Studies on Thin-shells and Thin-shell Wormholes

PhD THESIS DEFENSE, 2016

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Publications

4/19 of my publications which I will present today.

1. Existence of traversable wormholes in the spherical stellar systems
   A. Ovgun, M. Halilsoy. *Astrophys.Space Sci. 361 (2016) 214*

2. On a Particular Thin-shell Wormhole
   A. Ovgun, I. Sakalli. *arXiv:1507.03949 (accepted to publish in TMPh)*

3. Thin-shell wormholes from the regular Hayward black hole
   M. Halilsoy, A. Ovgun, S.H. Mazharimousavi *Eur.Phys.J. C74 (2014) 2796*

4. Tunnelling of vector particles from Lorentzian wormholes in 3+1 dimensions
   I. Sakalli, A. Ovgun. *Eur.Phys.J.Plus 130 (2015) no.6, 110*

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INTRODUCTION
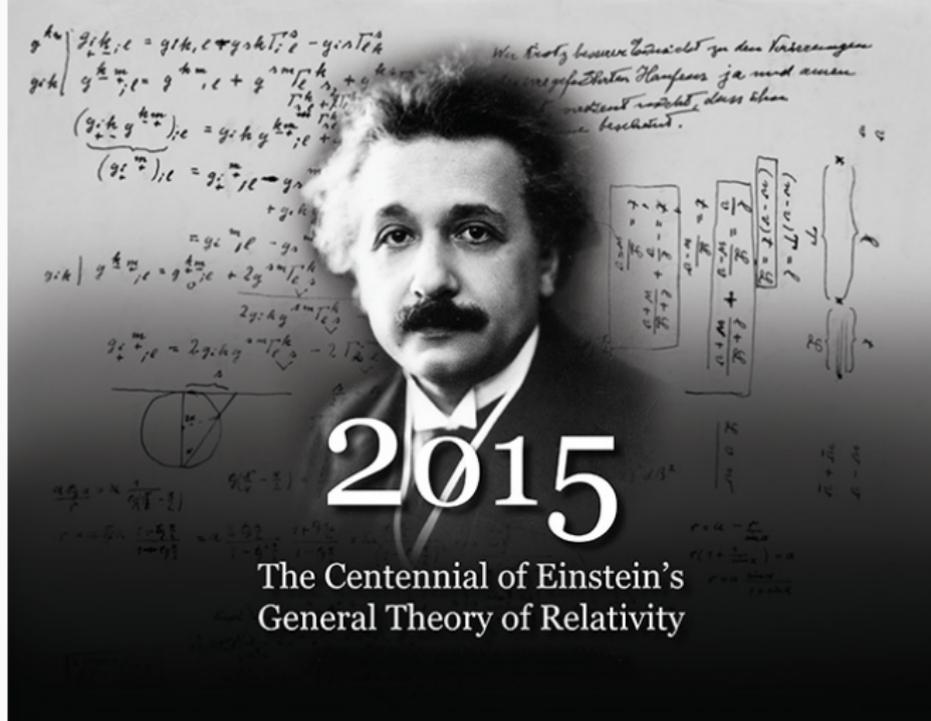
Figure 1: General relativity (GR) is a theory of gravitation that was developed by Albert Einstein between 1907 and 1915

Figure 2: Gravity

**Figure 3:** General relativity explains gravity as the curvature of spacetime

Figure 4: The inter-changeable nature of gravity and acceleration is known as the principle of equivalence.

a closed room on the Earth
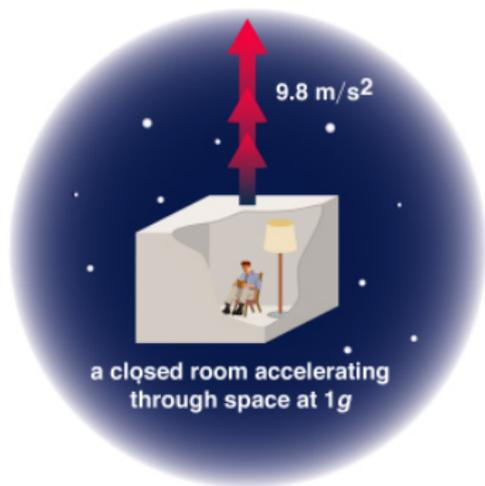
a closed room accelerating through space at 1*g*

9.8 m/s^2

zero motion   constant velocity   acceleration
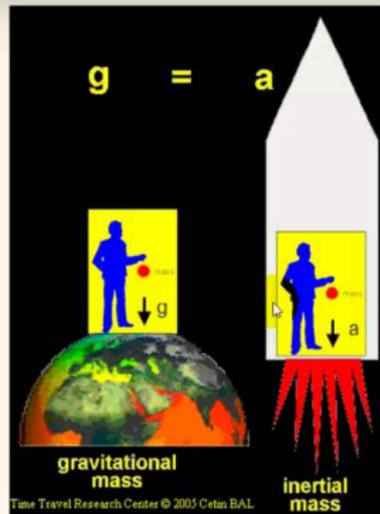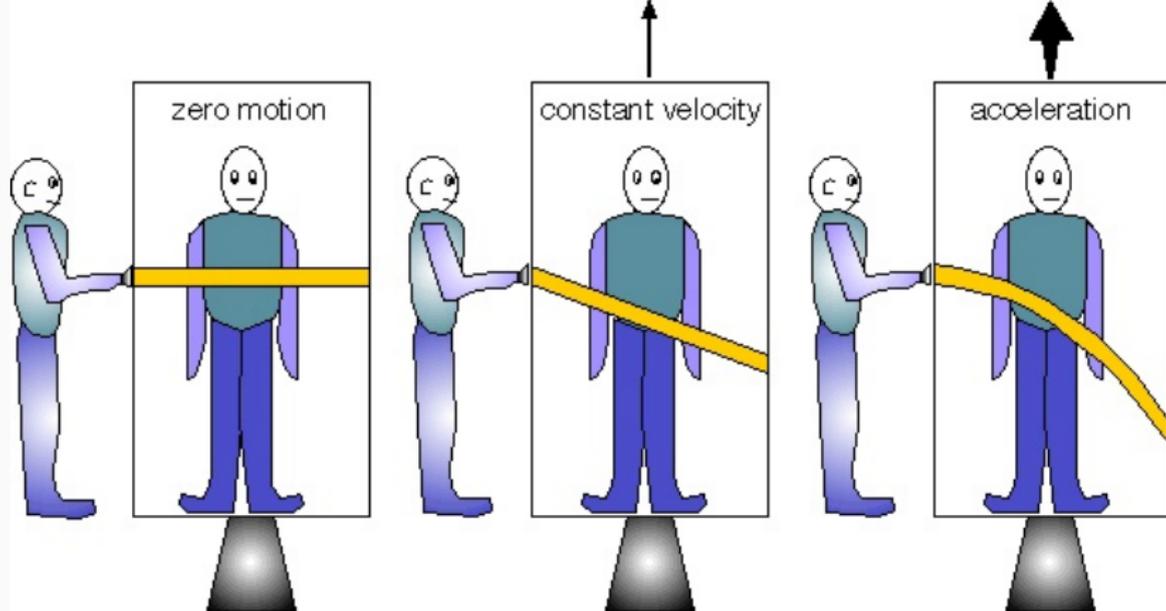
The path of a light beam in three different types of reference frames moving with respect to the person *outside* the elevator. The light path shown is what the person *inside* the elevator sees. Under large acceleration, the beam of light will curve downward. It should also do that in a region of strong gravity.

