Exponential Operators and Hermite Type Polynomials

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ABSTRACT

This thesis consists of five Chapters. Chapter 1 is devoted to the Introduction. We investigate some basic properties of the exponential operators, in Chapter 2. Chapter 3, gives the proves of some exponential operator identities such as Weyl, Sack, Hausdorff and Crofton identities. In Chapter 4, we study the monomiality principle and its properties.

Finally in the last chapter, as an application to Chapters 3 and 4, we investigate some properties of Hermite polynomials in two variables, Hermite-Kampe de Feriet polynomials, Laguerre polynomials in two variables and Hermite-Based Appell polynomials.

Keywords: Exponential operators, Weyl, Sack, Hausdorff and Crofton identities, Monomiality principle,Hermite-Kampe de Feriet polynomials, Laguerre polynomials in two variables.

ÖZ

Bu tez beş bölümden oluşmaktadır. Birinci bölüm giriş kısmına ayrılmıştır. İkinci bölümde üstel operatörlerin bazı özellikleri incelenmiştir. Üçüncü bölümde Weyl, Sack, Hausdorff ve Crofton özdeşlikleri ispatlanmıştır. Dördüncü bölümde tek terimlilik prensipleri çalışılmıştır. Son bölümde ise üçüncü ve dördüncü bölümün uygulamaları yapılmış, iki değişkenli Hermite polinomları, Hermite-Kampe de Feriet polinomları, iki değişkenli Laguerre polinomları ve Hermite-Based Appell polinomları gösterilmiştir.

Anahtar Kelimeler: Üstel operatörler, Weyl, Sack, Hausdorff ve Crofton özdeşlikleri, Monomiallik prensipleri, Hermite-Kampe de Feriet polinomları, iki değerli Laguerre polinomları

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Chapter 1

INTRODUCTION

In Special functions appear in the solution of physical and engineering problems. One of the most powerful tool in investigating the properties of special functions is the Operational Method.

In this thesis, we start with exponential operators and study some operational identities such as Weyl, Sack, Hausdorff and Crofton identities. We investigate some properties of Hermite polynomials by of the above identities.

On the other hand, inspiring from the fact that every polynomial is quasimonomial, we investigate the monomiality principle for one and t-variable. As an application of the operational identities and monomiality principle, we study Hermite-Kampe de Feriet polynomials, Laguerre polynomials in two variables and Hermite-based Appell polynomials.

Chapter 2

EXPONENTIAL OPERATORS

We give some basic properties, definitions and elementary properties of the Exponential Operators.

2.1 Shift Operators and Their Extensions

The Taylor expansion for the analytic function $G(y)$ is given by

$$G(y+\mu) = \sum_{m=0}^{\infty} \frac{\mu^m}{m!} G^{(m)}(y),$$

where the series converges to corresponding values of G in a neighborhood of y. If $\mu = 0$, the basic operator is defined in the following way:

$$G(y+\mu) = \sum_{m=0}^{\infty} \frac{G^{(m)}(y+\mu)}{m!} \mu^m = \sum_{k=0}^{\infty} \frac{G^{(m)}(y)}{m!} \mu^m = e^{\mu \frac{d}{dy}} G(y).$$

Therefore, we get

$$G(y+\mu) = e^{\mu \frac{d}{dy}} G(y). \tag{2.1.1}$$

In the following examples, we see some simple applications of the above result.

Example 2.1.1 *Considering*

$$e^{\mu y \frac{d}{dy}} G(y)$$

and setting $y = e^{\alpha}$ we get,

$$\frac{d}{d\alpha} = \frac{d}{dy}\frac{dy}{d\alpha} = e^{\alpha}\frac{d}{dy}.$$

Therefore using (2.1.1), we get

$$
\begin{aligned}
e^{\mu y \frac{d}{dy}} G(y) &= e^{\mu \frac{d}{d\alpha}} G(e^{\alpha}) \\
G\left(e^{\alpha+\mu}\right) &= G(e^{\alpha} e^{\mu}) = G(ye^{\mu}).
\end{aligned} \tag{2.1.2}
$$

Example 2.1.2 *Now consider $e^{\mu y^2 \frac{d}{dy}} G(y)$. Setting $y = -\frac{1}{w}$ we have*

$$
\frac{d}{dw} = \frac{d}{dy}\frac{dy}{dw} = \frac{1}{w^2}\frac{d}{dy} = y^2 \frac{d}{dy}.
$$

Hence, we get

$$
\begin{aligned}
e^{\mu y^2 \frac{d}{dy}} G(y) &= e^{\mu \frac{d}{dw}} G\left(-\frac{1}{w}\right) = G\left(-\frac{1}{w+\mu}\right) \\
&= G\left(\frac{1}{\frac{1}{y} - \mu}\right) = G\left(\frac{1}{\frac{1-\mu y}{y}}\right) = G\left(\frac{y}{1-\mu y}\right).
\end{aligned}
$$

Now using (2.1.1), we obtain

$$
e^{\mu y^2 \frac{d}{dy}} G(y) = G\left(\frac{y}{1-\mu y}\right), \tag{2.1.3}
$$

where $|y| < \frac{1}{|\mu|}$.

Example 2.1.3 *Considering $e^{\mu y^k \frac{d}{dy}} G(y)$ setting $y = \sqrt[k-1]{\frac{1}{\gamma}}$, we give the following general result*

$$
e^{\mu y^k \frac{d}{dy}} G(y) = G\left(\frac{y}{\sqrt[k-1]{1-\mu(k-1)y^{k-1}}}\right), |y| < \sqrt[k-1]{\frac{1}{\mu(k-1)}}. \tag{2.1.4}
$$

In proving (2.1.4) let $y = \frac{1}{\gamma^{\frac{1}{k-1}}}$. It is clear that

$$
\frac{d}{d\gamma} = \frac{d}{dy}\frac{dy}{d\gamma} = -\frac{1}{k-1}\gamma^{-\frac{k}{k-1}}\frac{d}{dy} = -\frac{1}{k-1}\left(\frac{1}{y^{k-1}}\right)^{-\frac{k}{k-1}}\frac{d}{dy},
$$

which gives

$$
-(k-1)\frac{d}{d\gamma} = \left(\frac{1}{y^{k-1}}\right)^{-\frac{k}{k-1}}\frac{d}{dy} = y^k \frac{d}{dy}.
$$

Finally from (2.1.1), we obtain

$$
e^{\mu y^k \frac{d}{dy}} G(y) = e^{-\mu(k-1)\frac{d}{d\gamma}} G\left(\frac{1}{\gamma^{\frac{1}{k-1}}}\right) = G\left(\frac{1}{(\gamma - \mu(k-1))^{\frac{1}{k-1}}}\right)
$$

$$= G\left(\frac{1}{\left(y^{-k+1}-\mu\left(k-1\right)\right)^{\frac{1}{k-1}}}\right) = G\left(\frac{1}{\left(y^{-k+1}\right)^{\frac{1}{k-1}}\left(1-\frac{\mu(k-1)}{y^{-k+1}}\right)^{\frac{1}{k-1}}}\right)$$

$$= G\left(\frac{1}{y^{-1}\left(1-\mu\left(k-1\right)y^{k-1}\right)^{\frac{1}{k-1}}}\right) = G\left(\frac{y}{\sqrt[k-1]{1-\mu\left(k-1\right)y^{k-1}}}\right).$$

2.1.1 An Extension Formula

For a given function $g\left(y\right)$, we consider a more general shift operator,

$$e^{\mu g(y)\frac{d}{dy}}. \tag{2.1.5}$$

Using the same procedure as in the preceding section, we choose $y=\varphi\left(\beta\right)$ such that

$$g\left(y\right)\frac{d}{dy} = \frac{d}{d\beta}$$

$$\frac{dy}{d\beta} = g\left(y\right). \tag{2.1.6}$$

Therefore,

$$\frac{d}{d\beta} = \frac{d}{dy}\frac{dy}{d\beta} = g\left(y\right)\frac{d}{dy}.$$

Since $y=\varphi\left(\beta\right)$, using (2.1.6) we obtain

$$\varphi^{'}\left(\beta\right) = g(\varphi(\beta)). \tag{2.1.7}$$

Assuming a suitable initial value in order to guarantee the local invertibility of $\varphi(\beta)$, we deduce the definition of the shift operator (2.1.5) as follows

$$e^{\mu g(y)\frac{d}{dy}} f\left(y\right) = e^{\mu\frac{d}{d\beta}} f\left(\varphi\left(\beta\right)\right) = f\left(\varphi\left(\beta+\mu\right)\right). \tag{2.1.8}$$

Letting $\beta=\varphi^{-1}\left(y\right)$, the inverse function of $\varphi\left(\beta\right)$, we rewrite (2.1.6) in the following form

$$e^{\mu g(y)\frac{d}{dy}} f\left(y\right) = f\left(\varphi\left(\varphi^{-1}\left(y\right)+\mu\right)\right). \tag{2.1.9}$$

2.2 Exponentials Relevant to the Sum of Operator

We take into account the following operator

$$E(y,\mu)=e^{\mu\left(r(y)+p(y)\frac{d}{dy}\right)}. \tag{2.2.1}$$

Now, set

$$e^{\mu\left(r(y)+p(y)\frac{d}{dy}\right)}y=\left(e^{\mu\left(r(y)+p(y)\frac{d}{dy}\right)}ye^{-\mu\left(r(y)+p(y)\frac{d}{dy}\right)}\right)e^{\mu\left(r(y)+p(y)\frac{d}{dy}\right)}$$

and

$$e^{\mu\left(r(y)+p(y)\frac{d}{dy}\right)}y=y(\mu)\,t(\mu). \tag{2.2.2}$$

Then we obtain the following theorem.

