Error Estimation Methods for the Finite-Difference Solution for Poisson's Equation

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ABSTRACT

The finite-difference method is universally used for the approximation of differential equations.

In this thesis two different approaches are reviewed for the error estimation of the approximation of the Dirichlet problem for elliptic equations, specifically Poisson's and Laplace's equations using various finite-difference schemes.

The first approach is based on the difference analogue of the maximum principle. Applying Gerschgorin's majorant method to the analysis , also the order of accuracy of the proposed scheme is obtained.

The second approach uses the difference analogue of Green's function and Green's third identity. In order to obtain an order of approximation, Gerschgorin's majorant method is applied in this approach also.

Both methods gave similar approximations.

Keywords: Finite-difference, maximum principle, Gerschgorin's majorant method, Green's function, Green's third identity.

ÖZ

Sonlu-farklar metodu, yakınsak çözümlemeler için evrensel olarak kullanılan bir metoddur.

Bu tezde, Poisson denklemi için Dirichlet probleminin sonlu-farklar analogu, iki farklı hata analizi yöntemi ile gözden geçirilmiştir.

Birinci yöntem, maksimum ilkesine (maximum principle) bağlıdır. Gerschgorin'in majorant metodunun da uygulanması ile sonlu farklar metodu analiz edilmiştir.

İkinci yöntemde ise, Green fonksiyonunun sonlu-farklar analogu, ve Green'in 3. denklemi analogu kullanılmıştır. Yakınsaklık derecesinin elde edilmesi için, Gerschgorin'in majorant metodu da kullanılmıştır.

İki yöntem de benzer sonuçlar vermiştir.

Anahtar kelimeler: sonlu farklar, maksimum ilkesi, Gerschgorin majorant metodu, Green fonksiyonu.

DEDICATION

I am dedicating this work to my family, specifically my brothers DLOVAN and AHMED.

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Chapter 1

INTRODUCTION

The finite-difference method is one of the most widely applied methods for the approximation of ordinary and partial differential equations.

This discretization method can be seen to be practiced in many applications of science such as in aerodynamics, dynamical meteorology and oceanography, mathematical physics, and many more disciplines. Hence, the error estimation and convergence analysis of this scheme carry practical, as well as theoretical importance.

An example of the application of finite-difference can also be seen in Richardson's extrapolation method. This method uses the finite-difference analogue of an equation to improve the order of convergence, thus resulting in a more accurate method. Hence, finite-difference can be viewed as the initial step for the improvement of error estimation.

When analysing the convergence and error estimation of the applied finite-difference scheme, the determination of the order of accuracy obtained by the proposed scheme is essential. Moreover, with investigation of the scheme, it might be possible to construct schemes with increased accuracy, therefore

the approach taken for error estimation carries a lot of importance. In this thesis, two different methods for the analysis of finite-difference schemes have been reviewed.

In Chapter 2, Gerschgorin's majorant method has been reviewed for the analysis of the difference analogue of the Dirichelet problem for Poisson's equation. It was shown that when the 5-point scheme is applied, second order accuracy is obtained for the approximate solution. Moreover, when the 9-point scheme was considered, an analysis with the majorant method proved that the scheme had an increased accuracy of $O(h^4)$, h is the mesh step.

In Chapter 3, a second approach was discussed for error analysis. First of all, the finite-difference analogues of problems were defied by the related finite-difference Green's function. Then, with the aid of the analogue of Green's third identity, error estimation was obtained. Greschgorin's majorant method was also applied when considering this approach.

Conclusion is given in Chapter 4.

Chapter 2

MAJORANT METHOD

2.1The Maximum Principle

2.1.1 The canonical form of finite difference equation

The maximum principle is frequently applied when considering the difference analogue of elliptic equations and is reviewed in this chapter.

We let ω be the set of interior nodes, and the set containing all grid nodes be $\bar{\omega} = \omega + \gamma$, where γ the set of boundary points. Now assume that, we have a point $S \in \omega$ and the point S satisfies the equation

$$N(S)y(S) = \sum_{K \in Patt'(s)} M(S,K)y(K) + Z(S), \qquad S \in \omega, \tag{2.1}$$

for grid function $y(S)$ defined on ω. Here the function $Z(S)$ and the coefficients of equation (1), $N(S)$ and $M(S,K)$ are given grid functions; and the neighborhood of the point S without the point S are denoted by $Patt'(S) \subset \omega$.

Suppose that, we have this condition for the coefficients $N(S)$ and $M(S,Q)$

$$\left.\begin{array}{l} N(S) > 0, \quad M(S,K) > 0 \quad \text{for all } \mathrm{S} \in \omega, K \in Patt'(S), \\ \mathrm{T}(S) = N(S) - \sum\limits_{K \in Patt'(S)} M(S,K) \geq 0. \end{array}\right\} \tag{2.2}$$

We call the point S boundary point of the grid ω if the value of $y(S)$ is known at this point:

$$y(S) = \mu(S) \qquad \text{for} \quad \mathrm{S} \in \gamma, \tag{2.3}$$

Now, if we compare (2.1) and (2.3) we will see for $S \in \gamma$ we have to set formally $N(S) = 1, M(S, Z) = 0$ and $Z(S) = \mu(S)$.

A point S is an interior node of the grid ω, if equation (2.1) satisfies conditions (2.2). When the boundary conditions are Neumann or Robin boundary conditions there are no boundary points, that is, $\bar{\omega} = \omega$. It is assumed that $\bar{\omega}$ is a connected grid, that is, for fixed points $S_0 \in \bar{\omega}$ and $S^* \in \bar{\omega}$ a continuation of neighborhoods $\{Patt'(S)\}$ are always available. We use the arbitrary points $S_1, S_2, ..., S_m$ of the grid ω such that

$$S_1 \in Patt'(S_0), \;\; S_2 \in Patt'(S_1), ..., \;\; S_m \in Patt'(S_{m-1}), \;\; P^* \in Patt'(P_m)$$

with

$$\begin{aligned} &M(S_i, S_{i+1}) \neq 0, \qquad i = 1, 2, ..., m-1, \\ &M(S_0, S_1) \neq 0, \qquad \mathrm{M}(S_m, S^*) \neq 0. \end{aligned} \tag{2.4}$$

The point S^* may be a boundary point, therefore by definition of connectedness it is to be understood that all points exist in at least one neighbourhood $Patt'(S)$ of an arbitrary interior point.

We use this notation

$$\mathcal{L}\, y(S) = N(S)y(S) - \sum_{K\in Patt'(S)} M(S,K)y(K), \qquad (2.5)$$

Since, from equation (2.1) we have $N(S)y(S) - \sum_{K\in Patt'(S)} M(S,K)y(K) = Z(S)$, then

$$\mathcal{L}\, y(S) = Z(S). \qquad (2.6)$$

From equation (2.5) we have

$$\mathcal{L}\, y(S) = N(S)y(S) - \sum_{K\in Patt'(S)} M(S,K)y(K) +$$

$$+ \sum_{K\in Patt'(P)} M(S,K)y(S) - \sum_{K\in Patt'(S)} M(S,K)y(S)$$

$$= \left(N(S) - \sum_{K\in Patt'(S)} M(S,K) \right) y(S) + \sum_{K\in Patt'(S)} M(S,K)\big(y(S) - y(K)\big).$$

Since, from equation the (2.2) we have $T(S) = N(S) - \sum_{K\in Patt'(S)} M(S,K) \ge 0$.