This rubber sheet represents spacetime curvature around the Sun today.
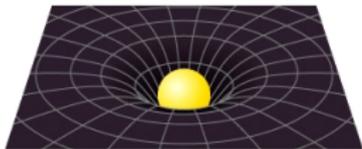
If the Sun became compressed, spacetime would become more curved near its surface (but unchanged farther away).
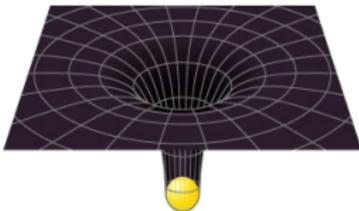
If compression of the Sun continued, the curvature would eventually become great enough to create a **black hole** in the universe.
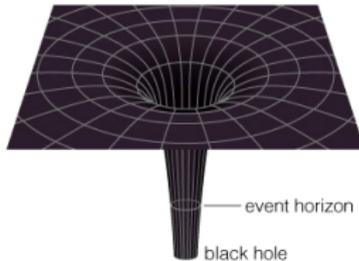
event horizon

black hole

WHAT IS A WORMHOLE?
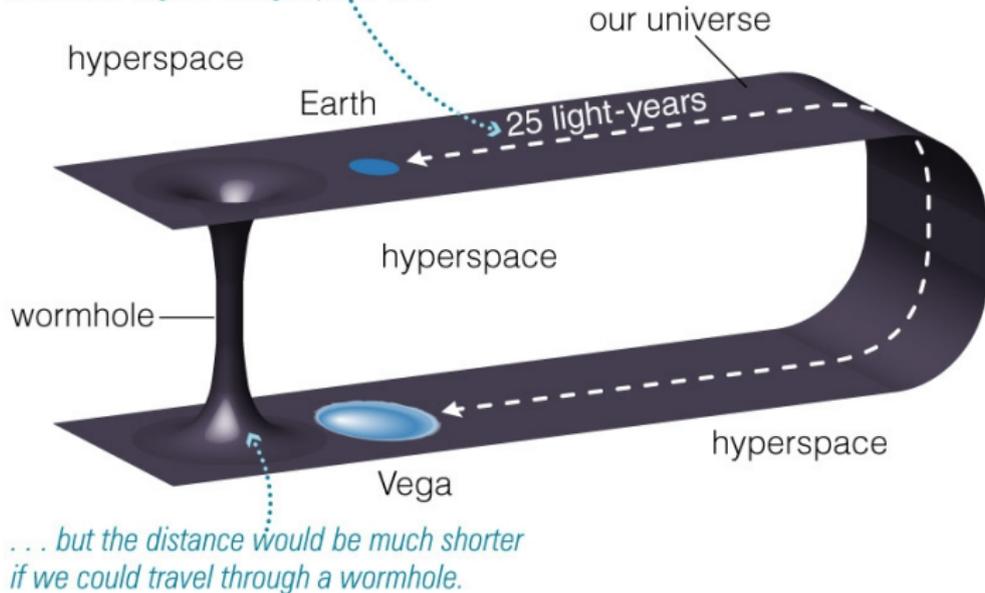
Figure 5: Hypothetical shortcut between distant points
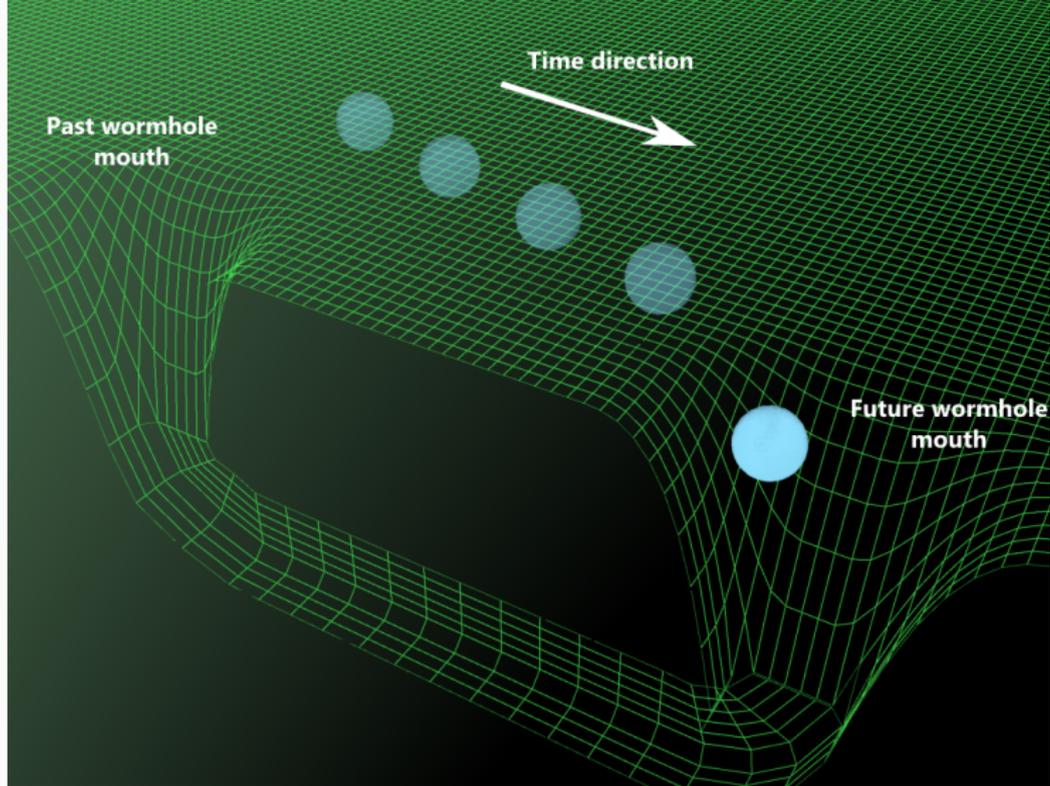
Figure 6: Can we make journeys to farther stars?
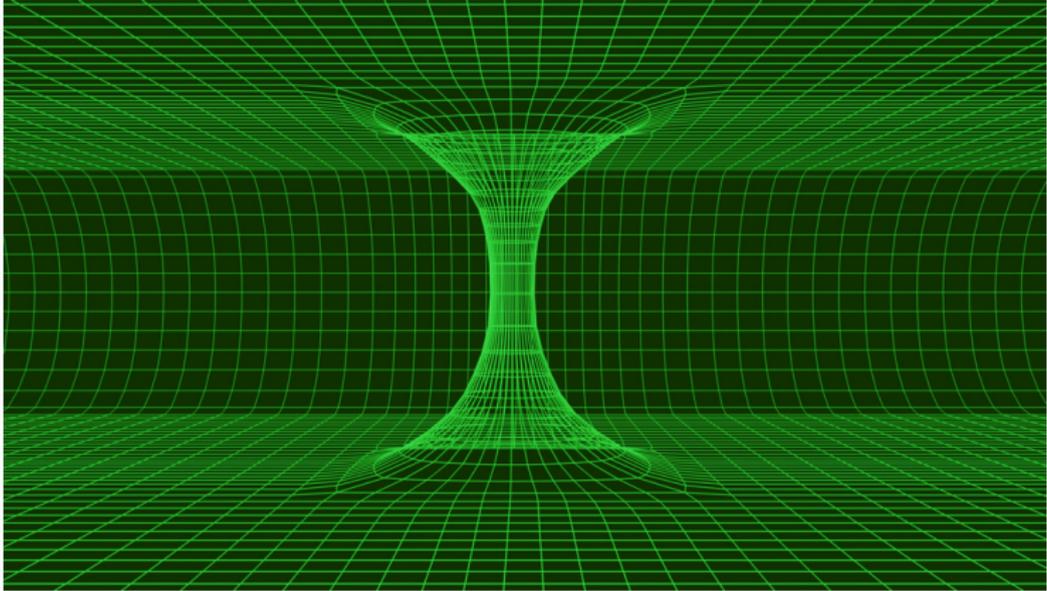
Figure 7: How can we open gate into space-time?

- How can we connect two regions of space-time?
- Can we make stable and traversable wormholes?

MOTIVATION
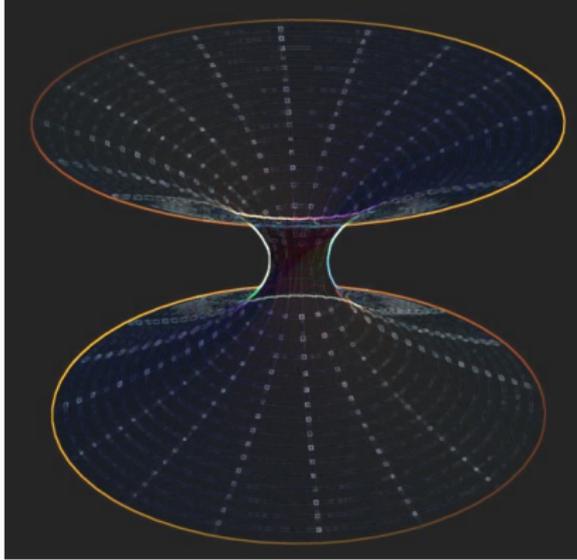
Figure 8: Wormhole

-We do not know how to open the throat without exotic matter.
-Thin-shell Methods with Israel junction conditions can be used to minimize the exotic matter needed.

However, the stability must be saved.