Theorem 2.2.1 *The functions $y(\mu)$ and $t(\mu)$, which are given in (2.2.2) satisfy the system of first-order differential equations*

$$\begin{cases} \frac{d}{d\mu}y(\mu)=p(y(\mu)), & y(0)=y, \\ \frac{d}{d\mu}t(\mu)=r(y(\mu))\,t(\mu), & t(0)=1. \end{cases} \tag{2.2.3}$$

Proof. In fact, using (2.2.2), with $g\equiv 1$, and $r(y)=0$ and then using (2.1.9) with $f(y)\equiv y$, we get,

$$e^{\mu p(y)\frac{d}{dy}}y=y(\mu),\;\; y(0)=y.$$

On the other hand $r(y)\neq 0$ and assume $f\equiv 1$ and therefore find

$$e^{\mu\left(r(y)+p(y)\frac{d}{dy}\right)}1=t(\mu),\;\; t(0)=1.$$

Differentiating both sides with respect to μ, we obtain

$$\begin{aligned} \frac{d}{d\mu}t(\mu) &= \frac{d}{d\mu}e^{\mu\left(r(y)+p(y)\frac{d}{dy}\right)}1 \\ &= \left(r(y)+p(y)\frac{d}{dy}\right)e^{\mu\left(r(y)+p(y)\frac{d}{dy}\right)}1 \\ &= r(y)\,t(\mu)+p(y)\frac{d}{dy}g(\mu) \end{aligned}$$

$$
\begin{aligned}
&= r(y)\,t(\mu) \\
&= r(y(\mu))\,t(\mu)\,.
\end{aligned}
$$

This completes the proof. ■

More generally, we have

$$
\begin{aligned}
e^{\mu\left(r(y)+p(y)\frac{d}{dy}\right)}y^2 &= e^{\mu\left(r(y)+p(y)\frac{d}{dy}\right)}ye^{-\mu\left(r(y)+p(y)\frac{d}{dxy}\right)}e^{\mu\left(r(y)+p(y)\frac{d}{dy}\right)}ye^{-\mu\left(r(y)+p(y)\frac{d}{dy}\right)} \\
&= y^2(\mu)\,t(\mu)\,.
\end{aligned}
$$

Finally, the following equation is satisfied for any analytic function h

$$
e^{\mu\left(r(y)+p(y)\frac{d}{dy}\right)}h(y) = h(y(\mu))\,t(\mu)\,. \tag{2.2.4}
$$

Chapter 3

DISENTANGLEMENT TECHNIQUES

Considering the exponential operators $\mathscr{C}$ and $\mathscr{D}$, we generally have

$$e^{\mathscr{C}+\mathscr{D}} \neq e^{\mathscr{C}} e^{\mathscr{D}}.$$

We study some cases of the operator $e^{\mathscr{C}+\mathscr{D}}$.

3.1 Weyl Identity

In the case $\mathscr{C} = \mu y$ and $\mathscr{D} = \mu \frac{d}{dy}$, we have the following theorem:

Theorem 3.1.1 *The following equality*

$$e^{\mu\left(y+\frac{d}{dy}\right)} = e^{\frac{\mu^2}{2}} e^{\mu y} e^{\mu \frac{d}{dy}} \tag{3.1.1}$$

holds true.

Proof. *Using (2.2.4), we have*

$$e^{\mu\left(y+\frac{d}{dy}\right)} h(y) = t(\mu) h(y(\mu)),$$

where $y(\mu)$ and $t(\mu)$ are given in (2.2.3).

Taking $p(y) = 1$ and $r(y) = y$ in (2.2.2) and using (2.2.3), we get

$$\frac{d}{d\mu} y(\mu) = 1, \;\; y(0) = y,$$

which gives $y(\mu) = \mu + y$. From the second equation of (2.2.3), we have

$$\frac{d}{d\mu} t(\mu) = (\mu + y) t(\mu),$$

which gives

$$t(\mu) = e^{\frac{\mu^2}{2} + y\mu} = e^{\frac{\mu^2}{2}} e^{\mu y}.$$

Using (2.2.4), we get

$$e^{\mu\left(y+\frac{d}{dy}\right)}h(y)=e^{\frac{\mu^2}{2}}e^{\mu y}h(y+\mu)=e^{\frac{\mu^2}{2}}e^{\mu y}e^{\mu\frac{d}{dy}}h(y).$$

The proof is completed. ■

The above result is substantially independent of the operators considered, provided that their commutators satisfy suitable properties. In fact, setting

$$\mathscr{C}=\mu y,\ \ \mathscr{D}=\mu\frac{d}{dy},$$

then we have the following commutation relation:

$$\begin{aligned}
[\mathscr{C},\mathscr{D}]F(y) &= \left[\mu y\left(\mu\frac{d}{dy}\right)-\mu\frac{d}{dy}\mu y\right]F(y)\\
&= \mu y\left(\mu F'\right)-\mu\frac{d}{dy}\mu yF(y)\\
&= \mu^2 yF'(y)-\mu\left[\mu F(y)+\mu yF'(y)\right]\\
&= \mu^2 yF'(y)-\mu^2F(y)-\mu^2 yF'(y)\\
&= -\mu^2F(y).
\end{aligned}$$

Therefore,

$$[\mathscr{C},\mathscr{D}]=-\mu^2.$$

Comparing with (3.1.1), we obtain

$$e^{\mathscr{C}+\mathscr{D}}=e^{\frac{\mu^2}{2}}e^{\mathscr{C}}e^{\mathscr{D}}.$$

Hence, we state the more general result in the following theorem:

Theorem 3.1.2 *[18] Let $\mathscr{C}$ and $\mathscr{D}$ be two operators satisfying the commutation relations*

$$[\mathscr{C},\mathscr{D}]=t,\ \ [t,\mathscr{C}]=[t,\mathscr{D}]=0.$$

Then the Weyl identity holds:

$$e^{\mathscr{C}+\mathscr{D}} = e^{-\frac{t}{2}} e^{\mathscr{C}} e^{\mathscr{D}}. \tag{3.1.2}$$

3.2 Sack Identity

Now consider

$$\mathscr{C} = \mu y, \ \ \mathscr{D} = \mu y \frac{d}{dy}.$$

Clearly,

$$\begin{aligned} [\mathscr{C}, \mathscr{D}] \mathscr{F}(y) &= \left[\mu y \left(\mu y \frac{d}{dy} \right) - \mu y \frac{d}{dy} \mu y \right] \mathscr{F}(y) \\ &= \mu^2 y^2 \mathscr{F}'(y) - \mu y \frac{d}{dy} \mu y \mathscr{F}(y) \\ &= \mu^2 y^2 \mathscr{F}'(y) - \mu y \left[\mu \mathscr{F}(y) + \mu y \mathscr{F}'(y) \right] \\ &= \mu^2 y^2 \mathscr{F}'(y) - \mu^2 y \mathscr{F}(y) - \mu^2 y^2 \mathscr{F}'(y) \\ &= -\mu^2 y \mathscr{F}(y) \\ &= -\mu \mathscr{C}. \end{aligned}$$

Considering the operator

$$e^{\mu \left(y + y \frac{d}{dy} \right)}, \tag{3.2.1}$$

we get $p(y) = y$ and $r(y) = y$ in (2.2.2) and using (2.2.3), we obtain the following system

$$\begin{cases} \frac{d}{d\mu} y(\mu) = y(\mu) & y(0) = y \\ \frac{d}{d\mu} f(\mu) = y(\mu) f(\mu) & f(0) = 1 \end{cases}. \tag{3.2.2}$$

From the first equation of (3.2.2) we have

$$y(\mu) = y e^{\mu}.$$

From the second equation of (3.2.2), we have

$$\frac{d}{d\mu} f(\mu) = y e^{\mu} f(\mu),$$

which gives

$$f(\mu) = e^{ye^{\mu}-y}.$$

Finally, using (2.2.4) we obtain

$$e^{\mu\left(y+y\frac{d}{dy}\right)}h(y) = e^{y(e^{\mu}-1)}h(ye^{\mu}) = e^{y(e^{\mu}-1)}e^{\mu y\frac{d}{dy}}h(y).$$

Theorem 3.2.1 *Let $\mathscr{C} = \mu y$ and $\mathscr{D} = \mu y\frac{d}{dy}$ be two operators. Then we have*

$$e^{\mathscr{C}+\mathscr{D}} = e^{\frac{e^{\mu}-1}{\mu}\mathscr{C}}e^{\mathscr{D}}.$$

Proof. Using (2.2.4), we have

$$\begin{aligned}
e^{\mathscr{C}+\mathscr{D}} &= e^{\mu\left(y+y\frac{d}{dy}\right)} \\
&= e^{y(e^{\mu}-1)}e^{\mu y\frac{d}{dy}} \\
&= e^{ye^{\mu}-y}e^{\mu y\frac{d}{dy}} \\
&= e^{\frac{\mu ye^{\mu}-\mu y}{\mu}}e^{\mu y\frac{d}{dy}} \\
&= e^{\frac{\mu y(e^{\mu}-1)}{\mu}}e^{\mu y\frac{d}{dy}} \\
&= e^{\frac{e^{\mu}-1}{\mu}\mathscr{C}}e^{\mathscr{D}}.
\end{aligned}$$

This completes the proof. ■

3.3 Hausdorff Identity and Applications

The Hausdorff identity (see [21]) is as follows:

Theorem 3.3.1 *Let $\mathscr{M}$ and $\mathscr{N}$ be two operators independent of the parameter μ. Then The Hausdorff identity*

$$\begin{aligned}
&e^{\mu\mathscr{M}}\mathscr{N}e^{-\mu\mathscr{M}} \\
&= \mathscr{N} + \mu[\mathscr{M},\mathscr{N}] + \frac{\mu^2}{2!}[\mathscr{M},[\mathscr{M},\mathscr{N}]] + \frac{\mu^3}{3!}[\mathscr{M},[\mathscr{M},[\mathscr{M},\mathscr{N}]]] + \ldots
\end{aligned} \tag{3.3.1}$$

holds true.

Proof. Firstly, let us notice that $\mathscr{M}$ and $e^{\mu\mathscr{M}}$ commute, since the latter operator is a power series in $\mathscr{M}$.

From the Taylor expansion of the left-hand side of (3.3.1),

$$e^{\mu\mathscr{M}}\mathscr{N}e^{-\mu\mathscr{M}} = \sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{d^n}{d\mu^n}\left(e^{\mu\mathscr{M}}\mathscr{N}e^{-\mu\mathscr{M}}\right)|_{\mu=0}.$$

On the other hand, obviously, we have

$$\left(e^{\mu\mathscr{M}}\mathscr{N}e^{-\mu\mathscr{M}}\right)|_{\mu=0} = \mathscr{N},$$

and

$$\begin{aligned}
&\frac{d}{d\mu}\left(e^{\mu\mathscr{M}}\mathscr{N}e^{-\mu\mathscr{M}}\right)|_{\mu=0} \\
= \; &\left(e^{\mu\mathscr{M}}\mathscr{M}\mathscr{N}e^{-\mu\mathscr{M}} - e^{\mu\mathscr{M}}\mathscr{N}\mathscr{M}e^{-\mu\mathscr{M}}\right)|_{\mu=0} \\
= \; &\left(e^{\mu\mathscr{M}}[\mathscr{M},\mathscr{N}]e^{-\mu\mathscr{M}}\right)|_{\mu=0} = [\mathscr{M},\mathscr{N}],
\end{aligned}$$

hence, the other coefficients of (3.3.1) can be obtained by induction. ■

Note that for every $p \in \mathbb{N}$, we have

$$\begin{aligned}
e^{\mu\frac{d^p}{dy^p}}(1) &= \sum_{n=0}^{\infty}\frac{\mu^n}{n!}\frac{d^{np}}{dy^{np}}(1) \\
&= \left(1+\mu\frac{d^p}{dy^p}+...\right)(1) \\
&= 1. \qquad (3.3.2)
\end{aligned}$$

