# On asymptotical behavior of solution of Riccati equation arising in linear filtering with shifted noises

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In this paper we consider a linear signal system together with the two linear observation systems. The observation systems differ from each other by the noise processes. The noise of one of them is a constant shift in time of the signal noise. In the other one the shift is neglected. Respectively, we consider two best estimates of the signal corresponding to two different observation systems. The following problem is investigated: whether the effect of the shift on the best estimate becomes negligible as time increases. This leads to a comparison of the asymptotical behaviors of the solutions of respective Riccati equations. It is proved that under a certain relation between the parameters, the effect of the shift is not negligible.

## 1 Introduction

Kalman filtering for both independent and correlated white noises (see, for example, [Ben92, CP78, LS98, Dav77, FR75]) and its modification to colored noises (see [BJ68]) are very powerful method of estimation in engineering, especially, in space engineering (see [BJ68, CJ04]). However, a detailed study of the nature of noises arising in guidance and control of spacecrafts shows that, more adequately, the noises disturbing the signal and the observations are shifted in time for some small value, while this shift is neglected in space engineering.

Indeed, let $\varepsilon$ be the time needed for electromagnetic signals to run the path ground radar–satellite–ground radar. Assume that the control action $u$ changes the parameter $x$ of the satellite in accordance with the linear equation $x' = ax + bu$ if noise effects and the distance to the satellite are neglected.



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Then at the time $t$ the ground radar detects the signal $z(t) = x(t-\varepsilon/2)+w(t)$ that is the useful information $x(t-\varepsilon/2)$ about the parameter of the satellite at $t-\varepsilon/2$ corrupted by white noise $w(t)$ due to atmospheric propagation. Furthermore, the parameter of the satellite at $t-\varepsilon/2$ is changed by the control action $u(t-\varepsilon)$ that is sent by the ground radar at the time moment $t-\varepsilon$. This control passing through the atmosphere is corrupted by the noise $w(t-\varepsilon)$. Hence, the equation for the parameter of the satellite must be written as $x'(t-\varepsilon/2) = ax(t-\varepsilon/2)+b[u(t-\varepsilon)+w(t-\varepsilon)]$. Substituting $\tilde{x}(t) = x(t-\varepsilon/2)$ and $\tilde{u}(t) = u(t-\varepsilon)$, we obtain the state-observation system

$$\begin{cases} \tilde{x}'(t) = a\tilde{x}(t) + b\tilde{u}(t) + bw(t-\varepsilon), \\ z(t) = \tilde{x}(t) + w(t), \end{cases}$$

disturbed by shifted white noises with the state noise delaying the observation noise. Since the Earth orbiting satellites have a nearly constant distance from the Earth, the value $\varepsilon$ of the shift for them is time independent.

New applications of the GPS such as measuring vertical and horizontal ground deformations aimed to study volcanos and earthquakes want getting a centimeter (or millimeter) accuracy of satellites' positions. Among different ways toward this aim, one may be the use of Kalman type optimal filter for shifted white noises, obtained in [Bash03] and [Bash05] (abbreviate this filter as KF$^*$). Note that in this case the observations are more informative than in the case of correlated noises since they depend on the future of the signal noise as well. Proper filtering with respect to such observations should produce an improvement in comparison with the Kalman filter for correlated white noises (abbreviate this filter as KF).

