Schwarz's Method for Differential and Difference Equations

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ABSTRACT

Schwarz introduced a method that permits to obtain a solutions for complicated domains by covering the domain with overlapping subdomains by realizing the solution of the Dirichlet problem for harmonic functions on these overlapping subdomains.

The present MS Thesis deals with analyzing the construction and justification of Schwarz's method and Schwarz-Neumann method for partial differential and finite-difference equations and how we can prove the convergence of them by using some theorems and assumptions.

Keywords: Dirichlet problem, Artificial boundary**,** Schwarz's method**,** Schwarz-Neumann method.

ÖZ

Schwarz, bu örtüşen alt etki alanında harmonik fonksiyonlar için Dirichlet probleminin çözümünü gerçekleştirerek alt alanları örtüşen etki kaplayarak karmaşık alanları için bir çözüm elde etmek için izin veren bir yöntem tanıttı.

Mevcut Yüksek Lisans Tezi, kısmi diferansiyel ve sonlu fark denklemlerinin ve nasıl bazı teoremleri ve varsayımlar kullanılarak bunların yakınsama ispat için Schwarz yöntem ve Schwarz-Neumann yönteminin inşaat ve gerekçesini analiz ile ilgilenir.

Anahtar Kelimeler: Dirichlet problemi, Yapay sınır, Schwarz metodu, Schwarz-Neumann yöntemi.

DEDICATION

This work is gently dedicated to my sweetheart, my wife AVAN and my dear children PARYA and DARYA

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First and predominant, all my praise worthiness to ALLAH (the most Gracious and essentially the most Merciful), I want to prolong to various persons who so generously contributed to the work awarded on this thesis.

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Chapter 1

INTRODUCTION

In over a hundred years ago Schwarz produced a method - which is known by "alternating" - to solve the Dirichlet problem for differential equations [1].

Schwarz's method requires covering a complicated domain by overlapping rectangles and solving each of these rectangles alternately. This has many computational advantages as fast direct methods, such as the Fast Fourier method, can be applied for the approximation of the solution on overlapping rectangles easily, thus eliminating the need for complicated algorithms and reducing the required time for the numerical calculation of the problem.

Another efficient method is the Schwarz-Neumann method. This method requires an irregular domain to be embedded in a domain which is the union of less complicated domains, such as rectangles. The irregular domain is placed at the intersection of these rectangles, and the solution is obtained with the approximation of the boundary value problem on the overlapping rectangles. Hence Schwarz-Neumann method is also an iterative method.

This thesis is concerned with the review of these methods, as well as their finite-difference analogues.

In Chapter 2 of this thesis, which begins with Section 2.1, we formulate Schwarz's method for partial differential equations on an L-shaped domain B, by solving the problem in two overlapping rectangles $B_1$and B_2 covering the domain B and finding a convergent analytic solution of the Dirichlet problem for arbitrary second order partial differential equations, by using successive approximations [2, 3]. In Section 2.2, we establish the finite difference analog of Schwarz's method for Laplace's equation and prove the convergence of the method [4, 5, 8]. For Section 2.3, we have calculated numerical example, where it is about finding the approximate solution of given problem by using Schwarz's alternating method.

In Chapter 3, in Section 3.1, we analyze the Schwarz-Neumann method for second-order partial differential equations on B'(which is the intersection of the two overlapping rectangles covering the L-shaped domain B) [2]. Section 3.2 is about the realization of finite difference equations on B', by solving two Dirichlet problem in two regions $B_1$and B_2 defined above, and finding the approximate solution by using the Schwarz-Neumann procedure. The approximate solutions are consistent with the theoretical results obtained.

Chapter 2

SCHWARZ'S METHOD FOR THE SOLUTION OF THE DIRICHLET PROBLEM FOR THE UNION OF TWO REGIONS

2.1 Partial Differential Equation of Schwarz's Method

On the $xy-plane$ let us have two regions $B_1$and B_2 that overlap one another, with the common part B'.

Schwarz showed how the Dirichlet problem for harmonic functions in each of the regions $B_1$and B_2, for any continuous or piece-wise continuous boundary functions can be solved. With the aid of the consecutive solution of problems for regions $B_1$and B_2, we can solve the Dirichlet problem for region B, covered by the overlapping regions $B_1$and B_2.

Schwarz's method allows one to get solutions for regions of more complicated character, and he regarded the application of the alternating method to the Dirichlet problem of Laplace's equation

$$\begin{aligned} \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad &\text{on } B \\ u = f \qquad &\text{on } L \end{aligned} \tag{2.1}$$

where L is the border of B.

This method was available to the determination of functions satisfying more common equations and more uneasy boundary conditions.

Let us take the partial differential equations of second order with the following form (2.2):

$$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}\right) = 0 \qquad (2.2)$$

On the $xy - plane$, we have the region B bounded by the border L and let the function $f(M)$ be piece-wise continuous at points M of border L.

From the Dirichlet problem we can solve the following problem. To find the function $u(x, y)$, subject to the requirements:

1- $u(x, y)$ is bounded in B,

2- In B , u satisfies equation (2.2),

3- At any points M of continuity of (M) , the values of u will coincide with value of $f(M)$ at this point.

With respect to equation (2.2), we will have the following assumptions:

Assumption I

Let us consider for a point (x, y), u and $\bar{u}$ two functions satisfying equation (2.2) in B, bounded and their values on the border L of region B equal, except, at a finite set of points, then they will be equal to each other everywhere in B.

B_1 = Region 1, B_2= Region 2, $B = B_1 \cup B_2$

B'= $B_1 \cap B_2$,

L_1 = Border of region B_1, L_2 = Border of region B_2

L = Border of region B

$\bar{\alpha}$= Part of L_1 in B_1 which lies within B_2, α = Remaining part of L_1 in B_1

$\bar{\beta}$= Part of L_2 in B_2 which lies within B_1, β = Remaining part of L_2 in B_2

γ = Intersection of α and β.

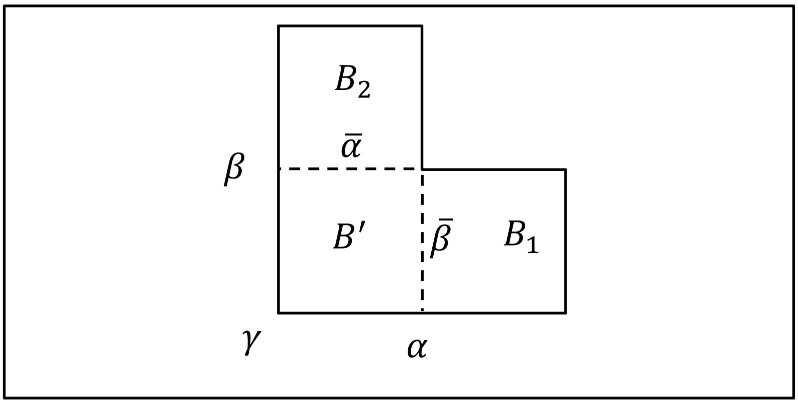

Figure 2.1: Figure of two regions that overlapped one another

We will consider at points of border L of region B the piece-wise continuous function $f(M)$, and we will find in region B the solution of equation (2.2) with the following boundary condition

$$u(x, y) = f(M) \quad \text{on } L \tag{2.3}$$

Now, let us provide a detailing to the alternating process of Schwarz, which allows us to construct successive approximations to the solution of the problem in B_1 and B_2.

We will start with the region B_1. On α which is a part of border L_1, we have given boundary values, and on $\bar{\alpha}$, we will define an arbitrary function $\varphi(M)$, both values of

$f(M)$ and $\varphi(M)$ on α and $\bar{\alpha}$ give a piece-wise continuous function on the whole border of the region B_1

By solving the Dirichlet problem for equation (2.2) in B_1 with the given boundary condition, we can construct u_1 as a first approximation to u

$$u_1 = \begin{cases} f(M) & \text{on } \alpha \\ \varphi(M) & \text{on } \bar{\alpha} \end{cases}. \tag{2.4}$$

By using u_1, we are able to construct the function v_1 for the point (x, y) by solving the Dirichlet problem in B_2 with the boundary conditions

$$v_1 = \begin{cases} f(M) & \text{on } \beta \\ u_1 & \text{on } \bar{\beta} \end{cases}. \tag{2.5}$$

By using v_1 we will construct the second approximation u_2 to u in B_1 as a solution of the Dirichlet problem for (2.2) with the following boundary condition

$$u_2 = \begin{cases} f(M) & \text{on } \alpha \\ v_1 & \text{on } \bar{\alpha} \end{cases} \tag{2.6}$$

and for v_2

$$v_2 = \begin{cases} f(M) & \text{on } \beta \\ u_2 & \text{on } \bar{\beta} \end{cases}. \tag{2.7}$$

By continuing this process we will get

$$u_n = \begin{cases} f(M) & \text{on } \alpha \\ v_{n-1} & \text{on } \bar{\alpha} \end{cases} \qquad v_n = \begin{cases} f(M) & \text{on } \beta \\ u_n & \text{on } \bar{\beta} \end{cases}. \tag{2.8}$$

For regions $B_1$and B_2, we have constructed a sequence of approximations to the function u

$$\begin{cases} u_1, u_2, \ \dots, \ u_n, \ \dots \ \text{ in } \ B_1 \\ \\ v_1, v_2, \ \dots, \ v_n, \ \dots \ \text{ in } \ B_2 \end{cases} . \tag{2.9}$$

We need to investigate the convergence of sequences (2.9) and show that the limit functions satisfy (2.2) and the boundary requisites that we supposed. We can succeed in doing this with additional assumptions.

Assumption II

On the $xy - plane$ we have region B with its border L; suppose that u and $\bar{u}$ are two bounded functions satisfying equation (2.2) in region B and on its border the boundary values are given, except a finite number of points. We will consider that if on border L the following inequality is true

$$\bar{u} \ \geq u, \tag{2.10}$$

then anywhere in region B we will also have

$$\bar{u} \ \geq u. \tag{2.11}$$

Assumption III

Let us have a sequence of solutions of equation (2.2) in B:

$$u_1, u_2, \ \dots, \ u_n, \dots . \tag{2.12}$$

Let this sequence be monotonic (increasing or decreasing) and uniformly bounded, it will converge everywhere within B

$$\lim_{n \to \infty} u_n = u. \tag{2.13}$$

So, we can say that the limit of any monotonic and bounded sequence of solutions of equation (2.2) will also be a solution of equation (2.2).

Assumption IV

In region B suppose that u is a solution of equation (2.2), bounded, and defined everywhere on border L, except at some of points. If for all boundary values of u on the border L of B following inequalities are true

$$-h \leq u \leq +g \quad , (h, g > 0) \tag{2.14}$$

then it is true everywhere within B that

$$-h \leq u \leq +g. \tag{2.15}$$

Assumption V

Let γ be a part of border L of region B. On γ, we have a continuous function $f(M)$ and P is an inner point of γ. Consider u as a solution of equation (2.2) which is bounded. Suppose that if at all points of γ, without the point P, u has boundary values equal to $f(M)$, then when the point (x, y) approach to P, u will approach to the limit value, and this limit value will be exactly $f(P)$.