For $p=2$, we have

$$e^{\mu\frac{d^2}{dx^2}}y = \left(e^{\mu\frac{d^2}{dy^2}}ye^{-\mu\frac{d^2}{dy^2}}\right)e^{\mu\frac{d^2}{dy^2}}(1) = e^{\mu\frac{d^2}{dy^2}}ye^{-\mu\frac{d^2}{dy^2}}(1). \qquad (3.3.3)$$

Let $\mathscr{M} = \frac{d^2}{dy^2}$ and $\mathscr{N} = y$. For any twice differentiable function $F(y)$, we have

$$\begin{aligned}
[\mathscr{M},\mathscr{N}]F(y) &= \left[\frac{d^2}{dy^2}y - y\frac{d^2}{dy^2}\right]F(y) \\
&= \frac{d^2}{dy^2}yF(y) - yF''(y)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{d}{dy}\left[\frac{d}{dy}yF(y)\right] - yF''(y) \\
&= \frac{d}{dy}\left[F(y) + yF'(y)\right] - yF''(y) \\
&= F'(y) + F'(y) + yF''(y) - yF''(y) \\
&= 2F'(y) = 2\frac{d}{dy}(F(y)).
\end{aligned}
$$

Hence,

$$
[\mathscr{M}, \mathscr{N}] = 2\frac{d}{dy}.
$$

Similarly,

$$
[[\mathscr{M}, \mathscr{N}], \mathscr{M}] F(y) = \left[2\frac{d}{dy}\frac{d^2}{dy^2} - \frac{d^2}{dy^2}2\frac{d}{dy}\right] F(y)
$$

$$
\begin{aligned}
&= 2F'''(y) - \frac{d}{dy}\left[\frac{d}{dy}2F'(y)\right] \\
&= 2F'''(y) - \frac{d}{dy}2F''(y) \\
&= 2F'''(y) - 2F'''(y) \\
&= 0,
\end{aligned}
$$

which implies

$$
[[\mathscr{M}, \mathscr{N}], \mathscr{M}] = 0.
$$

Using (3.3.1),

$$
e^{\mu \frac{d^2}{dy^2}} y e^{-\mu \frac{d^2}{dy^2}} = y + 2\mu \frac{d}{dy} \tag{3.3.4}
$$

$$
e^{\mu \frac{d^2}{dy^2}} y = \left(y + 2\mu \frac{d}{dy}\right)(1) = y \tag{3.3.5}
$$

$$
e^{\mu \frac{d^2}{dy^2}} y^n = \left(y + 2\mu \frac{d}{dy}\right)^n (1) = y^n \tag{3.3.6}
$$

Therefore, for an analytic function

$$\mathscr{F}(y)=\sum_{n=0}^{\infty}\frac{\mathscr{F}^{(n)}(0)}{n!}y^{n},$$

we have

$$e^{\mu\frac{d^2}{dy^2}}\mathscr{F}(y)=\sum_{n=0}^{\infty}\frac{\mathscr{F}^{(n)}(0)}{n!}\left(e^{\mu\frac{d^2}{dy^2}}y^{n}\right).$$

Using (3.3.6) and (2.1.5),

$$\sum_{n=0}^{\infty}\frac{\mathscr{F}^{(n)}(0)}{n!}\left(y+2\mu\frac{d}{dy}\right)^{n}(1)=\mathscr{F}\left(y+2\mu\frac{d}{dy}\right)(1)$$

$$e^{\mu\frac{d^2}{dy^2}}\mathscr{F}(y)=\mathscr{F}\left(y+2\mu\frac{d}{dy}\right)(1). \tag{3.3.7}$$

Let us choose $\mathscr{F}(x)=e^{ax}$ in (3.3.7),

$$e^{\mu\frac{d^2}{dx^2}}e^{ax}=e^{ax+2\mu a\frac{d}{dx}}(1),$$

$$\mathscr{M}=ax,\quad \mathscr{N}=2\mu a\frac{d}{dx},$$

$$\begin{aligned}[\mathscr{M},\mathscr{N}] &= \left[ax2\mu a\frac{d}{dx}-2\mu a\frac{d}{dx}ax\right]P(x)\\ &= 2a^2\mu xP'(x)-2\mu a\left[aP(x)+axP'(x)\right]\\ &= 2a^2\mu xP'(x)-2\mu a^2P(x)-2\mu a^2xP'(x)\\ &= -2\mu a^2.\end{aligned}$$

Therefore,

$$\begin{aligned}[[\mathscr{M},\mathscr{N}],\mathscr{M}] &= \left[-\mu a^2ax+ax2\mu a^2\right]\\ &= 0\end{aligned}$$

and

$$\begin{aligned}[[\mathscr{M},\mathscr{N}],\mathscr{N}]P(x) &= \left[-2\mu a^2 2\mu a\frac{d}{dx}-2\mu a\frac{d}{dx}\left(-2\mu a^2\right)\right]P(x)\\ &= -4\mu^2a^3P'(x)-2\mu a\left[0-2\mu a^2P'(x)\right]\end{aligned}$$

$$
\begin{aligned}
&= -4\mu^2 a^3 P'(x) + 4\mu^2 a^3 P'(x) \\
&= 0.
\end{aligned}
$$

Finally, using (3.1.3) we get the following result

$$e^{\mu \frac{d^2}{dx^2}} e^{ax} = e^{ax+2\mu a \frac{d}{dx}} (1) = e^{\mu a^2} e^{ax} e^{2\mu a \frac{d}{dx}} (1) = e^{a^2 \mu + ax}.$$

More generally we have,

$$e^{\mu \frac{d^n}{dx^n}} e^{ax} = \sum_{m=0}^{\infty} \frac{\mu^m}{m!} \frac{d^{mn}}{dx^{mn}} e^{ax} = \sum_{m=0}^{\infty} \frac{\mu^m}{m!} a^{nm} e^{ax} = e^{a^n \mu + ax}. \tag{3.3.8}$$

3.4 Crofton Identity

Definition 3.4.1 *A generalization of (3.3.7) gives the Crofton identitiy which is stated as follows:*

$$e^{\mu \frac{d^t}{dy^t}} \mathscr{G}(y) = \mathscr{G}\left(y + t\mu \frac{d^{t-1}}{dy^{t-1}} \right) (1).$$

Proof. Using (3.3.3), we get

$$
\begin{aligned}
e^{\mu \frac{d^t}{dy^t}} y &= \left(e^{\mu \frac{d^t}{dy^t}} y e^{-\mu \frac{d^t}{dy^t}} \right) e^{\mu \frac{d^t}{dy^t}} (1) \\
&= e^{\mu \frac{d^t}{dy^t}} y e^{-\mu \frac{dt}{dy^t}} (1),
\end{aligned}
$$

where $\mathscr{C} = \frac{d^t}{dy^t}$ and $\mathscr{D} = y$. Then, since $[\mathscr{C}, \mathscr{D}] = t \frac{d^{t-1}}{dx^{t-1}}$ and $[\mathscr{C}, [\mathscr{C}, \mathscr{D}]] = 0$, (3.3.1) implies that

$$
\begin{aligned}
[\mathscr{C}, \mathscr{D}] F(y) &= \left[\frac{d^t}{dy^t} y - y \frac{d^t}{dy^t} \right] F(y) \\
&= \frac{d^{t-1}}{dy^{t-1}} \left[\frac{d}{dy} y F(y) \right] - y F^{(t)}(y) \\
&= \frac{d^{t-1}}{dy^{t-1}} \left[F(y) + y F'(y) \right] - y F^{(t)}(y) \\
&= \frac{d^{t-2}}{dy^{t-2}} \left[\frac{d}{dy} \left(F(y) + y F'(y) \right) \right] - y F^{(t)}(y)
\end{aligned}
$$

$$= \frac{d^{t-2}}{dy^{t-2}} \left[2F'(y) + yF''(y) \right] - yF^{(t)}(y)$$

$$\vdots$$

$$\begin{aligned} &= tF^{(t-1)}(y) + yF^{(t)}(y) - yF^{(t)}(y) \\ &= t\frac{d^{t-1}}{dy^{t-1}}(F(y)), \end{aligned}$$

and

$$\begin{aligned} [\mathscr{C}, [\mathscr{C}, \mathscr{D}]] F(y) &= \left[\frac{d^t}{dy^t} t \frac{d^{t-1}}{dy^{t-1}} - t \frac{d^{t-1}}{dy^{t-1}} \frac{d^t}{dy^t} \right] F(y) \\ &= tF^{(2t-1)}(y) - tF^{(2t-1)}(y) \\ &= 0. \end{aligned}$$

Again, using (3.3.1),

$$e^{\mu \frac{d^t}{dy^t}} y e^{-\mu \frac{d^t}{dy^t}} = y + t\mu \frac{d^{t-1}}{dy^{t-1}}$$

$$e^{\mu \frac{d^t}{dy^t}} y = \left(y + t\mu \frac{d^{t-1}}{dy^{t-1}} \right)(1) = y.$$

Similarly, by use of the same techniques, we have

$$e^{\mu \frac{d^t}{dy^t}} y^p = \left(y + t\mu \frac{d^{t-1}}{dy^{t-1}} \right)^p (1) = y^p.$$

Hence, applying the operator (3.3.2) to the Taylor expansion of an analytic function $\mathscr{G}(x)$, we can write that,

$$\begin{aligned} \mathscr{G}(y) &= \sum_{k=0}^{\infty} \frac{\mathscr{G}^{(k)}(0)}{k!} y^k \\ e^{\mu \frac{d^t}{dy^t}} \mathscr{G}(y) &= \sum_{k=0}^{\infty} \frac{\mathscr{G}^{(k)}(0)}{k!} \left(e^{\mu \frac{d^t}{dy^t}} y^k \right) \\ &= \sum_{k=0}^{\infty} \frac{\mathscr{G}^{(k)}(0)}{k!} \left(y + t\mu \frac{d^{t-1}}{dy^{t-1}} \right)^k (1) \end{aligned}$$

$$= \mathscr{G}\left(y + t\mu \frac{d^{t-1}}{dy^{t-1}}\right)(1).$$

Whence the result. ■

Theorem 3.4.2 *If $\mathscr{C}$ and $\mathscr{D}$ are two operators independent of the parameter μ with the condition $[\mathscr{C}, \mathscr{D}] = 1$, then the Crofton identity*

$$e^{\mu \mathscr{C}^t} \mathscr{G}(\mathscr{D}) = \mathscr{G}\left(\mathscr{D} + t\mu \mathscr{C}^{t-1}\right) e^{\mu \mathscr{C}^t}$$

holds.