Thus, we have two filters KF and KF$^*$. The first one is easy in its realization, successfully tested in many applications and produces the best estimate if the shift in the model is neglected. But for the model with shifted noises it produces an estimate which is not the best one, being perhaps close to it. On the other hand, the second one produces the best estimate for the model with shifted noises, but it needs relatively more calculations for its realization and not yet used in applications. Whether the replacement of KF by KF$^*$ is reasonable? For this, let $\hat{x}_y(t)$ and $\hat{x}_z(t)$ be the estimates of the signal process $x(t)$ in accordance with KF$^*$ and KF, respectively. Denote $i(t) = \mathbf{E}[\hat{x}_y(t)-\hat{x}_z(t)]^2$, where $\mathbf{E}$ is a symbol for expectation, and call it an *improvement process*. From engineering point of view, regarding the guidance and control of satellites, the asymptotical behavior of $i(t)$ should be important since once a satellite is established on its approximate position in the orbit, $\lim_{t\to\infty} i(t)$ will say whether the improvement is valid at further time moments. If $\lim_{t\to\infty} i(t) = 0$, then the improvement provided by KF$^*$ in comparison with KF becomes negligible for large time moments and, therefore, this case does not support the replacement. Unlike, if $\lim_{t\to\infty} i(t) > 0$ or $\lim_{t\to\infty} i(t)$ does not exist, then the best estimate $\hat{x}_y(t)$ non-negligibly deviates from the estimate $\hat{x}_z(t)$ for large $t$ and, hence, the replacement is recommended.

In this paper we study $\lim_{t\to\infty} i(t)$, and use the respective Riccati equations of KF and KF$^*$. While KF is well discussed in the existing literature, KF$^*$ was found recently (see [Bash03, Bash05] together with Remark 1 in this paper). We proved that under certain relation between the parameters of the system, $\hat{x}_y(t)$ non-negligibly deviates from the estimate $\hat{x}_z(t)$ for large $t$. Moreover, numerical study of the respective Riccati equations shows that the error of estimation of KF$^*$ is greater than the same of KF. This also supports the replacement of KF by KF$^*$ because the greater is the error by KF$^*$, less reliable is the estimate by KF. Finally, note that the results of this paper are obtained for one dimensional systems. As far as the deviation of $\hat{x}_y(t)$ from $\hat{x}_z(t)$ is detected in an easy case, it should expectedly more valid for complicated cases as well.

## 2 Description of the problem

We will set the problem in one-dimensional case while the results can be extended to multidimensional case as well. Consider the one-dimensional linear signal system

$$x'(t) = ax(t) + bw(t),\ x(0) = x_0,\ t > 0, \tag{1}$$

and the two one-dimensional linear observation systems

$$z'(t) = cx(t) + w(t),\ z(0) = 0,\ t > 0, \tag{2}$$

$$y'(t) = cx(t) + w(t + \varepsilon),\ y(0) = 0,\ t > 0, \tag{3}$$

where $x(t)$ is a signal process, $y(t)$ and $z(t)$ are observation processes, $w(t)$ is a Gaussian white noise process with the mean $\mathbf{E}w(t) = 0$ and with the covariance $\mathrm{cov}(w(t), w(s)) = \delta(t - s)$, $\delta$ is the Dirac's delta function, $\varepsilon > 0$, $a, b, c$ are real numbers, $x_0$ is a Gaussian random variable with the mean $\mathbf{E}(x_0) = 0$ and with the variance $p_0$, $x_0$ and $w(t)$, $t \geq 0$, are independent.

Let $\hat{x}_z(t)$ and $\hat{x}_y(t)$ be the best estimates of the signal $x(t)$ based on the observations $z(s)$, $0 \leq s \leq t$, and $y(s)$, $0 \leq s \leq t$, respectively. Here, $\hat{x}_z(t)$ is the output of the well-known KF for the correlated white noises with the error of estimation

$$e_z(t) = \mathbf{E}[\hat{x}_z(t) - x(t)]^2 = f(t),$$

where $f(t)$ is a solution of the Riccati equation

$$f'(t) = 2(a - bc)f(t) - c^2 f(t)^2,\ f(0) = p_0,\ t > 0. \tag{4}$$

Adapting the results from [Bash05] for the estimate $\hat{x}_y(t)$, we can deduce that $\hat{x}_y(t)$ is the output of the KF$^*$ with the error of estimation

$$e_y(t) = \mathbf{E}[\hat{x}_y(t) - x(t)]^2 = p(t),$$

where $p(t)$ is a solution of the equation

$$\begin{cases} p'(t) = 2ap(t) + 2q(t,0) + b^2\chi_{(0,\varepsilon]}(t) - c^2p(t)^2, \\ p(0) = p_0,\ t > 0, \end{cases} \tag{5}$$

with $\chi_{(0,\varepsilon]}(t)$ being the indicator function of the interval $(0,\varepsilon]$. Here $q(t,\theta)$ is a solution of