$\mathcal{L}\, y(S)$ maybe written in the form:

$$\mathcal{L}\, y(S) = T(S)y(S) + \sum_{K\in Patt'(S)} M(S,K)(y(S) - y(K)) \; . \qquad (2.7)$$

We consider the finite-difference analogue of the heat conduction equation with weights, where the initial-value problem is given below.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \qquad x \in (0,1) \quad \text{and} \quad t > 0 \;,$$

$$u(x,0)=u_0(x), \qquad u(0,t)=\mu_1(t), \qquad u(1,t)=\mu_2(t).$$

We form the grid $\bar{\omega}_{hT}=\left\{\left(x_i=ih, t_j=jT\right),\ i=0,1,...,N,\ h=\frac{1}{N},\ j=0,1,...\right\}$, and for this scheme we have the form

$$\frac{y_i^{j+1}-y_i^j}{T}=\Lambda\left(\sigma y_i^{j+1}+(1-\sigma)y_i^j\right)+\varphi_i^j,$$
$$\Lambda y=y_{\bar{x}x}, \quad y_i^0=u_0(x_i), \quad y_0^j=\mu_1(t_j), \quad y_N^j=\mu_2(t_j). \tag{2.8}$$

Now we can write the canonical form as equation (2.1) for these scheme, giving S like a point of the grid $\bar{\omega}_{hk}$; $\mathrm{S}=S(x_i,t_{j+1})$, where the nodes $K_1=\left(x_i,t_j\right)$, $K_2=\left(x_{i-1},t_{j+1}\right)$, $K_3=\left(x_{i+1},t_{j+1}\right)$, $K_4=\left(x_{i-1},t_j\right)$, $K_5=\left(x_{i+1},t_j\right)$ belong to the $Patt'(S)$ and the set of boundary points γ consists of the nodes $\left(x_i,0\right)$ and $\left(0,t_j\right)$, $\left(1,t_j\right)$, where $i=0,1,...,N$, $j=0,1,....$ Now, rewrite equation (8) by fixing $t=t_{j+1}$ as

$$\left(\frac{1}{T}+\frac{2\sigma}{h^2}\right)y_i^{j+1}=\frac{\sigma}{h^2}\left(y_{i-1}^{j+1}+y_{i+1}^{j+1}\right)+\left(\frac{1}{T}+\frac{2(\sigma-1)}{h^2}\right)y_i^j+\frac{1-\sigma}{h^2}\left(y_{i-1}^j+y_{i+1}^j\right)+\varphi_i^j.$$

We can say that $M(S,K)\geq 0$ only if $T\leq\frac{h^2}{2(1-\sigma)}$ and $0\leq\sigma\leq 1$. Using the same reasoning, $T(S)=0$.

2.1.2 The Maximum Principle

Theorem 1: [8] Assume that $y(S)$ is a grid function defined on $\bar{\omega}$ defined above, $y(S)\neq$ constant, and let for $y(S)$ conditions (1.2) and (1.4) be satisfied. Then, if

$\mathcal{L}\, y(S) \le 0$ on the grid ω, then $y(S)$ will not take its maximal positive at the interior points $S \in \omega$, but if $\mathcal{L}\, y(S) \ge 0$ on the grid ω, then $y(S)$ will not take its minimal negative for $S \in \omega$.

Proof: Assume that $\mathcal{L}\, y(S) \le 0$ at every interior point $S \in \omega$. Also, suppose that the value of $y(S)$ takes its maximal positive at an interior point $S_0 \in \omega$, thus

$$y(S_0) = \max_{\bar{\omega}} y(S) = C_0 > 0 \ .$$

Now, we have to show that there exists an interior point S^* at which $\mathcal{L}\, y(S^*) > 0$, contradicting $\mathcal{L}\, y(S) \le 0$.

By equation (2.7) we have

$$\mathcal{L}\, y(S_0) = T(S_0) y(S_0) + \sum_{K \in Patt'(S)} M(S_0, K)\big(y(S_0) - y(K)\big).$$

Since $y(S)$ is a maximal positive value at the interior point $S_0 \in \omega$. So, $y(S_0) > 0$ and $y(S_0) \ge y(K)$ for all $K \in Patt'(S_0)$, from condition (2.2) we have $T(S_0) \ge 0$, then $T(S_0) y(S_0) \ge 0$ and

$$\mathcal{L}\, y(S_0) = T(S_0) y(S_0) + \sum_{K \in Patt'(S)} B(S_0, K)\big(y(S_0) - y(K)\big) \ge T(S_0) y(S_0) \ge 0 .$$

Hence, we have $\mathcal{L}\, y(\bar{S}) \geq 0$. However, we assumed that $\mathcal{L}\, y(S) \leq 0$ at every interior point $S \in \omega$. So, clearly it is correct only for the case $\mathcal{L}\, y(S_0) = 0$

By equation (2.7) it will be right only if $T(S_0) = 0$ and $y(K) = y(S_0)$ for all $K \in Patt'(S_0)$.

Now, we take the point $S_1 \in Patt(S_0)$ at which $y(S_1) = y(S_0) = C_0$. Since the grid is connected and $y(S) \neq$ constant on the grid ω, so that the connected grid has a sequence of points $S_1, S_2, ..., S_m, \bar{S}$, and condition (2.4) holds for those points such that

$$y(S_m) = y(S_0) = C_0, \qquad y(\bar{S}) < C_0$$

but

$$\bar{\mathrm{S}} \in Patt'(S_m), \qquad \mathrm{M}(S_m, \bar{S}) \neq 0.$$

$$\mathcal{L}\, y(S_m) \geq T(S_m) y(S_m) + M(S_m, \bar{S})\left(y(S_m) - y(\bar{S})\right)$$

then

$$\mathcal{L}\, y(S_m) \geq M(S_m, \bar{S})\left(y(S_0) - y(\bar{S})\right) > 0 .$$

Meaning $S^* = S_m$ we got contradiction. This proves that the first part of the theorem is valid. The remainder of the theorem will be proved with a similar method by replacing $y(S)$ to $-y(S)$.

Corollary 1: [8] Assume that the grid function $y(S)$ is defined on $\omega+\gamma$ and let $y(S)$ satisfies the conditions (2.2) and (2.4). If $y(S)\geq 0$ on γ and $\mathcal{L}\, y(S)\geq 0$ for $S\in\omega$ then $y(S)\geq 0$ on $S\in w+\gamma$, But if $y(S)\leq 0$ on $S\in\gamma$ and $\mathcal{L}\, y(S)\leq 0$ on $S\in\omega$, then $y(S)\leq 0$ for $S\in\omega+\gamma$.

Proof: Assume that $y(S)\geq 0$ for $S\in\gamma$ and $\mathcal{L}\, y(S)\geq 0$ for $S\in\omega$. Let at least one inner point belong to ω, that is $y(S_0)\ <0$ for $S_0\in\omega$. Then $y(S)$ should attain the minimal negative value on ω, but by Theorem 1 it is impossible, because $y(S)\neq$ constant on ω $\left(y(S_0)<0, y(S)\geq 0 \text{ on } \gamma\right)$. Thus, we have proved the first part of the corollary. The second part it will be proved in a similar method.

Corollary 2: [8] The homogenous equation (2.1) subject to the boundary condition

$$\mathcal{L}\, y(S)=0 \quad \text{on} \quad S\in\omega, \qquad y(S)=0 \quad \text{on} \quad S\in\gamma \tag{2.9}$$

has the unique solution $y(S)=0$.

Proof: It is straight forward affirm that $y(S)=0$ satisfies equation (2.9). In addition, for contradiction, suppose that $y(S)\neq 0$. If at least for one point $y(S)\neq 0$, we have $\mathcal{L}\, y(S)\leq 0$ on ω and $y(S)\leq 0$ on γ then from corollary 1 $y(S)\leq 0$ on $\omega+\gamma$. At the same time we have $y(S)\geq 0$ on $\omega+\gamma$. But it is impossible when $y(S)\neq 0$. So, we have proved Corollary 2.

Corollary 3: [8] problem (2.1)-(2.4) possesses only one solution.

Theorem 2: [8](comparison theorem) suppose that $y(S)$ satisfy problem (2.1)-(2.4) and assume that $Y(S)$ satisfy the following problem:

$$\mathcal{L}Y(S) = \bar{Z}(S), \qquad \mathrm{S} \in \omega \qquad Y(S) = \bar{\mu}(S), \qquad \mathrm{S} \in \gamma \ , \tag{2.10}$$

then the conditions

$$|Z(S)| \le \bar{Z}(S), \qquad \mathrm{S} \in \omega \qquad |\mu(S)| \le \bar{\mu}(S), \qquad \mathrm{S} \in \gamma \tag{2.11}$$

provide validity to the inequality

$$|y(S)| \le Y(S) \quad \text{for} \quad \mathrm{S} \in \omega + \gamma \tag{2.12}$$

Proof: since $\mathcal{L}Y(S) = \bar{Z}(S) \ge 0$ on the grid function ω, and $|\mu(S)| \le \bar{\mu}(S)$ on the boundary γ, then by Corollary 1 we can say that $Y(S) \ge 0$ on $\omega + \gamma$. for functions $u(S) = Y(S) + y(S)$ and $v(S) = Y(S) - y(S)$ we have the equations

$\mathcal{L}u(S) = Z_u(S) = \mathcal{L}\left(Y(S) + y(S)\right) = \bar{Z} + Z \ge 0$ for $S \in \omega$,and $u = (Y + y) = \bar{\mu} + \mu \ge 0$ on the boundary γ, then by corollary 1 we have $u \ge 0$ or $y \ge -Y$ on $\omega + \gamma$ the equations,

$$\mathcal{L}v(S) = Z_v(S) = \mathcal{L}\left(Y(S) - y(S)\right) = \bar{Z} - Z \ge 0$$

on the grid function ω, and $v = (Y - y) = \bar{\mu} - \mu \ge 0$ on the boundary γ, so by corollary 1we have $v \ge 0$ or $y \le Y$ on $\omega + \gamma$. Now, we have the inequality $-Y \le y \le Y$ or $|y(S)| \le Y(S)$ on $\omega + \gamma$. $Y(S)$ is the majorant of the solution of (1.1)-(1.3).

