non-linearities, such as Gauss–Bonnet (GB) or Lovelock have also been considered. These latter theories involve higher order invariants in such combinations that the field equations remain second order.

More recently there has been aroused interest about black hole solutions whose source is a power of the Maxwell scalar i.e., $(F_{\mu\nu}F^{\mu\nu})^q$, where q is an arbitrary positive real number [2]. Subsequently this will be developed easily into a hierarchies of YM terms. In the standard Maxwell theory we have $q = 1$, whereas now the choice $q \neq 1$ is also taken into account which adds to the theory a new dimension of non-linearity from the electromagnetism. Non-linear electrodynamics, such as Born–Infeld (BI) involves a kind of non-linearity that is more familiar for a long time [3]. From the outset we express that the non-linearity involved in the power-Maxwell formalism is radically different from that of BI. An infinite series expansion of the square root term in the latter reveals this fact. For the special choice $q = \frac{N}{4}$, where $N =$ dimension of the spacetime is a multiple of 4, it yields a traceless Maxwell's energy–momentum tensor which leads to conformal invariance. That is, in the absence of different fields such as self-interacting massless scalar field and/or a cosmological constant we have a vanishing scalar curvature. This implies a relatively simpler geometry under the invariance $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ and naturally attracts interest. The absence of black hole solutions in higher dimensions for a self-interacting scalar field was proved long time ago [4]. Self-interacting Maxwell field with a power of invariant, however, which conformally interacts with gravity admits black hole solutions [2].

Being motivated by the black holes sourced by the power of Maxwell's invariant we investigate in this work the existence of black holes with a power of YM source. That is we shall choose our source as $(F_{\mu\nu}^{(a)}F^{(a)\mu\nu})^q$ (and also $\sum_{k=0}^{q} b_k (F_{\mu\nu}^{(a)}F^{(a)\mu\nu})^k$, with constant coefficients b_k) where $F_{\mu\nu}^{(a)}$ is the YM field with its internal index $1 \leqslant a \leqslant \frac{1}{2}(N-1)(N-2)$ and q is a real number such that $q = 1$ recovers the EYM black holes. Similar to the power-Maxwell case we obtain the conformally invariant YM black holes with a zero trace for the energy–momentum tensor. It turns out in analogy that the dimensions of spacetimes are multiples of 4. The power q can be chosen arbitrary provided the conformal property is lost. It will also be shown that $q < 0$, will lead to the violation of the energy and causality conditions. This will restrict us only to the choice $q > 0$. As before, our magnetically charged YM field consists of the Wu–Yang ansatz in any higher dimensions [1]. The EYM metric function admits an integral proportional to $\sim \frac{\ln r}{r^2}$ for $N = 5$ and $\sim \frac{1}{r^2}$ for all $N > 5$. The fixed r-dependence for $N > 5$, was considered to be unusual i.e., a drawback or advantage, depending on the region of interest. Now with

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the choice of the power q on the YM invariant we obtain dependence on q as well, which brings extra r-dependence in the metric. The possible set of integer q values in each $N > 5$ is determined by the validity of the energy conditions. For $N = 4$ and 5 we show that $q = 1$, necessarily, but for $N > 5$ we can't accommodate $q = 1$ unless we violate some energy conditions.

We consider next the GB (i.e., second order Lovelock) and successively Lovelock's third order term added to the first order EH Lagrangian. Our source term throughout the Letter is the YM invariant raised to the power q (and its hierarchies). In each case, separately or together, we seek solutions to what we call, the Einstein-power-YM (EPYM) field with GB and Lovelock terms. It is remarkable that such a highly non-linear theory with non-linearities in various forms admits black hole solutions and in the appropriate limits, with $q = 1$, it yields all the previously known solutions. In the presence of both the second and third order Lovelock terms, however, we impose for technical reasons an algebraic condition between their parameters. This we do for the simple reason that the most general solution involving both the second and third order terms is technically far from being tractable. Useful thermodynamic quantities such as the Hawking temperature, specific heat and free energy are determined and briefly discussed.

Organization of the Letter is as follows. Section 2 contains the action, field equations, energy–momentum for EPYM gravity and solutions to the field equations. Sections 3 and 4 follow a similar pattern for the GB and third order Lovelock theories, respectively. Yang–Mills hierarchies are discussed in Section 5. We complete the Letter with Conclusion which appears in Section 6.

2. Field equations and the metric ansatz for EPYM gravity

The $N(= n + 2)$-dimensional action for Einstein-power-Yang–Mills (EPYM) gravity with a cosmological constant Λ is given by ($8\pi G = 1$)

$$I = \frac{1}{2} \int_{\mathcal{M}} d^{n+2}x \sqrt{-g} \left(R - \frac{n(n+1)}{3} \Lambda - \mathcal{F}^q \right), \tag{1}$$

in which $\mathcal{F}$ is the YM invariant

$$\mathcal{F} = \mathbf{Tr}\left(F_{\lambda\sigma}^{(a)} F^{(a)\lambda\sigma}\right), \quad \mathbf{Tr}(.) = \sum_{a=1}^{n(n+1)/2} (.), \tag{2}$$

R is the Ricci Scalar and q is a positive real parameter. Here the YM field is defined as

$$\mathbf{F}^{(a)} = \mathbf{dA}^{(a)} + \frac{1}{2\sigma} C_{(b)(c)}^{(a)} \mathbf{A}^{(b)} \wedge \mathbf{A}^{(c)} \tag{3}$$

in which $C_{(b)(c)}^{(a)}$ stands for the structure constants of $\frac{n(n+1)}{2}$-parameter Lie group G, σ is a coupling constant and $\mathbf{A}^{(a)}$ are the $SO(n+1)$ gauge group YM potentials. The determination of the components $C_{(b)(c)}^{(a)}$ has been described elsewhere [5]. We note that the internal indices $\{a, b, c, \ldots\}$ do not differ whether in covariant or contravariant form. Variation of the action with respect to the spacetime metric $g_{\mu\nu}$ yields the field equations

$$G^{\mu}{}_{\nu} + \frac{n(n+1)}{6} \Lambda \delta^{\mu}{}_{\nu} = T^{\mu}{}_{\nu}, \tag{4}$$

$$T^{\mu}{}_{\nu} = -\frac{1}{2} \left(\delta^{\mu}{}_{\nu} \mathcal{F}^q - 4q \, \mathbf{Tr}\left(F_{\nu\lambda}^{(a)} F^{(a)\,\mu\lambda}\right) \mathcal{F}^{q-1} \right), \tag{5}$$

where $G_{\mu\nu}$ is the Einstein tensor. Variation with respect to the gauge potentials $\mathbf{A}^{(a)}$ yields the YM equations

$$\mathbf{d}\left(^{\star}\mathbf{F}^{(a)} \mathcal{F}^{q-1}\right) + \frac{1}{\sigma} C_{(b)(c)}^{(a)} \mathcal{F}^{q-1} \mathbf{A}^{(b)} \wedge {}^{\star}\mathbf{F}^{(c)} = 0, \tag{6}$$

where * means duality. It is readily observed that for $q = 1$ our formalism reduces to the standard EYM theory. Our objective in this work therefore is to study the role of the parameter q in the black holes. Our metric ansatz for $N(= n + 2)$ dimensions, is chosen as

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\,d\Omega_n^2, \tag{7}$$

in which $f(r)$ is our metric function and

$$d\Omega_n^2 = d\theta_1^2 + \sum_{i=2}^{n} \prod_{j=1}^{i-1} \sin^2 \theta_j \, d\theta_i^2, \tag{8}$$

where

$$0 \leqslant \theta_n \leqslant 2\pi, \qquad 0 \leqslant \theta_i \leqslant \pi, \qquad 1 \leqslant i \leqslant n - 1.$$

The choice of these metrics can be traced back to the form of the stress–energy tensor (5), which satisfies $T_0^0 - T_1^1 = 0$ (see Eq. (12) below) and consequently $G_0^0 - G_1^1 = 0$, whose explicit form, on integration, gives $|g_{00}g_{11}| = C = \text{constant}$. We need only to choose the time scale at infinity to make this constant equal to unity.