To determine convergence of sequences (2.9) to the exact solution, we will begin by constructing majorant and minorant sequences for sequences (2.9). Let m be the exact upper bound of values taken by $|\varphi(M)|$ on $\bar{\alpha}$ and by $f(M)$ on L

$$m = max\left[sup|\varphi(M)|\, ,\, \sup f(M)\right] \tag{2.16}$$

By solving the Dirichlet problem in B_1 and B_2, we can construct the majorant sequences ${u_n}^+$ and ${v_n}^+$ for u_n and v_n respectively, by using

$$u_1{}^+ = \begin{cases} f(M) & \text{on } \alpha \\ m & \text{on } \bar{\alpha} \end{cases} \qquad v_1{}^+ = \begin{cases} f(M) & \text{on } \beta \\ u_1{}^+ & \text{on } \bar{\beta} \end{cases} \tag{2.17}$$

$$u_2{}^+ = \begin{cases} f(M) & \text{on } \alpha \\ v_1{}^+ & \text{on } \bar{\alpha} \end{cases} \qquad v_2{}^+ = \begin{cases} f(M) & \text{on } \beta \\ u_2{}^+ & \text{on } \bar{\beta} \end{cases} \tag{2.18}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$u_n{}^+ = \begin{cases} f(M) & \text{on } \alpha \\ v_{n-1}{}^+ & \text{on } \bar{\alpha} \end{cases} \qquad v_n{}^+ = \begin{cases} f(M) & \text{on } \beta \\ u_n{}^+ & \text{on } \bar{\beta} \end{cases} \tag{2.19}$$

Now, we will explain that $u_n{}^+$ and $v_n{}^+$ are majorant for u_n and v_n respectively.

On $\bar{\alpha}$, $u_1{}^+ = m \geq \varphi(M) = u_1$, and on α $u_1{}^+ = f(M) = u_1$, so, $u_1{}^+ \geq u_1$ on L_1. By Assumption II $u_1{}^+ \geq u_1$ on B_1.

On $\bar{\beta}$, $v_1{}^+ = u_1{}^+ \geq u_1 = v_1$ and on β, $v_1{}^+ = v_1 = f(M)$. Then on L_2, $v_1{}^+ \geq v_1$ and by Assumption II $v_1{}^+ \geq v_1$ on B_2. Continuing this we will get

$$u_n{}^+ \geq u_n \,,\, v_n{}^+ \geq v_n \tag{2.20}$$

Analogously we can produce the minorant sequences $u_n{}^-$ and $v_n{}^-$ for u_n and v_n respectively.

$$u_1{}^- = \begin{cases} f(M) & \text{on } \alpha \\ -m & \text{on } \bar{\alpha} \end{cases} \qquad v_1{}^- = \begin{cases} f(M) & \text{on } \beta \\ u_1{}^- & \text{on } \bar{\beta} \end{cases} \tag{2.21}$$

$$u_2{}^- = \begin{cases} f(M) & \text{on } \alpha \\ v_1{}^- & \text{on } \bar{\alpha} \end{cases} \qquad v_2{}^- = \begin{cases} f(M) & \text{on } \beta \\ u_2{}^- & \text{on } \bar{\beta} \end{cases} \tag{2.22}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$u_n{}^- = \begin{cases} f(M) & \text{on } \alpha \\ v_{n-1}{}^- & \text{on } \bar{\alpha} \end{cases} \qquad v_n{}^- = \begin{cases} f(M) & \text{on } \beta \\ u_n{}^- & \text{on } \bar{\beta} \end{cases} \tag{2.23}$$

On $\bar{\alpha}$, $u_1^- = -m \leq \varphi(M) = u_1$, and on α $u_1^- = f(M) = u_1$, so, $u_1^- \leq u_1$ on L_1. By Assumption II $u_1^- \leq u_1$ on B_1.

On $\bar{\beta}$, $v_1^- = u_1^- \leq u_1 = v_1$ and on β, $v_1^- = v_1 = f(M)$. Then on L_2, $v_1^- \leq v_1$ and by Assumption II $v_1^- \leq v_1$ on B_2. By similar procedures we can obtain

$$u_n^- \leq u_n \,,\ v_n^- \leq v_n \,. \tag{2.24}$$

So, we showed that $u_n^+$and v_n^+ are majorant sequences for u_n and v_n respectively and $u_n^-$and v_n^- are minorant sequences for u_n and v_n respectively.

Now, let us show that both auxiliary sequences are monotonic.

Firstly, we will begin with the majorant sequences, on $\bar{\alpha}$, $u_1^+ = m$ and on α $u_1^+ = f(M) \leq m$, so $u_1^+ \leq m$ on L_1, by Assumption II $u_1^+ \leq m$ everywhere on B_1.

On β $v_1^+ = f(M) \leq m$ and on $\bar{\beta}$ the value of u_1^+ is not greater than m, then on $\bar{\beta}$ $v_1^+ \leq m$, so, on L_2 $v_1^+ \leq m$, by Assumption II, $v_1^+ \leq m$ everywhere on B_2.

On $\bar{\alpha}$, $u_2^+ = v_1^+ \leq m = u_1^+$ and on α $u_2^+ = u_1^+ = f(M) \leq m$, so $u_2^+ \leq m$ on L_1, by Assumption II $u_2^+ \leq m$ everywhere on B_1, thus $u_1^+ \geq u_2^+$ on L_1, by Assumption II $u_1^+ \geq u_2^+$ everywhere on B_1.

By taking the difference between u_n^+, u_{n+1}^+ and v_n^+, v_{n+1}^+, $n = 1, 2, \ldots$ we can obtain the following result

$$u_1^+ - u_2^+ = \begin{cases} 0 & \text{on } \alpha \\ m - \varphi(M) & \text{on } \bar{\alpha} \end{cases} \tag{2.25}$$

$$v_1{}^+ - v_2{}^+ = \begin{cases} 0 & \text{on } \beta \\ u_1{}^+ - u_2{}^+ & \text{on } \bar{\beta} \end{cases} \tag{2.26}$$

$$\vdots$$

$$u_n{}^+ - u_{n+1}{}^+ = \begin{cases} 0 & \text{on } \alpha \\ v_{n-1}{}^+ - v_n{}^+ & \text{on } \bar{\alpha} \end{cases} \tag{2.27}$$

$$v_n{}^+ - v_{n+1}{}^+ = \begin{cases} 0 & \text{on } \beta \\ u_n{}^+ - u_{n+1}{}^+ & \text{on } \bar{\beta} \end{cases} \tag{2.28}$$

So, we showed that $u_n{}^+$and $v_n{}^+$ are monotonically decreasing sequences

$$u_1{}^+ \geq u_2{}^+ \geq \cdots, \tag{2.29}$$

$$v_1{}^+ \geq v_2{}^+ \geq \cdots. \tag{2.30}$$

By the same technique we can show that $u_n{}^-$and $v_n{}^-$ are monotonically increasing sequences

$$u_1{}^- \leq u_2{}^- \leq \cdots, \tag{2.31}$$

$$v_1{}^- \leq v_2{}^- \leq \cdots. \tag{2.32}$$

Thus

$$u_n{}^+ \geq u_n \geq u_n{}^- \text{ and } v_n{}^+ \geq v_n \geq v_n{}^-. \tag{2.33}$$

According to $u_n{}^+ \geq u_1{}^-$ and $v_n{}^+ \geq v_1{}^-$, we can say that $u_n{}^+$ and $v_n{}^+$ are bounded below, and by $u_1{}^+ \geq u_n{}^-$ and $v_1{}^+ \geq v_n{}^-$, both $u_n{}^-$ and $v_n{}^-$ are bounded above.

From monotonicity and boundedness proved above, $u_n{}^+, v_n{}^+$ and $u_n{}^-, v_n{}^-$ will be convergent in B_1 and B_2 respectively to some functions say u^+, v^+ and u^-, v^-respectively.

$$\lim_{n\to\infty} u_n{}^+ = u^+, \tag{2.34}$$

$$\lim_{n\to\infty} v_n{}^+ = v^+. \tag{2.35}$$

And

$$\lim_{n\to\infty} u_n{}^- = u^-, \tag{2.36}$$

$$\lim_{n\to\infty} v_n{}^- = v^-. \tag{2.37}$$

We must show that $u^+ = v^+$, (v^+ will be continuation of u^+), on $\bar{\alpha}$, $u_n{}^+ = v_{n-1}{}^+$, and on $\bar{\beta}$, $u_n{}^+ = v_n{}^+$, when n approach to ∞, u^+ and v^+ coincide on $\bar{\alpha}$ and $\bar{\beta}$.

For α and β, take $M \in \alpha$ or $M \in \beta$ with $f(M)$ continuous, we know that

$$u_1{}^+ \geq u^+ \geq u_1{}^-. \tag{2.38}$$

When (x, y) approaches M, both $u_1{}^+$ and $u_1{}^-$ approach $f(M)$ and by previous inequality, u^+ will approach to $f(M)$ when (x, y) approach to M. Similarly for v^+.

So, we proved that $u^+ = v^+$, (where v^+ is a continuation of u^+). By the same reason $u^- = v^-$, (v^- is a continuation of u^-).

On L, u^+ and u^- have similar values (with the exception of a finite number of points), therefore, $u^+ = u^- = u$ everywhere in B.

Consequently, u^+, v^+, u^- and v^- will converge in B_1 and B_2 respectively to the same solution u. In view of this the main successive approximations (2.9) will converge to the same solution u

$$\lim_{n\to\infty} u_n = u \,, \tag{2.39}$$

$$\lim_{n\to\infty} v_n = u \,. \tag{2.40}$$

So, convergence of the Schwarz algorithm is showed, with the five assumptions presented, for the Dirichlet problem in a domain of the union of two regions.

2.2 Finite-Difference Analog of Schwarz's Method

Theorem 2.2.1:

Let B_1 and B_2 be overlapping rectangles with sides parallel to the coordinate axes; L_1 and L_2 are their borders respectively, β_k is the part of L_k which belongs to B_l and not in L_l $(l \neq k)$, $\alpha_k = L_k / \beta_k$ (See Fig 2.2-2.5).