Corollary 4: [8] For $y(S)$, which is defined as the solution of

$$\mathcal{L}\, y(S) = 0 \quad \text{for} \quad \mathrm{S} \in \omega, \qquad y(S) = \mu(S) \quad \text{for} \quad \mathrm{S} \in \gamma \tag{2.13}$$

we have the estimate

$$\max_{S \in \omega + \gamma} |y(S)| = \|y\|_{\bar{C}} \le \|\mu\|_{C_\gamma} \tag{2.14}$$

where,

$$\|\mu\|_{C_\gamma} = \max_{S \in \gamma} |\mu(S)| \ .$$

Proof: Assume the majorant $Y(S)$ satisfies $\mathcal{L}Y = 0$ on the grid nodes ω and $Y = \|\mu\|_{C_\gamma} \ge 0$ on the boundary γ . Then by Corollary 1 $Y(S) \ge 0$ for $S \in \omega + \gamma$ and at some point of the grid $Y(S)$ takes its maximum. But if $Y(S) \ne const$ by Theorem 1 this point should be none of the interior points and, therefore,

$$\|Y\|_{\bar{C}} = \max_{P \in w + \gamma} Y(S) = \max_{P \in \gamma} Y(S) = \|\mu\|_{C_\gamma} \ .$$

If $Y(S) = const$, then $Y(S) = \|\mu\|_{C_\gamma}$. For both cases $\|Y\|_{\bar{C}} = \|\mu\|_{C_\gamma}$.

Then we can say that the inequality $\|y\|_{\bar{C}} \le \|Y\|_{\bar{C}}$ gives estimate (2.14).

2.1.3 Error Analysis of Nonhomogeneous Equations

We consider the solution of problem (2.1)-(2.3) in the form

$$y = y^* + w \ ,$$

where $y^* = y^*(S)$ satisfies

$$\mathcal{L}\, y^*(S) = 0 \quad \text{on} \quad \mathrm{S} \in \omega, \quad \bar{y}(S) = \mu(S) \quad \text{on} \quad \mathrm{S} \in \gamma . \tag{2.15}$$

We also have $w = w(S)$ as a solution to the nonhomogeneous equation with homogeneous boundary condition

$$\mathcal{L}\, w(S) = Z(S) \quad \text{for} \quad \mathrm{S} \in \omega, \quad w(S) = 0 \quad \text{for} \quad \mathrm{S} \in \gamma \tag{2.16}$$

In the previous Corollary we estimated the value of the function $y^{*}(S)$ by equation (2.14) and so we only need to consider the estimation of $w(S)$.

Theorem 3: [8] Assume that $T(S) > 0$ on ω. Then the solution of problem (2.16) is estimated by the inequality

$$\left\| w \right\|_C \le \left\| \frac{Z}{T} \right\|_C \tag{2.17}$$

Proof: Suppose that a majorant $Y(S)$ is defined as

$$\mathcal{L}Y(S) = \left| Z(S) \right| \quad \text{on} \quad \omega, \quad \mathrm{Y(S)} = 0 \quad \text{on} \quad \gamma,$$

then by Corollary 1 $Y(S) \ge 0$ on $\omega + \gamma$.

The function $Y(S)$ obtain the maximum at a node $S_0 \in \omega$. As far as $Y(S_0) = \left\| Y \right\|_C$ is concerned, the equation

$$\mathcal{L}Y(P_0) = \; T(S_0)Y(S_0) + \sum_{K \in Patt'(S_0)} M(S_0, K)(Y(S_0) - Y(K)) = \left| Z(S_0) \right|$$

Since, $Y(S_0)$ is a maximum so, $Y(S_0) - Y(K) \ge 0$ then we can say that

$$T(S_0)Y(S_0) \le \left| Z(S_0) \right|$$

Now, we have $T(S_0) \geq 0$, then $Y(S_0) \leq \frac{|Z(S_0)|}{T(S_0)} \leq \left\|\frac{Z}{T}\right\|_c$, $Y($ $Y(S_0) = \max_{\omega+\gamma} Y(P) = \|Y\|_c$

Since by comparison Theorem, we have $|w(P_0)| \leq Y(P_0) = \|Y\|_C \leq \left\|\frac{Z}{T}\right\|_C$, then

$\|w\|_C \leq \left\|\frac{Z}{T}\right\|_C$.

Remark: Estimate (2.17) is still appropriate for the equation (2.16) provided that instead of (2.2) other conditions

$$|N(S)| \neq 0, \qquad |M(S,K)| \neq 0,$$

$$\mathrm{T}(S) = |N(S)| - \sum_{K \in Patt'(S)} |M(S,K)| > 0$$

hold for $\|w\|_C \leq \left\|\frac{Z}{T}\right\|_C$.

Indeed, assume that $|w(S)| \geq 0$ is a maximal value at a point S_0 , so that

$$|N(S_0)| \cdot |w(S_0)| = \left| \sum_{K \in Patt'(S_0)} M(S_0,K) w(K) + Z(S_0) \right|$$

$$\leq \sum_{K \in Patt'(S_0)} |B(S_0,K)| \cdot |w(S_0)| + |Z(S_0)| \ .$$

We have now

$$|w(S_0)| \left[|N(S_0)| - \sum_{K \in Patt'(S_0)} |M(S_0,K)| \right] \leq |Z(P_0)| \, .$$

Since, $T(S_0) = \left|N(S_0)\right| - \sum_{K \in Patt'(S_0)} \left|M(S_0, K)\right| > 0$ then,

$$T(S_0)\ \left|w(S_0)\right| \le \left|Z(S_0)\right|, \qquad \left\|w\right\|_C = \left|w(S_0)\right| \le \frac{\left|Z(S_0)\right|}{T(S_0)} \le \left\| \frac{Z}{T} \right\|_C .$$

Theorem 4: [8] Suppose that Let $\bar{\omega}_h = \omega_h + \gamma_h$ be the set of regular and boundary points, the set of irregular points be denoted by ω_h^*, and ω_h^o be the set of all strictly interior points: $\omega_h = \omega_h^* + \omega_h^o$. And let the conditions

$$T(S) = 0 \quad \text{on} \quad \omega^o \qquad \text{and} \qquad \mathrm{T}(S) > 0 \quad \text{on} \quad \omega^*$$

hold, then for a solution of problem (2.16) with

$$Z(S) = 0 \quad \text{on} \quad \omega^o \qquad \text{and} \qquad \mathrm{Z}(S) \ne 0 \quad \text{on} \quad \omega^* .$$

we have the estimate

$$\left\|v\right\|_C \le \left\| \frac{Z}{T} \right\|_{C^*}, \tag{2.18}$$

where, $\left\|z\right\|_{C^*} = \max_{s \in \omega^*} \left|z(S)\right|$.

Proof: Assume that the function $Y(S)$ is a majorant and $\mathcal{L}Y(S) = \left|Z(S)\right|$ for $S \in \omega$ and $Y(S) = 0$ for $P \in \gamma$, $Y(S) \ge 0$. The function $Y(S)$ at some point of the finite set $w + \gamma$ must take the maximum, but it does not enter the boundary, because $Y(S) = 0$ for $S \in \gamma$ $\omega + \gamma$. Also, it must not belong to the grid ω^o because the connectedness of ω^o and the maximum principle. So that,

$$\max_{S\in\omega} Y(S) \;=\; \max_{S\in\omega^*} Y(S) \;=\; Y(S_0)$$

where S_0 is a point belonging to the set ω^*.

By the first assumption, $T(S_0) > 0$. By analogy to the proof of Theorem 3 we get inequality (2.18). The Remark given for Theorem 3 also applies by analogy to this case.

2.2 Analaysis of The Dirichlet Difference Problem

2.2.1 Approximation of The Dirichlet Problem

Consider $$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = -F(x),$$

where $x = (x_1, x_2) \in G$, G is a 2-dimensional finite domain with the boundary Γ. Let $\bar{\omega}_h = \omega_h + \gamma_h$ be the set of regular and boundary points, the set of irregular points be denoted by ω_h^*, and ω_h^o be the set of all strictly interior points: $\omega_h = \omega_h^* + \omega_h^o$.