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right)q(t,\theta) = aq(t,\theta) + r(t,0,\theta) - c^2p(t)q(t,\theta), \\ q(0,\theta) = 0,\ -\varepsilon \le \theta \le 0, \\ q(t,-\varepsilon) = -bcp(t),\ t > 0, \end{cases} \tag{6}$$

with $r(t,\tau,\theta)$ being a solution of

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \theta}\right)r(t,\tau,\theta) = -c^2q(t,\tau)q(t,\theta), \\ r(0,\tau,\theta) = 0,\ -\varepsilon \le \theta \le 0,\ -\varepsilon \le \tau \le 0, \\ r(t,-\varepsilon,\theta) = -bc(q(t,-\varepsilon) + q(t,\theta)),\ -\varepsilon \le \theta \le 0,\ t > 0, \\ r(t,\tau,-\varepsilon) = -bc(q(t,-\varepsilon) + q(t,\tau)),\ -\varepsilon \le \tau \le 0,\ t > 0. \end{cases} \tag{7}$$

*Remark 1.* There is a misprint in the formula (18) from Bashirov [Bash05]. The boundary condition

$$R\big(t,\tau,t-\lambda^{-1}(t)\big) = -Q^{\mathrm{T}}(t,\tau)C^{\mathrm{T}}F^{\mathrm{T}}$$

in this formula must be read as

$$R\big(t,\tau,t-\lambda^{-1}(t)\big) = -FCQ\big(t,t-\lambda^{-1}(t)\big) - Q^{\mathrm{T}}(t,\tau)C^{\mathrm{T}}F^{\mathrm{T}}.$$

Respectively, the boundary conditions

$$R(t,\tau,-\varepsilon) = -Q^{\mathrm{T}}(t,\tau)C^{\mathrm{T}}F^{\mathrm{T}},$$

and

$$R\big(t,\tau,t-c^{-1}t\big) = -Q^{\mathrm{T}}(t,\tau)C^{\mathrm{T}}F^{\mathrm{T}},$$

in the formulae (27) and (32) from Bashirov [Bash05] must also be read as

$$R(t,\tau,-\varepsilon) = -FCQ(t,-\varepsilon) - Q^{\mathrm{T}}(t,\tau)C^{\mathrm{T}}F^{\mathrm{T}},$$

and

$$R\big(t,\tau,t-c^{-1}t\big) = -FCQ\big(t,t-c^{-1}t\big) - Q^{\mathrm{T}}(t,\tau)C^{\mathrm{T}}F^{\mathrm{T}}.$$

## 3 The stability of the improvement

It is natural to call the mean square difference

$$i(t) = \mathbf{E}[\hat{x}_y(t) - \hat{x}_z(t)]^2$$

as an *improvement* provided by KF$^*$ in comparison with KF. We say that the improvement $i(t)$ is *unstable* if $\lim_{t\to\infty} i(t) = 0$. Otherwise, we say that it is *stable*. Note that the stability of the improvement should not be confused with the stability of the signal system or filters.