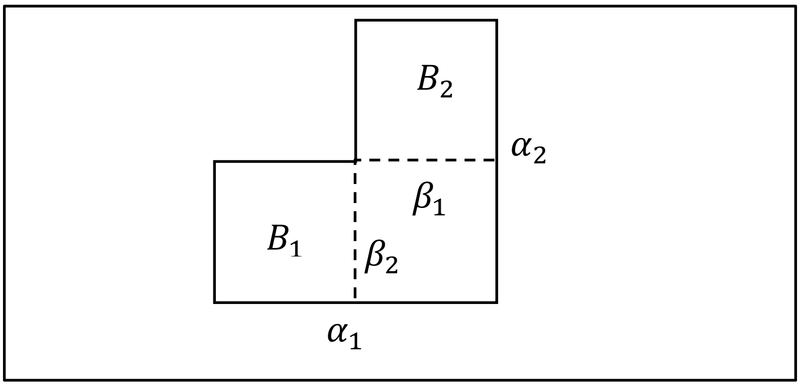

Figure 2.2: Figure of two regions that overlapped one another

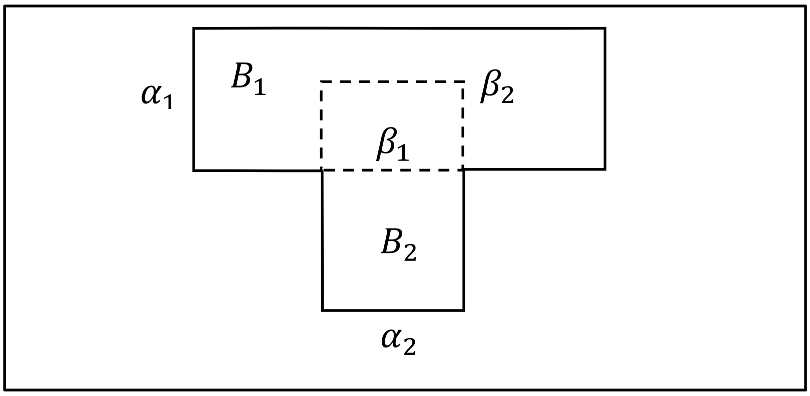

Figure 2.3: Figure of two regions that overlapped one another

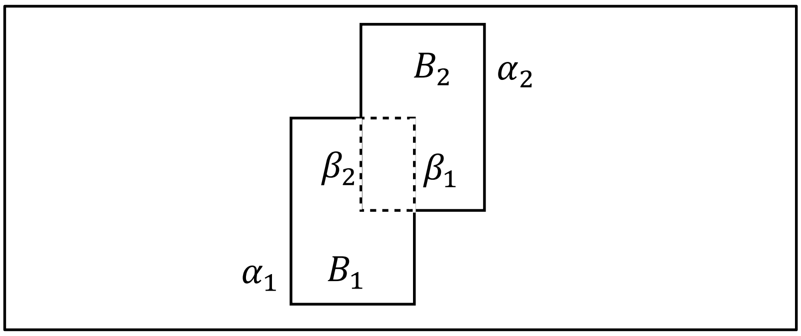

Figure 2.4: Figure of two regions that overlapped one another

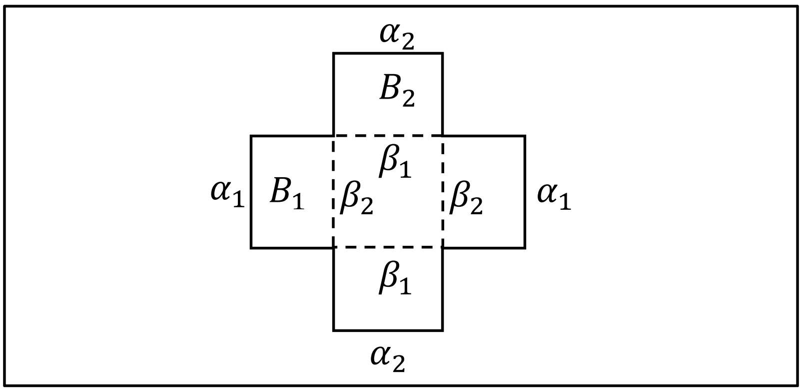

Figure 2.5: Figure of two regions that overlapped one another

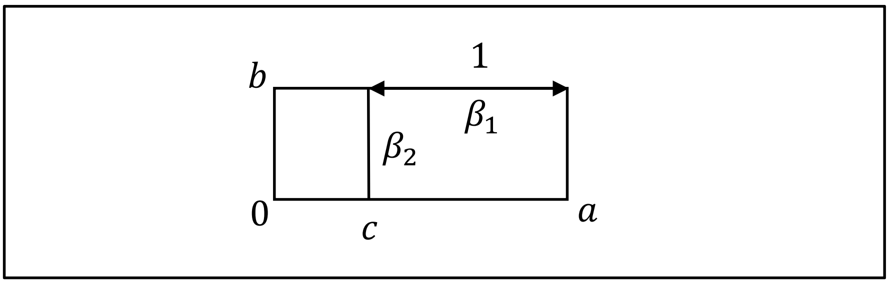

Figure 2.6: Figure of part of two regions that overlapped one another

Let us denote rectangular sets B_{1,h_1} and B_{2,h_2} on B_1 and B_2, respectively, with mesh-steps h_1 and h_2 such that the vertices B_1 and B_2, and the points of intersection of L_1 and L_2 be the nodes of the grids.

We consider the finite-difference problem

$$\begin{cases} -\left(\bar{\partial}_1\partial_1 u_i^{(k)} + \bar{\partial}_2\partial_2 u_i^{(k)}\right) = 0 \text{ for } x_i \in B_{k,h} \\ u_i^{(k)} = 0 \text{ for } x_i \in \alpha_k, \quad u_i^{(k)} = 1 \text{ for } x_i \in \beta_k \end{cases} \tag{2.41}$$

where $\bar{\partial}_1\partial_1 u_s$ is a finite difference approximation with second order accuracy for the function u_s in B_s dependent on x and $\bar{\partial}_2\partial_2 u_s$ is a finite difference approximation with the second order accuracy for the function u_s in B_s dependent on y.

Then there exists $q^{(k)}$, $0 < q^{(k)} < 1$ independent of h_1, h_2 such that for $x_i \in \beta_l$ where $(l \neq k)$

$$0 \leq u_i^{(k)} \leq q^{(k)}. \tag{2.42}$$

Proof:

Let us consider the case in Fig 2.2. It can be reduced to following problem

Let $B \equiv \{x: 0 < x_1 < a, 0 < x_2 < b\}$,

$\beta_1 \equiv \{x: 0 < c < x_1 < a, x_2 = b\}$, $\beta_2 \equiv \{x: x_1 = c, 0 < x_2 < b\}$,

The function v is a solution of problem

$$\begin{cases} -\left(\bar{\partial}_1\partial_1 v_i + \bar{\partial}_2\partial_2 v_i\right) = 0, x_i \in B_h \\ v_i = 1 \text{ for } x_i \in \beta_1 \\ v_i = 0 \text{ for } L_h \setminus \beta_1 \end{cases}. \tag{2.43}$$

Then

$$v_i \leq \max\left\{\frac{1}{2}, \frac{c}{a}\right\} \text{ for } x_i \in \beta_2. \tag{2.44}$$

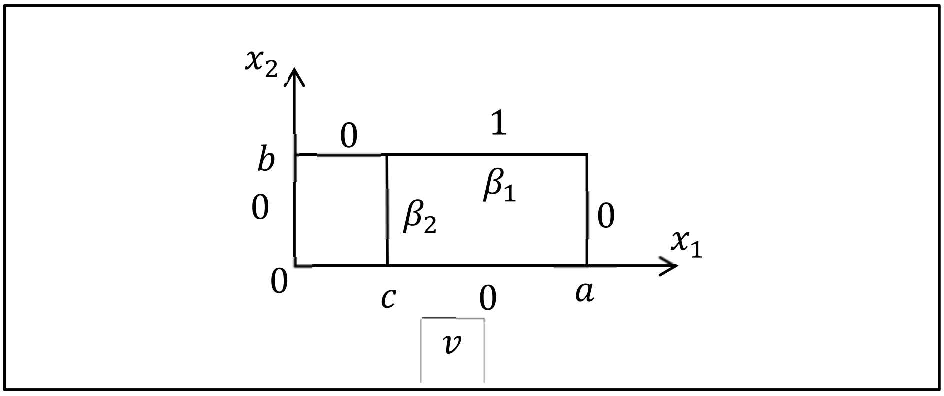

Figure 2.7: Figure of the function $\boldsymbol{v}$ which is a solution of problem (2.43)

We prove this statement.

Let w be a solution of problem

$$\begin{cases} -\left(\bar{\partial}_1 \partial_1 w_i + \bar{\partial}_2 \partial_2 w_i\right) = 0 \,, x_i \in B_h \\ w_i = 1 \text{ on the right of } x_1 = c \text{ on } L_h \\ w_i = 0 \text{ on the remainder nodes of } L_h \end{cases} \quad . \tag{2.45}$$

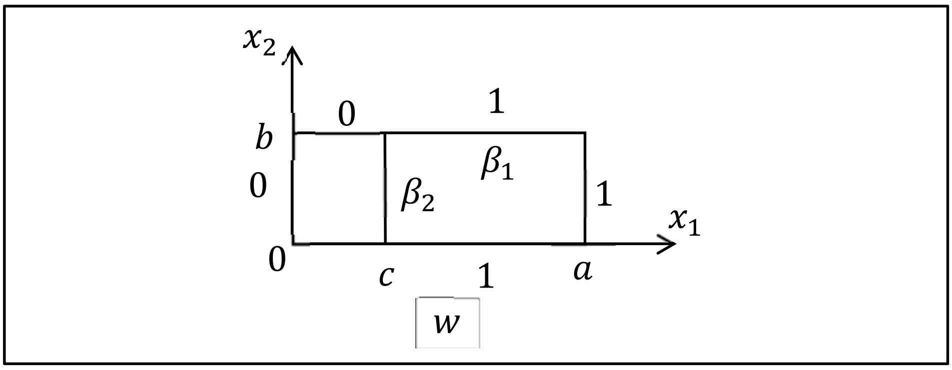

Figure 2.8: Figure of the function $\boldsymbol{w}$ which is a solution of problem (2.45)

Then $v_i \leq w_i$.

If $\frac{c}{a} \leq \frac{1}{2}$, then we define z as a solution of difference Laplace's equation on

$B' \equiv \{x: 0 < x_1 < 2c \,, 0 < x_2 < b\}$, $(B' \subset B)$ with the boundary condition on L_h

$z_i = 0$ for $i_1 h_1 < c \,, z_i = \dfrac{1}{2}$ for $i_1 h_1 = c$ and $z_i = 1$ for $i_1 h_1 > c$

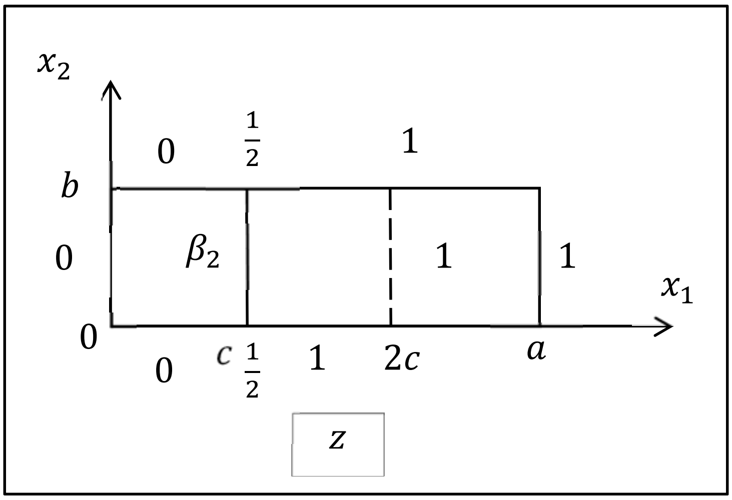

Figure 2.9: Figure of the function $\boldsymbol{z}$ when $\frac{\boldsymbol{c}}{\boldsymbol{a}} \leq \frac{\mathbf{1}}{\mathbf{2}}$

According to symmetry with respect to x_1 it follows that $z_i = \frac{1}{2}$ for $x_i \in \beta_2$.