So, at the regular points we have

$$\Lambda y + \phi(x) = 0 \tag{2.19}$$

at the irregular points we have

$$\Lambda y^* + \phi(x) = 0 \tag{2.20}$$

and at the boundary points we have

$$y = \mu(x), \qquad \text{on} \quad \gamma. \tag{2.21}$$

Now, we obtain a uniform estimate of the approximate solution of problem (2.19)-(2.21) with Dirichlet boundary conditions:

$$\begin{aligned} &\Lambda y = -\phi(x) && \text{at the regular points,} \\ &\Lambda y^* = -\phi(x) && \text{at the irregular points,} \\ &\mathrm{y} = \mu(x) && \text{at the boundary.} \end{aligned} \tag{2.22}$$

Where, $\Lambda y = \sum_{\alpha=1}^{2} \Lambda_\alpha y = \Lambda_1 y + \Lambda_2 y, \quad \Lambda_1 y = y_{\bar{x}_1 x_1}$ and $\Lambda_2 y = y_{\bar{x}_2 x_2}$, ,

$$\Lambda^* y = \sum_{\alpha=1}^{2} \Lambda^*_\alpha y = \Lambda^*_1 y + \Lambda^*_2 y, \quad \Lambda^*_1 y = y_{\bar{x}_1 x_1} \quad \text{and} \quad \Lambda^*_2 y = y_{\bar{x}_2 x_2} ,$$

$$\Lambda^*_\alpha y = \frac{1}{h_\alpha}\left(\frac{(y + h_{\alpha^+}) - y}{h_{\alpha^+}} - \frac{y - (y - h_{\alpha^-})}{h_{\alpha^-}} \right), \qquad \alpha = 1, 2 .$$

Leading to the alternative form

$$\bar{\Lambda} y = -\phi(x), \quad \text{for} \quad \mathrm{s} \in \omega_h, \qquad \mathrm{y} = \mu(x) \quad \text{for} \quad \mathrm{s} \in \gamma_h . \tag{2.23}$$

In conformity with given problem (1.22) can be defined as

$$N(x) y(x) = \sum_{\lambda \in Patt'(x)} M(x,\lambda) y(\lambda) + Z(x), \quad x \in \omega_h, \qquad \mathrm{y} = \mu(x) \quad \text{for} \quad x \in \gamma_h \tag{2.24}$$

where

$$\mathrm{N}(x) > 0, \quad \mathrm{M}(x,\lambda) > 0 \quad \text{for all } x \in \omega_h \text{ and } \lambda \in Patt'(s)$$

$$\mathrm{T}(x) = N(x) - \sum_{\lambda \in Patt'(x)} \mathrm{M}(x,\lambda) \geq 0 .$$

Let’s represent a solution of the sum of two functions

$$y = \breve{y} + \hat{y},$$

where $\breve{y}$ and $\hat{y}$ are two convenient functions of the problems

$$\bar{\Lambda}\breve{y} = 0 \quad \text{for} \quad x \in \omega_h, \quad \breve{y} = \mu \quad \text{on} \quad \gamma_h \tag{2.25}$$

$$\bar{\Lambda}\hat{y} = -\phi \quad \text{for} \quad x \in \omega_h, \quad \hat{y} = 0 \quad \text{on} \quad \gamma_h \ . \tag{2.26}$$

So, by the corollary of the comparison theorem we have estimation of (2.25) comparable to

$$\|\breve{y}\|_{\bar{C}} \le \|\mu\|_{C_\gamma} \ . \tag{2.27}$$

We decomposed the right-hand side ϕ of problem (2.26) as

$$\phi = \phi^o + \phi^* \ ,$$

where $\phi^o = \phi$ and $\phi^* = 0$ are defined at the strictly interior points $x \in \omega_h^o$, and $\phi^o = 0$ and $\phi^* = \phi$ at the near-boundary points $x \in \omega_h^*$ we have

$$\hat{y} = u + k$$

where u and k are two convenient functions of the problems

$$\bar{\Lambda}u = -\phi^o \quad \text{on} \ \ \omega_h \ , \quad u = 0 \quad \text{on} \quad \gamma_h \tag{2.28}$$

$$\bar{\Lambda}k = -\phi^* \quad \text{on} \ \ \omega_h \ , \quad k = 0 \quad \text{on} \quad \gamma_h \tag{2.29}$$

Now we evaluate individually the functions $u(x)$ and $k(x)$. With the intention to estimate $u(x)$, it's needful to have a majorant $Y(x)$. Choosing the domain G so that the origin belongs to it, we construct this function as

$$Y(x) = L(R^2 - r^2), \qquad r^2 = \sum_{\alpha=1}^{p} x_\alpha^2 \quad ,$$

where R is the radius of a p-dimensional ball, or a circle when $p = 2$, centered at the origin, and the whole of G is contained in it, and L>0 is a constant that can be chosen later.

By using $\Lambda_\alpha x_\beta^2 = 0$ for $\alpha \neq \beta$

$$\Lambda_\alpha L R^2 = \frac{(LR^2)^2 - 2(LR^2)^2 + (LR^2)^2}{h_\alpha^2} = 0$$

$$\Lambda_\alpha x_\alpha^2 = \frac{(x_\alpha + h_\alpha)^2 - 2x_\alpha^2 + (x_\alpha - h_\alpha)^2}{h_\alpha^2} = 2 \ ,$$

$$\Lambda_\alpha^* x_\alpha^2 = \frac{1}{h_\alpha}\left[\frac{(x_\alpha + h_\alpha)^2 - x_\alpha^2}{h_{\alpha^+}} - \frac{x_\alpha^2 - (x_\alpha - h_\alpha)^2}{h_{\alpha^-}}\right] = 2\theta_\alpha \quad , \qquad \theta = \frac{h_{\alpha^+} + h_{\alpha^-}}{2h_\alpha} .$$

We determine that

$$\Lambda Y = \Lambda L R^2 - \Lambda L r^2 = -L\sum_{\alpha=1}^{p} \Lambda_\alpha x_\alpha^2 = -2pL \qquad \text{for} \quad x \in \omega^o{}_h \ ,$$

$$\Lambda^* Y = -2p\theta L \qquad \text{for} \quad x \in \omega_h^* \ ,$$

where $\theta = p^{-1}\sum_{\alpha=1}^{p} \theta_\alpha$. Here $\theta_\alpha = 1$ if the point $x \in \omega_h^*$ is regular with respect to x_α .

Hence, the function Y satisfies the problem

$$\bar{\Lambda} Y = -\bar{F}(x), \qquad \mathrm{Y}(x) = L(R^2 - r^2) \geq 0 \quad \text{for} \quad x \in \gamma_h,$$

where, $\bar{F}(x) = 2pL$ on ω_h^o and $\bar{F}(x) = 2p\theta L$ on ω_h^* . Comparison with problem (2.28), where $F = \phi^o$, that is, $F = 0$ on ω_h^*, and $u = 0$ for $x \in \gamma_h$, shows that $\bar{F}(x) \geq |F(x)| = |\phi^o(x)|$ with the constant L chosen as $L = \frac{1}{2p}\|\phi^o\|_C$.

Now by the comparison theorem we have $\bar{F}(x) \geq |F(x)| = 0$ for $x \in \omega_h^*$, hence providing the inequality $\|u\|_C \leq \|Y\|_C$. From the expression of Y we can say that $\|Y\|_C \leq LR^2$. So, the estimation of a solution of the function $u(x)$ for problem (2.28) comparable to

$$\|u\|_C \leq \frac{R^2}{2p}\|\phi^o\|_C = \frac{R^2}{2p}\|\phi\|_{C^o} \tag{2.30}$$

is appropriate in the following norm $\|\phi\|_{C^o} = \max_{x \in \omega_h^o} |\phi(x)|$.

Now we are going to find the estimation of the function $k(x)$. First, for problem (2.29)

$$T(x) \geq \frac{1}{h^2} \quad \text{for} \quad x \in \omega_h^*, \quad \text{where} \quad h = \max_{\alpha} h_\alpha \tag{2.31}$$

$$T(x) = 0 \qquad on \quad \omega_h^o \ . \tag{2.32}$$

After that, we consider the problem (2.29) where x belong to the near-boundary point ω_h^*

$$N(x)k(x) = \sum_{\lambda \in Patt'(x)} M(x,\lambda)k(\lambda) + F(x),$$
$$F(x) = \phi^*(x) \quad \text{for } x \in \omega_h^* \quad \text{and} \quad k = 0 \quad \text{for} \quad x \in \gamma_h \qquad (2.33)$$

If one of the points $\lambda = \lambda_0$, say $\lambda_0 = (x + h_{\alpha^+})$, happens to be a boundary point, then $k(\lambda_0) = 0$ and λ_0 does not belong to the $Patt'(x)$.