Table 1
Energy conditions WEC, SEC and DEC and the causality condition (CC) versus the admissible ranges of parameter q.

	WEC	SEC	DEC	CC
$q < 0$	no	no	no	no
$0 \leqslant q < \frac{n}{4}$	yes	no	no	no
$\frac{n}{4} \leqslant q < \frac{n+1}{4}$	yes	yes	no	no
$\frac{n+1}{4} \leqslant q < \frac{n+1}{2}$	yes	yes	yes	yes
$\frac{n+1}{2} < q$	yes	yes	yes	no

2.1. Energy–momentum tensor

Recently we have introduced and used the higher dimensional version of the Wu–Yang ansatz in EYM theory of gravity [1,5]. In this ansatz we express the Yang–Mills magnetic gauge potential one-forms as

$$\mathbf{A}^{(a)} = \frac{Q}{r^2} C^{(a)}_{(i)(j)} x^i \, dx^j, \quad Q = \text{YM magnetic charge}, \, r^2 = \sum_{i=1}^{n+1} x_i^2, \tag{9}$$

$$2 \leqslant j + 1 \leqslant i \leqslant n + 1, \quad \text{and} \quad 1 \leqslant a \leqslant n(n+1)/2,$$

$$x_1 = r \cos\theta_{n-1} \sin\theta_{n-2} \cdots \sin\theta_1, \qquad x_2 = r \sin\theta_{n-1} \sin\theta_{n-2} \cdots \sin\theta_1,$$

$$x_3 = r \cos\theta_{n-2} \sin\theta_{n-3} \cdots \sin\theta_1, \qquad x_4 = r \sin\theta_{n-2} \sin\theta_{n-3} \cdots \sin\theta_1,$$

$$\cdots$$

$$x_n = r \cos\theta_1.$$

One can easily show that these ansaetze satisfy the YM equations [1,5]. In consequence, the energy–momentum tensor (5), with

$$\mathcal{F} = \frac{n(n-1)Q^2}{r^4}, \tag{10}$$

$$\mathbf{Tr}\left(F^{(a)}_{\theta_i \lambda} F^{(a)\theta_i \lambda}\right) = \frac{(n-1)Q^2}{r^4} = \frac{1}{n}\mathcal{F} \tag{11}$$

becomes

$$T^a_{\ b} = -\frac{1}{2}\mathcal{F}^q \, \text{diag}[1, 1, \kappa, \kappa, \ldots, \kappa], \quad \text{and} \quad \kappa = \left(1 - \frac{4q}{n}\right). \tag{12}$$

We observe that the trace of $T^a_{\ b}$ is $T = -\frac{1}{2}\mathcal{F}^q(N - 4q)$ which vanishes for the particular case $q = \frac{N}{4}$. It is also remarkable to give the intervals of q in which the Weak Energy Condition (WEC), Strong Energy Condition (SEC), Dominant Energy Condition (DEC) and Causality Condition (CC) are satisfied [6]. It is observed from Table 1 that the physically meaningful range for q is $\frac{n+1}{4} \leqslant q < \frac{n+1}{2}$, which satisfies all the energy and causality conditions. The choice $q < 0$, violates all these conditions so it must be discarded. In the sequel we shall use this energy–momentum tensor to find black hole solutions for the EPYM, EPYMGB and EPYMGBL field equations with the cosmological constant Λ.

2.2. EPYM black hole solution for $N \geqslant 5$ dimensions

In $N(= n + 2) \geqslant 5$ dimensions the rr component of Einstein equation reads

$$\frac{3(n(n-1)Q^2)^q}{r^{2(2q-1)}} + 3n\left[rg'(r) + (n-1)g(r) + \frac{\Lambda}{3}(n+1)r^2\right] = 0, \tag{13}$$

in which $f(r) = 1 + g(r)$. Direct integration leads to the following solutions

$$f(r) = \begin{cases} 1 - \frac{4m}{nr^{n-1}} - \frac{Q_1}{r^{4q-2}} - \frac{\Lambda}{3}r^2, & q \neq \frac{n+1}{4}, \\ 1 - \frac{4m}{nr^{n-1}} - \frac{Q_2 \ln r}{r^{n-1}} - \frac{\Lambda}{3}r^2, & q = \frac{n+1}{4}, \end{cases}$$

$$Q_1 = \frac{((n-1)nQ^2)^q}{n(n+1-4q)}, \qquad Q_2 = \frac{((n-1)nQ^2)^{\frac{n+1}{4}}}{n}, \tag{14}$$

where m is the ADM mass of the black hole. It is observed that physical properties of such a black hole depends on the parameter q. The location of horizons, $f(r_h) = 0$, involves an algebraic equation whose roots can be found numerically. The entropy, Hawking temperature and other thermodynamics properties all can be calculated accordingly and they are dependent on q. Table 1 shows that the minimum possible value for q which provides all the energy conditions to be satisfied is given by $q_{\min} = \frac{n+1}{4}$, that is, the case of solution with logarithmic term. In 5 dimensions $q_{\min} = 1$, which recovers the usual EYM solution found in [1,5]. With the exception of $N = 5$ where $q = 1$ is part of possible q's (which satisfy all the energy conditions), in higher dimensions q must be greater than one. For instance, in 6 dimensions $\frac{5}{4} \leqslant q < \frac{5}{2}$ and in 7 dimensions $\frac{3}{2} \leqslant q < 3$. If one constrains q to be an integer, Table 2 gives the possible q values in some dimensions. From this table we can identify the dimensions in which the logarithmic term appears naturally. These are $N = 5, 9, 13, \ldots,$

Table 2
List of some possible integer q values versus N.

Dimensions N	**5**	6	7	8	**9**	10	11	12	**13**
possible integer q	**1**	2	2	2, 3	**2**, 3	3, 4	3, 4	3, 4, 5	**3**, 4, 5

for which $q_{\min} = \frac{N-1}{4}$ is an integer. Let us remark that since for $N = 4$ our YM field gauge transforms to an Abelian form [7], our results become automatically valid also for $N = 4$.

We observe that although the metric function $f(r)$ at infinity goes to $-\frac{\Lambda}{3}r^2$ its behavior about the origin is quite different and strongly depends on q i.e.,

$$\lim_{r \to 0} f(r) \to \begin{cases} -\frac{4m}{nr^{n-1}} \to -\infty, & q < \frac{n+1}{4}, \\ \frac{((n-1)nQ^2)^{\frac{n+1}{4}}}{nr^{n-1}} \ln(\frac{1}{r}) \to +\infty, & q = \frac{n+1}{4}, \\ \frac{((n-1)nQ^2)^q}{n(4q-n-1)r^{4q-2}} \to +\infty, & q > \frac{n+1}{4}. \end{cases} \tag{15}$$

This is important because for the case of $q \geqslant \frac{n+1}{4}$ one may adjust the mass and charge to have a metric function in contradiction with the Cosmic Censorship Conjecture (CCC). One statement of this conjecture is that all singularities (here $r = 0$) are hidden behind event horizons. Of course, nature may restrict Q and m in order not to violate this conjecture.