Chapter 4

THE MONOMIALITY PRINCIPLE

4.1 Definition and Basic properties

We start to this section by giving the definition of the monomiality:

Definition 4.1.1 *[18] For the derivative operator R and the multiplication operator S, a quasi-monomial polynomial set is the set* $\{q_k(y)\}_{n\varepsilon\mathbb{N}}$ *which satisfies the following relations for all* $n\varepsilon\mathbb{N}$ *:*

$$R(q_k(y)) = kq_{k-1}(y), \quad S(q_k(y)) = q_{k+1}(y). \tag{4.1.1}$$

The commutation relation below is satisfied for the operators R and S and therefore a Weyl group structure is gained.

$$\begin{aligned} [R,S]\,q_k(y) &= R(S(q_k(y))) - S(R(q_k(y))) \\ &= R(q_{k+1}(y)) - S(kq_{k-1}(y)) \\ &= (k+1)\,q_k(y) - kq_k(y) \\ &= kq_k(y) + q_k(y) - kq_k(y) \\ &= 1q_k(y). \end{aligned} \tag{4.1.2}$$

If the considered polynomial set $\{q_k(y)\}$ is quasi-monomial, its properties can easily be derived from the operators R and S. For instance,

(i) if R and S have a differential realization, then the polynomial $q_k(y)$ satisfies the differential equation

$$\begin{aligned} SR\left(q_k\left(y\right)\right) &= S\left(kq_{k-1}\left(y\right)\right) \\ &= kq_k\left(y\right). \end{aligned} \tag{4.1.3}$$

(ii) Let $q_0\left(y\right)=1$, then $q_k\left(y\right)$ can be explicitly composed as

$$\begin{aligned} q_1\left(y\right) &= R^1\left(1\right) \\ q_2\left(y\right) &= R\left(R^1\left(1\right)\right)=R^2\left(1\right) \\ &\vdots \\ q_k\left(y\right) &= R^k\left(1\right), \end{aligned} \tag{4.1.4}$$

(iii) the last identity in (4.1.4) shows that the exponential generating function of $q_k\left(y\right)$ can be stated as

$$\begin{aligned} e^{hR}\left(1\right) &= \sum_{k=0}^{\infty}\frac{\left(hR\right)^k}{k!}\left(1\right)=\sum_{k=0}^{\infty}\frac{h^k}{k!}R^k\left(1\right) \\ &= \sum_{k=0}^{\infty}\frac{h^k}{k!}q_k\left(y\right), \end{aligned}$$

and therefore,

$$e^{hR}\left(1\right)=\sum_{k=0}^{\infty}\frac{h^k}{k!}q_k\left(y\right). \tag{4.1.5}$$

4.2 Construction of the Derivative and Multiplication Operators

Theorem 4.2.1 *[18] The relevant exponential generating function $Z\left(t,y\right)$, corresponding to the quasi-monomial set $q_k\left(y\right)$ w.r.t. the operators R and S, satisfies the following condition:*

$$RZ\left(t,y\right)=tZ\left(t,y\right)$$

or equivalently:

$$Re^{tS}\left(1\right)=te^{tS}\left(1\right).$$

Proof. In fact, we can write

$$\begin{aligned} RZ(t,y) &= Re^{tS}(1) = R\sum_{k=0}^{\infty}\frac{(tS)^k}{k!} \\ &= R\sum_{k=0}^{\infty}\frac{t^k}{k!}q_k(y) = \sum_{k=0}^{\infty}\frac{t^k}{k!}R(q_k(y)) \\ &= \sum_{k=0}^{\infty}\frac{t^k}{k!}kq_{k-1}(y) = t\sum_{k=1}^{\infty}\frac{t^{k-1}}{k(k-1)!}kq_{k-1}(y) \\ &= t\sum_{k=0}^{\infty}\frac{t^k}{k!}q_k(y) = te^{tS}(1) = tZ(t,y). \end{aligned}$$

■

Now, we aim to extend the concept of quasi-monomiality to more general setting. Let us consider $\{q_k(y)\}_{k\varepsilon\mathbb{N}}$ with $q_0 = 1$ as a quasi-monomial family and R_0 denotes the corresponding derivative operator.

Theorem 4.2.2 *[18] Assume that there exists an operator Ψ commuting with R_0 such that*

$$e^{z\Psi}(q_k(y)) = \Phi_k(y,z), \tag{4.2.1}$$

and moreover, for a suitable operator $S_1(z)$, it satisfies the condition

$$e^{z\Psi}(q_k(y)) = \Phi_k(y,z) = (S_1(z))^k(1), \tag{4.2.2}$$

where $[R_0, S_1(z)] = 1$ for all z. Then the polynomial family $\{\Phi_k(y,z)\}_{k\varepsilon\mathbb{N}}$ is quasi-monomial with respect to the operators $R_1 \equiv R_0$ and $S_1(z)$.

Proof. In fact, since R_0 commutes with Ψ, it also commutes with $e^{z\Psi}$, so that

$$\begin{aligned} R_0(\Phi_k(y,z)) &= R_0e^{z\Psi}(q_k(y)) = e^{z\Psi}R_0(q_k(y)) \\ &= ke^{z\Psi}(q_{k-1}(y)) = k\Phi_{k-1}(y,z) \end{aligned}$$

and the operator $R_1 \equiv R_0$ satisfy the first monomiality condition. Furthermore, we obviously have

$$S_1(z)(\Phi_k(y,z)) = S_1(z)(S_1(z))^k(1) = (S_1(z))^{k+1} = \Phi_{k+1}(y,z)$$

and the second condition also holds. ■

Theorem 4.2.3 *Consider the polynomial set* $\{q_k(y)\}_{k\varepsilon\mathbb{N}}$ *with* $q_0 = 1$*, and assume that this family is quasi-monomial with respect to the operators* R_0 *and* S_0*. Also consider an operator* Ψ *satisfying* $[\Psi, R_0] = 0$ *and* $e^{z\Psi}(1) = 1$*, and set*

$$\Phi_k(y,z) = e^{z\Psi}(q_k(y)).$$

Then the polynomial family $\{\Phi_k(y,z)\}_{k\varepsilon\mathbb{N}}$ *has the "derivative operator"* $R_1 \equiv R_0$*. Moreover, the "multiplication operator"* $S_1(z)$ *is given by*

$$S_1(z) = S_0 + z[\Psi, S_0] + \frac{z^2}{2!}[\Psi, [\Psi, S_0]] + \ldots \tag{4.2.3}$$

Theorem 4.2.4 *Consider the polynomial set* $\left\{q_k\left(y, z_1, ..., z_j\right)\right\}_{k\varepsilon\mathbb{N}}$*,* $q_0 = 1$*, and assume that this family is quasi-monomial with respect to the operators* R_j *and* $S_j = S_j\left(z_1, ..., z_j\right)$*. Consider an operator* Φ *satisfying* $\left[\Phi, R_j\right] = 0$*, and* $e^{z_{j+1}\Phi}(1) = 1$*, with*

$$C_k\left(y, z_1, ..., z_j\right) = e^{z_{j+1}\Phi} q_k\left(y, z_1, ..., z_j\right).$$

Then the polynomial family $\left\{C_k\left(y, z_1, ..., z_{j+1}\right)\right\}_{n\varepsilon\mathbb{N}}$ *has the "derivative operator"* $R_{j+1} \equiv R_j$*. Moreover, the "multiplication operator"* $S_{j+1} = S_{j+1}\left(z_1, ..., z_{j+1}\right)$ *is given by*

$$S_{j+1}\left(z_1, ..., z_{j+1}\right) = S_j + z_j\left[\Phi, S_j\right] + \frac{z_j^2}{2!}\left[\Phi, \left[\Phi, S_j\right]\right] + ... \tag{4.2.4}$$

Proof. Recalling (4.0.3), we have $q_k\left(y, z_1, ..., z_j\right) = S_j^k\left(y_1, ..., y_j\right)(1)$, and consequently (4.1.1) can be written as follows:

$$e^{z_{j+1}\Phi} S_j^k\left(y_1 ..., y_j\right)(1) = C_k\left(y, z_1, ..., z_{j+1}\right).$$

Applying the Hausdorff identity, we find

$$e^{z_{j+1}\Phi} S_j(z_1, \ldots, z_k) = \left(e^{z_{j+1}\Phi} S_j\left(z_1, ..., z_j\right) e^{-z_{j+1}\Phi}\right) e^{z_{j+1}\Phi}(1)$$

and

$$e^{z_{j+1}\Phi} S_j\left(z_1, ..., z_j\right) e^{-z_{j+1}\Phi} = S_j + z_j\left[\Phi, S_j\right] + \frac{z_j^2}{2!}\left[\Phi, \left[\Phi, S_j\right]\right] + ...$$

$$= S_{j+1}\left(z_1,...,z_{j+1}\right).$$

Therefore

$$\begin{aligned}
e^{z_{j+1}\Phi}S_j\left(z_1,...,z_j\right) &= S_{j+1}\left(z_1,...,z_{j+1}\right) = q_{j+1} \\
e^{z_{j+1}\Phi}S_j^2\left(z_1,...,z_j\right) &= e^{z_{j+1}\Phi}S_j\left(q_{j+1}\right) = S_{j+1}\left(q_{j+1}\right) = q_{j+1}^2 \\
&\vdots \\
e^{z_{j+1}\Phi}S_j^k\left(z_1,...,z_j\right) &= \left(S_{j+1}(z_1,...,z_{j+1})\right)^k(1) = C_k\left(y,z_1,...,z_{j+1}\right).
\end{aligned}$$

This completes the proof. ■

Remark 4.2.5 *Let A denote the space of analytic functions. The monomial set* $\left\{y^k\right\}$ *can be transformed into the set* $\left\{\frac{y^k}{k!}\right\}$ *by substituting the Laguerre derivative* $\mathscr{D}$ *with its antiderivative* $\mathscr{D}_y^{-1}$ *defined as below:*

$$\mathscr{D}_y^{-k}(1) = \frac{y^k}{k!}, \ \ k = 0,1,2,... \tag{4.2.5}$$

The linear transformation T is denoted as a differential isomorphism; since it preserves linear differential operators.