Figure 9: How to realize Wormholes in real life

Figure 10: Firstly , Flamm's work on the WH physics using the Schwarzschild solution (1916).
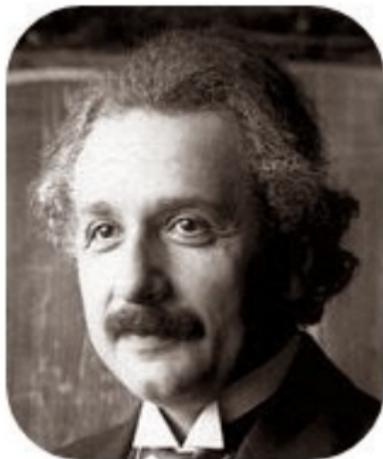
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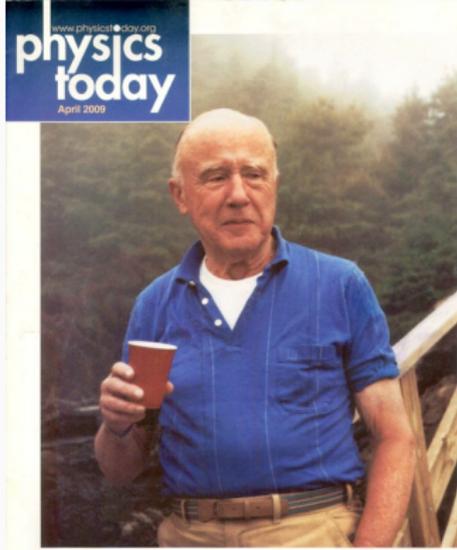
Figure 12: J.Wheeler used "geons" (self-gravitating bundles of electromagnetic fields) by giving the first diagram of a doubly-connected-space (1955).

Wheeler added the term "wormhole" to the physics literature at the quantum scale.

Figure 13: First traversable WH, Morris-Thorne (1988).

Figure 14: Then Morris, Thorne, and Yurtsever investigated the requirements of the energy condition for traversable WHs.

Figure 15: A technical way to make thin-shell WHs by Visser (1988).

Traversable Wormhole Construction Criteria

- Spherically symmetric and static metric
- Obey the Einstein field equations.
- Must have a throat that connects two asymptotically flat regions of spacetime.
- No horizon, since a horizon will prevent two-way travel through the wormhole.
- Tidal gravitational forces experienced by a traveler must be bearably small.
- Traveler must be able to cross through the wormhole in a finite and reasonably small proper time.
- Physically reasonable stress-energy tensor generated by the matter and fields.
- Solution must be stable under small perturbation.
- Should be possible to assemble the wormhole. ie. assembly should require much less than the age of the universe.

Traversable Lorentzian Wormholes

The first defined traversable WH is Morris Thorne WH with a the red-shift function $f(r)$ and a shape function $b(r)$:

$$ds^2 = -e^{2f(r)}dt^2 + \frac{1}{1 - \frac{b(r)}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \qquad (1)$$

- Spherically symmetric and static
- Radial coordinate r such that circumference of circle centered around throat given by $2\pi r$
- r decreases from $+\infty$ to $b = b_0$ (minimum radius) at throat, then increases from b_0 to $+\infty$
- At throat exists coordinate singularity where r component diverges
- Proper radial distance $l(r)$ runs from $-\infty$ to $+\infty$ and vice versa

HAWKING RADIATION OF THE TRAVERSABLE WORMHOLES
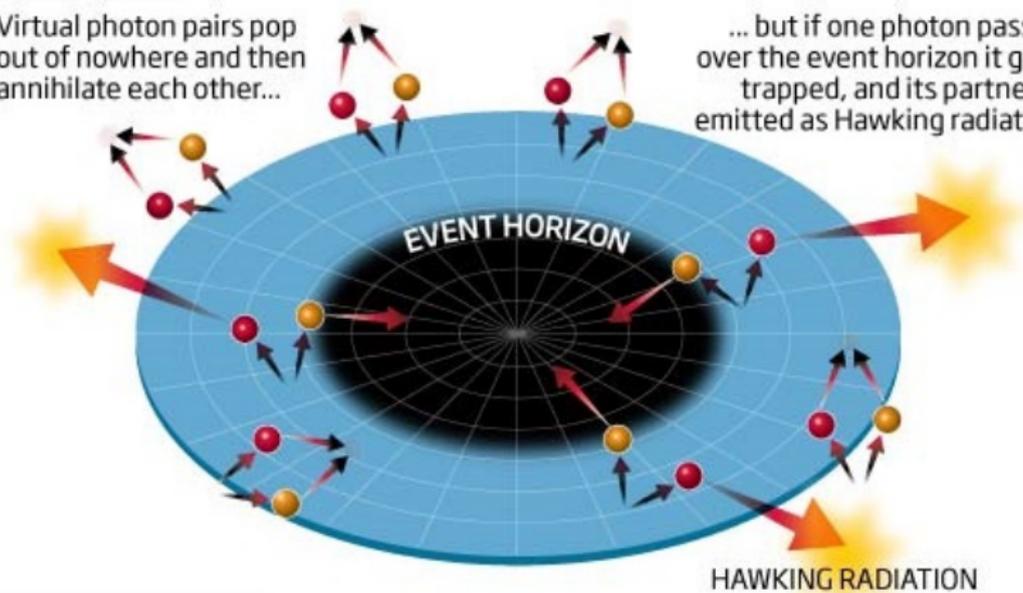
Figure 16: Hawking radiation

DISAPPEARANCE When the hole evaporates, all the information disappears with it.
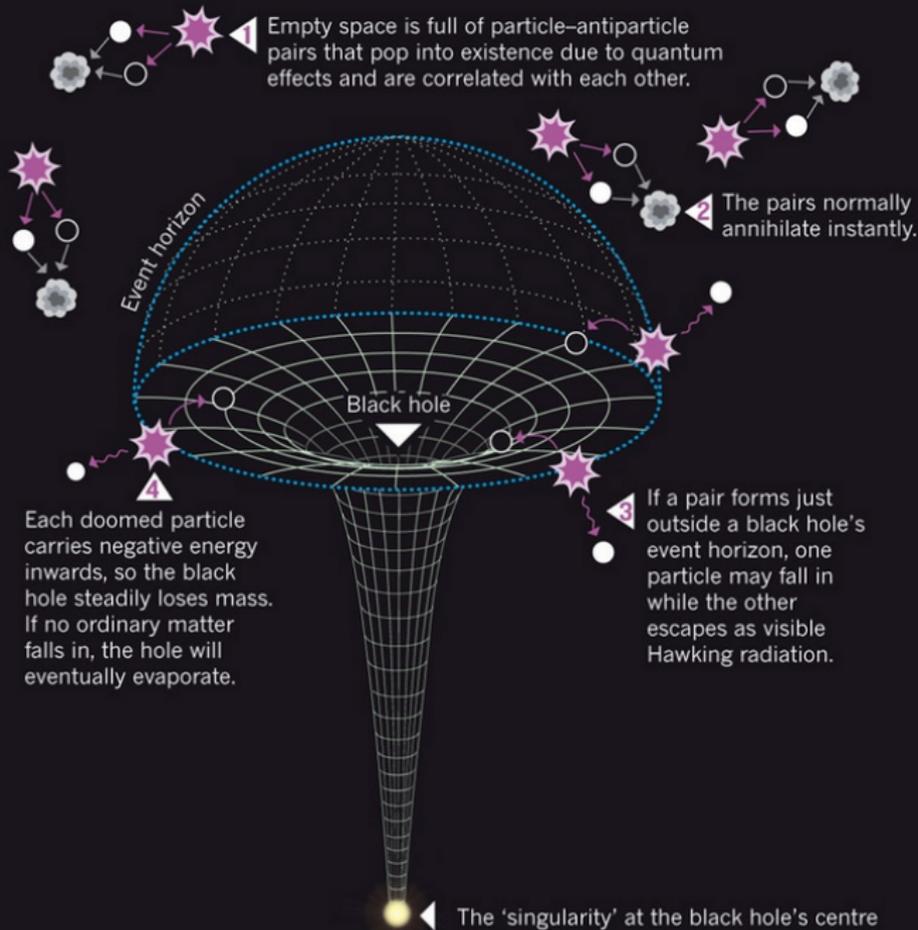
Empty space is full of particle–antiparticle pairs that pop into existence due to quantum effects and are correlated with each other.

The pairs normally annihilate instantly.

Event horizon

Black hole

Each doomed particle carries negative energy inwards, so the black hole steadily loses mass. If no ordinary matter falls in, the hole will eventually evaporate.