Note that $r = 0$ is a singularity for the metric whose Ricci scalar is given by

$$R = \begin{cases} \frac{((n-1)nQ^2)^q(n+2-4q)}{nr^{4q}}, & q \neq \frac{n+1}{4}, \\ \frac{((n-1)nQ^2)^{\frac{n+1}{4}}}{nr^{n+1}}, & q = \frac{n+1}{4}. \end{cases} \tag{16}$$

2.3. Extremal black holes

Closely related with a usual black hole is an extremal black hole whose horizons coincide. As it is well known to get extremal solution one should solve $f(r) = 0$, and $f'(r) = 0$ simultaneously. This set of equations for the solution (14), without cosmological constant, leads to

$$r_e = \left(n(n-1)\right)^{\frac{q-1}{2(2q-1)}} Q^{\left(\frac{q}{2q-1}\right)}, \tag{17}$$

$$m_e = \begin{cases} \frac{(2q-1)Q^{\left(\frac{q}{2q-1}\right)}}{2(4q-n-1)} n^{\frac{3q-1-n+qn}{2(2q-1)}} (n-1)^{\frac{(n-1)(q-1)}{2(2q-1)}}, & q \neq \frac{n+1}{4}, \\ \frac{(n-1)^{\frac{n-3}{4}}Q^{\frac{n+1}{2}}}{8}\{n^{\frac{n+1}{4}}(2 + \frac{3}{2}\ln\frac{n(n-1)}{Q^{2/3}}) - \frac{n^{\frac{n+5}{4}}}{2}\ln(n(n-1)Q^2)\}, & q = \frac{n+1}{4}, \end{cases} \tag{18}$$

where r_e is the radius of degenerate horizon and m_e and Q are the extremal mass and charge of the black hole, respectively. One may check the case of $q = 1$, resulting in

$$r_e = Q, \qquad m_e = \frac{n}{2(3-n)}Q, \tag{19}$$

which clearly in 4 dimensions gives $r_e = Q = m_e$, as it should.

2.4. Thermodynamics of the EPYM black hole

In this section we present some thermodynamical properties of EPYM black hole solution with cosmological constant. Here it is convenient to rescale our quantities in terms of some different powers of radius of the horizon r_h, i.e., we introduce

$$\check{T}_H = T_H r_h, \qquad \check{M}_{\text{ADM}} = M_{\text{ADM}}/r_h^{n-1}, \qquad \check{\Lambda} = \Lambda r_h^2, \qquad \check{Q}_i = Q_i/r_h^{2(2q-1)}, \qquad \check{C} = C/r_h^n \quad \text{and} \quad \check{F} = F/r^{n-1}, \tag{20}$$

where $T_H = f'(r_h)/4\pi$ is the Hawking temperature, $C = C_Q = T_H(\frac{\partial S}{\partial T_H})_Q$ is the heat capacity for constant Q and $F = M_{\text{ADM}} - T_H S$ is the free energy of the black hole as a thermodynamical system. Therein

$$S = \frac{A}{4} = \frac{(n+1)\pi^{\left(\frac{n+1}{2}\right)}}{4\Gamma(\frac{n+3}{2})}r_h^n \tag{21}$$

is the Bekenstein–Hawking entropy where $\Gamma(.)$ stands for the gamma function. As one may notice in (14) m represent the ADM mass of the black hole. This helps us to write

$$\check{M}_{\text{ADM}} = \check{m} = \begin{cases} \frac{n}{4}(1 - \check{Q}_1 - \frac{\check{\Lambda}}{3}), & q \neq \frac{n+1}{4}, \\ \frac{n}{4}(1 - \check{Q}_2 \ln(r_h) - \frac{\check{\Lambda}}{3}), & q = \frac{n+1}{4}, \end{cases} \tag{22}$$

which imposes some restrictions on $\check{Q}_i$ and $\check{\Lambda}$ in order to have a positive and physically acceptable $\check{M}_{\text{ADM}}$.

In terms of the event horizon r_h Hawking temperature becomes

$$\check{T}_H = \begin{cases} \dfrac{-\check{Q}_1(n+1-4q)-\frac{\check{\Lambda}}{3}(n+1)+(n-1)}{4\pi}, & q \neq \frac{n+1}{4}, \\[2mm] \dfrac{-\check{Q}_2-\frac{\check{\Lambda}}{3}(n+1)+(n-1)}{4\pi}, & q = \frac{n+1}{4}. \end{cases} \tag{23}$$

For the case of $q \neq \frac{n+1}{4}$ clearly by imposing $\check{M}_{ADM}, \check{T}_H > 0$ one finds $\frac{\check{\Lambda}}{3} < (1 - \frac{1}{2q})$ and for the case of $q = \frac{n+1}{4}$ and choosing $r_h = 1$, one gets $\frac{\check{\Lambda}}{3} < 1 - \frac{\check{Q}_2+2}{n+1}$. The heat capacity $\check{C}$ is given by

$$\check{C} = \begin{cases} \dfrac{n}{2} \dfrac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \dfrac{\frac{(n+1)\check{\Lambda}}{3}+(n+1-4q)\check{Q}_1-(n-1)}{\frac{\check{\Lambda}}{3}(n+1)-\check{Q}_1(n+1-4q)(4q-1)+n-1}, & q \neq \frac{n+1}{4}, \\[4mm] \dfrac{n}{2} \dfrac{\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \dfrac{\frac{(n+1)\check{\Lambda}}{3}-(n-1)+\check{Q}_2}{(n+1)\frac{\check{\Lambda}}{3}-n\check{Q}_2+(n-1)}, & q = \frac{n+1}{4}, \end{cases} \tag{24}$$

which reveals the thermodynamic instability of the black hole. In fact the possible roots of denominator of $\check{C}$ present a phase transition which can be interpreted as thermodynamical instability.

For completeness we give also the free energy F of our black hole as a thermodynamical system, which is

$$\check{F} = \begin{cases} \dfrac{[\check{Q}_1(n+1-4q)+\frac{\check{\Lambda}(n+1)}{3}+1-n]\pi^{\frac{n-1}{2}}-2n(\check{Q}_1+\frac{\check{\Lambda}}{3}-1)\Gamma(\frac{n+1}{2})}{8\Gamma(\frac{n+1}{2})}, & q \neq \frac{n+1}{4}, \\[4mm] \dfrac{[\check{Q}_2+\frac{\check{\Lambda}(n+1)}{3}-(n-1)]\pi^{\frac{n-1}{2}}-2n(\check{Q}_2\ln(r_h)+(\frac{\check{\Lambda}}{3}-1))\Gamma(\frac{n+1}{2})}{8\Gamma(\frac{n+1}{2})}, & q = \frac{n+1}{4}. \end{cases} \tag{25}$$