On the basis of maximum principle, we have $w_i \leq z_i$ on B'.

Therefore, $w_i \leq \frac{1}{2}$ on $x_i \in \beta_2$.

If $\frac{c}{a} \geq \frac{1}{2}$, then we define z on $B' \subset B$ as a solution of difference Laplace's equation with the boundary condition

$$z = \begin{cases} -\dfrac{x_1}{a} & \text{for } x_1 < c \\ 0 & \text{for } x_1 = c \\ \dfrac{2c}{a} - \dfrac{x_1}{a} & \text{for } x_1 > c \end{cases} \tag{2.46}$$

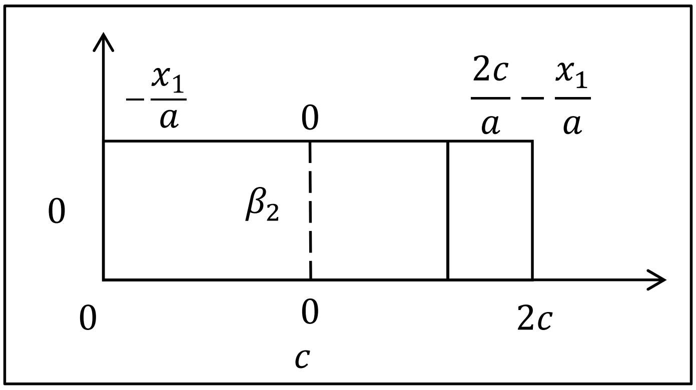

Figure 2.10: Figure of the function $\boldsymbol{z}$ when $\frac{\boldsymbol{c}}{\boldsymbol{a}} \geq \frac{\mathbf{1}}{\mathbf{2}}$

By symmetry $z_i = 0$ for $x_i \in \beta_2$ and $z_i \geq 0$ for $i_1 h_1 > c$.

The function $\frac{x_1}{a}$ satisfies Laplace's difference equation, therefore $\omega = z + \frac{x_1}{a}$ also satisfies this equation on B' . Compare it with w on domain B, $w \leq \omega$ and therefore $w_i = 0 + \frac{c}{a}$ for $x_i \in \beta_2 \Longrightarrow w_i \leq \frac{c}{a}$ for $x_i \in \beta_2$.

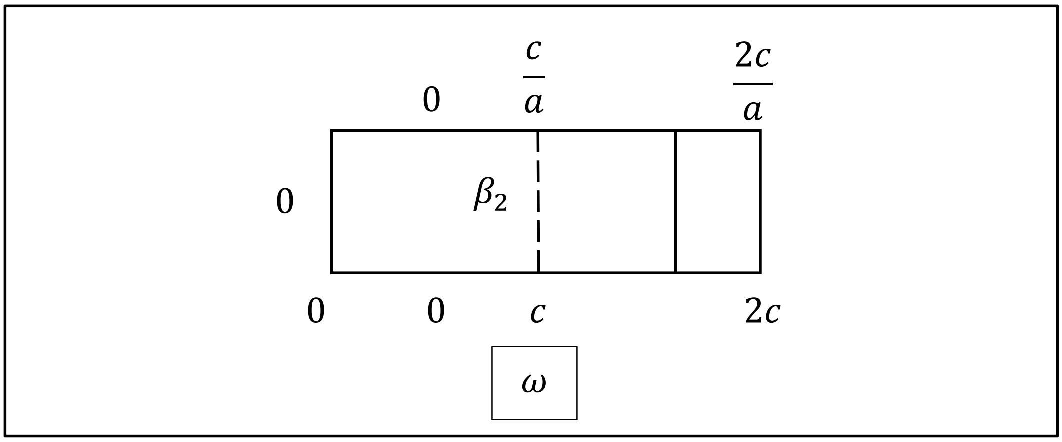

Figure 2.11: Figure of the function $\boldsymbol{\omega}$ which is a solution of problem (2.47), (2.48)

$$-\left(\bar{\partial}_1 \partial_1 \omega_i + \bar{\partial}_2 \partial_2 \omega_i\right) = 0 \tag{2.47}$$

$$\begin{cases} \omega = 0 & \text{for } x_1 < c \\ \omega = \dfrac{x_1}{a} & \text{for } x_1 = c \\ \omega = \dfrac{2c}{a} & \text{for } x_1 > c \end{cases} \tag{2.48}$$

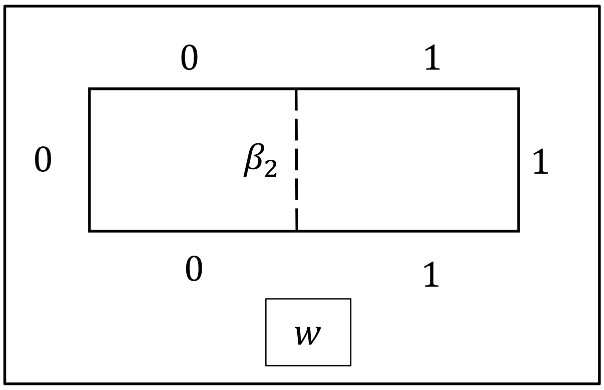

Figure 2.12: Figure of the function $\boldsymbol{w}$

Therefore, Theorem 2.2.1 in the case Fig.2.2 is proved.

The remainder case reduces to the considered case by the maximum principle. For example in the case Fig.2.14

$$\begin{cases} -\left(\bar{\partial}_1\partial_1 z_i + \bar{\partial}_2\partial_2 z_i\right) = 0 \text{ on } B_1 \\ z_i = 1 \text{ on the right of vertical } \beta_2 \text{ of } L_h \\ z_i = 0 \text{ on the remainder of } L_h \end{cases} \tag{2.49}$$

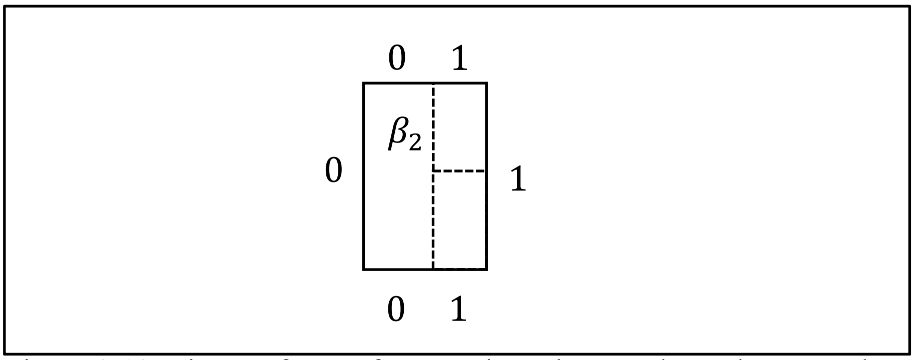

Figure 2.13: Figure of part of two regions that overlapped one another

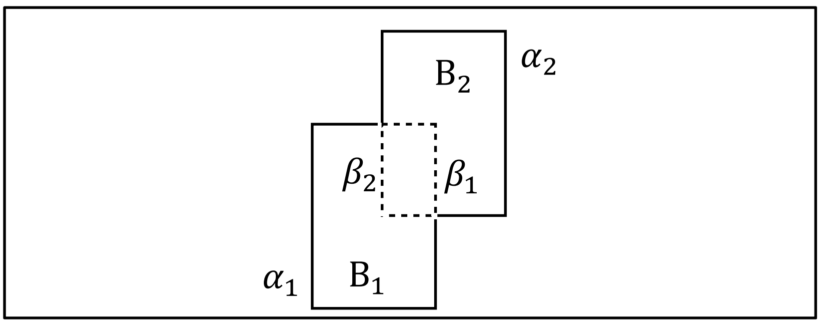

Figure 2.14: Figure of two regions that overlapped one another

Then on vertical part of β_2 , $u_i^{(1)} \le z_i \le q^{(1)}$.

By similarity the inequality is true on the horizontal part of β_2.

Theorem 2.2.1 can be generalized, first of all, for the case of finite number (of rectangles) space variables, and second in the case of cubic grids and for the domains as in Figure 2.15, 2.16 and so on.

Furthermore, the operator can be replaced with 9-point difference operator, i.e.,

$$-\sum_{r=1}^{2} \bar{\partial}_r \partial_r \backsim -\sum_{r=1}^{2} \bar{\partial}_r \partial_r - \frac{h_1 + h_2}{6} \bar{\partial}_1 \partial_1 \bar{\partial}_2 \partial_2 \tag{2.50}$$

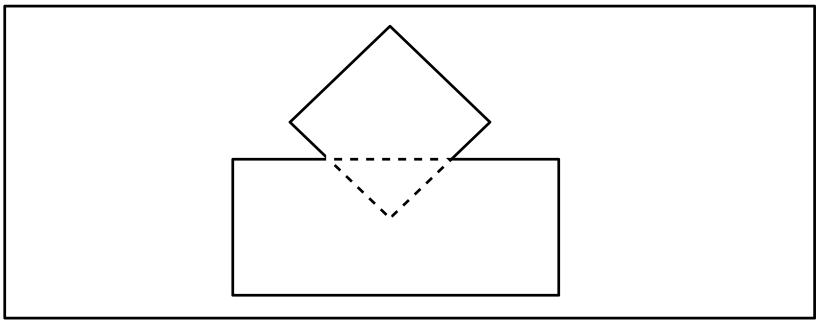

Figure 2.15: Figure of two regions that overlapped one another

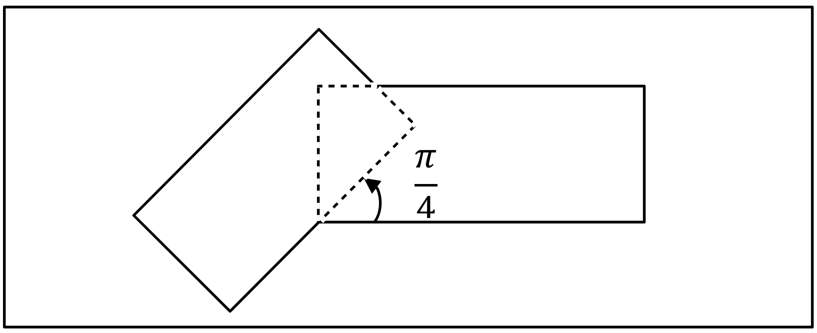

Figure 2.16: Figure of two regions that overlapped one another

Theorem 2.2.1 helps us to show the convergence of Schwarz's method for the solution of the problem

$$\begin{gathered} -\sum_{r=1}^{2} \bar{\partial}_r \partial_r u_i = 0 \,, x_i \in B_h \\ u_i = \varphi_i \,, x_i \in L_h \end{gathered} \tag{2.51}$$

where $B \equiv B_1 \cup B_2$ (see Theorem 2.2.1).