If a pair forms just outside a black hole's event horizon, one particle may fall in while the other escapes as visible Hawking radiation.
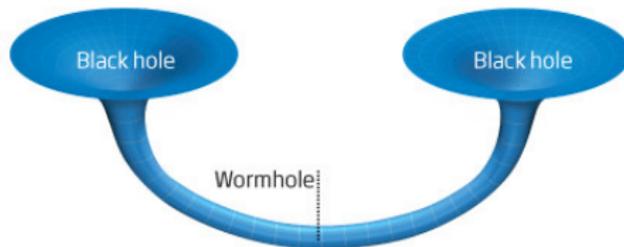
The 'singularity' at the black hole's centre
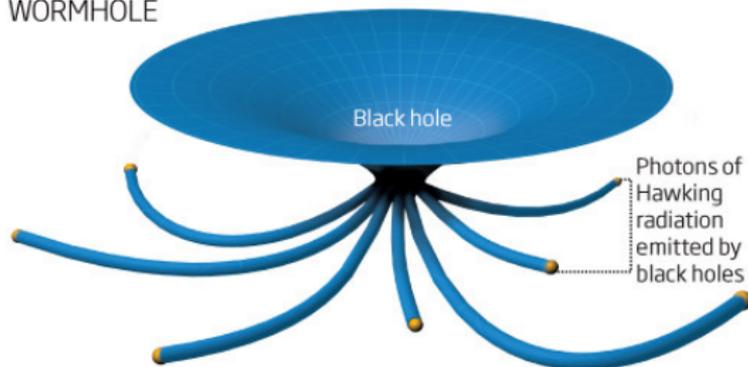
Figure 18: ER=EPR
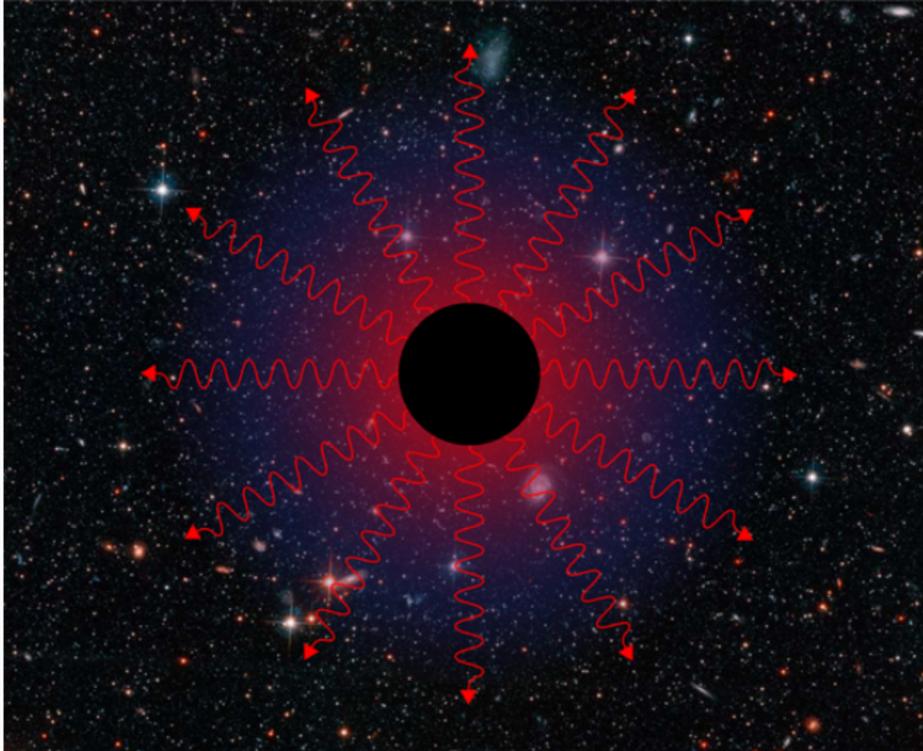
Figure 19: The radiation is due to the black hole capturing one of a particle-antiparticle pair created spontaneously near to the event horizon.

- For studying the HR of traversable WHs, we consider a general spherically symmetric and dynamic WH with a past outer trapping horizon.

- The traversable WH metric can be transformed into the generalized retarded Eddington-Finkelstein coordinates as following

$$ds^2 = -C du^2 - 2 du dr + r^2 \left(d\theta^2 + B d\varphi^2\right), \tag{2}$$

  where $C = 1 - 2M/r$ and $B = \sin^2\theta$.

- Proca equation in a curved space-time :

$$\frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g}\psi^{\nu;\mu}\right) + \frac{m^2}{\hbar^2}\psi^\nu = 0, \tag{3}$$

  in which the wave functions are defined as $\psi_\nu = (\psi_0, \psi_1, \psi_2, \psi_3)$.

- Use WKB approximation, the following HJ ansätz is substituted into Eq. (3)

$$\psi_\nu = (c_0, c_1, c_2, c_3)\, e^{\frac{i}{\hbar} S(u, r, \theta, \phi)}, \tag{4}$$

- Furthermore, we define the action $S(u, r, \theta, \phi)$ as following

$$S(u, r, \theta, \phi) = S_0(u, r, \theta, \phi) + \hbar S_1(u, r, \theta, \phi) + \hbar^2 S_2(u, r, \theta, \phi) + ... \quad (5)$$

- Then one can use the separation of variables method to the action $S_0(u, r, \theta, \phi)$:

$$S_0 = Eu - W(r) - j\theta - k\phi, \quad (6)$$

- It is noted that E and (j, k) are energy and real angular constants, respectively.

- After inserting Eqs. (4), (5), and (6) into Eq. (3), a matrix equation

$\Delta\left(c_0, c_1, c_2, c_3\right)^T = 0$ (to the leading order in $\hbar$) is obtained by

$$
\begin{aligned}
\Delta_{11} &= 2B\left[\partial_r W(r)\right]^2 r^2, \\
\Delta_{12} &= \Delta_{21} = 2m^2 r^2 B + 2B\partial_r W(r)Er^2 + 2Bj^2 + 2k^2, \\
\Delta_{13} &= -\frac{2\Delta_{31}}{r^2} = -2Bj\partial_r W(r), \\
\Delta_{14} &= \frac{\Delta_{41}}{Br^2} = -2k\partial_r W(r), \\
\Delta_{22} &= -2BCm^2 r^2 + 2E^2 r^2 B - 2j^2 BC - 2k^2 C, \qquad (7) \\
\Delta_{23} &= \frac{-2\Delta_{32}}{r^2} = 2jBC\partial_r W(r) + 2EjB, \\
\Delta_{24} &= \frac{\Delta_{42}}{Br^2} = 2kC\partial_r W(r) + 2kE, \\
\Delta_{33} &= m^2 r^2 B + 2BEr^2\partial_r W(r) + r^2 BC\left[\partial_r W(r)\right]^2 + k^2, \\
\Delta_{34} &= \frac{-\Delta_{43}}{2B} = -kj, \\
\Delta_{44} &= -2r^2 BC\left[\partial_r W(r)\right]^2 - 4BEr^2\partial_r W(r) - 2B(m^2 r^2 + j^2).
\end{aligned}
$$

The determinant of the Δ-matrix ($\det\Delta = 0$) is used to get

$$\det\Delta = 64Bm^2r^2 \left\{ \frac{1}{2}r^2BC\left[\partial_r W(r)\right]^2 + BEr^2\partial_r W(r) + \frac{B}{2}\left(m^2r^2 + j^2\right) + \frac{k^2}{2} \right\}^3 \tag{8}$$

- Then the Eq. (8) is solved for $W(r)$

$$W_\pm(r) = \int \left(\frac{-E}{C} \pm \sqrt{\frac{E^2}{C^2} - \frac{m^2}{C} - \frac{j^2}{CB^2r^2} - \frac{k^2}{Cr^2}} \right) dr. \tag{9}$$

- The above integral near the horizon ($r \to r_0$) reduces to

$$W_\pm(r) \simeq \int \left(\frac{-E}{C} \pm \frac{E}{C} \right) dr. \tag{10}$$

- The probability rate of the ingoing/outgoing particles only depend on the imaginary part of the action.
- Eq. (10) has a pole at $C = 0$ on the horizon.
- Using the contour integration in the upper r half-plane, one obtains

$$W_\pm = i\pi \left(\frac{-E}{2\kappa|_H} \pm \frac{E}{2\kappa|_H} \right). \tag{11}$$

2

From which

$$ImS = ImW_{\pm}, \tag{12}$$

that the $\kappa|_H = \partial_r C/2$ is the surface gravity.

- Note that the $\kappa|_H$ is positive quantity because the throat is an outer trapping horizon.
- When we define the probability of incoming particles W_+ to 100% such as $\Gamma_{absorption} \approx e^{-2ImW} \approx 1$.
- Consequently W_- stands for the outgoing particles.
- Then we calculate the tunneling rate of the vector particles as

$$\Gamma = \frac{\Gamma_{emission}}{\Gamma_{absorption}} = \Gamma_{emission} \approx e^{-2ImW_-} = e^{\frac{2\pi E}{\kappa|_H}}. \tag{13}$$

- The Boltzmann factor $\Gamma \approx e^{-\beta E}$ where β is the inverse temperature is compared with the Eq. (13) to obtain the Hawking temperature $T|_H$ of the traversable WH as

$$T|_H = -\frac{\kappa|_H}{2\pi}, \tag{14}$$

- Surprisingly, we derive the the negative $T|_H$ that past outer trapping horizon of the traversable WH radiate thermal phantom energy (i.e. dark energy)

- Additionally, the radiation of phantom energy has an effect of reduction of the size of the WH's throat and its entropy.