By letting $q = 1$ and $n = 2$ for the 4-dimensional Reissner–Nordström metric, the foregoing expressions become

$$M_{ADM} = m = \frac{r_h}{2}\left(1 + \frac{Q^2}{r_h^2}\right), \tag{26}$$

$$T_H = \frac{f'(r_h)}{4\pi} = \frac{1}{4\pi r_h}\left(1 - \frac{Q^2}{r_h^2}\right), \tag{27}$$

$$S_{BH} = \pi r_h^2, \tag{28}$$

$$C_Q = -\frac{2\pi[1 - \frac{Q^2}{r_h^2}]r_h^2}{[1 - 3\frac{Q^2}{r_h^2}]}, \tag{29}$$

$$F = \left(1 + \frac{3Q^2}{r_h^2}\right)\frac{r_h}{4}. \tag{30}$$

3. Field equations and the metric ansatz for EPYMGB gravity

The EPYMGB action in $N(=n+2)$ dimensions is given by $(8\pi G = 1)$

$$I = \frac{1}{2}\int_{\mathcal{M}} d^{n+2}x \sqrt{-g}\left(R - \frac{n(n+1)}{3}\Lambda + \alpha\mathcal{L}_{GB} - \mathcal{F}^q\right), \tag{31}$$

where α is the GB parameter and $\mathcal{L}_{GB}$ is given by

$$\mathcal{L}_{GB} = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2. \tag{32}$$

Variation of the new action with respect to the spacetime metric $g_{\mu\nu}$ yields the field equations

$$G^E_{\mu\nu} + \alpha G^{GB}_{\mu\nu} + \frac{n(n+1)}{6}\Lambda g_{\mu\nu} = T_{\mu\nu}, \tag{33}$$

where

$$G^{GB}_{\mu\nu} = 2\left(-R_{\mu\sigma\kappa\tau}R^{\kappa\tau\sigma}{}_\nu - 2R_{\mu\rho\nu\sigma}R^{\rho\sigma} - 2R_{\mu\sigma}R^\sigma{}_\nu + RR_{\mu\nu}\right) - \frac{1}{2}\mathcal{L}_{GB}g_{\mu\nu}, \tag{34}$$

and $T_{\mu\nu}$ is given by (12).

3.1. EPYMGB black hole solution for $N \geqslant 5$ dimensions

As before, the rr component of Einstein equation (33) can be written as

$$\frac{3(n(n-1)Q^2)^q}{r^{4(q-1)}} + 3n\left[(r^3 - 2\tilde{\alpha}_2 rg(r))g'(r) - \tilde{\alpha}_2(n-2)g(r)^2 + r^2(n-1)g(r) + \frac{\Lambda}{3}(n+1)r^4\right] = 0, \tag{35}$$

in which $\tilde{\alpha}_2 = (n-1)(n-2)\alpha_2$. This equation admits a solution as

$$f_\pm(r) = \begin{cases} 1 + \frac{r^2}{2\tilde{\alpha}_2}(1 \pm \sqrt{1 + \frac{4}{3}\Lambda\tilde{\alpha}_2 + \frac{16m\tilde{\alpha}_2}{nr^{n+1}} + \frac{4\tilde{\alpha}_2 Q_1}{r^{4q}}}), & q \neq \frac{n+1}{4}, \\ 1 + \frac{r^2}{2\tilde{\alpha}_2}(1 \pm \sqrt{1 + \frac{4}{3}\Lambda\tilde{\alpha}_2 + \frac{16m\tilde{\alpha}_2}{nr^{n+1}} + \frac{4\tilde{\alpha}_2 Q_2 \ln(r)}{r^{n+1}}}), & q = \frac{n+1}{4}. \end{cases} \tag{36}$$

The asymptotic behavior of the metric reveals that

$$\lim_{r\to\infty} f_\pm(r) \to \begin{cases} 1 + \frac{r^2}{2\tilde{\alpha}_2}(1 \pm \sqrt{1 + \frac{4}{3}\Lambda\tilde{\alpha}_2}), & q < \frac{n+1}{4}, \\ 1 + \frac{r^2}{2\tilde{\alpha}_2}(1 \pm \sqrt{1 + \frac{4}{3}\Lambda\tilde{\alpha}_2}), & q = \frac{n+1}{4}, \\ 1 + \frac{r^2}{2\tilde{\alpha}_2}(1 \mp \sqrt{1 + \frac{4}{3}\Lambda\tilde{\alpha}_2}), & q > \frac{n+1}{4}, \end{cases} \tag{37}$$

which depending on Λ it is de Sitter, Anti-de Sitter or flat. Abiding by the (anti) de Sitter limit for $\tilde{\alpha}_2 \to 0$, we must choose the $(-)$ sign.

3.2. Thermodynamics of the EPYMGB black hole

By using the above rescaling plus $\check{\alpha}_2 = \tilde{\alpha}_2/r_h^2$, one can find the Hawking temperature of the EPYMGB black hole solutions (36) as

$$\check{T}_H(-) = \begin{cases} -\frac{\check{Q}_1(n+1-4q) + \frac{\check{\Lambda}}{3}(n+1) - (n-1) - \check{\alpha}_2(n-3)}{4\pi(1+2\check{\alpha}_2)}, & q \neq \frac{n+1}{4}, \\ -\frac{\check{Q}_2 + \frac{\check{\Lambda}}{3}(n+1) - (n-1) - \check{\alpha}_2(n-3)}{4\pi(1+2\check{\alpha}_2)}, & q = \frac{n+1}{4}, \end{cases} \tag{38}$$

$$\check{T}_H(+) = \begin{cases} \frac{\check{Q}_1\check{\alpha}_2(n+1-4q) + \frac{\check{\Lambda}}{3}\check{\alpha}_2(n+1) - \check{\alpha}_2^2(n-3) - \check{\alpha}_2(n-5) + 2}{4\pi\check{\alpha}_2(1+2\check{\alpha}_2)}, & q \neq \frac{n+1}{4}, \\ \frac{\check{Q}_2\check{\alpha}_2 + \frac{\check{\Lambda}}{3}\check{\alpha}_2(n+1) - \check{\alpha}_2^2(n-3) - \check{\alpha}_2(n-5) + 2}{4\pi\check{\alpha}_2(1+2\check{\alpha}_2)}, & q = \frac{n+1}{4}, \end{cases} \tag{39}$$

here $(\pm)$ state the correspondence branches. Here we observe that $\check{T}_H(-)$ in the limit of $\check{\alpha}_2 \to 0$ correctly reduces to the Hawking temperature of EPYM black hole (23) as expected. It is remarkable to observe that $\check{\alpha}_2 = -\frac{1}{2}$ is a point of infinite temperature, or instability of the black hole. This means that if $\tilde{\alpha}_2/r_h^2 = -\frac{1}{2}$, the black hole will be unstable. For the positive branch one should be careful about $\check{\alpha}_2 \to 0$ which is not applicable.

In the sequel we give the other thermodynamical properties of the BH solution (36) in separate cases.