Here $T(x)$ is defined as

$$T(x) = N(x) - \sum_{\substack{\lambda \in Patt'(x) \\ \lambda \neq \lambda_0}} M(x,\lambda)$$

$$= N(x) - \left[\sum_{\lambda \in Patt'(x)} M(x,\lambda) - M(x,\lambda_0) \right] = M(x,\lambda_0) \ ,$$

since for the Laplace equation we have $N(x) \ = \sum_{\lambda \in Patt'(x)} M(x,\lambda)$, we can form the inequality

$$T(x) = M(x, x + h_{\alpha^+}) \ > 0 \ .$$

If a point x is near-boundary node not only with respect to of x_α, but also in other directions, then sum of the equation (2.33) contains no other terms for $\lambda = \lambda_1, \lambda_2, ..., \lambda_k$ then

$$T(x) = M(x,\lambda_0) + M(x,\lambda_1) + \ ... \ + M(x,\lambda_k) \ > 0 \ .$$

Suppose that the point $x \in x_h^*$ is an irregular near-boundary point only in some direction x_α and $\lambda_0 = (x + h_{\alpha^+}) \in \gamma_h$, $(x - h_{\alpha_-}) \in \omega_h$. By the equation

$$\Lambda_\alpha^* k + \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{p} \Lambda_\beta k = -\phi^*(x),$$

where

$$\Lambda_\beta y = y_{\bar{x}_\beta x_\beta} \ ,$$

$$\Lambda_\alpha^* k = \frac{1}{h_\alpha}\left(\frac{(k + h_{\alpha^+}) - k}{h_{\alpha^+}} - \frac{k - (k - h_{\alpha^-})}{h_\alpha} \right)$$

$$= -\frac{1}{h_\alpha}\left(\frac{k}{h_{\alpha^+}} + \frac{k - (k - h_{\alpha^-})}{h_\alpha} \right).$$

so we have

$$N(x) = \frac{1}{h_\alpha h_{\alpha^+}} + \frac{1}{h_\alpha^2} + \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{p} \frac{2}{h_\beta^2} \ ,$$

$$\sum_{\lambda \in Patt'(x)} M(x,\lambda) = \frac{1}{h_\alpha^2} + \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{p} \frac{2}{h_\beta^2} \ ,$$

$$T(x) = \frac{1}{h_\alpha h_{\alpha^+}} \geq \frac{1}{h^2} \ .$$

If a point x is irregular in some directions and a regular point only with respect to x_α , then

$$T(x) = \frac{1}{h_\alpha^2} \geq \frac{1}{h^2} \ .$$

From Theorem 4 of section 1 to evaluate solution of (2.26) we have

$$\|k\|_C \le \left\|\frac{\phi^*}{T}\right\|_{C^*} \le h^2 \|\phi^*\|_C . \qquad (2.34)$$

Combining the estimates (2.27), (2.30), (2.34) we have $\|y\|_{\bar{C}} \le \|\bar{y}\|_{\bar{C}} + \|u\|_C + \|k\|_C$.

2.2.2 The Uniform Convergence And The order of Accuracy of a Difference Scheme

In this section we study the convergence and accuracy of scheme (2.23). We start by finding the error between difference and exact solution for (2.22), and assume that

$$z = \bar{w} - w \ ,$$

where $\bar{w}$ is a difference solution of (2.22) and we have $w = w(x)$ as an exact solution of (2.22). Putting $z = \bar{w} + w$ into (2.22) or (2.23) yields

$$\bar{\Lambda} z = -\psi(x) \quad \text{on } \omega \quad \text{and} \qquad z = 0 \quad \text{on} \quad \gamma \qquad (2.35)$$

where $\psi(x) = \bar{\Lambda} w + \phi(x)$, and

$$\psi(x) = O(|h|^2) = O(h^2) , \qquad \text{for regular points,}$$

$$\psi(x) = O(1) , \qquad \text{for irregular points,}$$

or, more particularly,

$$|\psi| \le \frac{M_4 |h|^2}{12} \le p \frac{M_4}{12} h^2 , \qquad \text{for regular points}$$

$$|\psi| \le pM_2 , \qquad \text{for irregular points}$$

where

$$M_k = \max_{\substack{x\in\bar{G} \\ 1\le \alpha \le p}} \left| \frac{\partial^2 w}{\partial x_\alpha^k} \right|, \qquad k=2, 3, 4, \ldots, \qquad |h|^2 = \sum_{\alpha=1}^{p} h_\alpha^2, \qquad h = \max_{1\le\alpha\le p} h_\alpha .$$

So

$$\|z\|_C \le \frac{R^2}{2p} \|\psi\|_{\overset{0}{C}} + h^2 \|\psi\|_{C^*} .$$

Putting the estimate of $|\theta|$ at the irregular point and regular point s into the above inequality results in;

$$\|z\|_C = \|\bar{w} - w\|_C \le \left(\frac{R^2}{24} M_4 + pM_2 \right) h^2 . \tag{2.36}$$

Theorem 2: [7] Assume that $w(x)$ has continuous fourth derivatives in the space $\bar{G}$, $w(x) \in C^4(\bar{G})$ where $\bar{G} = G + \Gamma$ then the difference scheme is of second-order accuracy.

2.3 Higher-Accurate Schemes

2.3.1 The Dirichlet Difference Problem with Higher Accuracy

On the bases of the 5-point scheme, we can construct operators giving an error approximation of $O(|h|^4)$ or $O(|h|^6)$ for a solution within the square (cube) grid. Consider $w = w(x)$ satisfying the equation

$$\Delta w = \sum_{\alpha=1}^{p} \frac{\partial^2 w}{\partial x_\alpha^2} = -f(x) . \tag{2.37}$$

For p=2 (2D case) we have

$$\Delta u = (L_1 + L_2)w = L_1 w + L_2 w, \qquad \mathrm{L}_\alpha w = \frac{\partial^2 w}{\partial x_\alpha^2}, \qquad \alpha = 1,2 \ ,$$

by appealing to the difference operator

$$\Lambda u = (\Lambda_1 + \Lambda_2)u = \Lambda_1 u + \Lambda_2 u \ , \qquad \Lambda_\alpha u = u_{\bar{x}_\alpha x_\alpha}, \qquad \alpha = 1,2 \ ,$$

let $w = w(x)$ possess all necessary derivatives. So that

$$\Lambda w - Lw = \frac{h_1^2}{12} L_1^2 w + \frac{h_2^2}{12} L_2^2 w + O(|h|^4) \ . \qquad (2.38)$$

By the equation $L_1 w + L_2 w = -f(x)$ we obtain

$$L_1^2 w = -L_1 f - L_1 L_2 w, \qquad L_2^2 w = -L_2 f - L_1 L_2 w,$$

in order that

$$\Lambda w = Lw - \frac{h_1^2}{12} L_1 f - \frac{h_2^2}{12} L_2 f - \frac{h_1^2 + h_2^2}{12} L_1 L_2 w + O(|h|^4) \qquad (2.39)$$