- The main reason of this negativeness is the phantom energy which is located at the throat of WH.

- Moreover, as a result of the phantom energy, the ordinary matter can travel backward in time.

- Nonetheless, this does not create a trouble. The total entropy of universe always increases, hence it prevents the violation of the second law of thermodynamics.

- Moreover, in our different work, we show that the gravitino also tunnels through WH and we calculate the tunneling rate of the emitted gravitino particles from traversable WH.

THIN-SHELL WORMHOLES
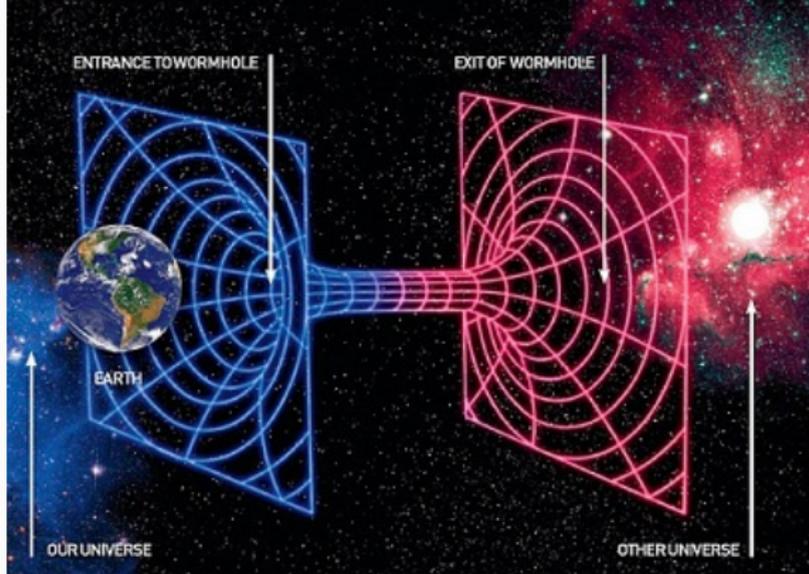
- Constructing WHs with non-exotic (normal matter) source is a difficult issue in General Relativity.
- On this purpose, firstly , Visser use the thin-shell method to construct WHs by minimizing the exotic matter on the throat of the WHs.

- We need to introduce some conditions on the
  energy-momentum tensor.

$\mathcal{M}_+$

$\mathcal{M} = \mathcal{M}_+ + \mathcal{M}_-$ $\Sigma = \Sigma_+ + \Sigma_-$

$\mathcal{M}_-$
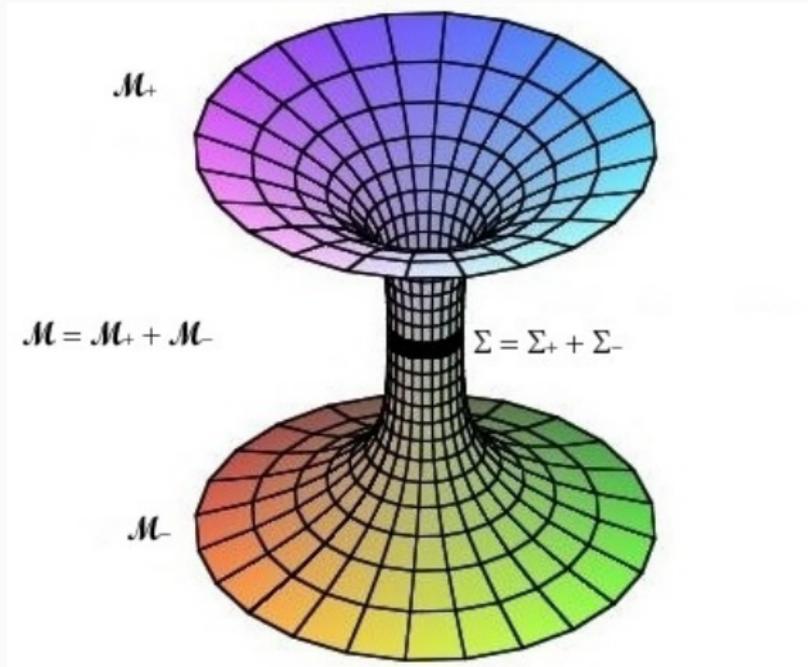
Input: Two space-times

-Use the Darmois –Israel formalism and match an interior spacetime to an exterior spacetime

-Use the Lanczos equations to find a surface energy density σ and a surface pressure *p*.

-Use the energy conservation to find the equation of motion of particle on the throat of the thin-shell wormhole

-Check the stability by using different EoS equations.

-Check Stability by using the

Outputs Thin-shell wormhole

$\sigma < 0$ with extrinsic curvature $K > 0$ of the throat means it required exotic matter.

ON A PARTICULAR THIN-SHELL WH

- The line element of a Scalar Hairy Black Hole (SHBH) investigated by Mazharimousavi and Halilsoy is

$$ds^2 = -f(r)dt^2 + \frac{4r^2 dr^2}{f(r)} + r^2 d\theta^2, \qquad (15)$$

  where

$$f(r) = \frac{r^2}{l^2} - ur. \qquad (16)$$

- Here u and l are constants.

- Event horizon of the BH is located at $r_h = u\ell^2$.

- It is noted that the singularity located at $r = 0$.

- Firstly we take two identical copies of the SHBHs with $(r \geq a)$:

$$M^\pm = (x|r \geq 0),$$

- The manifolds are bounded by hypersurfaces M^+ and M^-, to get the single manifold $M = M^+ + M^-$,

- We glue them together at the surface of the junction

$$\Sigma^{\pm} = (x|r = a).$$

  where the boundaries Σ are given.

- The spacetime on the shell is

$$ds^2 = -d\tau^2 + a(\tau)^2 d\theta^2, \tag{17}$$

  where τ represents the proper time .

- Setting coordinates $\xi^i = (\tau, \theta)$, the extrinsic curvature formula connecting the two sides of the shell is simply given by

$$K_{ij}^{\pm} = -n_{\gamma}^{\pm} \left(\frac{\partial^2 x^{\gamma}}{\partial \xi^i \partial \xi^j} + \Gamma_{\alpha\beta}^{\gamma} \frac{\partial x^{\alpha}}{\partial \xi^i} \frac{\partial x^{\beta}}{\partial \xi^j} \right), \tag{18}$$

  where the unit normals ($n^{\gamma} n_{\gamma} = 1$) are

$$n_{\gamma}^{\pm} = \pm \left| g^{\alpha\beta} \frac{\partial H}{\partial x^{\alpha}} \frac{\partial H}{\partial x^{\beta}} \right|^{-1/2} \frac{\partial H}{\partial x^{\gamma}}, \tag{19}$$

with $H(r) = r - a(\tau)$.

- The non zero components of $n_\gamma^\pm$ are calculated as

$$n_t = \mp 2a\dot{a}, \tag{20}$$

$$n_r = \pm 2\sqrt{\frac{al^2(4\dot{a}^2l^2a - l^2u + a)}{(l^2u - a)}}, \tag{21}$$

where the dot over a quantity denotes the derivative with respect to τ.

- Then, the non-zero extrinsic curvature components yield

$$K_{\tau\tau}^\pm = \mp \frac{\sqrt{-al^2}(8\dot{a}^2l^2a + 8\ddot{a}l^2a^2 - l^2u + 2a)}{4a^2l^2\sqrt{-4\dot{a}^2l^2a - l^2u + a}}, \tag{22}$$

$$K_{\theta\theta}^\pm = \pm \frac{1}{2a^{\frac{3}{2}}l}\sqrt{4\dot{a}^2l^2a - l^2u + a}. \tag{23}$$

- Since K_{ij} is not continuous around the shell, we use the Lanczos equation:

$$S_{ij} = -\frac{1}{8\pi}\left([K_{ij}] - [K]g_{ij}\right). \tag{24}$$

where K is the trace of K_{ij}, $[K_{ij}] = K_{ij}^+ - K_{ij}^-$.

- Firstly, $K^+ = -K^- = [K_{ij}]$ while $[K_{ij}] = 0$.

- For the conservation of the surface stress–energy $S_j^{ij} = 0$.

- S_{ij} is stress energy-momentum tensor at the junction which is given in general by

$$S_j^i = diag(\sigma, -p), \qquad (25)$$

  with the surface pressure p and the surface energy density σ.

- Due to the circular symmetry, we have

$$K_j^i = \begin{pmatrix} K_\tau^\tau & 0 \\ 0 & K_\theta^\theta \end{pmatrix}. \qquad (26)$$

  Thus, from Eq.s (25) and (24) one obtains the surface pressure and surface energy density .