3.2.1. Negative branch $q \neq \frac{n+1}{4}$

The ADM mass:

$$\check{M}_{\text{ADM}} = \check{m} = \frac{n}{4}\left(1 + \check{\alpha}_2 - \check{Q}_1 - \frac{\check{\Lambda}}{3}\right). \tag{40}$$

The heat capacity:

$$\check{C} = \frac{n}{2}\pi^{\frac{n+1}{2}}\frac{(\check{\alpha}_2 + \frac{1}{2})}{\Gamma(\frac{n+1}{2})}\frac{\frac{\check{\Lambda}}{3}(n+1) + (n+1-4q)\check{Q}_1 - (n-3)\check{\alpha}_2 - (n-1)}{\{(n+1)(\check{\alpha}_2 + \frac{1}{6})\check{\Lambda} + (n-3)\check{\alpha}_2^2 - 4[(q-\frac{3}{4})\check{\alpha}_2 + \frac{1}{2}(q-\frac{1}{4})]\check{Q}_1(1+n-4q) + (\frac{n-7}{2})\check{\alpha}_2 + (\frac{n-1}{2})\}}. \tag{41}$$

The free energy:

$$\check{F} = \frac{[\check{Q}_1(n+1-4q) + \frac{\check{\Lambda}(n+1)}{3} - (n-3)\check{\alpha}_2 + 1 - n]\pi^{\frac{n-1}{2}} - 4n(\check{Q}_1 + \frac{\check{\Lambda}}{3} - 1 - \check{\alpha}_2)(\check{\alpha}_2 + \frac{1}{2})\Gamma(\frac{n+1}{2})}{8(1+2\check{\alpha}_2)\Gamma(\frac{n+1}{2})}. \tag{42}$$

3.2.2. Negative branch $q = \frac{n+1}{4}$

The ADM mass:

$$\check{M}_{\text{ADM}} = \check{m} = \frac{n}{4}\left(1 + \check{\alpha}_2 - \check{Q}_2 \ln(r_h) - \frac{\check{\Lambda}}{3}\right), \tag{43}$$

The heat capacity:

$$\check{C} = n\pi^{\frac{n+1}{2}}\frac{(\check{\alpha}_2 + \frac{1}{2})}{\Gamma(\frac{n+1}{2})}\frac{\frac{(n+1)\check{\Lambda}}{3} - (n-1) - (n-3)\check{\alpha}_2 + \check{Q}_2}{(n+1)(1+6\check{\alpha}_2)\frac{\check{\Lambda}}{3} - (2(n-2)\check{\alpha}_2 + n)\check{Q}_2 + 2(n-3)\check{\alpha}_2^2 + (n-7)\check{\alpha}_2 + (n-1)}. \tag{44}$$

The free energy:

$$\check{F} = \frac{[\check{Q}_2 + \frac{\check{\Lambda}(n+1)}{3} - (n-3)\check{\alpha}_2 - (n-1)]\pi^{\frac{n-1}{2}} - 4n(\check{Q}_2 \ln(r_h) + (\frac{\check{\Lambda}}{3} - 1) - \check{\alpha}_2)(\check{\alpha}_2 + \frac{1}{2})\Gamma(\frac{n+1}{2})}{8(1+2\check{\alpha}_2)\Gamma(\frac{n+1}{2})}. \tag{45}$$

3.2.3. Positive branch $q \neq \frac{n+1}{4}$

The ADM mass:

$$\check{M}_{\text{ADM}} = \check{m} = \frac{n}{4}\left(1 + \check{\alpha}_2 - \check{Q}_1 - \frac{\check{\Lambda}}{3}\right).$$
(46)

The heat capacity:

$$\check{C} = -\frac{n}{2}\pi^{\frac{n+1}{2}}\frac{(\check{\alpha}_2 + \frac{1}{2})}{\Gamma(\frac{n+1}{2})}$$

$$\times \frac{-\frac{\check{\Lambda}}{3}(n+1)\check{\alpha}_2 - (n+1-4q)\check{\alpha}_2\check{Q}_1 + (n-3)\check{\alpha}_2^2 + (n-5)\check{\alpha}_2 - 2}{\{(n+1)(\check{\alpha}_2 + \frac{1}{6})\check{\alpha}_2\check{\Lambda} + (n-3)\check{\alpha}_2^3 - 4[(q-\frac{3}{4})\check{\alpha}_2 + \frac{1}{2}(q-\frac{1}{4})]\check{\alpha}_2\check{Q}_1(1+n-4q) + (\frac{n+1}{2})\check{\alpha}_2^2 + (\frac{n+7}{2})\check{\alpha}_2 + 1\}}.$$
(47)

The free energy:

$$\check{F} = \frac{[-\check{Q}_1\check{\alpha}_2(n+1-4q) - \frac{\check{\Lambda}(n+1)}{3}\check{\alpha}_2 + (n-3)\check{\alpha}_2^2 + (n-5)\check{\alpha}_2 - 2]\pi^{\frac{n-1}{2}} - 4n\check{\alpha}_2(\check{Q}_1 + \frac{\check{\Lambda}}{3} - 1 - \check{\alpha}_2)(\check{\alpha}_2 + \frac{1}{2})\Gamma(\frac{n+1}{2})}{8\check{\alpha}_2(1 + 2\check{\alpha}_2)\Gamma(\frac{n+1}{2})}.$$
(48)

3.2.4. Positive branch $q = \frac{n+1}{4}$

The ADM mass:

$$\check{M}_{\text{ADM}} = \check{m} = \frac{n}{4}\left(1 + \check{\alpha}_2 - \check{Q}_2\ln(r_h) - \frac{\check{\Lambda}}{3}\right).$$
(49)

The heat capacity:

$$\check{C} = n\pi^{\frac{n+1}{2}}\frac{(\check{\alpha}_2 + \frac{1}{2})}{\Gamma(\frac{n+1}{2})}\frac{\frac{\check{\Lambda}}{3}(n+1)\check{\alpha}_2 - (n-5)\check{\alpha}_2 - (n-3)\check{\alpha}_2^2 + \check{Q}_2\check{\alpha}_2 + 2}{\{2(n+1)(\check{\alpha}_2 + \frac{1}{6})\check{\alpha}_2\check{\Lambda} + 2(n-3)\check{\alpha}_2^3 - [2(n-2)\check{\alpha}_2 + n]\check{\alpha}_2\check{Q}_2 + (n+1)\check{\alpha}_2^2 + (n+7)\check{\alpha}_2 + 2\}}.$$
(50)

The free energy:

$$\check{F} = \frac{[-\check{Q}_2\check{\alpha}_2 - \frac{\check{\Lambda}(n+1)}{3}\check{\alpha}_2 + (n-3)\check{\alpha}_2^2 + (n-5)\check{\alpha}_2 - 2]\pi^{\frac{n-1}{2}} - 4n\check{\alpha}_2(\check{Q}_2\ln r_h + (\frac{\check{\Lambda}}{3} - 1 - \check{\alpha}_2))(\check{\alpha}_2 + \frac{1}{2})\Gamma(\frac{n+1}{2})}{8\check{\alpha}_2(1 + 2\check{\alpha}_2)\Gamma(\frac{n+1}{2})}.$$
(51)

Finally in this section we look at $\check{C}$ which clearly, in general, vanishes at $\check{\alpha}_2 = -\frac{1}{2}$. Also any possible root for the denominator of $\check{C}$ gives instability point or a phase transition.