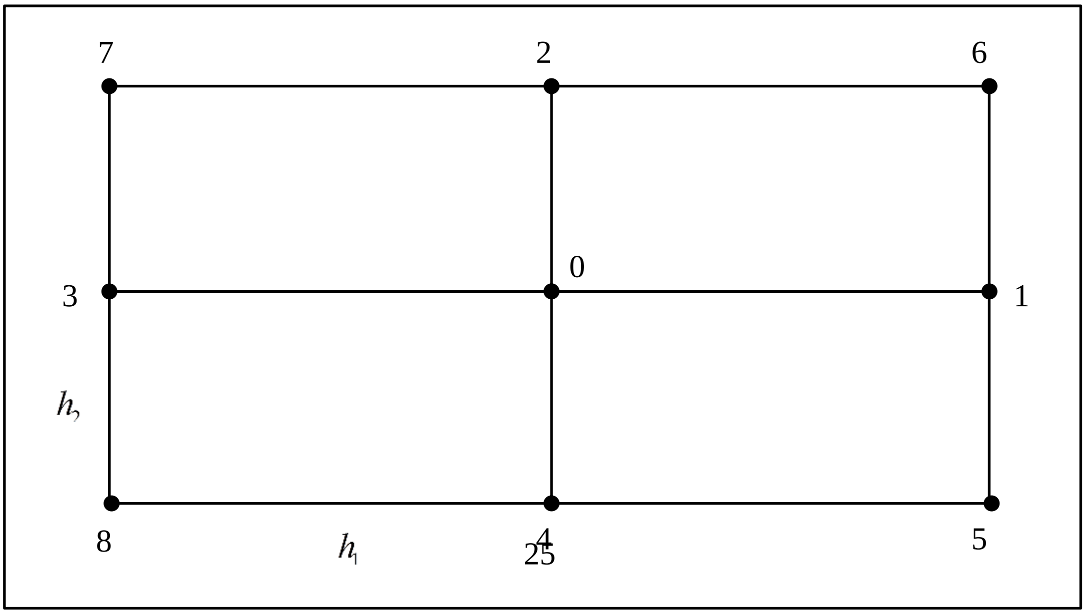

Figure 1: 9-points stencil

We substitute here $-f$ in place of Lw and change L_1L_2w by the difference operator,

$$\Lambda_1\Lambda_2 w = w_{\bar{x}_1 x_1 \bar{x}_2 x_2} \sim L_1L_2 w = \frac{\partial^4 w}{\partial x_1^2 \partial x_2^2}\,.$$

This operator is defined on the 9-point pattern given in Figure1 and we have $\Lambda_1\Lambda_2 w$, as follows,

$$\Lambda_1\Lambda_2 w = \Lambda_1\left[\frac{w(x_1,x_2-h_2)-2w(x_1,x_2)+w(x_1,x_2+h^2)}{h_2{}^2}\right]$$

$$=\frac{1}{h_1{}^2 h_2{}^2}\{w(x_1-h_1,x_2-h_2)-2w(x_1,x_2-h_2)$$

$$+w(x_1+h_1,x_2-h_2)+\ 4w(x_1,x_2)$$

$$-2w(x_1-h_1,x_2)+\ \mathrm{w}(x_1-h_1,x_2+h_2)-2w(x_1,x_2+h_2)$$

$$-2w(x_1+h_1,x_2)+\ \mathrm{w}(x_1+h_1,x_2+h_2)\}$$

is required within the estimation of the error of approximation to $\Lambda_1\Lambda_2 w - L_1L_2 w$ through advantage of the good-established expansion

$$\Lambda r = r_{\bar{x}x} = \frac{r(x+h)-2r(x)+r(x-h)}{h^2} r(\lambda), \qquad \lambda = x+\theta h, \qquad |\theta|\le 1. \qquad (2.40)$$

Suppose that $r(x)\in C^2\left[x-h,x+h\right]$, so that

$$\Lambda r = r_{\bar{x}x} = r''(x) + \frac{h^2}{12} r^{(4)}(\lambda^*), \qquad \lambda^* = x + \theta^* h, \qquad |\theta^*| \le 1, \tag{2.41}$$

$r(x) \in C^4\ [x-h, x+h]$. By using pertaining x_1 to be fixed we could have

$$\Lambda_2 r = L_2 r(x_1, x_2) + \frac{h_2^2}{12} \frac{\partial^4 r}{\partial x_2^4}(x_1, \lambda_2), \qquad \lambda_2 = x_2 + \theta_2 h_2, \qquad |\theta_2| \le 1.$$

$$\Lambda_1 \Lambda_2 w(x_1, x_2) = \Lambda_1 L_2 w(x_1, x_2) + \frac{h_2^2}{12}\ \Lambda_1 \frac{\partial^4 w}{\partial x_2^4}(x_1, \lambda_2).$$

Applying equation (1.41) with $r = L_2 w$ and $x = x_1$ to the first summand yields

$$\Lambda_1 L_2 w(x_1, x_2) = L_1 L_2 w(x_1, x_2) + \frac{h_1^2}{12}\ \Lambda_1 \frac{\partial^4 w}{\partial x_1^4}(\lambda_1^*, x_2), \qquad \lambda_1^* = x_1 + \theta_1^* h_1, \qquad |\theta_1^*| \le 1$$

by the similar method for the second summand with respect to equation (2.39):

$$\frac{h_2^2}{12}\ \Lambda_1 \frac{\partial^4 w}{\partial x_2^4}(x_1, \lambda_2) = \frac{h_2^2}{12}\ \Lambda_1 \frac{\partial^6 w}{\partial x_1^2 \partial x_2^4}(\lambda_1, \lambda_2), \qquad \lambda_1 = x_1 + \theta_1 h_1, \qquad |\theta_1| \le 1\ .$$

What must be done is to bring together the outcomes acquired:

$$(\Lambda_1 \Lambda_2 - L_1 L_2) w(x_1, x_2) = \Lambda_1 \Lambda_2 w(x_1, x_2) - L_1 L_2 w(x_1, x_2) = O(h_1^2) + O(h_2^2) = O(|h|^2)\ .$$

Substituting into equation (2.39) the difference operator $\Lambda_1 \Lambda_2 w$ in to place of $L_1 L_2 w$,

$$L_1 L_2 w = \Lambda_1 \Lambda_2 w + O(|h|^2),$$

and $-f(x)$ in to place of Lw, we finally obtain

$$\Lambda w = Lw - \frac{h_1^2 + h_2^2}{12}\Lambda_1\Lambda_2 w - \frac{h_1^2}{12}L_1 f - \frac{h_2^2}{12}L_2 f + O(|h|^4)$$
$$= -\left(f + \frac{h_1^2}{12}L_1 f + \frac{h_2^2}{12}L_2 f \right) - \frac{h_1^2 + h_2^2}{12}\Lambda_1\Lambda_2 w + O(|h|^4)$$

(2.42)

Since, the equation

$$\Lambda' y = -\phi, \qquad \Lambda' y = \Lambda y + \frac{h_1^2 + h_2^2}{12}\Lambda_1\Lambda_2 y,$$
$$\phi = f + \frac{h_1^2}{12}L_1 f + \frac{h_2^2}{12}L_2 f, \tag{2.43}$$

provides an approximation of order 4 for a solution $w = w(x)$ of Poisson's equation (2.37). In fact, equation (2.42) gives

$$\Lambda' w + \phi = \Lambda' u + \phi - Lw - f = O(|h|^4), \qquad L = L_1 + L_2 .$$

The operator Λ' formed using the nodes in Figure 1 $(x_1 + m_1 h_1, x_2 + m_2 h_2)$;

$m_1, m_2 = -1, 0, 1$, and used in (2.43) is represented by

$$\frac{5}{3}\left(\frac{1}{h_1^2}+\frac{1}{h_2^2}\right)w=\frac{1}{6}\left(\frac{5}{h_1^2}-\frac{1}{h_2^2}\right)\left(w^{(+1_1)}+w^{(-1_1)}\right)+$$

$$+\frac{1}{6}\left(\frac{5}{h_2^2}-\frac{1}{h_1^2}\right)\left(w^{(+1_2)}+w^{(-1_2)}\right)+$$

$$+\frac{1}{12}\left(\frac{1}{h_1^2}+\frac{1}{h_2^2}\right)\left(w^{(+1_1,+1_2)}+w^{(+1_1,-1_2)}\right)+ \tag{2.44}$$

$$+\left(w^{(-1_1,-1_2)}+w^{(-1_1,+1_2)}\right)+\phi,$$

where, $w^{(+1_1)}=w(x_1+h_1,x_2),\ \mathrm{w}^{(-1_1)}=w(x_1-h_1,x_2),\ \mathrm{w}^{(+1_1,-1_2)}=w(x_1+h_1,x_2-h_2)$.

When an equidistant grid is considered in all directions $(h_1=h_2=h)$ the equation is obtained as:

$$w_0=\frac{4(w_1+w_2+w_3+w_4)+w_5+w_6+w_7+w_8}{20}+\frac{3}{10}h^2\phi$$

(See Figure 1).

To avoid exhaustive computations, we put $\Lambda_1 f$ in place of $\mathrm{L}_1 f$ and $\Lambda_2 f$ in place of $\mathrm{L}_2 f$ into the equation of φ and replace φ by $O(|h|^4)$, as $\psi=\Lambda' w+\phi=O(|h|^4)$, so that $\phi= f+\frac{h_1^2}{12}\Lambda_1 f+\frac{h_2^2}{12}\Lambda_2 f$.