- Using the cut and paste technique, we can demount the interior regions $r < a$ of the geometry, and links its exterior parts.

- The energy density and pressure are

$$\sigma = -\frac{1}{8\pi a^{\frac{3}{2}} l}\sqrt{4\dot{a}^2 l^2 a - l^2 u + a}, \tag{27}$$

$$p = \frac{1}{16\pi a^{\frac{3}{2}} l}\frac{(8\dot{a}^2 l^2 a + 8\ddot{a} l^2 a^2 - l^2 u + 2a)}{\sqrt{4\dot{a}^2 l^2 a - l^2 u + a}}. \tag{28}$$

Then for the static case ($a = a_0$), the energy and pressure quantities reduce to

$$\sigma_0 = -\frac{1}{8\pi a_0^{\frac{3}{2}} l}\sqrt{-l^2 u + a_0}, \tag{29}$$

$$p_0 = \frac{1}{16\pi a_0^{\frac{3}{2}} l}\frac{(-l^2 u + 2a_0)}{\sqrt{-l^2 u + a_0}}. \tag{30}$$

Once $\sigma \geq 0$ and $\sigma + p \geq 0$ hold, then WEC is satisfied.

- It is obvious from Eq. (24) that negative energy density violates the WEC, and consequently we are in need of the exotic matter for constructing thin-shell WH.

- We note that the total matter supporting the WH is given by

$$\Omega_\sigma = \int_0^{2\pi} [\rho\sqrt{-g}]\big|_{r=a_0} \, d\phi = 2\pi a_0 \sigma(a_0) = -\frac{1}{4a_0^{\frac{1}{2}}|l|}\sqrt{-l^2 u + a_0}. \tag{31}$$

- Stability of the WH is investigated using the linear perturbation so that the EoS is

$$p = \psi(\sigma), \tag{32}$$

  where $\psi(\sigma)$ is an arbitrary function of σ.
- It can be written in terms of the pressure and energy density:

$$\frac{d}{d\tau}(\sigma a) + \psi \frac{da}{d\tau} = -\dot{a}\sigma. \tag{33}$$

- From above equation, one reads

$$\sigma' = -\frac{1}{a}(2\sigma + \psi), \tag{34}$$

  and its second derivative yields

$$\sigma'' = \frac{2}{a^2}(\tilde{\psi} + 3)(\sigma + \frac{\psi}{2}). \tag{35}$$

2

where prime and tilde symbols denote derivative with respect to a and σ, respectively.

- The conservation of energy for the shell is in general given by

$$\dot{a}^2 + V = 0, \tag{36}$$

where the effective potential V is found from Eq. (27)

$$V = \frac{1}{4l^2} - \frac{u}{4a} - 16a^2\sigma^2\pi^2. \tag{37}$$

- In fact, Eq. (36) is nothing but the equation of the oscillatory motion in which the stability around the equilibrium point $a = a_0$ is conditional on $V'(a_0) = 0$ and $V''(a_0) \geq 0$.

- We finally obtain

$$V'' = -\frac{1}{2a^3}\left[64\pi^2 a^5\left((\sigma\sigma')' + 4\sigma'\frac{\sigma}{a} + \frac{\sigma^2}{a^2}\right) + u\right]\Bigg|_{a=a_0}, \tag{38}$$

or equivalently,

$$V'' = \frac{1}{2a^3} \{ -64\pi^2 a^3 \left[(2\psi' + 3)\sigma^2 + \psi(\psi' + 3)\sigma + \psi^2 \right] - u \} \Big|_{a=a_0}.$$
(39)

- The equation of motion of the throat, for a small perturbation becomes

$$\dot{a}^2 + \frac{V''(a_0)}{2}(a - a_0)^2 = 0.$$

- Note that for the condition of $V''(a_0) \geq 0$, TSW is stable where the motion of the throat is oscillatory with angular frequency $\omega = \sqrt{\frac{V''(a_0)}{2}}$.

Some Models of EoS Supporting Thin-Shell WH

In this section, we use particular gas models (linear barotropic gas (LBG) , chaplygin gas (CG) , generalized chaplygin gas (GCG) and logarithmic gas (LogG)) to explore the stability of TSW.

Stability analysis of Thin-Shell WH via the LBG

The equation of state of LBG is given by

$$\psi = \varepsilon_0 \sigma, \tag{40}$$

and hence

$$\psi'(\sigma_0) = \varepsilon_0, \tag{41}$$

where ε_0 is a constant parameter. By changing the values of l and u in Eq. (35), we illustrate the stability regions for TSW, in terms of ε_0 and a_0.
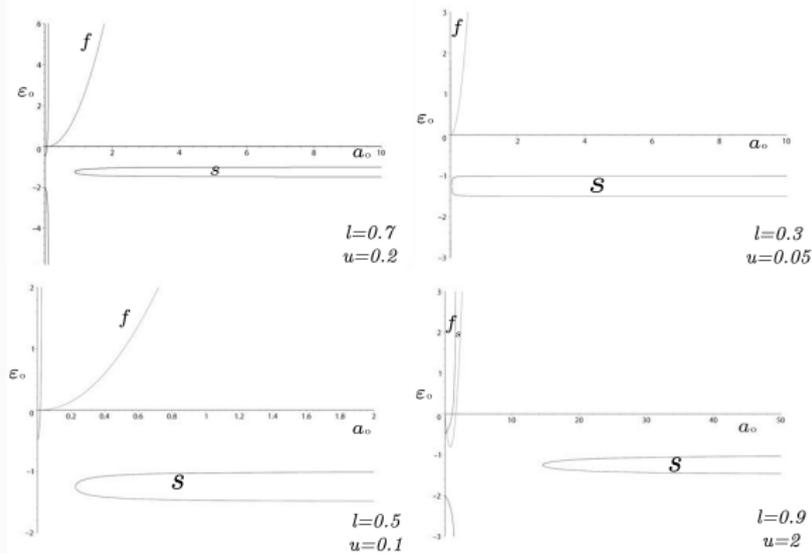
Figure 20: Stability Regions via the LBG

Stability analysis of Thin-Shell WH via CG

The equation of state of CG that we considered is given by

$$\psi = \varepsilon_0(\frac{1}{\sigma} - \frac{1}{\sigma_0}) + p_0, \tag{42}$$

and one naturally finds

$$\psi'(\sigma_0) = \frac{-\varepsilon_0}{\sigma_0^2}. \qquad (43)$$

After inserting Eq. (39) into Eq. (35), The stability regions for thin-shell WH supported by CG is plotted in Fig.
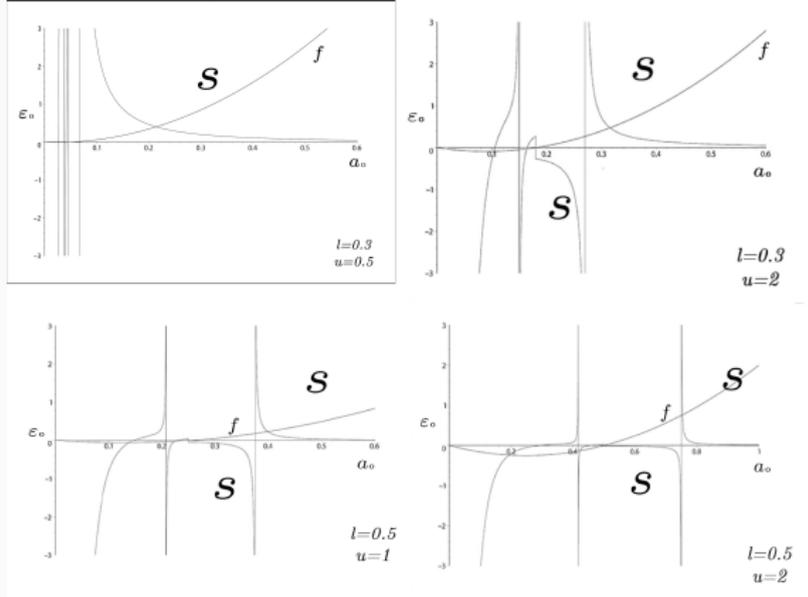
Figure 21: Stability Regions via the CG