Theorem 2.2.2

Let $\beta_{k,h}$ be the set of nodes of the grid on β_k, and let the operators V and W, where the operator V is defined for the function w^{n-1} on $\beta_{1,h}$ which transform to the function $v^n \equiv V(w^{n-1})$ defined on $B_{1,h}$as a solution of the problem

$$\begin{aligned} &-\textstyle\sum_{r=1}^{2} \bar{\partial}_r \partial_r v_i = 0\,, x_i \in B_{1,h} \\ &v_i^n = \varphi_i \text{ for } x_i \in \alpha_1\,, v_i^n = w_i^{n-1} \text{ for } x_i \in \beta_{1,h} \end{aligned} \tag{2.52}$$

and the operator W is defined as $w^n \equiv W(v^n)$ on $\beta_{2,h}$in $B_{2,h}$ from the condition:

$$\begin{aligned} &-\textstyle\sum_{r=1}^{2} \bar{\partial}_r \partial_r w_i = 0\,, x_i \in B_{2,h} \\ &w_i^n = \varphi_i \text{ for } x_i \in \alpha_2\,, w_i^n = v_i^n \text{ for } x_i \in \beta_{2,h} \end{aligned} \tag{2.53}$$

Let for the solution of problem (2.50) the following iterative process be applied: for the given $w^{-1} \equiv v^0$ on $\beta_{1,h}$ we define $v^0 = V(w^{-1}) = V(v^0)$ on $B_{1,h}$ and on $\beta_{2,h}$; then we find $w^0 = W(v^0)$, $v^1 = V(w^0)$, $w^1 = W(v^1)$, … .

Then

$$\begin{gathered} \|u - v^n\|_{C(\beta_2)} \le q^n \|u - v^0\|_{C(\beta_2)} \\ \|u - w^n\|_{C(\beta_1)} \le q^n q^{(2)} \|u - v^0\|_{C(\beta_2)} \\ n \ge 0\,,\ q = q^{(1)} q^{(2)} \end{gathered} \tag{2.54}$$

Proof:

The function $u - v^n$ satisfies on B_{1,h_1}, the difference Laplace equation and is equal to 0 on α_1. Then, by Theorem 2.2.1

$$\|u - v^n\|_{C(\beta_2)} \leq q^{(1)}\|u - v^n\|_{C(\beta_1)} = q^{(1)}\|u - w^{n-1}\|_{C(\beta_1)} \,. \qquad (2.55)$$

Similarly (by analogously)

$$\|u - w^{n-1}\|_{C(\beta_1)} \leq q^{(2)}\|u - v^{n-1}\|_{C(\beta_2)} \,. \qquad (2.56)$$

Then, from (2.55), (2.56) and the principle of maximum, we obtain

$$\begin{gathered} \|u - v^n\|_{C(\beta_2)} \leq q^{(1)}q^{(2)}\|u - v^{n-1}\|_{C(\beta_2)} \leq q^2\|u - v^{n-2}\|_{C(\beta_2)} \leq \cdots \leq \\ \leq q^n\|u - v^0\|_{C(\beta_2)}, \qquad q = q^{(1)}q^{(2)} \end{gathered} \qquad (2.57)$$

and

$$\|u - w^n\|_{C(\beta_1)} \leq q^{(2)}\|u - w^n\|_{C(\beta_2)} = q^{(2)}\|u - v^n\|_{C(\beta_2)} \,. \qquad (2.58)$$

From (2.57) and (2.58), we have

$$\|u - w^n\|_{C(\beta_1)} \leq q^{(2)}q^n\|u - v^0\|_{C(\beta_2)} \qquad (2.59)$$

Theorem 2.2.2 is proved.

Let us consider the case when functions v^n and w^n are defined approximately, for example by using some iterative (or other approximate) methods. Then for real values we have the functions $\overline{v}^0, \overline{w}^0, \overline{v}^1, \overline{w}^1, \ldots$, where

$$\begin{aligned} &\overline{v}^0 = V(v^0) + \xi^0, \overline{w}^0 = W(\overline{v}^0) + \eta^0, \\ &\overline{v}^1 = V(\overline{w}^0) + \xi^1, \overline{w}^1 = W(\overline{v}^1) + \eta^1, \\ &\qquad\qquad\qquad \vdots \\ &\overline{v}^n = V(\overline{w}^{n-1}) + \xi^n, \overline{w}^n = W(\overline{v}^n) + \eta^n \end{aligned} \qquad (2.60)$$

and the functions are defined by errors of iterative methods.

Theorem 2.2.3

Let in (2.59)

$$\|\xi^k\|_{C(B_1)} \leq \bar{\varepsilon}\,, \|\eta^k\|_{C(B_2)} \leq \bar{\varepsilon} \tag{2.61}$$

and

$$\rho_n = max\{\|u - \bar{v}^{\,n}\|_{C(\beta_2)}\,, \|u - \bar{w}^n\|_{C(\beta_1)}\} \tag{2.62}$$

Then

$$\rho_n \leq q^n \|u - v^0\|_{C(\beta_1)} + \frac{1 - q^n}{1 - q}\,(1 + \max(q^{(1)}, q^{(2)}))\bar{\varepsilon} \tag{2.63}$$

Proof:

We have

$u - \bar{v}^{\,n} = u - V(\bar{w}^{n-1}) - \xi^n.$

Therefore

$$\|u - \bar{v}^{\,n}\|_{C(\beta_2)} \leq \|u - V(\bar{w}^{n-1})\|_{C(\beta_2)} + \bar{\varepsilon} \leq q^{(1)}\|u - \bar{w}^{n-1}\|_{C(\beta_1)} + \bar{\varepsilon} =$$

$$= q^{(1)}\|u - W(\bar{v}^{\,n-1}) - \eta^{n-1}\|_{C(\beta_1)} + \bar{\varepsilon} \leq$$

$$\leq q^{(1)}\|u - W(\bar{v}^{\,n-1})\|_{C(\beta_1)} + q^{(1)}\bar{\varepsilon} + \bar{\varepsilon}$$

$$\leq q^{(1)}q^{(2)}\|u - \bar{v}^{\,n-1}\|_{C(\beta_2)} + \bar{\varepsilon}\left(1 + q^{(1)}\right) =$$

$$= q\|u - \bar{v}^{\,n-1}\|_{C(\beta_2)} + \bar{\varepsilon}\,(1 + q^{(1)}) \leq$$

$$\leq q^n\|u - V(v^0)\|_{C(\beta_2)} + \bar{\varepsilon} + (1 + q + q^2 + \cdots + q^{n-1})\left(1 + q^{(1)}\right)\bar{\varepsilon}$$

$$= q^n\|u - V(v^0)\|_{C(\beta_2)} + \bar{\varepsilon} + \frac{1 - q^n}{1 - q}\left(1 + q^{(1)}\right)\bar{\varepsilon} \leq$$

$$\leq q^{(1)} q^n \|u - v^0\|_{C(\beta_1)} + \bar{\varepsilon} + \frac{1 - q^n}{1 - q}\left(1 + q^{(1)}\right)\bar{\varepsilon}.$$

By analogs we estimate $\|u - \overline{w}^n\|_{C(\beta_1)}$, and we obtain (2.63)

2.3 Numerical Experiment

In this section, we will present a numerical experiment as an illustration of the application of Schwarz's method. For the approximate solution of the problem on the overlapping rectangles covering the domain the Gauss Seidel methods has been used, where the computations are carried out by using MATLAB programming language.

Example 1: Let B be an L-Shaped region bounded by the border L on the $xy-plane$, which is covered by two overlapping rectangles B_1 and B_2 (see Figure 2.17)

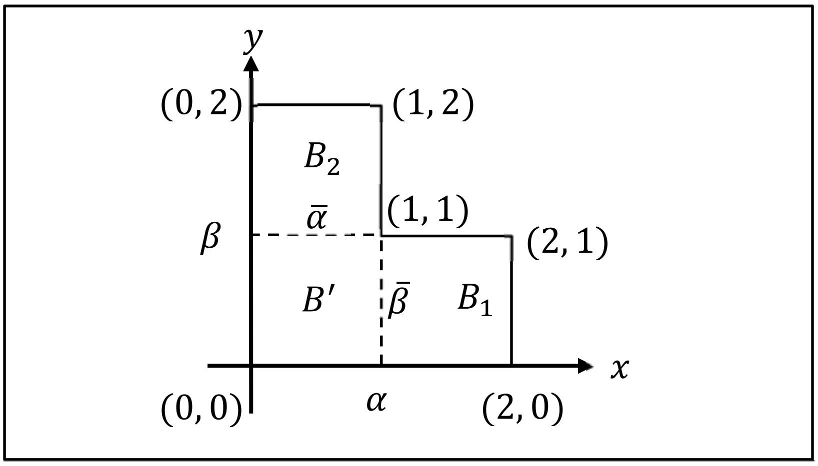

Figure 2.17: Figure of two regions that overlapped one another

with the common part B' bounded by the border L', where

$B_1 = \{(x, y): 0 < x < 2 \text{ and } 0 < y < 1\}$

$B_2 = \{(x, y): 0 < x < 1 \text{ and } 0 < y < 2\}$

$B = \{(x, y): 0 < x < 2 \text{ and } 0 < y < 2\} / \{(x, y): 1 < x < 2 \text{ and } 1 < y < 2\}$

$B' = \{(x, y): 0 < x < 1 \text{ and } 0 < y < 1\}.$

We consider the boundary value problem

$$\begin{aligned} \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad & \text{on} \quad B, \\ u = e^x \sin y \quad & \text{on} \quad L, \end{aligned} \tag{2.64}$$

where $u(x, y) = e^x \sin y$ is assumed to be the exact solution. For the approximate solution we have the grid set,

$B_h = \{(x_i, y_j): x_i = ih, y_j = jh$ for $i = 0, 1, \dots, n, j = 0, 1, \dots, n,$ where $0 < x < 2$ and $0 < y < 2\} / \{(x_i, y_j): 1 < x < 2$ and $1 < y < 2\}$.

A Schwarz iteration was completed by obtaining an approximate solution on each rectangle with the use of the 5-point scheme, where the system of finite-difference equations were solved by the Gauss Seidel method.