We have,

$$\begin{aligned}
\frac{d}{dy}y\frac{d}{dy}\frac{y^k}{k!} &= \frac{d}{dy}yk\frac{y^{k-1}}{k!} \\
&= \frac{d}{dy}k\frac{y^k}{k!} = \frac{ky^{k-1}}{(k-1)!},
\end{aligned} \tag{4.2.6}$$

which corresponds to

$$\frac{d}{dy}y^k = ky^{k-1}, \tag{4.2.7}$$

and furthermore

$$\mathscr{D}_y^{-1}\left(\frac{y^k}{k!}\right) = \frac{y^{k+1}}{(k+1)k!} = \frac{y^{k+1}}{(k+1)!} \tag{4.2.8}$$

which corresponds to

$$yy^k = y^{k+1}. \tag{4.2.9}$$

4.3 $t-$Variable Monomiality Principle

Definition 4.3.1 *An t-variable, t-index polynomial family* $\left\{q_{k_1,\dots,k_t}(y_1,\dots,y_t)\right\}$ *is said to be quasi-monomial if* $2t$ *operators* $R_{y_1},\dots,R_{y_t}$*, and* $S_{y_1},\dots,S_{y_t}$ *exist such that*

$$\begin{cases} R_{y_1} q_{k_1,\dots k_t}(y_1,\dots,y_t) = k_1 q_{k_1-1,k_2,\dots,k_t}(y_1,\dots,y_t), \\ \vdots \\ R_{y_t} q_{k_1,\dots,k_t}(y_1,\dots y_t) = k_t q_{k_1,\dots,k_t-1}(y_1,\dots,y_t), \end{cases}$$

$$\begin{cases} S_{y_1} q_{k_1,\dots,k_t}(y_1,\dots,y_t) = q_{k_1+1,k_2,\dots,k_t}(y_1,\dots,y_t), \\ \vdots \\ S_{y_t} q_{k_1,\dots,k_t}(y_1,\dots,y_t) = q_{k_1,\dots,k_t+1}(y_1,\dots,y_t). \end{cases}$$

From the above formulas it follows that

$$\begin{aligned} & [R_{y_1}, S_{y_1}]\, q_{k_1,\dots,k_t}(y_1,\dots,y_t) \\ = \;& R_{y_1}\left(S_{y_1}\left(q_{k_1,\dots,k_t}(y_1,\dots,y_t)\right)\right) - S_{y_1}\left(R_{y_1}\left(q_{k_1,\dots,k_t}(y_1,\dots,y_t)\right)\right) \\ = \;& R_{y_1}\left(q_{k_1+1,k_2,\dots k_t}(y_1,\dots y_t)\right) - S_{y_1}\left(k_1 q_{k_1-1,k_2,\dots,k_t}(y_1,\dots y_t)\right) \\ = \;& (k_1+1)\, q_{k_1,k_2,\dots,k_t}(y_1,\dots y_t) - k_1 q_{k_1,k_2,\dots,k_t}(y_1,\dots,y_t) \\ = \;& 1 q_{k_1,k_2,\dots k_t}(y_1,\dots y_t), \end{aligned}$$

which gives

$$[R_{y_1}, S_{y_1}] = 1, \cdots, [R_{y_t}, S_{y_t}] = 1.$$

One can observe that the main properties of a polynomial family can be gained using the above relations.

(i) Having a differential realization between the derivative and multiplication operators, we get:

$$\begin{aligned} & S_{y_1} R_{y_1} q_{k_1,\dots k_t}(y_1,\dots y_t) \\ = \quad & S_{y_1}\left[k_1 q_{k_1-1,k_2,\dots,k_t}(y_1,\dots,y_t)\right] = k_1 q_{k_1,k_2,\dots,k_t}(y_1,\dots y_t) \end{aligned}$$

$$\vdots$$

$$\begin{aligned} & S_{y_t} R_{y_t} q_{k_1,\dots,k_t}(y_1,\dots,y_t) \\ = \quad & S_{y_t}\left[k_t q_{k_1,\dots,k_{t-1}}(y_1,\dots,y_t)\right] = k_t q_{k_1,\dots,k_t}(y_1,\dots,y_t) \end{aligned}$$

i.e., we find t (independent) differential equations satisfied by the polynomial family.

(ii) Let $q_{0,\dots 0}(y_1,\dots y_t) \equiv 1$, the explicit expression of $\left\{q_{k_1,\dots,k_t}(y_1,\dots y_t)\right\}$ is given by

$$q_{k_1,\dots,k_t}(y_1,\dots y_t) = S_{y_1}^{k_1}\cdots S_{y_t}^{k_t}(1)$$

(iii) The exponential generating function of $\left\{q_{k_1,\dots,k_t}(y_1,\dots y_t)\right\}$, assuming again

$$q_{0,\dots 0}(y_1,\dots y_t) \equiv 1,$$

is given by

$$\begin{aligned} e^{z_1 S_{y_1}+\cdots+z_t S_{y_t}}(1) &= \sum_{k_1=0}^{\infty}\cdots\sum_{k_t=0}^{\infty}\frac{(z_1 S_{y_1})^{k_1}\dots(z_t S_{y_t})^{k_t}}{k_1!k_2!\dots k_t!} \\ &= \sum_{k_1=0}^{\infty}\cdots\sum_{k_t=0}^{\infty}\frac{z_1^{k_1}}{k_1!}\frac{z_2^{k_2}}{k_2!}\cdots\frac{z_t^{k_t}}{k_t!}\left(S_{y_1}^{k_1}\cdots S_{y_t}^{k_t}\right)(1) \\ &= \sum_{k_1=0}^{\infty}\cdots\sum_{k_t=0}^{\infty}\frac{z_1^{k_1}}{k_1!}\frac{z_2^{k_2}}{k_2!}\cdots\frac{z_t^{k_t}}{k_t!}q_{k_1,\dots k_t}(y_1,\dots y_t). \end{aligned}$$

Theorem 4.3.2 *Let $\mathscr{B}_1,\dots,\mathscr{B}_t$ be commuting operators (i.e., $\left[\mathscr{B}_i,\mathscr{B}_j\right]=0$ for all i,j) independent of the parameters $z_1,\dots z_t$. Then the Hausdorff identity holds:*

$$
\begin{aligned}
& e^{z_1\mathscr{B}_1+\cdots+z_t\mathscr{B}_t}\mathscr{K}\,e^{-z_1\mathscr{B}_1-\cdots-z_t\mathscr{B}_t} \\
= \;& \mathscr{K}+\left(\sum_{i=0}^{t} z_i\,[\mathscr{B}_i,\mathscr{K}]\right)+\frac{1}{2!}\left(\sum_{i,j=0}^{t} z_i z_j\,[\mathscr{B}_i,[\mathscr{B}_j,\mathscr{K}]]\right) \\
& +\frac{1}{3!}\left(\sum_{i,j,k=0}^{t} z_i z_j z_k\,[\mathscr{B}_i,[\mathscr{B}_j,[\mathscr{B}_k,\mathscr{K}]]]\right)+\cdots
\end{aligned}
$$

Proof. By using Hausdorff identity,

$$
\begin{aligned}
& e^{z_1\mathscr{B}_1+\ldots+z_t\mathscr{B}_t}\mathscr{K}\,e^{-z_1\mathscr{B}_1-\ldots-z_t\mathscr{B}_t} \\
= \;& \sum_{k=0}^{\infty}\frac{(z_1\ldots z_t)^k}{k!}\frac{\partial^k}{\partial z^k}\left(e^{z_1\mathscr{B}_1+\cdots+z_t\mathscr{B}_t}\mathscr{K}\,e^{-z_1\mathscr{B}_1-\cdots-z_t\mathscr{B}_t}\right)|_{z=0}
\end{aligned}
$$

$$
\left(e^{z_1\mathscr{B}_1+\ldots+z_t\mathscr{B}_t}\mathscr{K}\,e^{-z_1\mathscr{B}_1+\ldots+z_t\mathscr{B}_t}\right)|_{z=0}=\mathscr{K},
$$

$$
\begin{aligned}
& \frac{\partial}{\partial z}\left(e^{z_1\mathscr{B}_1+\ldots+z_t\mathscr{B}_t}\mathscr{K}\,e^{-z_1\mathscr{B}_1+\ldots+z_t\mathscr{B}_t}\right)|_{z=0} \\
= \;& [(\mathscr{B}_1+\ldots+\mathscr{B}_t)\,e^{z_1\mathscr{B}_1+\ldots+z_t\mathscr{B}_t}\mathscr{K}\,e^{-z_1\mathscr{B}_1-\ldots-z_t\mathscr{B}_t} \\
& -e^{z_1\mathscr{B}_1+\ldots+z_t\mathscr{B}_t}\mathscr{K}\,(\mathscr{B}_1+\ldots+\mathscr{B}_t)\,e^{-z_1\mathscr{B}_1-\ldots-z_t\mathscr{B}_t}]|_{z=0} \\
= \;& \sum_{i=0}^{t}[\mathscr{B}_i,\mathscr{K}]
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial^2}{\partial z^2}\left(e^{z_1\mathscr{B}_1+\ldots+z_t\mathscr{B}_t}\mathscr{K}\,e^{-z_1\mathscr{B}_1-\ldots-z_t\mathscr{B}_t}\right) \\
= \;& [(\mathscr{B}_1+\cdots+\mathscr{B}_t)\,e^{z_1\mathscr{B}_1+\cdots+z_t\mathscr{B}_t}(\mathscr{B}_1+\cdots+\mathscr{B}_t)\,\mathscr{K}\,e^{-z_1\mathscr{B}_1-\cdots-z_t\mathscr{B}_t} \\
& -e^{z_1\mathscr{B}_1+\cdots+z_t\mathscr{B}_t}\mathscr{K}\,(\mathscr{B}_1+\mathscr{B}_2+\ldots+\mathscr{B}_t)\,e^{-z_1\mathscr{B}_1-\cdots-z_t\mathscr{B}_t}(\mathscr{B}_1+\mathscr{B}_2+\ldots+\mathscr{B}_t)] \\
= \;& \sum_{i,k=0}^{t}[\mathscr{B}_i,[\mathscr{B}_k,\mathscr{K}]],
\end{aligned}
$$

$$
\begin{aligned}
& e^{z_1\mathscr{B}_1+\cdots+z_t\mathscr{B}_t}\mathscr{K}\,e^{-z_1\mathscr{B}_1-\cdots-z_t\mathscr{B}_t} \\
= \;& \mathscr{K}+\left(\sum_{i=0}^{t} z_i\,[\mathscr{B}_i,\mathscr{K}]\right)+\frac{1}{2!}\left(\sum_{i,j=0}^{t} z_i z_j\,[\mathscr{B}_i,[\mathscr{B}_j,\mathscr{K}]]\right) \\
& +\frac{1}{3!}\left(\sum_{i,j,k=0}^{t} z_i z_j z_k\,[\mathscr{B}_i,[\mathscr{B}_j,[\mathscr{B}_k,\mathscr{K}]]]\right)+\cdots
\end{aligned}
$$

This completes the proof. ■

Theorem 4.3.3 *Consider t operators $\Psi_{y_1},\ldots,\Psi_{y_t}$ commuting respectively with $R_{y_1},\ldots,R_{y_t}$, and set*

$$\mathscr{O}_{k_1,\ldots k_t}(y_1,\ldots,y_t;z_1,\ldots,z_t)=e^{z_1\Psi_{y_1}+\cdots+z_t\Psi_{y_t}}p_{k_1,\ldots,k_t}(y_1,\ldots,y_t).$$