Stability analysis of Thin-Shell WH via GCG

By using the equation of state of GCG

$$\psi = p_0 \left(\frac{\sigma_0}{\sigma}\right)^{\varepsilon_0}, \tag{44}$$

and whence

$$\psi'(\sigma_0) = -\varepsilon_0 \frac{p_0}{\sigma_0}, \tag{45}$$

Substituting Eq. (41) in Eq. (35), one can illustrate the stability regions of thin-shell WH supported by GCG as seen in Fig.
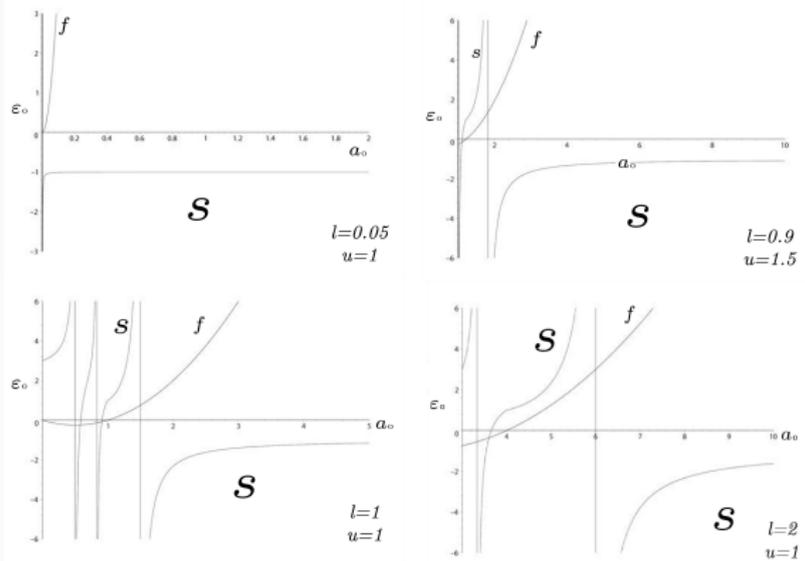
Figure 22: Stability Regions via the GCG

Stability analysis of Thin-Shell WH via LogG

- In our final example, the equation of state for LogG is selected as follows ($\varepsilon_0, \sigma_0, p_0$ are constants)

$$\psi = \varepsilon_0 \ln(\frac{\sigma}{\sigma_0}) + p_0, \tag{46}$$

  which leads to

$$\psi'(\sigma_0) = \frac{\varepsilon_0}{\sigma_0}. \tag{47}$$

- After inserting the above expression into Eq. (35), we show the stability regions of thin-shell WH supported by LogG in Fig.
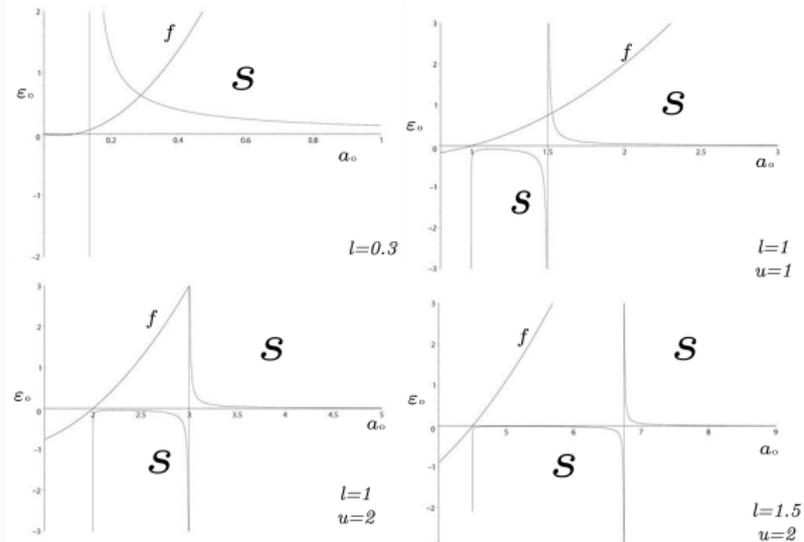
Figure 23: Stability Regions via the LogG

- In summary, we have constructed thin-shell WH by gluing two copies of SHBH via the cut and paste procedure.
- To this end, we have used the fact that the radius of throat must be greater than the event horizon of the metric given: ($a_0 > r_h$).
- We have used LBG, CG, GCG, and LogG EoS to the exotic matter.
- Then, the stability analysis ($V''(a_0) \geq 0$) is plotted.
- We show the stability regions in terms a_0 and ε_0

HAYWARD THIN-SHELL WH IN 3+1-D

- The metric of the Hayward BH is given by

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2. \tag{48}$$

  with the metric function

$$f(r) = \left(1 - \frac{2mr^2}{r^3 + 2ml^2}\right) \tag{49}$$

  and

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{50}$$

- It is noted that m and l are free parameters.
- At large r, the metric function behaves as a Schwarzschild BH

$$\lim_{r\to\infty} f(r) \to 1 - \frac{2m}{r} + \mathcal{O}\left(\frac{1}{r^4}\right), \tag{51}$$

  whereas at small r becomes a de Sitter BH

$$\lim_{r\to 0} f(r) \to 1 - \frac{r^2}{l^2} + \mathcal{O}\left(r^5\right). \tag{52}$$

- Thin-shell is located at $r = a$.
- The throat must be outside of the horizon ($a > r_h$).
- Then we paste two copies of it at the point of $r = a$.
- For this reason the thin-shell metric is taken as

$$ds^2 = -d\tau^2 + a(\tau)^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \tag{53}$$

  where τ is the proper time on the shell.
- The Einstein equations on the shell are

$$\left[K_i^j \right] - [K]\, \delta_i^j = -S_i^j \tag{54}$$

  where $[X] = X_2 - X_1,$.
- It is noted that the extrinsic curvature tensor is K_i^j.
- Moreover, K stands for its trace.
- The surface stresses, i.e., surface energy density σ and surface pressures $S_\theta^\theta = p = S_\phi^\phi$, are determined by the surface stress-energy tensor S_i^j.

2

- The energy and pressure densities are obtained as

$$\sigma = -\frac{4}{a}\sqrt{f(a) + \dot{a}^2} \tag{55}$$

$$p = 2\left(\frac{\sqrt{f(a) + \dot{a}^2}}{a} + \frac{\ddot{a} + f'(a)/2}{\sqrt{f(a) + \dot{a}^2}}\right). \tag{56}$$

- Then they reduce to simple form in a static configuration $(a = a_0)$

$$\sigma_0 = -\frac{4}{a_0}\sqrt{f(a_0)} \tag{57}$$

and

$$p_0 = 2\left(\frac{\sqrt{f(a_0)}}{a_0} + \frac{f'(a_0)/2}{\sqrt{f(a_0)}}\right). \tag{58}$$

- Stability of such a WH is investigated by applying a linear perturbation with the following EoS

$$p = \psi(\sigma) \tag{59}$$

2

- Moreover the energy conservation is

$$S^{ij}_{;j} = 0 \qquad (60)$$

which in closed form it equals to

$$S^{ij}_{,j} + S^{kj}\Gamma^{i\mu}_{kj} + S^{ik}\Gamma^{j}_{kj} = 0 \qquad (61)$$

after the line element in Eq.(53) is used, it opens to

$$\frac{\partial}{\partial \tau}\left(\sigma a^2\right) + p\frac{\partial}{\partial \tau}\left(a^2\right) = 0. \qquad (62)$$

- The 1-D equation of motion is

$$\dot{a}^2 + V(a) = 0, \qquad (63)$$

in which $V(a)$ is the potential,

$$V(a) = f - \left(\frac{a\sigma}{4}\right)^4. \qquad (64)$$

- The equilibrium point at $a = a_0$ means $V'(a_0) = 0$ and $V''(a_0) \geq 0$.

- Then it is considered that $f_1(a_0) = f_2(a_0)$, one finds $V_0 = V'_0 = 0$.
- To obtain $V''(a_0) \geq 0$ we use the given $p = \psi(\sigma)$ and it is found as follows

$$\sigma'\left(=\frac{d\sigma}{da}\right) = -\frac{2}{a}(\sigma + \psi) \tag{65}$$

and

$$\sigma'' = \frac{2}{a^2}(\sigma + \psi)(3 + 2\psi'), \tag{66}$$

where $\psi' = \frac{d\psi}{d\sigma}$. After we use $\psi_0 = p_0$, finally it is found that

$$V''(a_0) = f''_0 - \frac{1}{8}\left[(\sigma_0 + 2p_0)^2 + 2\sigma_0(\sigma_0 + p_0)(1 + 2\psi'(\sigma_0))\right] \tag{67}$$

Some models of EoS

Now we use some models of matter to analyze the effect of the parameter of Hayward in the stability of the constructed thin-shell WH.