**Lemma 1.** *Let $f(t)$ and $p(t)$ be solutions of the equations* (4) *and* (5), *respectively. The following statements hold:*

(a) $\left(\sqrt{f(t)} - \sqrt{p(t)}\right)^2 \le i(t) \le \left(\sqrt{f(t)} + \sqrt{p(t)}\right)^2$.
(b) *If both* $\lim_{t\to\infty} f(t)$ *and* $\lim_{t\to\infty} p(t)$ *exist and equal to* 0, *then the improvement* $i(t)$ *is unstable.*
(c) *If both* $\lim_{t\to\infty} f(t)$ *and* $\lim_{t\to\infty} p(t)$ *exist and equal to different values, then the improvement* $i(t)$ *is stable.*
(d) *If* $\lim_{t\to\infty} f(t)$ *exists, while* $\lim_{t\to\infty} p(t)$ *does not exist, then the improvement* $i(t)$ *is stable.*

*Proof.* By Cauchy–Schwarz inequality,

$$\begin{aligned} i(t) &= \mathbf{E}[\hat{x}_z(t) - \hat{x}_y(t)]^2 = \mathbf{E}[(\hat{x}_z(t) - x(t)) - (\hat{x}_y(t) - x(t))]^2 \\ &= f(t) + p(t) - 2\mathbf{E}[(\hat{x}_z(t) - x(t))(\hat{x}_y(t) - x(t))] \\ &\ge f(t) + p(t) - 2\sqrt{\mathbf{E}[\hat{x}_z(t) - x(t)]^2\mathbf{E}[\hat{x}_y(t) - x(t)]^2} \\ &= f(t) + p(t) - 2\sqrt{f(t)p(t)} = \left(\sqrt{f(t)} - \sqrt{p(t)}\right)^2, \end{aligned}$$

and

$$\begin{aligned} i(t) &= f(t) + p(t) - 2\mathbf{E}[(\hat{x}_z(t) - x(t))(\hat{x}_y(t) - x(t))] \\ &\le f(t) + p(t) + 2|\mathbf{E}[(\hat{x}_z(t) - x(t))(\hat{x}_y(t) - x(t))]| \\ &\le f(t) + p(t) + 2\sqrt{\mathbf{E}[\hat{x}_z(t) - x(t)]^2\mathbf{E}[\hat{x}_y(t) - x(t)]^2} \\ &= f(t) + p(t) + 2\sqrt{f(t)p(t)} = \left(\sqrt{f(t)} + \sqrt{p(t)}\right)^2, \end{aligned}$$

proving part (a). Part (b) follows from part (a) by the squeeze principle. Also, parts (c) and (d) follow from the first inequality in part (a).

Parts (c) and (d) of Lemma 1 give sufficient conditions for stability of the improvement, while Lemma 1(b) presents a sufficient condition for being unstable. Also, Lemma 1(a) tells us that the expression $(\sqrt{f(t)} - \sqrt{p(t)})^2$ is a lower bound of $i(t)$ and in case of stability it can be used to approximate the improvement from below at different instants.

Concerning the Riccati equation (4), it has a trivial solution $f(t) = 0$ if $p_0 = 0$. In case $p_0 > 0$, its solution can explicitly be expressed as

$$f(t) = \frac{p_0}{1 + p_0 c^2 t} \tag{8}$$

if $a = bc$, and

$$f(t) = \left[\frac{c^2}{2(a-bc)} + \left(\frac{1}{p_0} - \frac{c^2}{2(a-bc)}\right)e^{-2(a-bc)t}\right]^{-1} \tag{9}$$

if $a \ne bc$. One particular subcase of (9) is $f(t) \equiv p_0$ if $2(a - bc) = p_0 c^2$. Hence, the following asymptotical behavior of $f(t)$ can easily be deduced:

$$\lim_{t\to\infty} f(t) = \begin{cases} 0, & \text{if } a \le bc \text{ or } p_0 = 0, \\ 2(a-bc)/c^2, & \text{if } a > bc \text{ and } p_0 > 0. \end{cases} \tag{10}$$

Numerical investigation of the equations (5)–(7) allows to conjecture that

$$\lim_{t\to\infty} p(t) \begin{cases} = 0, \text{ if } a \le bc \text{ or } p_0 = 0, \\ > 0, \text{ if } a > bc \text{ and } p_0 > 0. \end{cases}$$

Therefore, in the next section in order to prove that there are the values of the parameters $a$, $b$ and $c$, which make the improvement $i(t)$ stable, we will concentrate on the case when $p_0 > 0$ and $a > bc$, assuming that

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} f(t) = 2(a-bc)/c^2. \tag{11}$$

Then we will deduce a necessary condition for this assumption. The negation of the necessary condition will produce a sufficient condition for the improvement $i(t)$ to be stable.