4. Field equations and the metric ansatz for EPYMGBL gravity

In this section we consider a more general action which involves, beside the GB term, the third order Lovelock term [8,9]. The EPYMGBL action in $N(=n+2)$ dimensions is given by $(8\pi G = 1)$

$$I = \frac{1}{2}\int_{\mathcal{M}} d^{n+2}x\sqrt{-g}\left(R - \frac{n(n+1)}{3}\Lambda + \alpha_2\mathcal{L}_{\text{GB}} + \alpha_3\mathcal{L}_{(3)} - \mathcal{F}^q\right),$$
(52)

where α_2 and α_3 are the second and third order Lovelock parameters respectively, and [8]

$$\mathcal{L}_{(3)} = 2R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\rho\tau}R^{\rho\tau}{}_{\mu\nu} + 8R^{\mu\nu}{}_{\sigma\rho}R^{\sigma\kappa}{}_{\nu\tau}R^{\rho\tau}{}_{\mu\kappa} + 24R^{\mu\nu\sigma\kappa}R_{\sigma\kappa\nu\rho}R^{\rho}{}_{\mu} + 3RR^{\mu\nu\sigma\kappa}R_{\sigma\kappa\mu\nu}$$

$$+ 24R^{\mu\nu\sigma\kappa}R_{\sigma\mu}R_{\kappa\nu} + 16R^{\mu\nu}R_{\nu\sigma}R^{\sigma}{}_{\mu} - 12RR^{\mu\nu}R_{\mu\nu} + R^3,$$
(53)

is the third order Lovelock Lagrangian. Variation of the new action with respect to the spacetime metric $g_{\mu\nu}$ yields the field equations

$$G_{\mu\nu} + \alpha_2 G^{\text{GB}}_{\mu\nu} + \alpha_3 G^{(3)}_{\mu\nu} + \frac{n(n+1)}{6}\Lambda g_{\mu\nu} = T_{\mu\nu},$$
(54)

where

$$G^{(3)}_{\mu\nu} = -3\big(4R^{\tau\rho\sigma\kappa}R_{\sigma\kappa\lambda\rho}R^{\lambda}{}_{\nu\tau\mu} - 8R^{\tau\rho}{}_{\lambda\sigma}R^{\sigma\kappa}{}_{\tau\mu}R^{\lambda}{}_{\nu\rho\kappa} + 2R_{\nu}{}^{\tau\sigma\kappa}R_{\sigma\kappa\lambda\rho}R^{\lambda\rho}{}_{\tau\mu} - R^{\tau\rho\sigma\kappa}R_{\sigma\kappa\tau\rho}R_{\nu\mu} + 8R^{\tau}{}_{\nu\sigma\rho}R^{\sigma\kappa}{}_{\tau\mu}R^{\rho}{}_{\kappa}$$

$$+ 8R^{\sigma}{}_{\nu\tau\kappa}R^{\tau\rho}{}_{\sigma\mu}R^{\kappa}{}_{\rho} + 4R_{\nu}{}^{\tau\sigma\kappa}R_{\sigma\kappa\mu\rho}R^{\rho}{}_{\tau} - 4R_{\nu}{}^{\tau\sigma\kappa}R_{\sigma\kappa\tau\rho}R^{\rho}{}_{\mu} + 4R^{\tau\rho\sigma\kappa}R_{\sigma\kappa\tau\mu}R_{\nu\rho} + 2RR_{\nu}{}^{\kappa\tau\rho}R_{\tau\rho\kappa\mu}$$

$$+ 8R^{\tau}{}_{\nu\mu\rho}R^{\rho}{}_{\sigma}R^{\sigma}{}_{\tau} - 8R^{\sigma}{}_{\nu\tau\rho}R^{\tau}{}_{\sigma}R^{\rho}{}_{\mu} - 8R^{\tau\rho}{}_{\sigma\mu}R^{\sigma}{}_{\tau}R_{\nu\rho} - 4RR^{\tau}{}_{\nu\mu\rho}R^{\rho}{}_{\tau} + 4R\tau\rho R_{\rho\tau}R_{\nu\mu} - 8R^{\tau}{}_{\nu}R_{\tau\rho}R^{\rho}{}_{\mu}$$

$$+ 4RR_{\nu\rho}R^{\rho}{}_{\mu} - R^2 R_{\nu\mu}\big) - \frac{1}{2}\mathcal{L}_{(3)}g_{\mu\nu}.$$
(55)

4.1. EPYMGBL black hole solution for $N(=n+2) \geqslant 7$ dimensions

As before we start with the rr component of Einstein equation which reads

$$\frac{3(n(n-1)Q^2)^q}{r^{4(q-1)}} + 3n\bigg[\big(r^5 - 2\tilde{\alpha}_2 r^3 g(r) + 3rg^2\big)g'(r) + \tilde{\alpha}_3(n-5)r^2 g(r)^3$$

$$- \tilde{\alpha}_2(n-3)r^2 g(r)^2 + r^4(n-1)g(r) + \frac{\Lambda}{3}(n+1)r^6\bigg] = 0, \tag{56}$$

where $\tilde{\alpha}_3 = (n-1)(n-2)(n-3)(n-4)\alpha_3$.

4.1.1. The particular case of $\tilde{\alpha}_3 = \tilde{\alpha}_2^2/3$

In the third order Lovelock theory we first prefer to impose a condition on Lovelock's parameters such as $\tilde{\alpha}_3 = \tilde{\alpha}_2^2/3$. This helps us to work with less complicity and in the sequel for the sake of completeness we shall present the general solution without this restriction as well. The metric function after this condition is given by

$$f(r) = \begin{cases} 1 + \frac{r^2}{\tilde{\alpha}_2}(1 - \sqrt[3]{1 + \Lambda\tilde{\alpha}_2 + \frac{12m\tilde{\alpha}_2}{nr^{n+1}} + \frac{3\tilde{\alpha}_2 Q_1}{r^{4q}}}), & q \neq \frac{n+1}{4}, \\ 1 + \frac{r^2}{\tilde{\alpha}_2}(1 - \sqrt[3]{1 + \Lambda\tilde{\alpha}_2 + \frac{12m\tilde{\alpha}_2}{nr^{n+1}} + \frac{3\tilde{\alpha}_2 Q_2 \ln r}{r^{n+1}}}), & q = \frac{n+1}{4}, \end{cases} \tag{57}$$

where as usual m is the mass of the black hole. One may find

$$\lim_{r \to \infty} f(r) \to 1 + \frac{r^2}{\tilde{\alpha}_2}\big(1 - \sqrt[3]{1 + \Lambda\tilde{\alpha}_2}\big) \quad (\Lambda > 0) \tag{58}$$

which gives the asymptotical behavior of the metric such as de Sitter, Anti-de Sitter or flat ($\Lambda = 0$). We note that in the limit $\tilde{\alpha}_2 \to 0$, we have $f(r) \to 1 - \frac{\Lambda}{3}r^2$, as it should.

4.1.2. The case of arbitrary $\tilde{\alpha}_2, \tilde{\alpha}_3$

The general solution of the metric function for the case of EPMGBL is given by

$$f(r) = \begin{cases} 1 + \frac{\tilde{\alpha}_2 r^2}{3\tilde{\alpha}_3}(1 + \frac{\sqrt[3]{\Delta}}{2\omega n\tilde{\alpha}_2 r^{n+1+2q}} + \frac{2\omega n(\tilde{\alpha}_2^2 - 3\tilde{\alpha}_3)r^{n+1+2q}}{\sqrt[3]{\Delta}\tilde{\alpha}_2}), & q \neq \frac{n+1}{4}, \\ 1 + \frac{\tilde{\alpha}_2 r^2}{3\tilde{\alpha}_3}(1 + \frac{\sqrt[3]{\tilde{\Delta}}}{2n\tilde{\alpha}_2 r^{n+1}} + \frac{2n(\tilde{\alpha}_2^2 - 3\tilde{\alpha}_3)r^{n+1}}{\sqrt[3]{\tilde{\Delta}}\tilde{\alpha}_2}), & q = \frac{n+1}{4}, \end{cases} \tag{59}$$

where

$$\Delta = 36\omega^2 n^2 r^{2(1+n+q)}\big(\sqrt{\delta}\tilde{\alpha}_3 - 3Q_1^2\tilde{\alpha}_3^2 r^{1+n} - \omega r^{4q}\zeta\big),$$