We use the difference approximation for Laplace's equation

$$\begin{aligned} -(\bar{\partial}_1 \partial_1 u_s + \bar{\partial}_2 \partial_2 u_s) &= 0 \quad \text{on} \quad B_{h_s}, s = 1, 2 \\ u_s &= e^{ih_s} \sin jh_s \quad \text{on} \quad L_{h_s} \end{aligned} \tag{2.65}$$

The exact and numerical solutions are shown in Table 2.1, where $\|\varepsilon_h\|_{C(B)} = \max\{|u - u_h|\}$, is difference between the exact and approximate solution, in the maximum norm, $h = 2^{-m}$, $m = 3, 4, 5, 6$ and $R_h = \frac{\|u - u_{2^{-m}}\|_{C(B)}}{\|u - u_{2^{-(m+1)}}\|_{C(B)}}$ is the order of convergence of the approximate solution.

Table 2.1: The maximum error between exact solution and approximate solution and the order of convergence of the approximate solution.

h	$\|\varepsilon_h\|_{C(B)}$	R_h	$\simeq$
$\frac{1}{8}$	4.950960176828279e-04		3.931757779541467
$\frac{1}{16}$	1.259223089120631e-04		3.977278259276012
$\frac{1}{32}$	3.166042220414944e-05		4.014597237097936
$\frac{1}{64}$	7.886325908756930e-06		

Chapter 3

THE SCHWARZ-NEUMANN'S METHOD FOR THE SOLUTION OF THE DIRICHLET PROBLEM FOR THE INTERSECTION OF TWO REGIONS

3.1 Schwarz-Neumann Method for the Solution of Partial Differential Equations

In Chapter 2 we have discussed Schwarz's method for the solution of the Dirichlet problem in a region that is the union of two regions. We can apply a similar idea to the solution of the Dirichlet problem in a region that is the common part of the two overlapping domains.

Let us explain the idea of this method. We will start with an arbitrary linear homogeneous partial differential equation of second order that satisfies the assumptions mentioned in Chapter 2. Let us take an equation with the following form:

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = 0 \qquad (3.1)$$

and on the $xy - plane$ we will take a region B, bounded by the border L .

For equation (3.1), we will consider the same assumptions will be satisfied, that we formed in Chapter 2.

We have two regions B_1 and B_2 with same properties that we explained in Chapter 2, the Dirichlet problem for any continuous or piece-wise continuous boundary values of the sought functions can be solved in these regions, and let it be wanted to find a function $w(x,y)$ satisfing equation (3.1) in B' and obtaining the specified values on its border L'

$$w(x,y) = f(M) \quad \text{on} \quad L' \tag{3.2}$$

In view of (3.2), $f(M)$ is a piece-wise continuous function at the point Mon L'. To obtain this function, Neumann decided to form a function $w(x,y)$ which is the sum of two functions

$$w(x,y) = u(x,y) + v(x,y) \tag{3.3}$$

where u and v are determined and satisfy equation (3.1) in $B_1$and B_2 respectively. We must choose them so that the sum of u and v satisfies boundary condition (3.2).

For clarity, consider the specific case given in Figure 2.1.

In region B, for the point (x,y), let us define a function ω as a solution of (3.1), determined by its boundary values on the border L of B. We want to represent function w in the following form

$$w = [u + \omega] + [v - \omega] \tag{3.4}$$

Now, on the part $\alpha - \gamma$ on L_1, there is no restriction on the boundary values of u, so we can take arbitrary values of ω for the first summand $u + \omega$. Analogously, for the second summand $v - \omega$ on the part $\beta - \gamma$ of the border L_2, we can choose arbitrary boundary values.

It remains to consider part γ of region B, we can manage the choice of ω that $u + \omega$ has arbitrary values. The second summand, $v - \omega$, will be fully determined by the values $f(M) - (u + \omega)$ because the values of $w = u + v$ are given.

In accordance with this idea, on α, we can take any values for the function u. For the function v, arbitrary boundary values can be defined only on $\beta - \gamma$, and on γ its values equal $f(M) - u$.

Now, we must show that both summands u and v, if they exist, are unique. Let us resolve w as follows

$$w = u' + v' = u'' + v'' \tag{3.5}$$

where u' and u'' are in B_1 and coincide on α, and v' and v'' are in B_2 and they coincide on β. We will take the differences $u' - u''$and $-(v' - v'')$. They are also solutions of equation in B_1 and B_2 respectively. On parts α and β the values of the difference equal to zero. From equation (3.5), in part B' they will coincide

$$u' - u'' = -(v' - v'')\,. \tag{3.6}$$

Thus we see that the difference $v'' - v'$ is a continuation of the solution $u' - u''$ of equation (3.1) from B_1 into B_2. Additionally the solution on the border L of the region B has null values. By Assumption I, it is equal to zero everywhere in B, therefore, $u' = u''$ and $v' = v''$. Then both summands u and v are uniquely determined by specified boundary conditions.

So, we have reduced the proof of existence and the construction of function w which is a solution for equation (3.1), to the construction of functions u and v. To check them we will be able to follow the method of successive approximations in a form fairly special from that proposed with the aid C. Neumann.

Now let us select the boundary values of u on α and v on $\beta-\gamma$. On $\alpha-\gamma$ we can assign a function $\varphi(M)$, so values of $\varphi(M)$ together with values $f(M)$on $\bar{\alpha}+\gamma$ will form a piece-wise continuous function at a point M on L_1 keeping in mind

$$u=\begin{cases}\varphi(M) & \text{on } \alpha-\gamma \\ f(M) & \text{on } \gamma\end{cases} \tag{3.7}$$

By choosing values of u on α we can determine values of v on γ so that $v=0$ on γ. On $\beta-\gamma$, we can assign arbitrary values of v that we explained above. For simplicity we can set them so that they have null values, then for v we obtain the boundary values as follows

$$v=0 \quad \text{on } \beta \tag{3.8}$$

Boundary values of u and v on α and β are arbitrarily was chosen by us for clarity and simplicity. In addition it is evident that for convergence of successive approximations such a choice of the boundary values of u and v is inessential: if successive approximations converge for our choice of boundary values, they will converge for any other choice of them, supplied simplest that they be piece-wise continuous and such that $u+v=f(M)$ on γ. So, for any other way of selecting boundary values will be reduced to ours by the representation

$$u'=u+\omega \text{ and } v'=v-\omega \tag{3.9}$$

On part $\bar{\alpha}$ the values of u are unknown to us. We can arbitrarily define piece-wise continuous values $\bar{\varphi}(M)$ on $\bar{\alpha}$, both values of u and $\bar{\varphi}(M)$ on α and $\bar{\alpha}$ respectively they form piece-wise continuous values on the whole border L_1.

We will construct first approximation u_1 to u, as a solution by solving the Dirichlet problem for equation (3.1) with the following boundary conditions

$$u_1 = \begin{cases} f(M) & \text{on } \alpha - \gamma \\ \varphi(M) & \text{on } \gamma \\ \bar{\varphi}(M) & \text{on } \bar{\alpha} \end{cases} . \tag{3.10}$$

By using the values u_1 takes on $\bar{\beta}$ and subtracting from $f(M)$, from the values of v on β, and from solving the Dirichlet problem for equation (3.1) in B_2, we can construct the function v_1 which is the first approximation to v in B_2 with boundary values

$$v_1 = \begin{cases} 0 & \text{on } \beta \\ f(M) - u_1 & \text{on } \bar{\beta} \end{cases} . \tag{3.11}$$

By using v_1, we are able to construct second approximation u_2 to u in B_1 as a solution of the Dirichlet problem for (3.1) with the following boundary condition

$$u_2 = \begin{cases} f(M) & \text{on } \alpha - \gamma \\ \varphi(M) & \text{on } \gamma \\ f(M) - v_1 & \text{on } \bar{\alpha} \end{cases} \tag{3.12}$$

and for v_2

$$v_2 = \begin{cases} 0 & \text{on } \beta \\ f(M) - u_1 & \text{on } \bar{\beta} \end{cases} . \tag{3.13}$$

We can continue this process to get successive approximations u_{n+1} and v_{n+1} for u and v respectively, by solving the Dirichlet problem of (3.1) in regions B_1 and B_2 under the boundary conditions

$$u_{n+1} = \begin{cases} f(M) & \text{on} \quad \alpha - \gamma \\ \varphi(M) & \text{on} \quad \gamma \\ f(M) - v_n & \text{on} \quad \bar{\alpha} \end{cases} \tag{3.14}$$

$$v_{n+1} = \begin{cases} 0 & \text{on} \quad \beta \\ f(M) - u_{n+1} & \text{on} \quad \bar{\beta} \end{cases}. \tag{3.15}$$

So, we have constructed a sequence of approximations to functions u and v in each of B_1 and B_2 respectively

$$\begin{cases} u_1, u_2, \ \dots, \ u_{n+1}, \ \dots \ \text{in} \ B_1 \\ \\ v_1, v_2, \ \dots, \ v_{n+1}, \ \dots \ \text{in} \ B_2 \end{cases} \tag{3.16}$$

The concept of checking convergence of successive approximations is the same process as in Chapter 2. For each of the sequences we must construct a majorant and a minorant sequence and prove that they converge to same limit.

In Chapter 2, for the convergence of Schwarz's method, we have constructed majorants ${u^+}_n$ and ${v^+}_n$ for u_n and v_n together. But here, the construction of the majorant ${u^+}_n$ for u_n will be conducted parallel with construction for minorant ${v^-}_n$ for v_n. The reason of this is that construction of the majorant for one term for the sum must be connected with minorant for the other term.

Let N be a positive number greater than $max|\bar{\varphi}(M)|$. Now let us begin with constructing the sequence of functions $u^{+}{}_{n}$ and $v^{-}{}_{n}$, by solving the Dirichlet problem for (3.1) in $B_1$and B_2 respectively with the following boundary values

$$u^{+}{}_{1} = \begin{cases} f(M) & \text{on } \alpha - \gamma \\ \varphi(M) & \text{on } \gamma \\ +N & \text{on } \bar{\alpha} \end{cases} \tag{3.17}$$

$$v^{-}{}_{1} = \begin{cases} 0 & \text{on } \beta \\ f(M) - u^{+}{}_{1} & \text{on } \bar{\beta} \end{cases} \tag{3.18}$$

$$\vdots$$

$$u^{+}{}_{n+1} = \begin{cases} f(M) & \text{on } \alpha - \gamma \\ \varphi(M) & \text{on } \gamma \\ f(M) - v^{-}{}_{n} & \text{on } \bar{\alpha} \end{cases} \tag{3.19}$$

$$v^{-}{}_{n+1}(x,y) = \begin{cases} 0 & \text{on } \beta \\ f(M) - u^{+}{}_{n+1}(x,y) & \text{on } \bar{\beta} \end{cases}. \tag{3.20}$$

We must prove that the sequence $u^{+}{}_{1}$, $u^{+}{}_{2}$, ... will be majorant for u_1, u_2 , ..., and sequence $v^{-}{}_{1}$, $v^{-}{}_{2}$, ... will be minorant for v_1, v_2 , ...