## 4 The system Riccati equation (5)–(7)

Let $p_0 > 0$ and $a > bc$ and assume that (11) holds. To investigate $\lim_{t\to\infty} p(t)$ we need an explicit representation for the solution $(p(t), q(t,\theta), r(t,\tau,\theta))$ of the system (5)–(7). While it will not be used in the sequel, it is interesting to note that $r(t,\tau,\theta) = 0$ if $0 \le t \le \max(\varepsilon+\theta, \varepsilon+\tau)$, and

$$r(t,\tau,\theta) = -bc \begin{Bmatrix} q(t-\theta-\varepsilon, \tau-\theta-\varepsilon) + q(t-\theta-\varepsilon, -\varepsilon), \ -\varepsilon \le \theta \le \tau \le 0 \\ q(t-\tau-\varepsilon, \theta-\tau-\varepsilon) + q(t-\tau-\varepsilon, -\varepsilon), \ -\varepsilon \le \tau \le \theta \le 0 \end{Bmatrix}$$
$$-c^2 \int_{\max(t-\theta-\varepsilon, t-\tau-\varepsilon)}^{t} q(s, s-t+\tau) q(s, s-t+\theta)\, ds$$

if $t > \max(\varepsilon+\theta, \varepsilon+\tau)$. Moreover, $r(t,\tau,\theta)$ is a continuous kernel of the nonnegative integral operator (see [Bash05]) and, therefore, it satisfies $r(t,\tau,\theta) = r(t,\theta,\tau)$ and

$$r(t,\tau,\theta) \ge 0,\ t \ge 0,\ -\varepsilon \le \theta \le 0,\ -\varepsilon \le \tau \le 0, \tag{12}$$

that will be used later.

Furthermore, the solution of the equation (6) can be represented as $q(t,\theta) = 0$ if $0 \le t \le \varepsilon+\theta$, and

$$q(t,\theta) = -bce^{a(\varepsilon+\theta) - c^2 \int_{t-\varepsilon-\theta}^{t} p(\alpha)\, d\alpha} p(t-\theta-\varepsilon)$$
$$+ \int_{t-\theta-\varepsilon}^{t} e^{a(t-s) - c^2 \int_s^t p(\alpha)\, d\alpha} r(s, 0, s-t+\theta)\, ds \tag{13}$$

if $t > \varepsilon + \theta$.

Regarding $p(t)$, it satisfies the initial condition $p(0) = p_0$ together with the differential equation $p'(t) = 2ap(t) + b^2 - c^2 p(t)^2$, if $0 < t \le \varepsilon$, and

$$p'(t) = 2(a - bc)p(t) - c^2 p(t)^2 + 2[q(t, 0) - q(t, -\varepsilon)]$$

if $t > \varepsilon$. Therefore, for a given $\delta > 0$, we can represent $p(t)$ in the form

$$\begin{aligned} p(t) = {} & e^{2(a-bc)\delta - c^2 \int_{t-\delta}^{t} p(\alpha)\, d\alpha} p(t - \delta) \\ & + 2 \int_{t-\delta}^{t} e^{2(a-bc)(t-s) - c^2 \int_{s}^{t} p(\alpha)\, d\alpha} [q(s, 0) - q(s, -\varepsilon)]\, ds \end{aligned} \tag{14}$$

for sufficiently large $t$.