$$\delta = \big(3Q_1\tilde{\alpha}_3^2 r^{1+n}\big)^2 + 6Q_1\omega r^{4q+1+n}\zeta + \frac{\omega^2 r^{8q}}{\tilde{\alpha}_3^2}\bigg\{\zeta^2 - \big(\tilde{\alpha}_2^2 - 3\tilde{\alpha}_3\big)^3\bigg(\frac{2n}{9}r^{1+n}\bigg)^2\bigg\},$$

$$\zeta = \lambda n r^{1+n} + \tilde{\alpha}_3^2 m, \qquad \omega = 1 + n - 4q, \qquad \lambda = \alpha_3^2\Lambda + \tilde{\alpha}_2\tilde{\alpha}_3 - \frac{2}{9}\tilde{\alpha}_2^3, \qquad Q_1 = n^q(n-1)^q Q^{2q}, \tag{60}$$

and

$$\tilde{\Delta} = 36n^2 r^{2(1+n)}\big(3\tilde{\alpha}_3\sqrt{\tilde{\delta}} - \tilde{\zeta}\big), \qquad \tilde{\delta} = -\bigg(\frac{2nr^{1+n}}{3\tilde{\alpha}_3}\bigg)^2\big(\tilde{\alpha}_2^2 - 3\tilde{\alpha}_3\big)^3 + \frac{9}{\tilde{\alpha}_3^2}\tilde{\zeta}^2,$$

$$\tilde{\zeta} = \tilde{\alpha}_3^2\chi + \lambda n r^{1+n}, \qquad \chi = 3Q_2 \ln r + m, \qquad Q_2 = n^{\frac{1+n}{4}}(n-1)^{\frac{1+n}{4}} Q^{\frac{1+n}{2}}. \tag{61}$$

Occurrence of the roots naturally restricts the ranges of parameters since the results must be real and physically admissible.

Here also one can find the nature of metric at infinity, namely

$$\lim_{r \to \infty} f(r) \to 1 + \Lambda_{\text{eff}} r^2 \tag{62}$$

where

$$\Lambda_{\text{eff}} = \frac{1}{9\tilde{\alpha}_3}\bigg(-9\sqrt[3]{\frac{\lambda}{6}} + \big(3\tilde{\alpha}_3 - \tilde{\alpha}_2^2\big)\sqrt[3]{\frac{6}{\lambda}} + 3\tilde{\alpha}_2\bigg). \tag{63}$$

4.2. Thermodynamics of the EPYMGBL black hole

As before, we complete this chapter by giving some thermodynamical properties of the EPYMGB black hole solution. Clearly, working analytically with the arbitrary $\tilde{\alpha}_2, \tilde{\alpha}_3$ may not be possible therefore we only stress on the specific case of $\tilde{\alpha}_3 = \tilde{\alpha}_2^2/3$. Given this particular choice, we start with the ADM mass of the BH which reads

$$\check{M}_{\text{ADM}} = \check{m} = \begin{cases} \frac{n}{4}(1 + \check{\alpha}_2(\frac{\check{\alpha}_2}{3} + 1) - \check{Q}_1 - \frac{\check{\Lambda}}{3}), & q \neq \frac{n+1}{4}, \\ \frac{n}{4}(1 + \check{\alpha}_2(\frac{\check{\alpha}_2}{3} + 1) - \check{Q}_1 \ln(r_h) - \frac{\check{\Lambda}}{3}), & q = \frac{n+1}{4}, \end{cases} \tag{64}$$

whose Hawking temperature is given by

$$\check{T}_H = \begin{cases} \dfrac{-\check{Q}_1(n+1-4q)-\frac{\check{\Lambda}}{3}(n+1)+(n-5)\frac{\check{\alpha}_2^2}{3}+\check{\alpha}_2(n-3)+n-1}{4\pi(1+\check{\alpha}_2)^2}, & q \neq \frac{n+1}{4}, \\[4mm] \dfrac{-\check{Q}_1-\frac{\check{\Lambda}}{3}(n+1)+(n-5)\frac{\check{\alpha}_2^2}{3}+\check{\alpha}_2(n-3)+n-1}{4\pi(1+\check{\alpha}_2)^2}, & q = \frac{n+1}{4}. \end{cases} \tag{65}$$

We notice here that the Hawking temperature diverges as $\check{\alpha}_2$ approaches to -1.

4.2.1. $q \neq \frac{n+1}{4}$

The heat capacity:

$$\check{C} = \frac{n}{2}\pi^{\frac{n+1}{2}}\frac{(\check{\alpha}_2+1)}{\Gamma(\frac{n+1}{2})}\frac{\frac{\check{\Lambda}}{3}(n+1)+(n+1-4q)\check{Q}_1-(n-5)\frac{\check{\alpha}_2^2}{3}-(n-3)\check{\alpha}_2-(n-1)}{\{(1+n)(5\check{\alpha}_2+1)\frac{\check{\Lambda}}{3}-4[(q-\frac{5}{4})\check{\alpha}_2+(q-\frac{1}{4})]\check{Q}_1(1+n-4q)+(n-5)\frac{\check{\alpha}_2^3}{3}+\frac{2}{3}(n-8)\check{\alpha}_2^2-6\check{\alpha}_2+n-1\}}. \tag{66}$$

The free energy:

$$\check{F} = \frac{\{[\check{Q}_1(n+1-4q)+\frac{\check{\Lambda}(n+1)}{3}-(n-5)\frac{\check{\alpha}_2^2}{3}-(n-3)\check{\alpha}_2+1-n]\pi^{\frac{n-1}{2}}-2n(\check{Q}_1+\frac{\check{\Lambda}}{3}-1-\check{\alpha}_2-\frac{\check{\alpha}_2^2}{3})(\check{\alpha}_2+1)^2\Gamma(\frac{n+1}{2})\}}{8(1+\check{\alpha}_2)^2\Gamma(\frac{n+1}{2})}. \tag{67}$$

4.2.2. $q = \frac{n+1}{4}$

The heat capacity:

$$\check{C} = \frac{n}{2}\pi^{\frac{n+1}{2}}\frac{(\check{\alpha}_2+1)}{\Gamma(\frac{n+1}{2})}\frac{\frac{\check{\Lambda}}{3}(n+1)+\check{Q}_2-(n-5)\frac{\check{\alpha}_2^2}{3}-(n-3)\check{\alpha}_2-(n-1)}{\{(1+n)(5\check{\alpha}_2+1)\frac{\check{\Lambda}}{3}-[(n-4)\check{\alpha}_2+n]\check{Q}_2+(n-5)\frac{\check{\alpha}_2^3}{3}+\frac{2}{3}(n-8)\check{\alpha}_2^2-6\check{\alpha}_2+n-1\}}. \tag{68}$$

The free energy:

$$\check{F} = \frac{[\check{Q}_2+\frac{\check{\Lambda}(n+1)}{3}-(n-5)\frac{\check{\alpha}_2^2}{3}-(n-3)\check{\alpha}_2+1-n]\pi^{\frac{n-1}{2}}-2n(\check{Q}_2\ln(r_h)+\frac{\check{\Lambda}}{3}-1-\check{\alpha}_2-\frac{\check{\alpha}_2^2}{3})(\check{\alpha}_2+1)^2\Gamma(\frac{n+1}{2})}{8(1+\check{\alpha}_2)^2\Gamma(\frac{n+1}{2})}. \tag{69}$$

In the foregoing expressions it is observed that for $\check{\alpha}_2 = -1$, the free energy diverges, signalling the occurrence of a critical point. Further, the sign of the heat capacity can be investigated to see whether thermodynamically the system is stable ($\check{C} > 0$) or unstable ($\check{C} < 0$), which will be ignored in this Letter.