Assume that there exist t operators $S_{1,y_1}(z_{1,\ldots,}z_t),\ldots,S_{1,y_t}(z_1,\ldots,z_t)$ such that

$$\mathscr{O}_{k_1\ldots,k_t}(y_1,\ldots y_t;z_1,\ldots,z_t)=\left(S_{1,y_1}(z_1,\ldots,z_t)\right)^{k_1}\left(S_{1,y_t}(z_1,\ldots,z_t)\right)^{k_t}(1)$$

and, furthermore, for all $z_{1,\cdots,}z_t$,

$$\left[R_{y_1},S_{1,y_1}(z_1,\ldots,z_t)\right]=\cdots=\left[R_{y_t},S_{1,y_t}(z_1,\ldots,z_t)\right]=1.$$

Then the polynomial family $\mathscr{O}_{k_1\ldots,k_t}(y_1,\ldots y_t;z_1,\ldots,z_t)$ is quasi-monomial with respect to the operators

$$R_{y_1},\cdots R_{y_t},S_{1,y_1}(z_1,\ldots,z_t),\ldots,S_{1,y_t}(z_1,\ldots,z_t).$$

Proof. It is straightforward that

$$\begin{aligned}
R\left(\mathscr{O}_{k_1\ldots,k_t}(y_1,\ldots y_t;z_1,\ldots,z_t)\right) &= Re^{z_1\Psi_{y_1}+\cdots+z_t\Psi_{y_t}}q_{k_1,\cdots,k_t}(y_1,\ldots,y_t)\\
&= k_1e^{z_1\Psi_{y_1}+\cdots+z_t\Psi_{y_t}}q_{k_1-1,\ldots,k_t}(y_1,\ldots,y_t)\\
&= k_1\mathscr{O}_{k_1-1,\ldots,k_t}(y_{1,\cdots,}y_t;z_1,\cdots,z_r),
\end{aligned}$$

and

$$\begin{aligned}
&\left(S_{1,y_1}(z_1,\cdots,z_t)\right)\mathscr{O}_{k_1,\ldots,k_t}(y_{1,\cdots,}y_t;z_{1,\cdots,}z_t)\\
=&\left(S_{1,y_1}(z_1,\cdots,z_t)\right)\left[\left(S_{1,y_1}(z_1,\cdots,z_t)\right)^{k_1}\cdots\left(S_{1,y_t}(z_1,\cdots,z_t)\right)^{k_t}\right](1)\\
=&\left[\left(S_{1,y_1}(z_1,\cdots,z_t)\right)^{k_1+1}\cdots\left(S_{1,y_t}(z_1,\cdots,z_t)\right)^{k_t}\right](1)\\
=&\mathscr{O}_{k_1+1,\cdots,k_t}(y_{1,\cdots,}y_t;z_1,\cdots,z_t).
\end{aligned}$$

■

Theorem 4.3.4 *[18] Consider the quasi-monomial set* $\left\{q_{k_1,\ldots,k_t}(y_1,\ldots y_t)\right\}$ *w.r.t. the operators* $R_{y_1},\ldots,R_{y_t}$, *and* $S_{y_1},\ldots,S_{y_t}$. *Set* $q_{0,\ldots,0}(y_1,\ldots,y_t)\equiv 1$ *in the polynomial family* $\left\{q_{k_1,\ldots,k_t}(y_1,\ldots y_t)\right\}$. *For the operators* $\Psi_{y_1,\ldots,}\Psi_{y_t}$, *independent of the parameters* $z_1,\ldots,z_t$, *assume that:*

$$[\Psi_{y_1},R_{y_1}]=\cdots=[\Psi_{y_t},R_{y_t}]=0,\ e^{z_1\Psi_{y_1}+\cdots+z_t\Psi_{y_t}}(1)=1.$$

Define the polynomial set

$$\mathscr{O}_{k_1,\ldots,k_t}(y_1,\ldots,y_t;z_1,\ldots,z_t)=e^{z_1\Psi_{y_1}+\cdots+z_t\Psi_{y_t}}q_{k_1,\ldots,k_t}(y_1,\ldots,y_t).$$

Then, the Hausdorff expansions below results the desired multiplication operators.

$$\begin{aligned} S_{1,y_1} &= S_{y_1}+\left\{z_1[\Psi_{y_1},S_{y_1}]+\cdots+z_t[\Psi_{y_t},S_{y_1}]\right\} \\ &\quad +\frac{1}{2!}\left(\sum_{i,j=0}^{t} z_iz_j\left[\Psi_{y_i},\left[\Psi_{y_j},S_{y_1}\right]\right]\right)+\cdots, \end{aligned}$$

$$\vdots$$

$$\begin{aligned} S_{1,y_t} &= S_{y_t}+\left\{z_1[\Psi_{y_1},S_{y_t}]+\cdots+z_t[\Psi_{y_t},S_{y_t}]\right\} \\ &\quad +\frac{1}{2!}\left(\sum_{i,j=0}^{t} z_iz_j\left[\Psi_{y_i},\left[\Psi_{y_j},S_{y_t}\right]\right]\right)+\cdots. \end{aligned}$$

Proof. Clearly,

$$q_{k_1,\ldots,k_t}(y_1,\ldots,y_t)=S_{y_1}^{k_1},\ldots,S_{y_t}^{k_t}(1)$$

$$e^{z_1\Psi_{y_1}+\cdots+z_t\Psi_{y_t}}q_{k_1,\ldots,k_t}(y_1,\ldots,y_t)=\mathscr{O}_{k_1,\ldots,k_t}(y_1,\ldots,y_t;z_1,\ldots,z_t).$$

Applying the Hausdorff identity, we find

$$\begin{aligned} & e^{z_1\Psi_{y_1}+\cdots+z_t\Psi_{y_t}}S_{y_1},\ldots,S_{y_t} \\ = & \left(e^{z_1\Psi_{y_1}+\cdots+z_t\Psi_{y_t}}S_{y_1},\ldots,S_{y_t}e^{-z_1\Psi_{y_1}-\cdots-z_t\Psi_{y_t}}\right)e^{z_1\Psi_{y_1}+\cdots+z_t\Psi_{y_t}}(1) \end{aligned}$$

$$
\begin{aligned}
& e^{z_1\Psi_{y_1}+\cdots+z_t\Psi_{y_t}} S_{y_1} e^{-z_1\Psi_{y_1}-\cdots-z_t\Psi_{y_t}} \\
= \; & S_{y_1} + \left(\sum_{i=0}^{t} z_i \left[\Psi_i, S_{y_1}\right]\right) + \frac{1}{2!}\left(\sum_{i,j=0}^{t} z_i, z_j \left[\Psi_i, \left[\Psi_j, S_{y_1}\right]\right]\right) + \cdots \\
= \; & S_{1,y_1}
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
& e^{z_1\Psi_{y_1}+\cdots+z_t\Psi_{y_t}} S_{y_t} e^{-z_1\Psi_{y_1}-\cdots-z_t\Psi_{y_t}} \\
= \; & S_{y_t} + \left(\sum_{i=0}^{t} z_i \left[\Psi_i, S_{y_t}\right]\right) + \frac{1}{2!}\left(\sum_{i,j=0}^{t} z_i, z_j \left[\Psi_i, \left[\Psi_j, S_{y_t}\right]\right]\right) + \cdots \\
= \; & S_{1,y_t}.
\end{aligned}
$$

The proof is completed. ■

Chapter 5

APPLICATIONS

5.1 Hermite Polynomials in Two Variables

Firstly, let us give the definition of Hermite polynomials in two variables which is due to P. Appell and J. Kampe de Feriet [2]and followed by G. Dattoli et al. [10].

Definition 5.1.1 (i) *The Hermite polynomials in two variables* $\mathscr{H}_k^{(1)}(a,b)$ *are simply the powers defined by*

$$\mathscr{H}_k^{(1)}(a,b) = (a+b)^k. \tag{5.1.1}$$

(ii) The Hermite polynomials in two variables $\mathscr{H}_k^{(2)}(a,b)$ are defined by

$$\mathscr{H}_k^{(2)}(a,b) = \sum_{p=0}^{\left[\frac{k}{2}\right]} \frac{k!}{p!\,(k-2p)!} b^p a^{k-2p}. \tag{5.1.2}$$

(iii) The t^{th} order Hermite polynomials in two variables $\mathscr{H}_k^{(t)}(a,b)$ are defined by

$$\mathscr{H}_k^{(t)}(a,b) = \sum_{p=0}^{\left[\frac{k}{t}\right]} \frac{k!}{p!\,(k-pt)!} b^p a^{k-pt}. \tag{5.1.3}$$

Now, setting $\mathscr{D} = \frac{d}{da}$ we get from (2.1.1) that

$$e^{b\mathscr{D}}\mathscr{G}(a) = \sum_{k=0}^{\infty} \frac{b^k}{k!}\mathscr{G}^{(k)}(a) = \mathscr{G}(a+b). \tag{5.1.4}$$

Remark 5.1.2 (i) *If we choose* $\mathscr{G}(a) = a^k$ *in (5.1.4), we have*

$$\begin{aligned} e^{bD}a^k &= \sum_{t=0}^{\infty} \frac{b^t}{t!}\mathscr{D}^t a^k \\ &= \sum_{t=0}^{k} \frac{b^t}{t!}\frac{k!}{(k-t)!}a^{k-t} \\ &= (a+b)^k. \end{aligned}$$

(ii) *If we choose* $\mathscr{G}(a) = \sum\limits_{k=0}^{\infty} c_k a^k$ *in (5.1.4), we get*

$$e^{bD}\mathscr{G}(a) = \sum_{k=0}^{\infty} c_k e^{b\mathscr{D}} a^k = \sum_{k=0}^{\infty} c_k (a+b)^k.$$

(iii) If $\mathscr{G}(a) = \sum_{k=0}^{\infty} c_k a^k$ *in (5.1.4), then*

$$e^{bD}\mathscr{G}(a) = \sum_{k=0}^{\infty} c_k a^k e^{b\mathscr{D}} = \sum_{k=0}^{\infty} c_k (a+b)^k = \sum_{k=0}^{\infty} c_k \mathscr{H}_k^{(1)}(a,b)$$

$$e^{bD}\mathscr{G}(a) = \sum_{k=0}^{\infty} c_k \mathscr{H}_k^{(1)}(a,b).$$

Taking into account the exponential operator with second derivative, we have

$$e^{bD^2}\mathscr{G}(a) = \sum_{k=0}^{\infty} \frac{b^k}{k!} \frac{d^{2k}}{da^{2k}} \mathscr{G}(a) = \sum_{k=0}^{\infty} \frac{b^k}{k!} \mathscr{G}^{(2k)}(a)$$

$$e^{bD^2}\mathscr{G}(a) = \sum_{k=0}^{\infty} \frac{b^k}{k!} \mathscr{G}^{(2k)}(a). \tag{5.1.5}$$