Applying the assumption (11) to (14), we obtain

$$\lim_{t\to\infty} \int_{t-\delta}^{t} [q(s, 0) - q(s, -\varepsilon)]\, ds = 0.$$

Since $\delta > 0$ is arbitrary and $q(t, 0) - q(t, -\varepsilon)$ is continuous in $t$,

$$\lim_{t\to\infty} [q(t, 0) - q(t, -\varepsilon)] = 0. \tag{15}$$

By (6) and (13),

$$\begin{aligned} q(t, 0) - q(t, -\varepsilon) = {} & bc\Big[p(t) - e^{a\varepsilon - c^2 \int_{t-\varepsilon}^{t} p(\alpha)\, d\alpha} p(t - \varepsilon)\Big] \\ & + \int_{t-\varepsilon}^{t} e^{a(t-s) - c^2 \int_{s}^{t} p(\alpha)\, d\alpha} r(s, 0, s - t)\, ds. \end{aligned}$$

Hence, from (15) and by the assumption (11),

$$\lim_{t\to\infty} \int_{t-\varepsilon}^{t} e^{(2bc-a)(t-s)} r(s, 0, s - t)\, ds = \frac{2b(a - bc)}{c} \Big[e^{(2bc-a)\varepsilon} - 1\Big].$$

Thus, by (12),

$$\frac{2b(a - bc)}{c} \Big[e^{(2bc-a)\varepsilon} - 1\Big] \ge 0. \tag{16}$$

Since are in the case $a > bc$, we find out that the inequality (16) does not hold if $0 < 2bc < a$. Hence, the following is proved.

**Theorem 1.** *If* $0 < 2bc < a$ *and* $p_0 > 0$*, then the improvement* $i(t)$ *is stable.*

## 5 Concluding remarks

In this paper the asymptotical behaviors of solutions of two related Riccati equations are compared. These solutions represent the errors of estimation

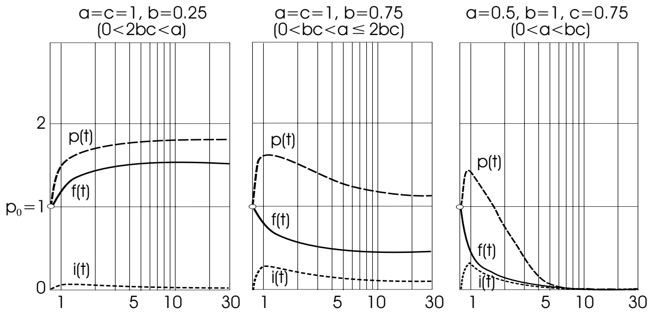


**Fig. 1.** Graphs of $f(t)$, $p(t)$ and $i(t)$ in different cases.

by KF* and KF, producing the best estimates when the noises in the linear filtering problem are shifted in time and not shifted, respectively. The aim was to detect the cases of the parameters $a$, $b$ and $c$ for which the best estimates by KF* and KF asymptotically deviate from each other, and wherethrough to support the significance of KF* for models with shifted noises. In the paper we prove that if $0 < 2bc < a$ and $p_0 > 0$ the improvement $i(t)$ provided by KF* in comparison with KF is asymptotically non-negligible while it may be small. The graphs of functions $f(t)$, $p(t)$ and $i(t)$ from Fig. 1 (left), obtained by numerical methods, are typical for the case $0 < 2bc < a$. Numerical study of other cases allows to conjecture that $i(t)$ is still asymptotically non-negligible in the case $0 < bc < a \leq 2bc$ (see Fig. 1 (center)), while it is negligible when $0 < a < bc$ (Fig. 1 (right)). One more observation from Fig. 1 is that in any case $p(t) > f(t)$, i.e., the error of estimation by KF* is greater than the same of KF. Since in applications we never get the exact values of unknown parameters and make decision about them on the base of estimations and respective errors, this tells us that the preciseness of estimation by KF is indeed less reliable. Although the results of the paper are obtained for a one dimensional system, the complication of the system should expectedly make the deviation of $\hat{x}_y(t)$ from $\hat{x}_z(t)$ more valid. All this supports KF* in applications required delicate estimations, especially for positioning satellites with extreme precision.

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