Chapter 3

GREEN FUNCTION METHOD

3.1 Second Order Estimates

During this Chapter we will be able to consider the approximation of the following problem.

$$\begin{aligned}\Delta w(x_1, x_2) &= \frac{\partial^2 w}{\partial x_1^{\ 2}} + \frac{\partial^2 w}{\partial x_2^{\ 2}} = F(x_1, x_2) \quad \text{for} \quad (x_1, x_2) \in \omega \\ \mathrm{w}(x_1, x_2) &= f(x_1, x_2) \quad \text{for} \quad (x_1, x_2) \in \gamma.\end{aligned} \tag{3.1}$$

Suppose that ω and γ are defined the same as in Chapter 1. We form an (x_1, x_2) plane with a square grid, a distance h apart in both x and y directions. These are called "mesh" nodes. Assume that the set of all those mesh nodes in ω which are regular are in ω_h. Those nodes in ω which are not in ω_h are denoted by γ_h^*. The remainder of the nodes forms the set γ_h.

For any node S belonging to $\omega_h + \gamma_h^* + \gamma_h$ neighboring nodes $N(S)$ are defined the same as in Chapter 1.

If $w(x_1, x_2)$ is any grid function defined on $\omega_h + \gamma_h^* + \gamma_h$ satisfying the finite difference operator Δ_h, then for $(x_1, x_2) \in \omega_h$

$$\Delta_h w(x_1, x_2) \equiv h^{-2} \begin{Bmatrix} w(x_1 + h, x_2) + w(x_1, x_2 + h) + \\ + w(x_1 - h, x_2) + w(x_1, x_2 + h) - 4w(x_1, x_2) \end{Bmatrix} \tag{3.2}$$

which is the second order approximation of Δ for the function

$w(x_1, x_2) \in C^4$ in ω. More precisely, from equation (3.1) and (3.2) we have

$$\left|\Delta_h w(x_1, x_2) - \Delta w(x_1, x_2)\right| = \begin{vmatrix} w(x_1 + h, x_2) + w(x_1, x_2 + h) + w(x_1 - h, x_2) + w(x_1, x_2 + h) \\ -4w(x_1, x_2) - F(x_1, x_2) \end{vmatrix}$$

By Taylor's formula for $w(x_1, x_2) \in C^4$

$$w(x_1 + h, x_2)) = w(x_1, x_2) + h\frac{\partial w}{\partial x_1} + \frac{h^2}{2}\frac{\partial^2 w}{\partial x_1^{\ 2}} + \frac{h^3}{6}\frac{\partial^3 w}{\partial x_1^{\ 3}} + \frac{h^4}{24}\frac{\partial^4 w(\chi_1, x_2)}{\partial x_1^{\ 4}},$$

where $\chi_1 \in (x_1, x_1 + h)$,

$$w(x_1 - h, x_2) = w(x_1, x_2) - h\frac{\partial w}{\partial x_1} + \frac{h^2}{2}\frac{\partial^2 w}{\partial x_1^{\ 2}} - \frac{h^3}{6}\frac{\partial^3 w}{\partial x_1^{\ 3}} + \frac{h^4}{24}\frac{\partial^4 w(\chi_2, x_2)}{\partial x_1^{\ 4}},$$

where $\chi_2 \in (x_1 - h, x_1)$,

$$w(x_1, x_2 + h) = w(x_1, x_2) + h\frac{\partial w}{\partial y} + \frac{h^2}{2}\frac{\partial^2 w}{\partial x_2^{\ 2}} + \frac{h^3}{6}\frac{\partial^3 w}{\partial x_2^{\ 3}} + \frac{h^4}{24}\frac{\partial^4 w(x_1, \xi_1)}{\partial x_2^{\ 4}},$$

where $\xi_1 \in (x_2, x_2 + h)$,

$$w(x_1, x_2 - h) = w(x_1, x_2) - h\frac{\partial w}{\partial x_2} + \frac{h^2}{2}\frac{\partial^2 w}{\partial x_2^{\ 2}} - \frac{h^3}{6}\frac{\partial^3 w}{\partial x_2^{\ 3}} + \frac{h^4}{24}\frac{\partial^4 w(x_1, \xi_2)}{\partial x_2^{\ 4}},$$

where $\xi_2 \in (x_2 - h, x_2)$, so that

$$\left|\Delta_h w(x_1,x_2)-\Delta w(x_1,x_2)\right| \le \left\|\left[\left(\frac{\partial^2 w}{\partial x_1{}^2}+\frac{\partial^2 w}{\partial x_2{}^2}\right)-Fw(x_1,x_2)\right]\right\| +$$

$$+\left|\frac{h^2}{24}\left[\frac{\partial^4 w(\chi_1,x_2)}{\partial x_1{}^4}+\frac{\partial^4 w(\chi_2,x_2)}{\partial x_1{}^4}+\frac{\partial^4 w(x_1,\xi_1)}{\partial x_2{}^4}+\frac{\partial^4 w(x_1,\xi_2)}{\partial x_2{}^4}\right]\right|$$

since, from equation (3.1) we have

$$\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-F(x_1,x_2)=0 \text{ , so that}$$

$$\left|\Delta_h w(x_1,x_2)-\Delta w(x_1,x_2)\right| \le \frac{h^2}{12}\left[\left|\frac{\partial^4 w(\chi,x_2)}{\partial x_1{}^4}\right|+\left|\frac{\partial^4 w(x_1,\xi)}{\partial x_2{}^4}\right|\right],$$

where, $\chi \in (\chi_1,\chi_2)$ and $\xi \in (\xi_1,\xi_2)$

since $w \in C^4$, let $M_4 = \max\left\{\max\left|\frac{\partial^4 w(\chi,x_2)}{\partial x_1{}^4}\right|,\ \max\left|\frac{\partial^4 w(x_1,\xi)}{\partial x_2{}^4}\right|\right\}$, so that

$$\left|\Delta_h w(x_1,x_2)-\Delta w(x_1,x_2)\right| \le \frac{h^2}{6}M_4, \qquad \text{for} \quad (x_1,x_2)\in\omega_h. \tag{3.3}$$

At nodes of γ_h^*, Δ_h is defined for any point $(\bar{x}_1,\bar{x}_2)\in\gamma_h^*$ as

$$\Delta_h w(\bar{x}_1,\bar{x}_2) = 2h^{-2}\left\{\left(\frac{1}{\alpha+1}\right)w(\bar{x}_1+h,\bar{x}_2)+\left(\frac{1}{\alpha(\alpha+1)}\right)w(\bar{x}_1-\alpha h,\bar{x}_2)\right.$$
$$\left.+\left(\frac{1}{\beta+1}\right)w(\bar{x}_1,\bar{x}_2+h)+\left(\frac{1}{\beta(\beta+1)}\right)w(\bar{x}_1,\bar{x}_2-\beta h)-\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)w(\bar{x}_1,\bar{x}_2)\right\} \tag{3.4}$$

Δ_h returns to the form (3.2) when $\alpha=\beta=1$. We note that when Δ_h is defined as (3.5), it approximates Δ to first order accuracy for $w(\bar{x}_1,\bar{x}_2)\in C^3$ in ω, i.e.

$$\left|\Delta w(\bar{x}_1,\bar{x}_2)-\ \Delta_h w(\bar{x}_1,\bar{x}_2)\right|\le \frac{h}{3}\left[\left|\frac{\partial^3 w(\chi,\bar{x}_2)}{\partial \bar{x}_1{}^3}\right|+\left|\frac{\partial^3 w(\bar{x}_1,\xi)}{\partial \bar{x}_2{}^3}\right|\right],$$

with, $\bar{x}_1-h<\chi<\bar{x}_1+h$ and, $\bar{x}_2-h<\xi<\bar{x}_2+h$,

since $w\in C^3$, let us have $M_3=\max\left\{\max\left|\frac{\partial^3 w(\chi,\bar{x}_2)}{\partial \bar{x}_1{}^3}\right|,\ \max\left|\frac{\partial^3 w(\bar{x}_1,\xi)}{\partial \bar{x}_2{}^3}\right|\right\}$, so that

$$\left|\Delta w(\bar{x}_1,\bar{x}_2)-\ \Delta_h w(\bar{x}_1,\bar{x}_2)\right|\leqq \frac{2M_3 h}{3}. \tag{3.5}$$

We recall the finite difference analogue of equation (3.1),

$$\begin{aligned}\Delta_h W(x_1,x_2)&=F(x_1,x_2), \quad \text{for} \quad (x_1,x_2)\in\omega_h+\gamma_h^*,\\ \mathrm{W}(x_1,x_2)&=f(x_1,x_2), \quad \text{for} \quad (x_1,x_2)\in\gamma_h.\end{aligned} \tag{3.6}$$

This gives a system of linear equations for the determination of the grid function $W(x_1,x_2)$. The solution for (3.6) exists and is unique.