(iv) If $\mathscr{G}(a) = a^k$ *then for* $p = 0,1,2\ldots,\left[\frac{k}{2}\right]$ *it follows that*

$$\begin{aligned} e^{b\mathscr{D}^2} a^k &= \sum_{p=0}^{\left[\frac{k}{2}\right]} \frac{b^p}{p!} \mathscr{D}^{2p} a^k \\ &= \sum_{p=0}^{\left[\frac{k}{2}\right]} \frac{b^p}{p!} \frac{k!}{(k-2p)!} a^{k-2p} = H_k^{(2)}(a,b). \end{aligned} \tag{5.1.6}$$

(v) If $\mathscr{G}(a) = \sum_{k=0}^{\infty} c_k a^k$*, then*

$$\begin{aligned} e^{b\mathscr{D}^2}\mathscr{G}(a) &= \sum_{k=0}^{\infty} c_k a^k e^{b\mathscr{D}^2} = \sum_{k=0}^{\infty} c_k \sum_{p=0}^{\left[\frac{k}{2}\right]} \frac{k!}{p!\,(k-2p)!} a^{k-2p} b^p \\ &= \sum_{k=0}^{\infty} c_k \mathscr{H}_k^{(2)}(a,b). \end{aligned}$$

More generally,

$$e^{bD^t}\mathscr{G}(a) = \sum_{p=0}^{\infty} \frac{b^p}{p!} \mathscr{G}^{(tp)}(a), \tag{5.1.7}$$

and hence

(vi) If $\mathscr{G}(a) = a^k$ *then for* $p = 0,1,\ldots\left[\frac{k}{t}\right]$

$$
\begin{aligned}
e^{b\mathscr{D}^t} a^k &= \sum_{p=0}^{\left[\frac{k}{t}\right]} \frac{b^p}{p!} \mathscr{D}^{(pt)} a^k \\
&= \sum_{p=0}^{\left[\frac{k}{t}\right]} \frac{b^p}{p!} \frac{k!}{(k-pt)!} a^{k-pt}. \qquad (5.1.8)
\end{aligned}
$$

(vii) If $\mathscr{G}(a) = \sum\limits_{k=0}^{\infty} c_k a^k$

$$
\begin{aligned}
e^{b\mathscr{D}^t} \mathscr{G}(a) &= \sum_{k=0}^{\infty} c_k a^k e^{b\mathscr{D}^t} = \sum_{k=0}^{\infty} c_k \sum_{p=0}^{\left[\frac{k}{t}\right]} \frac{k!}{p!\,(k-pt)!} b^p a^{k-pt} \\
&= \sum_{k=0}^{\infty} c_k \mathscr{H}_k^{(t)}(a,b).
\end{aligned}
$$

Remark 5.1.3 *Note that, taking into account the $t-th$ iteration of power, we have for $G(a) = \sum\limits_{k=0}^{\infty} c_k a^k$ that ([5])*

$$
\left(\left(\left(e^{b\mathscr{D}}\right)^{b\mathscr{D}}\right)^{\cdots}\right)^{b\mathscr{D}} \mathscr{G}(a) = e^{b^t\mathscr{D}^t} \mathscr{G}(a) = \sum_{p=0}^{\infty} \frac{b^{pt}}{p!} \mathscr{G}^{(pt)}(a)
$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} c_k e^{b^t\mathscr{D}^t} a^k = \sum_{k=0}^{\infty} c_k \sum_{p=0}^{\left[\frac{k}{t}\right]} \frac{k!}{p!\,(k-pt)!} b^{pt} a^{k-pt} \\
&= \sum_{k=0}^{\infty} c_k \mathscr{H}_k^{(t)}\left(a, b^t\right). \qquad (5.1.9)
\end{aligned}
$$

5.1.1 Second-Level Exponentials

The second level exponentials are operators of the type

$$
e^{\left(e^{b\mathscr{D}}\right)}, \quad |b| < 1, \qquad (5.1.10)
$$

with

$$
\begin{aligned}
e^{\left(e^{b\mathscr{D}}\right)} \mathscr{H}(a) &= \sum_{l=0}^{\infty} \frac{\left(e^{b\mathscr{D}}\right)^l}{l!} \mathscr{H}(a) = \sum_{l=0}^{\infty} \frac{e^{lb\mathscr{D}}}{l!} \mathscr{H}(a) \\
&= \sum_{l=0}^{\infty} \frac{1}{l!} \mathscr{H}(a+lb). \qquad (5.1.11)
\end{aligned}
$$

A result relevant to this subject can be found in [7]. A different result is obtained by considering the series

$$
\sum_{p=0}^{\infty} \frac{e^{b^p\mathscr{D}^p}}{p!} \mathscr{H}(a).
$$

In fact assuming $\mathscr{H}(a)=\sum\limits_{k=0}^{\infty}c_k a^k$ and using Remark (5.1.3),

$$\begin{aligned}\sum_{p=0}^{\infty}\frac{e^{b^p\mathscr{D}^p}}{p!}\mathscr{H}(a) &= \sum_{p=0}^{\infty}\frac{1}{p!}\sum_{k=0}^{\infty}c_k\mathscr{H}_k^{(p)}(a,b^p)\\ &= \sum_{k=0}^{\infty}c_k\sum_{p=0}^{\infty}\frac{1}{p!}\mathscr{H}_k^{(p)}(a,b^p).\end{aligned}\tag{5.1.12}$$

If we choose $\mathscr{H}(a)=a^k$, for $p=0,1,\ldots,\left[\frac{k}{p}\right]$, we have

$$\begin{aligned}e^{b^t\mathscr{D}^p}a^k &= \sum_{t=0}^{\left[\frac{k}{p}\right]}\frac{b^{tp}}{t!}\mathscr{D}^{(tp)}a^k\\ &= \sum_{t=0}^{\left[\frac{k}{p}\right]}\frac{k!}{(k-tp)!t!}b^{tp}a^{k-tp}\end{aligned}$$

and, therefore

$$\mathscr{H}_k^{(p)}(a,b^p)=\sum_{t=0}^{\left[\frac{k}{p}\right]}\frac{k!}{t!(k-tp)!}b^{tp}a^{k-tp}.\tag{5.1.13}$$

5.1.2 Connection with the Heat Problem

The polynomials $\mathscr{H}_k^{(2)}(a,b)$ is related with the following heat problem considering the analytic function $\mathscr{F}(a)=\sum\limits_{k=0}^{\infty}c_k a^k$:

$$\begin{cases}\frac{\partial S}{\partial b}=\frac{\partial^2 S}{\partial a^2} & \text{in the half-plane } b>0\\ S(a,0)=\mathscr{F}(a).\end{cases}\tag{5.1.14}$$

The heat problem given in (5.1.14) admits the formal solution as

$$S(a,b)=e^{b\mathscr{D}^2 a}\mathscr{F}(a).\tag{5.1.15}$$

It is also known (see [20]) that the solution of (5.1.14) can be represented by the Gauss-Weierstrass transform as follows

$$S(a,b)=\frac{1}{2\sqrt{\pi b}}\int_{-\infty}^{\infty}\mathscr{F}(\gamma)e^{-\frac{(a-\gamma)^2}{4b}}d\gamma.\tag{5.1.16}$$

Comparing (5.1.15) and (5.1.16), we get the following integral representation:

$$e^{b\mathscr{D}^2 a}\mathscr{F}(a)=\frac{1}{2\sqrt{\pi b}}\int_{-\infty}^{\infty}\mathscr{F}(\gamma)e^{-\frac{(a-\gamma)^2}{4b}}d\gamma\tag{5.1.17}$$

Expanding an analytic function $\mathscr{F}(a)$ in a series $\mathscr{F}(a) = \sum\limits_{k=0}^{\infty} c_k a^k$, we get

$$e^{b\mathscr{D}^2 a}\mathscr{F}(a) = \sum_{k=0}^{\infty} c_k \mathscr{H}_k^{(2)}(a,b). \tag{5.1.18}$$

Furthermore, the Gauss-Weierstrass transform representation of the Hermite polynomials $\mathscr{H}_k^{(2)}(a,b)$ is given as follows:

$$\mathscr{H}_k^{(2)}(a,b) = \frac{1}{2\sqrt{\pi b}} \int_{-\infty}^{\infty} \gamma^k e^{-\frac{(a-\gamma)^2}{4b}} d\gamma, \tag{5.1.19}$$

since $e^{b\mathscr{D}_a^2} a^k = \mathscr{H}_k^{(2)}(a,b)$.

5.2 Hermite-Kampe de Feriet Polynomials

The Hermite-Kampe de Feriet polynomials are $\mathscr{H}_k^{(2)}(a,b)$ and they are denoted for simplicity by $\mathscr{H}_k(a,b)$:

$$\mathscr{H}_k(a,b) = k! \sum_{p=0}^{\left[\frac{k}{2}\right]} \frac{b^p a^{k-2p}}{p!\,(k-2p)!}. \tag{5.2.1}$$

Clearly for $b = 0$,

$$\mathscr{H}_k(a,0) = k!\frac{a^k}{k!} + k! \sum_{p=1}^{\left[\frac{k}{2}\right]} \frac{b^p a^{k-2p}}{p!\,(k-2p)!} = a^k.$$

The relation between the Hermite-Kampe de Feriet polynomials and the ordinary one variable Hermite polynomials is given in the following equation:

$$\mathscr{H}_k\left(a, -\frac{1}{2}\right) = \mathscr{H}e_k(a) = k! \sum_{p=0}^{\left[\frac{k}{2}\right]} \frac{\left(-\frac{1}{2}\right)^p a^{k-2p}}{p!(k-2p)!}$$

$$\mathscr{H}_k(2a,-1) = \mathscr{H}_k(a) = k! \sum_{p=0}^{\left[\frac{k}{2}\right]} \frac{(-1)^p (2a)^{k-2p}}{p!\,(k-2p)!}.$$

Theorem 5.2.1 *The polynomials $\mathscr{H}_k(a,b)$ are quasi-monomials with respect to the operators*

$$L = a + 2b\frac{\partial}{\partial a}, \quad K = \frac{\partial}{\partial a}. \tag{5.2.2}$$

Proof. Assuming $q_k(a) = a^k$ we note that $K = \frac{\partial}{\partial a}$ commutes with $\Psi = \frac{\partial^2}{\partial a^2}$. Furthermore, since

$$
\begin{aligned}
e^{b\frac{\partial^2}{\partial a^2}}a^k &= \sum_{p=0}^{\infty}\frac{b^p}{p!}\frac{\partial^{2p}}{\partial a^{2p}}a^k \\
&= \sum_{p=0}^{\left[\frac{k}{2}\right]}\frac{b^p}{p!}\frac{k!}{(k-2p)!}a^{k-2p} = \mathscr{H}_k(a,b)
\end{aligned}
$$

and using (3.3.6),

$$
e^{b\frac{\partial^2}{\partial a^2}}a^k = \mathscr{H}_k(a,b) = \left(a+2b\frac{\partial}{\partial a}\right)^k(1) = L^k(1). \tag{5.2.3}
$$

For any twice differentiable function $f(a,b)$,

$$
[L,K]f(a,b) = \left(\frac{\partial}{\partial a}\left(a+2b\frac{\partial}{\partial a}\right) - \left(a+2b\frac{\partial}{\partial a}\right)\frac{\partial}{\partial a}\right)f(a,b)
$$

$$
\begin{aligned}
&= \frac{\partial}{\partial a}af + \frac{\partial}{\partial a}2b\frac{\partial}{\partial a}f - af_a - 2bf_{aa} \\
&= f(a,b) + af_a + 2bf_{aa} - af_a - 2bf_{aa} = f(a,b).
\end{aligned}
$$

Hence

$$
[L,K] \equiv 1.
$$

Therefore the hypotheses of Theorem 4.2.2 are satisfied for the operators in (5.2.2) and thus (4.1.1) holds resulting quasi-monomial polynomials $\mathscr{H}_k(a,b)$.