Linear gas

For a LG, EoS is choosen as

$$\psi = \eta_0 \left(\sigma - \sigma_0 \right) + p_0 \tag{68}$$

in which η_0 is a constant and $\psi'\left(\sigma_0\right) = \eta_0$.
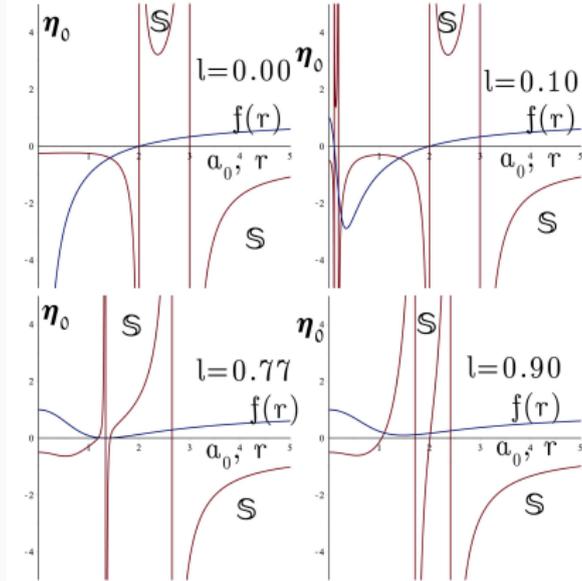
Figure 24: Stability of Thin-Shell WH supported by LG.

Fig. shows the stability regions in terms of η_0 and a_0 with different Hayward's parameter. It is noted that the S shows the stable regions.

Chaplygin gas

For CG, we choose the EoS as follows

$$\psi = \eta_0 \left(\frac{1}{\sigma} - \frac{1}{\sigma_0} \right) + p_0 \tag{69}$$

where η_0 is a constant and $\psi'(\sigma_0) = -\frac{\eta_0}{\sigma_0^2}$.
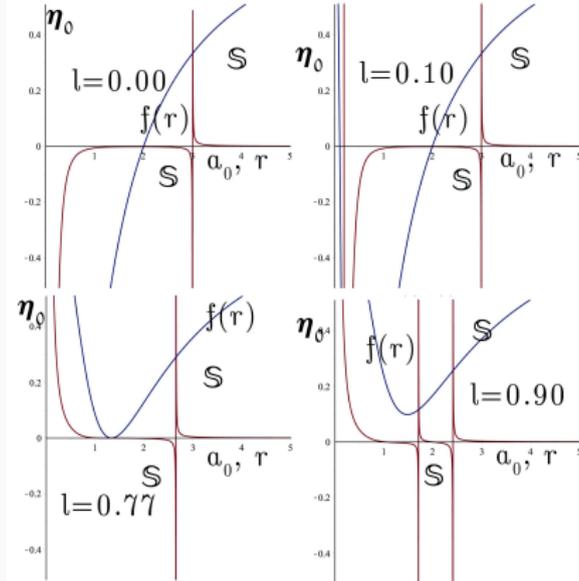
Figure 25: Stability of Thin-Shell WH supported by CG.

In Fig., the stability regions are shown in terms of η_0 and a_0 for different values of ℓ. The effect of Hayward's constant is to increase the stability of the Thin-Shell WH.

Generalized Chaplygin gas

The EoS of the GCG is taken as

$$\psi\left(\sigma\right) = \eta_0 \left(\frac{1}{\sigma^{\nu}} - \frac{1}{\sigma_0^{\nu}}\right) + p_0 \tag{70}$$

where ν and η_0 are constants. We check the effect of parameter ν in the stability and ψ becomes

$$\psi\left(\sigma\right) = p_0 \left(\frac{\sigma_0}{\sigma}\right)^{\nu}. \tag{71}$$
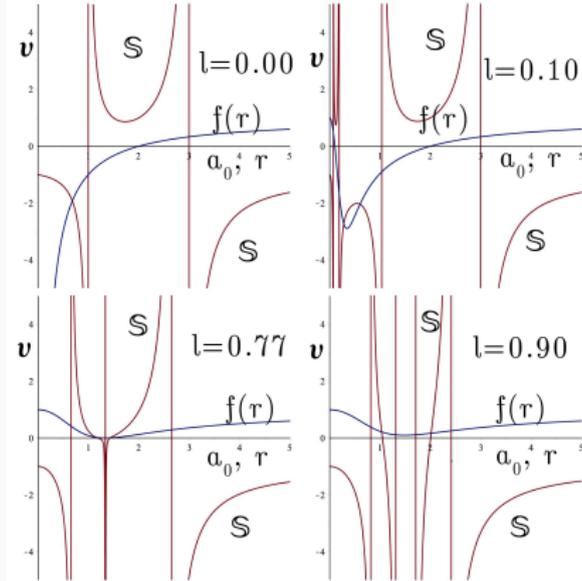
Figure 26: Stability of Thin-Shell WH supported by GCG.

We find $\psi'(\sigma_0) = -\frac{p_0}{\sigma_0}\nu$. In Fig., the stability regions are shown in terms of ν and a_0 with various values of ℓ.

Modified Generalized Chaplygin gas

In this case, the MGCG is

$$\psi\left(\sigma\right) = \xi_0\left(\sigma - \sigma_0\right) - \eta_0\left(\frac{1}{\sigma^\nu} - \frac{1}{\sigma_0^\nu}\right) + p_0 \tag{72}$$

in which ξ_0, η_0 and ν are free parameters. Therefore,

$$\psi'\left(\sigma_0\right) = \xi_0 + \eta_0\frac{\eta_0\nu}{\sigma_0^{\nu+1}}. \tag{73}$$
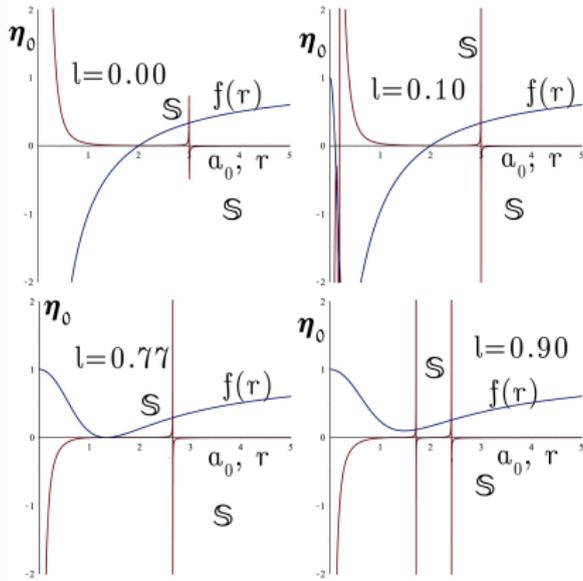
Figure 27: Stability of Thin-Shell WH supported by MGCG.

To go further we set $\xi_0 = 1$ and $\nu = 1$. In Fig., the stability regions are plotted in terms of η_0 and a_0 with various values of ℓ. The effect of Hayward's constant is to increase the stability of the Thin-Shell WH.

Logarithmic gas

Lastly LogG is choosen by follows

$$\psi\left(\sigma\right) = \eta_0 \ln \left|\frac{\sigma}{\sigma_0}\right| + p_0 \tag{74}$$

in which η_0 is a constant. For LogG, we find that

$$\psi'\left(\sigma_0\right) = \frac{\eta_0}{\sigma_0}. \tag{75}$$
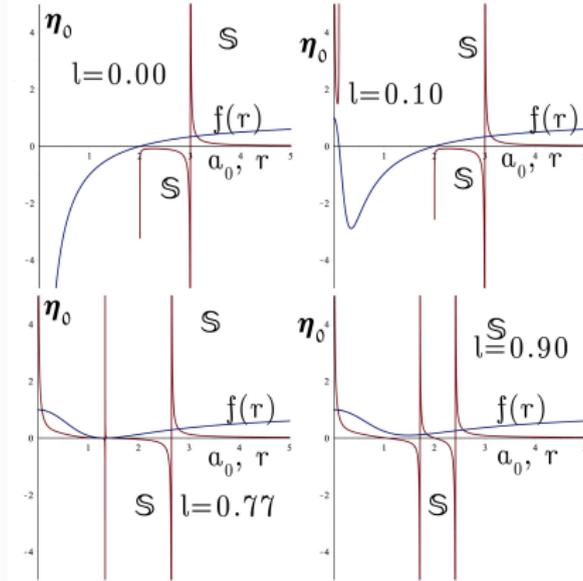
Figure 28: Stability of Thin-Shell WH supported by LogG.

In Fig., the stability regions are plotted to show the effect of Hayward's parameter clearly. The effect of Hayward's constant is to increase the stability of the Thin-Shell WH.

- In this section we construct thin-shell WHs from the Hayward BH.
- On the thin-shell we use the different type of EoS with the form $p = \psi(\sigma)$ and plot possible stable regions.
- We show the stable and unstable regions on the plots.
- Stability simply depends on the condition of $V''(a_0) > 0$.
- We show that the parameter ℓ, which is known as Hayward parameter has a important role.
- Moreover, for higher ℓ value the stable regions are increased.
- It is checked the small velocity perturbations for the throat.
- It is found that throat of the thin-shell WH is not stable against such kind of perturbations.
- Hence, energy density of the WH is found negative so that we need exotic matter.