We will now show that

$\varepsilon(P)=w(P)-W(P),$ for $P\in\omega_h+\gamma_h^*+\gamma_h$,

where L is a constant independent of P and h, satisfies the inequality

$$\left|\varepsilon\right|_M\le Lh^2, \tag{3.7}$$

where

$$\phi_M=\sup_{S\in Q\subset\bar{\omega}}\phi(S) \tag{3.8}$$

for each function ϕ defined on a subset Q of $\bar{\omega}$. Now we define the finite difference analogue of the Green's function, $G_h(S,K)$ in the form

$$\begin{aligned} \Delta_{h,S} G_h(S,K) &= -\delta(S,K)h^{-2}, \quad &\text{for} \quad \mathrm{S} \in \omega_h + \gamma_h^* \\ G_h(S,K) &= \delta(S,K), \quad &\text{for} \quad \mathrm{S} \in \gamma_h, \end{aligned} \tag{3.9}$$

where $K \in \omega_h + \gamma_h^* + \gamma_h$.

Here

$$\delta(S,K) = \begin{cases} 1, & \mathrm{S} = K \\ 0, & \mathrm{S} \neq K. \end{cases} \tag{3.10}$$

Lemma 2.1 (maximum principle): [2]

Assume that the function $R(S)$ is any grid function defined on $\omega_h + \gamma_h^* + \gamma_h$ and $\Delta_h R(S) \geq 0$ on $\omega_h + C_h$, then $R(S)$ will take its maximal on γ_h .

Lemma 2.2 (Green's third identity): [2]

Suppose that the function $R(S)$ is any grid function defined on $\omega_h + \gamma_h^* + \gamma_h$, so that

$$R(S) = h^2 \sum_{K \in \omega_h + \gamma_h^*} G_h(S,K)\left[-\Delta_h R(K)\right] + \sum_{K \in \gamma_h} G_h(S,K) R(S) \tag{3.11}$$

where $S \in \omega_h + \gamma_h^* + \gamma_h$.

Proof: we can prove it by the finite difference analogue of Green's second identity.

Assume that $Z(S)$ is the RHS of equation (3.11), with the use of the Green's function $G_h(S,K)$, we obtain

$$\Delta_h Z(S) = \Delta_h R(S) \quad \text{on} \quad \omega_h + \gamma_h^* \tag{3.12}$$

$$Z(S) = R(S) \quad \text{on} \quad \gamma_h . \tag{3.13}$$

So, from the uniqueness of the solution of equation (3.6), we can say that $Z(S) = R(S)$.

Lemma 2.3: [2]

$$G_h(S,K) \geq 0 \qquad \text{for} \qquad K \in \omega_h + \gamma_h^* + \gamma_h . \tag{3.14}$$

Proof: Apply the maximum principle (lemma 2.1) to $-G_h(S,K)$ for the randomly selected but fixed $K \in \omega_h + \gamma_h^* + \gamma_h$.

Lemma 2.4: [2]

$$\sum_{K \in \gamma_h^*} G_h(S,K) \leq 1 \quad \text{for} \quad \mathrm{S} \in \omega_h + \gamma_h^* + \gamma_h . \tag{3.15}$$

Proof: suppose that the mesh function $Z(S)$ is given by

$$Z(K)=\begin{cases}1, & \text{for } K\in\omega_h+\gamma_h^*,\\ 0, & \text{for } K\in\gamma_h\ .\end{cases} \qquad (3.16)$$

Then, $\Delta_h Z(K)=0,$ where $\mathrm{K}\in\omega_h$. By the definition of Δ_h on γ_h^* we can obtain the inequality $-\Delta_h Z(K)\geq h^{-2}$.

Now, by equations (3.11) and (3.16) it follows that for $K\in\omega_h+\gamma_h^*$

$$h^2\sum_{K\in\gamma_h^*}G_h(S,K)\left[-\Delta_h Z(K)\right]=1$$

Since, $-\Delta_h Z(K)\geq\dfrac{1}{h^2}$. Then $\sum_{K\in\gamma_h^*}G_h(S,K)\leq 1$,

where, $S\in\gamma_h$ the inequality (3.15) is satisfied.

Lemma 2.5: [2] Assume that $\bar{D}$ is the diameter of the smallest circle containing ω then

$$h^2\sum_{K\in\omega_h+\gamma_h^*}G_h(S,K)\leq\frac{\bar{D}^2}{16} \quad \text{for} \quad \mathrm{S}\in\omega_h+\gamma_h^*+\gamma_h\,. \qquad (3.17)$$

Proof: suppose that $\bar{C}$ is the center of the circle about ω of diameter $\bar{D}$.

Assume that $Z(S)=\frac{\gamma(S)^2}{4}$ for $S\in\omega_h+\gamma_h^*+\gamma_h$ where $\gamma(S)$ is the Euclidean distance from $\bar{C}$ to S, so that

$$\Delta_h Z(S)=1,\quad \text{for}\quad S\in\omega_h+\gamma_h^*.$$

Now define the grid function

$$R(S)=h^2\sum_{K\in\omega_h+\gamma_h^*} G_h(S,K)\ ,$$

by equation (3.9) we have

$$\Delta_h R(S)=-1,\quad \text{for}\quad S\in\omega_h+\gamma_h^*$$

$$R(S)=0,\qquad \text{for}\quad S\in\gamma_h.$$

Hence, $\Delta_h\left[R(S)+Z(S)\right]=\Delta_h R(S)+\Delta_h Z(S)=0$ for $S\in\omega_h+\gamma_h^*$,

and $R(S)+Z(S)\le\frac{\bar{D}^2}{16}$ for $S\in\gamma_h$.

By Lemma 2.1,in view that, $Z\ge 0$, it follows that

$$R(P)=h^2\sum_{K\in\omega_h+\gamma_h^*} G_h(S,K)\le\frac{\bar{D}^2}{16},\qquad \text{for}\quad S\in\omega_h+\gamma_h^*+\gamma_h.$$

Theorem1: [2] If $W(x_1,x_2)$ and $w(x_1,x_2)$ are the solutions of equations (3.6) and (3.1), respectively, then the inequality

$$|\varepsilon|_M \leqq \frac{M_4}{96}h^2 + \frac{2M_3}{3}h^3. \qquad (3.19)$$

holds for the truncation error $\varepsilon(S) = w(S) - W(S)$.

Proof: Since $\varepsilon(S) = 0$ for the boundary γ_h by Lemma 2.2 we have

$$\varepsilon(S) = h^2 \sum_{K \in \omega_h + \gamma_h^*} G_h(S,K)\left[-\Delta_h \varepsilon(K)\right]. \qquad (3.20)$$

By equation (3.1) and (3.6) we have

$$\left|-\Delta_h \varepsilon(K)\right| = \left|\Delta_h w(K) - \Delta w(K)\right|. \qquad (3.21)$$

substituting equation (3.21) to (3.20) and using inequality (3.3) and (3.5) we have

$$|\varepsilon(S)| \leq \left|h^2 \sum_{K \in \omega_h} G_h(S,K)\right| \frac{h^2 M_4}{6} + \left|\sum_{K \in \gamma_h^*} G_h(S,K)\right| \frac{2M_3}{3}h^3.$$

Hence from Lemmas (3.4) and (3.5), we get equation (3.20).

Chapter 4

CONCLUSION

In this thesis, the use of the finite-difference method has been discussed for the approximation of elliptic equations.

Section 2.1 in Chapter 2 has been devoted to the statement of the difference analogue of the maximum principle, and Gerschgorin's majorant method. With the review of these, the necessary tools were provided for the convergence analysis and error estimation for the finite-difference analogues of various problems.

Gerschgorin's majorant method has been applied in Section 2.2 for the error estimation of the difference analogue of Poisson's equation, with Dirichlet boundary conditions. It has been shown that with the use of the 5-point scheme it is possible to obtain an accuracy of $O(h^2)$, where h is mesh step.

Furthermore the applications of different schemes have been provided, where it is possible to obtain higher accuracy.

In Chapter 3, another approach for error estimation was reviewed. First of all, a finite-difference Green's function for the Dirichlet problem for Poisson's equation was introduced. Using this, an analogue of Green's identity has been obtained, along with Gerschgorin's majorant method, the error analysis was carried out. This method also provided second-error accuracy when the 5-point scheme was applied.

Both of these methods can be generalized to mixed boundary conditions.

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