Let us take differences between functions of these sequences $u^{+}{}_{n+1}$ and u_{n+1} and between $v^{-}{}_{n+1}$ and v_{n+1}. They will also be solutions of (3.1) in $B_1$and B_2 respectively, and their boundary values will be connected under the following relations

$$u^{+}{}_{1} - u_1 = \begin{cases} 0 & \text{on } \alpha \\ +N - \bar{\varphi}(M) & \text{on } \bar{\alpha} \end{cases} \tag{3.21}$$

$$v^{-}{}_{1} - v_1 = \begin{cases} 0 & \text{on } \beta \\ u_1 - u^{+}{}_{1} & \text{on } \bar{\beta} \end{cases} \tag{3.22}$$

$$\vdots$$

$$u^+{}_{n+1} - u_{n+1} = \begin{cases} 0 & \text{on} \quad \alpha \\ v_n - v^-{}_n & \text{on} \quad \bar{\alpha} \end{cases} \tag{3.23}$$

$$v^-{}_{n+1} - v_{n+1} = \begin{cases} 0 & \text{on} \quad \beta \\ u_{n+1} - u^+{}_{n+1} & \text{on} \quad \bar{\beta} \end{cases}. \tag{3.24}$$

If we see the difference $u^+{}_1 - u_1$, we can decide that boundary values of it on L_1 are non-negative. By Assumption II, $u^+{}_1 - u_1 \geq 0$ everywhere in B_1. The values of $v^-{}_1 - v_1 \leq 0$ on L_2, from Assumption II, that $v^-{}_1 - v_1 \leq 0$ everywhere in B_2. Continuing this procedure, we can obtain that for any n the following inequalities are true

$$u^+{}_n \geq u_n \,, \; v^-{}_n \leq v_n \,. \tag{3.25}$$

Analogously we can construct sequences $u^-{}_1, u^-{}_2, \ldots$, as a minorant for $u_1, u_2, \ldots$, and $v^+{}_1, v^+{}_2, \ldots$, as a majorant for $v_1, v_2, \ldots$, by solving the Dirichlet problem for (3.1) in B_1or B_2 respectively, with the boundary conditions

$$u^-{}_1 = \begin{cases} f(M) & \text{on} \quad \alpha - \gamma \\ \varphi(M) & \text{on} \quad \gamma \\ -N & \text{on} \quad \bar{\alpha} \end{cases} \tag{3.26}$$

$$v^+{}_1 = \begin{cases} 0 & \text{on} \quad \beta \\ f(M) - u^-{}_1 & \text{on} \quad \bar{\beta} \end{cases} \tag{3.27}$$

$$\vdots$$

$$u^-{}_{n+1} = \begin{cases} f(M) & \text{on} \quad \alpha - \gamma \\ \varphi(M) & \text{on} \quad \gamma \\ f(M) - v^+{}_n & \text{on} \quad \bar{\alpha} \end{cases} \tag{3.28}$$

$$v^+{}_{n+1} = \begin{cases} 0 & \text{on} \quad \beta \\ f(M) - u^-{}_{n+1} & \text{on} \quad \bar{\beta} \end{cases}. \tag{3.29}$$

We can show that for any n the following inequalities will be true:

$$u^{-}{}_{n} \leq u_{n}\, ,\, v^{+}{}_{n} \geq v_{n}\, . \tag{3.30}$$

So, we have obtained

$$u^{+}{}_{n} \geq u_{n} \geq u^{+}{}_{n}\, , \tag{3.31}$$

$$v^{-}{}_{n} \leq v_{n} \leq v^{+}{}_{n}\, . \tag{3.32}$$

The choice of the number N will be decided such that the auxiliary majorant and minorant sequences of functions are monotonic. It will be done with an additional Assumption V, which is related not only with partial differential equation, but also with regions B_1 and B_2.

Assumption V: Let functions $U_{\bar{\alpha},f}(x, y)$ and $U_{\alpha,f}(x, y)$ be solutions of the Dirichlet problem for equation (3.1) in region B_1 with boundary conditions

$$U_{\bar{\alpha},f} = \begin{cases} 0 & \text{on} \quad \alpha \\ f(M) & \text{on} \quad \bar{\alpha} \end{cases} \tag{3.33}$$

$$U_{\alpha,f} = \begin{cases} f(M) & \text{on} \quad \alpha - \gamma \\ \varphi(M) & \text{on} \quad \gamma \\ 0 & \text{on} \quad \bar{\alpha} \end{cases} \tag{3.34}$$

and, function $V_{\bar{\beta},f}$ is a solution of the Dirichlet problem in B_2 with following boundary condition

$$V_{\bar{\beta},f} = \begin{cases} 0 & \text{on} \quad \beta \\ f(M) & \text{on} \quad \bar{\beta} \end{cases}. \tag{3.35}$$

We will use the specific case of function $U_{\bar{\alpha},f}$ by solving the Dirichlet problem in B_1 for boundary values equal to zero on α and equal to one on $\bar{\alpha}$, and for $U_{\alpha,f}$ the boundary values equal to the values of u or $u^{+}{}_{1}$ on α and equal to zero on $\bar{\alpha}$

$$U_{\bar{\alpha},1} = \begin{cases} 0 & \text{on} \quad \alpha \\ 1 & \text{on} \quad \bar{\alpha} \end{cases}, \tag{3.36}$$

$$U_{\alpha,u} = \begin{cases} u & \text{on} \quad \alpha \\ 0 & \text{on} \quad \bar{\alpha} \end{cases}. \tag{3.37}$$

For function $V_{\bar{\beta},U_{\bar{\alpha},1}}$, we will solve the Dirichlet problem in B_2 for boundary values equal to zero on β and equal to $U_{\bar{\alpha},1}$ on $\bar{\beta}$

$$V_{\bar{\beta},U_{\bar{\alpha},1}} = \begin{cases} 0 & \text{on} \quad \beta \\ U_{\bar{\alpha},1} & \text{on} \quad \bar{\beta} \end{cases}. \tag{3.38}$$

By Assumption II we can say that all values obtained by $U_{\bar{\alpha},1}$ in B_1 will lie in the interval [0, 1]. So, values for $V_{\bar{\beta},U_{\bar{\alpha},1}}$ will also lie in the same interval, and therefore all values that this function will obtain in B belongs to [0, 1].

In view of the values that $V_{\bar{\beta},U_{\bar{\alpha},1}}$ acquires on $\bar{\alpha}$ of the border of B_1, they are between zero and one. Using this and by Assumption V, we will require that none of them will overtake some proper fraction

$$V_{\bar{\beta},U_{\bar{\alpha},1}} \leq \vartheta < 1 \ \text{ for } \ (x, y) \in \bar{\alpha}. \tag{3.39}$$

Now let us return to the auxiliary sequences, and look for the difference between ${u^+}_1$ and ${u^+}_2$

$${u^+}_1 - {u^+}_2 = \begin{cases} 0 & \text{on} \quad \alpha \\ N - f(M) + {v^-}_1 & \text{on} \quad \bar{\alpha} \end{cases}. \tag{3.40}$$

We can represent ${u^+}_1(x, y)$ in the form of the sum of two solutions of equation (3.1)

$${u^+}_1 = NU_{\bar{\alpha},1} + U_{\alpha,u}. \tag{3.41}$$

First of them, $NU_{\bar{\alpha},1}$, acquires on $\bar{\alpha}$ values equal to N, and on α its values equal to zero; for second, $U_{\alpha,u}$, acquires on α same values as does ${u^+}_1$ or u, and on $\bar{\alpha}$ its values equal to zero.

Accordingly for ${v^-}_1$, by its boundary values, we can represent it in the form

$$ {v^-}_1 = V_{\bar{\beta},f} - V_{\bar{\beta},u_1} = V_{\bar{\beta},f} - NV_{\bar{\beta},U_{\bar{\alpha},1}} - V_{\bar{\beta},U_{\alpha,u}} \,. \tag{3.42} $$

Then values of difference ${u^+}_1 - {u^+}_2$on $\bar{\alpha}$ are equal to

$$ N - f(M) + {v^-}_1 = N\left(1 - V_{\bar{\beta},U_{\bar{\alpha},1}}\right) + V_{\bar{\beta},f} - V_{\bar{\beta},U_{\alpha,u}} - f(M). \tag{3.43} $$

The last three terms ($V_{\bar{\beta},f} - V_{\bar{\beta},U_{\alpha,u}} - f(M)$), are bounded functions at the point (x, y) on $\bar{\beta}$. The coefficient of N, (by Assumption V), is positive and not less than $1 - \vartheta$. Therefore, we must choose a large N such that following inequality is true

$$ N\left(1 - V_{\bar{\beta},U_{\bar{\alpha},1}}\right) \geq \left|V_{\bar{\beta},f} - V_{\bar{\beta},U_{\alpha,u}} - f(M)\right| \,. \tag{3.44} $$

Values of ${u^+}_1 - {u^+}_2$ on α will not be negative. By the choice of N, ${u^+}_1 - {u^+}_2$ will not be less than zero everywhere on L_1, and by Assumption II we can say that everywhere in B_1 following inequality is satisfed

$$ {u^+}_1 - {u^+}_2 \geq 0 \,. \tag{3.45} $$

With ${u^+}_n$ and ${v^-}_n$ and their boundary values, we will obtain the following inequalities

$$ {u^+}_n - {u^+}_{n+1} = \begin{cases} 0 & \text{on} \quad \alpha \\ {v^-}_n - {v^-}_{n-1} & \text{on} \quad \bar{\alpha} \end{cases}, \tag{3.46} $$

$$ {v^-}_n - {v^-}_{n+1} = \begin{cases} 0 & \text{on} \quad \beta \\ {u^+}_{n+1} - {u^+}_n & \text{on} \quad \beta \end{cases}, \tag{3.47} $$

where $n = 1, 2, 3, \ldots$. From the above inequalities it is not difficult to show that sequences $u^+{}_n$ and $v^-{}_n$ are monotonic

$$u^+{}_1 \geq u^+{}_2 \geq \cdots, \tag{3.48}$$

$$v^-{}_1 \leq v^-{}_2 \leq \cdots. \tag{3.49}$$

Analogously by taking same value of N, we are able to show monotonicity of sequences $u^-{}_n$ and $v^+{}_n$too

$$u^-{}_1 \leq u^-{}_2 \leq \cdots, \tag{3.50}$$

$$v^+{}_1 \geq v^+{}_2 \leq \cdots. \tag{3.51}$$

We have showed the monotonicity of auxiliary sequences, by using $u^+{}_n \geq u_n \geq u^-{}_n$ and $v^-{}_n \leq v_n \leq v^+{}_n$ and we obtained the following inequalities

$$u^+{}_1 \geq u^+{}_n \geq u_n \geq u^-{}_n \geq u^-{}_1, \tag{3.52}$$

$$v^-{}_n \leq v^-{}_1 \leq v_n \leq v^+{}_1 \leq v^+{}_n\,. \tag{3.53}$$

We can say that sequences $u^+{}_n$ and $u^-{}_n$, are bounded. By assumption III, they will converge to some functions and their limit functions

$$u^+ = \lim_{n\to\infty} u^+{}_n \text{ and } u^- = \lim_{n\to\infty} u^-{}_n \tag{3.54}$$

will satisfy equation (3.1) in B_1.