5. Yang–Mills hierarchies

In this section we investigate the possible black hole solutions for the case of a superposition of the different power of the YM invariant $\mathcal{F}$ and any further investigation in this line is going to be part of our future study. It is our belief that a detailed analysis of the energy conditions for the YM hierarchy exceeds the limitations of the present Letter, we shall therefore ignore it. The YM hierarchies in d dimensions has been studied by Tchrakian et al. [10] in a different sense. Here we start with an action in the form of

$$I = \frac{1}{2}\int_{\mathcal{M}} d^{n+2}x\sqrt{-g}\left(R+\alpha_2\mathcal{L}_{GB}+\alpha_3\mathcal{L}_{(3)}-\sum_{k=0}^{q}b_k\mathcal{F}^k\right), \tag{70}$$

in which $\mathcal{F}$ is the YM invariant, $b_0 = \frac{n(n+1)}{3}\Lambda$ and $b_{k\geqslant 1}$ is a coupling constant. Variation of the action with respect to the spacetime metric $g_{\mu\nu}$ yields the field equations

$$G^{\mu}{}_{\nu}+\alpha_2 G^{\mu\,GB}{}_{\nu}+\alpha_3 G^{\nu\,(3)}{}_{\mu} = T^{\mu}{}_{\nu}, \tag{71}$$

$$T^{\mu}{}_{\nu} = -\frac{1}{2}\sum_{k=0}^{q}b_k\left(\delta^{\mu}{}_{\nu}\mathcal{F}^k-4k\mathbf{Tr}\left(F^{(a)}_{\nu\lambda}F^{(a)\,\mu\lambda}\right)\mathcal{F}^{k-1}\right), \tag{72}$$

and variation with respect to the gauge potentials $\mathbf{A}^{(a)}$ yields the YM equations

$$\sum_{k=0}^{q}b_k\left\{\mathbf{d}\left({}^{\star}\mathbf{F}^{(a)}\mathcal{F}^{k-1}\right)+\frac{1}{\sigma}C^{(a)}_{(b)(c)}\mathcal{F}^{k-1}\mathbf{A}^{(b)}\wedge{}^{\star}\mathbf{F}^{(c)}\right\} = 0. \tag{73}$$

Our metric ansatz for $N(=n+2)$ dimensions, is given by (7) and the YM field ansatz is as before such that the new energy–momentum tensor reads as

$$T^a{}_b = -\frac{1}{2}\sum_{k=0}^{q}b_k\mathcal{F}^k\,\mathrm{diag}[1,1,\gamma,\gamma,\ldots,\gamma], \quad\text{and}\quad \gamma = \left(1-\frac{4k}{n}\right). \tag{74}$$

The solution of Einstein equation for $\alpha_2 = \alpha_3 = 0$ reveals that

$$f(r) = 1 - \frac{4m}{nr^{n-1}} - \frac{1}{nr^{n-1}}\Psi \tag{75}$$

where m is the ADM mass of the black hole and

$$\Psi = \int r^n \sum_{k=0}^{q} b_k \mathcal{F}^k \, dr = \begin{cases} \sum_{k=0}^{q} b_k \frac{(n(n-1)Q^2)^k}{(n-4k+1)r^{4k-n-1}}, & k \neq \frac{n+1}{4}, \\ b_{\frac{n+1}{4}} (n(n-1)Q^2)^{\frac{n+1}{4}} \ln r + \sum_{k=0 \neq \frac{n+1}{4}}^{q} b_k \frac{(n(n-1)Q^2)^k}{(n-4k+1)r^{4k-n-1}}, & k' = \frac{n+1}{4}. \end{cases} \tag{76}$$

The case of GB which comes after $\alpha_3 = 0$ reveals

$$f_{\pm}(r) = 1 + \frac{r^2}{2\tilde{\alpha}_2}\left(1 \pm \sqrt{1 + \frac{16m\tilde{\alpha}_2}{nr^{n+1}} + \frac{4\tilde{\alpha}_2}{nr^{n+1}}\Psi}\right). \tag{77}$$

For the case of $\alpha_2, \alpha_3 \neq 0$ first we give a solution for the specific choice of $\tilde{\alpha}_3 = \tilde{\alpha}_2^2/3$ which admits

$$f(r) = 1 + \frac{r^2}{\tilde{\alpha}_2}\left(1 - \sqrt[3]{1 + \frac{12m\tilde{\alpha}_2}{nr^{n+1}} + \frac{3\tilde{\alpha}_2}{nr^{n+1}}\Psi}\right), \tag{78}$$

and then the most general solution for $\alpha_2, \alpha_3 \neq 0$ yields a general metric function as

$$f(r) = 1 + \frac{\tilde{\alpha}_2 r^2}{3\tilde{\alpha}_3}\left(1 + \frac{\sqrt[3]{\Delta}}{2n\tilde{\alpha}_2 r^{n+1}} + \frac{2n(\tilde{\alpha}_2^2 - 3\tilde{\alpha}_3)r^{n+1}}{\sqrt[3]{\Delta}\tilde{\alpha}_2}\right), \tag{79}$$

where

$$\Delta = 36n^2 r^{2(1+n)}\left(\frac{\tilde{\alpha}_3}{3}\sqrt{3}\sqrt{\delta} + \left(\tilde{\alpha}_3 - \frac{2}{9}\tilde{\alpha}_2^2\right) - 6\tilde{\alpha}_3^2\left(\frac{1}{2}\Psi + m\right)\right), \tag{80}$$

and

$$\delta = \left(4\tilde{\alpha}_3 - \tilde{\alpha}_2^2\right)n^2 r^{2(n+1)} + 36\tilde{\alpha}_2 nr^{n+1}\left(\tilde{\alpha}_3 - \frac{2}{9}\tilde{\alpha}_2^2\right)\left(\frac{1}{2}\Psi + m\right) + 108\tilde{\alpha}_3^2\left(\frac{1}{2}\Psi + m\right)^2. \tag{81}$$

6. Conclusion

Clearly, the YM invariant/source $\mathcal{F}^q$ becomes simplest for $q = 1$. Beside simplicity there is no valid argument that prevents us from choosing $q \neq 1$. As a result, the latter modifies many black holes obtained from YM field as a source and it specifies also in higher dimensions, which q values are consistent with the energy conditions. For electric type fields there is a drawback that $\mathcal{F}^q$ may not be real for any q, however, this doesn't arise for our pure magnetic type YM field. We note that the same situation is valid also in the power-Maxwell case. In spite of so much non-linearities, including even a YM source of the form $\sum_{k=0}^{q} b_k \mathcal{F}^k$, with the requirement of spherical symmetry we obtained exact black hole solutions to the Lovelock's third order theory. In analogy with the non-linear electrodynamics, the requirement of conformal invariance puts further restrictions on q and the spacetime, namely the dimension of spacetime turns out to be a multiple of 4. Physically, the power q modifies the strength of fields both for $r \to 0$ and $r \to \infty$. It is observed that asymptotically $(r \to \infty)$, irrespective of q the effect of Lovelock gravity, whether at second or third order, becomes equivalent to an effective cosmological constant.

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