By a similar method we can prove that the sequences $v^+{}_n$ and $v^-{}_n$ are also convergent, and their limit functions

$$v^+ = \lim_{n\to\infty} v^+{}_n \text{ and } v^- = \lim_{n\to\infty} v^-{}_n \tag{3.55}$$

also satisfy equation (3.1) in B_2.

So, u^+ and v^- are determined in regions $B_1$and B_2 respectively. Their sum $w = u^+ + v^-$ is determined in $B'$and satisfies equation (2.1) there.

Now, we want to find limit values on $L' = \bar{\alpha} + \bar{\beta} + \gamma$.

Let us take any point M on $\bar{\alpha}$, and let the defined function $f(M)$ be continuous at this point. By our construction of $u^+{}_{n+1}$and $v^-{}_n$we have

$$u^+{}_{n+1}(x, y) = f(M) - v^-{}_n(x, y) \tag{3.56}$$

Starting with this and by simple logic we can show that approaching from (x, y) to M, w will tend to a limit value and this limit value is $f(M)$. Let a point (x, y) tend to a point M. By $u^+ \leq u^+{}_n$, we will have

$$\lim_{(x,y)\to M} u^+(x, y) \leq \lim_{(x,y)\to M} u^+{}_{n+1}(x, y) = f(M) - v^-(M). \tag{3.57}$$

For any n above inequality is true. As n tends to infinity the right side will approach to $f(M) - v^-(M)$, and the left side of the inequality does not change by changing n , so, we obtain the following inequality

$$\overline{\lim_{(x,y)\to M}} u^+(x, y) \leq f(M) - v^-(M). \tag{3.58}$$

On the other hand, suppose function $u_*(x, y)$, solving the Dirichlet problem in B_1 with boundary condition

$$u_*(x, y) = \begin{cases} \varphi(M) & \text{on} \quad \alpha - \gamma \\ f(M) & \text{on} \quad \gamma \\ f(M) - v^-(M) & \text{on} \quad \bar{\alpha} \end{cases} . \tag{3.59}$$

Because of $v^-{}_n \leq v^-$, on border L_1 and by Assumption II everywhere in B_1, $u^+{}_{n+1} \geq u_*$. Then

$$\lim_{n\to\infty} u^+{}_{n+1} = u^+ \geq u_*. \tag{3.60}$$

In this inequality, if the point (x, y) approaches M, we get

$$\varliminf_{(x,y)\to M} u^+(x,y) \leq \lim_{(x,y)\to M} u_*(x,y) = f(M) - v^-(M). \tag{3.61}$$

If we see inequalities (3.55) and (3.58), the limit value of $u^+(x,y)$ at the point M, we obtain the following equation

$$\lim_{n\to\infty} u^+{}_{n+1} = u^+ = f(M) - v^-(M). \tag{3.62}$$

From this we can say that $w = u^+ + v^-$ will have a limit value equal to $f(M)$ when a point (x, y) tends to M.

By a similar method we can show that when the point (x, y) tends to any point M on $\bar{\beta}$ with continuous $f(M)$, w will approach to a limit value equal to $f(M)$.

It remains only on γ to investigate limit values of w. Suppose that M is an inner point of γ, and suppose the function $f(M)$ is continuous at M. We must prove that for an approach of a point (x, y) to a point M, w will approach to $f(M)$. We have

$$u^+{}_1 \geq u^+ \geq u^-{}_1 \tag{3.63}$$

From construction of $u^+{}_1$ and $u^-{}_1$, when a point (x, y) tends to M, $u^+{}_1$ and $u^-{}_1$ will tend to $f(M)$. So, $u^+ \to f(M)$ as $(x, y) \to M$.

On $\alpha - \gamma$ we will take a point M which $\varphi(M)$ is continuous at M. When (x, y) approaches M, ${u^+}_1$ and ${u^-}_1$will approach $\varphi(M)$. From previous inequality it follows that u^+ will tend to $\varphi(M)$.

By the inequality

$$ {v^-}_1 \leq v^- \leq {v^+}_1 , \tag{3.64} $$

and by the actuality that when (x, y) approach to any inner point of β of border L_2, ${v^-}_1$ and ${v^+}_1$tend to zero, v^- has a limit value equal to zero at any inner point of β.

Consequently $w = u^+ + v^-$ has limit values equal to $f(M)$ at any points of the border L' of B'.

Similarly, we can show that for $u^-$and v^+ their sum $u^- + v^+$ is also a solution of (3.1) and at any points of the border L' of B' has limit values equal to $f(M)$.

Because of the Dirichlet problem in B' can have only a unique solution, and hence we have

$$ w = u^+ + v^- = u^- + v^+. \tag{3.65} $$

So,

$$ u^+ = u^- = u, \tag{3.66} $$

$$ v^+ = v^- = v, \tag{3.67} $$

and u_n , v_n must converge to the same function

$$ \lim_{n\to\infty} u_n = u \tag{3.68} $$

$$ \lim_{n\to\infty} v_n = v \tag{3.69} $$

$$w = u + v\,. \tag{3.70}$$

From this convergence, the Schwarz-Neumann algorithm is formed, with the previous five assumptions, in the Dirichlet problem for the intersection of two regions.

3.2 Finite-Difference Analog of Schwarz-Neumann Method with Numerical Experiment

In this section, we will demonstrate the finite-difference analog of Schwarz-Neumann method by using the following numerical experiment and we will present numerical experiments as an illustration of the application of Schwarz-Neumann method. For the approximate solution of the problem on the overlapping rectangles covering the domain the Gauss Seidel methods has been used, where the computations are carried out by using MATLAB programming language.

Example 1: Let B be an L-Shaped region bounded by the border L on the $xy-plane$, that consists of two overlapping rectangles B_1 and B_2 at the origin point (see Figure 3.1).

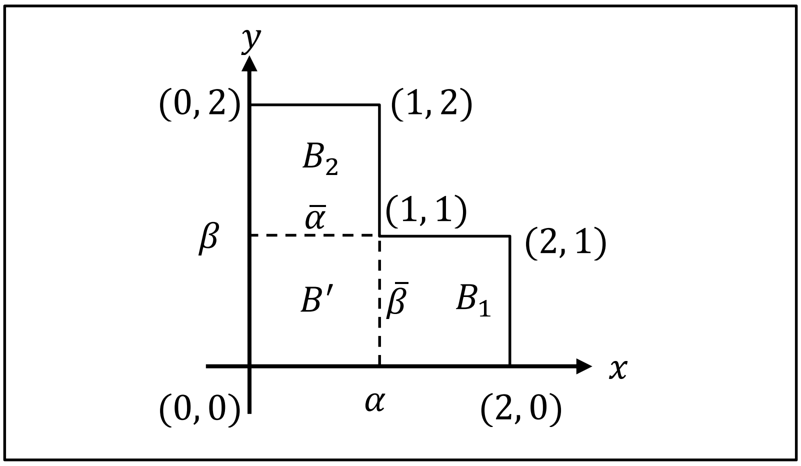

Figure 3.1: Figure of two regions that overlapped one another

Their common part is B' bounded by its border L', where

$B_1 = \{(x, y): 0 < x < 2 \text{ and } 0 < y < 1\}$, $B_2 = \{(x, y): 0 < x < 1 \text{ and } 0 < y < 2\}$

$B = \{(x, y): 0 < x < 2 \text{ and } 0 < y < 2\} / \{(x, y): 1 < x < 2 \text{ and } 1 < y < 2\}$

$B' = \{(x, y): 0 < x < 1 \text{ and } 0 < y < 1\}$

$$\begin{aligned} \Delta u = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \quad &\text{on} \quad B' \\ w = e^x \sin y \qquad\qquad &\text{on} \quad L' \end{aligned} \quad . \tag{3.71}$$

For the approximate solutions we have the set of nodes

$B'_h = \{(x_i, y_j) : x_i = ih, y_j = jh \text{ for } i = 0, 1, \dots, n, j = 0, 1, \dots, n, \ 0 < x < 1 \text{ and } 0 < y < 1\}$.

We will use the finite-difference problem with their boundary conditions.

To solve this difference equation we have applied the Gauss Seidel method. The exact and numerical solutions are shown in Table 3.1.

We use the difference approximation for Laplace's equation

$$\begin{aligned} -(\bar{\partial}_1 \partial_1 u_s + \bar{\partial}_2 \partial_2 u_s) = 0 \qquad &\text{on} \qquad B'_{h_s}, s = 1, 2 \\ \sum_{s=1}^{2} u_s = e^{ih_s} \sin jh_s \qquad &\text{on} \qquad L'_{h_s} \end{aligned} \quad . \tag{3.72}$$

The exact and numerical solutions are shown in Table 3.1, where $\|\varepsilon_h\|_{C(B)} = \max\{|w - \sum_{s=1}^{2} u_{s,h}|\}$, is difference between the exact and approximate solution, in the maximum norm, $h = 2^{-m}$, $m = 2, \dots, 6$ and $R_h = \frac{\|w - \sum_{s=1}^{2} u_{s,2^{-m}}\|_{C(B)}}{\|w - \sum_{s=1}^{2} u_{s,2^{-(m+1)}}\|_{C(B)}}$ is the order of convergence of the approximate solution.

Table 3.1: The maximum error between exact solution and approximate solution and the order of convergence of the approximate solution.

h	$\|\varepsilon_h\|_{C(B')}$	R_h	$\simeq$
$\frac{1}{4}$	5.788943433201466e-04		3.617961095277416
$\frac{1}{8}$	1.600056849908604e-04		3.935755982929401
$\frac{1}{16}$	4.065437127831473e-05		3.987760849778389
$\frac{1}{32}$	1.019478670105656e-05		3.993895824293641
$\frac{1}{64}$	2.552592042848190e-06		

Chapter 4

CONCLUSION

Schwarz's method gave us the unbounded possibility of extending the class of regions for which the explicit solution of the first boundary-value problem can be constructed.

In this thesis, we have discussed the Schwarz and the Schwarz-Neumann methods for solving the Dirichlet problem for partial differential equations on an L-shaped domain. The convergence of the solution has also been reviewed.

Numerical experiments have been provided to demonstrate the application of the finite-difference analogue of these methods. The approximate solutions are consistent with the theoretical results.

Both of these methods can be applied for the approximation of the solution of boundary value problems in domains covered by more than two sub-domains. Schwarz's method can also be applied in three dimensional domains.

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