■

Take into account that, the operational definition (5.2.3) implies the $\mathscr{H}_k(a,b)$ which satisfy the partial differential equation as follows

$$
\frac{\partial}{\partial a}\mathscr{H}_k(a,b) = \frac{\partial^2}{da^2}\mathscr{H}_k(a,b). \tag{5.2.4}
$$

Also,

$$
e^{-b\frac{\partial^2}{\partial a}}\mathscr{H}_k(a,b) = a^k.
$$

5.2.1 Differential Equation

From (i) of Section 4.1, we have

$$LK\left(\mathscr{H}_k(a,b)\right) = k\mathscr{H}_k(a,b),$$

which gives

$$\left(a+2b\frac{\partial}{\partial a}\right)\left(\frac{\partial}{\partial a}\right)\mathscr{H}_k(a,b) = a\frac{\partial}{\partial a}\mathscr{H}_k(a,b) + 2b\frac{\partial^2}{\partial a^2}\mathscr{H}_k(a,b)$$

or equivalently to

$$a\frac{\partial}{\partial a}\mathscr{H}_k(a,b) + 2a\frac{\partial^2}{\partial a^2}\mathscr{H}_k(a,b) = k\mathscr{H}_k(a,b). \tag{5.2.5}$$

5.2.2 Exponential Generating Function

From item (iii) of Section 4.1,

$$\sum_{p=0}^{\infty}\frac{g^p}{p!}\mathscr{H}_k(a,b) = e^{gK}(1) = e^{g\left(a+2b\frac{\partial}{\partial a}\right)}(1),$$

so we can use (3.1.3),

$$\begin{aligned}
\mathscr{C} &= ag \quad \mathscr{D} = 2bg\frac{\partial}{\partial a} \\
[\mathscr{C},\mathscr{D}]\,f(a,b) &= \left(ag2bg\frac{\partial}{\partial a} - 2bg\frac{\partial}{\partial a}ag\right)f \\
&= 2abg^2 f_a - 2bg\,(gf + agf_a) \\
&= 2abg^2 f_a - 2bg^2 f - 2abg^2 f_a = -2bg^2 f \\
[\mathscr{C},\mathscr{D}] &= -2bg^2
\end{aligned}$$

and

$$\begin{aligned}
[[\mathscr{C},\mathscr{D}],\mathscr{C}]\,f(a,b) &= \left(-2bg^2ag + ag2bg^2\right)f = -2abg^3 f + 2abg^3 f = 0 \\
[[\mathscr{C},\mathscr{D}],\mathscr{D}]\,f(a,b) &= \left(-2bg^2 2bg\frac{\partial}{\partial a} + 2bg\frac{\partial}{\partial a}2bg^2\right)f
\end{aligned}$$

$$
\begin{aligned}
&= -4b^2g^3f_a + 2bg(0+2bg^2f_a) \\
&= -4b^2g^3f_a + 4b^2g^3f_a = 0
\end{aligned}
$$

$$[[\mathscr{C},\mathscr{D}],\mathscr{C}] = [[\mathscr{C},\mathscr{D}],\mathscr{D}] = 0.$$

Therefore, we can write

$$e^{g\left(a+2b\frac{\partial}{\partial a}\right)}(1) = e^{bg^2}e^{ag}e^{2bg\frac{\partial}{\partial a}}(1) = e^{ag+bg^2}.$$

We have found the exponential generating function as

$$\sum_{p=0}^{\infty}\frac{g^p}{p!}\mathscr{H}_k(a,b) = e^{ag+bg^2}. \tag{5.2.6}$$

5.2.3 Recurrence Relation

From (5.2.3), we have

$$
\begin{aligned}
\mathscr{H}_{k+1}(a,b) &= K\mathscr{H}_k(a,b) = \left(a+2b\frac{\partial}{\partial a}\right)\mathscr{H}_k(a,b) \\
&= a\mathscr{H}_k(a,b) + 2b\frac{\partial}{\partial a}\mathscr{H}_k(a,b) \\
&= a\mathscr{H}_k(a,b) + 2bL\mathscr{H}_k(a,b) \\
&= a\mathscr{H}_k(a,b) + 2bk\mathscr{H}_{k-1}(a,b).
\end{aligned}
$$

Hence the recurrence relation is obtained as

$$\mathscr{H}_{k+1}(a,b) = a\mathscr{H}_k(a,b) + 2bk\mathscr{H}_{k-1}(a,b).$$

5.2.4 Burchnall Identity

Theorem 5.2.2

$$\left(a+2b\frac{\partial}{\partial a}\right)^m = \sum_{l=0}^{m}\binom{m}{l}\mathscr{H}_{m-l}(a,b)\left(2b\frac{\partial}{\partial a}\right)^l. \tag{5.2.7}$$

Proof. In this proof we need to use the Weyl identity. Multiplying the left-hand side of (5.2.7) by $\frac{g^p}{p!}$ and summing over p, we find

$$\sum_{p=0}^{\infty} \frac{g^p}{p!} \left(a + 2b\frac{\partial}{\partial a}\right)^p = e^{g\left(a+2b\frac{\partial}{\partial a}\right)}.$$

Letting

$$K = ga \quad L = 2bg\frac{\partial}{\partial a},$$

we see that

$$[K,L] = -2bg^2,$$

therefore we can write

$$e^{g\left(a+2b\frac{\partial}{\partial a}\right)}(1) = e^{bg^2} e^{ga} e^{2bg\frac{\partial}{\partial a}}(1).$$

Now, by using (5.2.6) and expanding the exponential function, we obtain

$$\begin{aligned} e^{bg^2+ag} e^{2bg\frac{\partial}{\partial a}} &= \sum_{p=0}^{\infty} \frac{g^p}{p!} \mathscr{H}_p(a,b) \sum_{l=0}^{\infty} \frac{\left(2bg\frac{\partial}{\partial a}\right)^l}{l!} \\ &= \sum_{p=0}^{\infty} \sum_{l=0}^{\infty} \frac{g^{p+l}}{p!l!} \mathscr{H}_p(a,b) \left(2b\frac{\partial}{\partial a}\right)^l. \end{aligned}$$

Therefore

$$\begin{aligned} &\sum_{p=0}^{\infty} \sum_{l=0}^{p} \frac{g^p p!}{(p-l)!l!p!} \mathscr{H}_{p-l}(a,b) \left(2b\frac{\partial}{\partial a}\right)^l \\ = &\sum_{p=0}^{\infty} \frac{g^p}{p!} \sum_{l=0}^{p} \binom{p}{l} \mathscr{H}_{p-l}(a,b) \left(2b\frac{\partial}{\partial a}\right)^l \end{aligned}$$

$$\Rightarrow \sum_{p=0}^{\infty} \frac{g^p}{p!} \left(a + 2b\frac{\partial}{\partial a}\right)^p = \sum_{p=0}^{\infty} \frac{g^p}{p!} \sum_{l=0}^{p} \binom{p}{l} \mathscr{H}_{p-l}(a,b) \left(2b\frac{\partial}{\partial a}\right)^l,$$

implying,

$$\left(a + 2b\frac{\partial}{\partial a}\right)^p = \sum_{l=0}^{p} \binom{p}{l} \mathscr{H}_{p-l}(a,b) \left(2b\frac{\partial}{\partial a}\right)^l.$$

■

The Hermite-Kampe de Feriet (or Gould-Hopper) polynomials $\mathscr{H}_p^{(k)}(a,b)$ satisfy the conditions of the quasi-monomiality where,

$$L=\frac{\partial}{\partial a}=\mathscr{D}_a,\ \ K=a+kb\frac{\partial^{k-1}}{\partial a^{k-1}}.$$

It is clear that

$$e^{b\frac{\partial^k}{da^k}}a^p=\mathscr{H}_p^{(k)}(a,b)=\left(a+kb\frac{\partial^{k-1}}{\partial a^{k-1}}\right)^p(1).$$

The explicit expression of the polynomials $\mathscr{H}_p^{(k)}(a,b)$ can be derived from the definition, since

$$e^{b\frac{\partial^k}{\partial a^k}}a^p=\sum_{t=0}^{\infty}\frac{b^t}{t!}\frac{\partial^{kt}}{\partial a^{kt}}a^p.$$

Using (5.1.8), we see that

$$e^{b\frac{\partial^k}{\partial a^k}}a^p=\sum_{t=0}^{\left[\frac{p}{k}\right]}\frac{b^t}{t!}\frac{p!}{(p-kt)!}a^{p-kt}=\mathscr{H}_p^{(k)}(a,b). \tag{5.2.8}$$

The exponential generating function can be found by multiplying both sides of (5.2.8) and by $\frac{g^p}{p!}$ summing over p,

$$\begin{aligned}\sum_{p=0}^{\infty}\frac{g^p}{p!}\mathscr{H}_p^{(k)}(a,b) &= e^{b\frac{\partial^k}{\partial a^k}}\sum_{p=0}^{\infty}\frac{g^p}{p!}a^p=e^{b\frac{\partial^k}{\partial a^k}}\sum_{p=0}^{\infty}\frac{(ag)^p}{p!}=e^{b\frac{\partial^k}{\partial a^k}}e^{ag}\\ &= \sum_{r=0}^{\infty}\frac{a^r}{r!}\frac{\partial^{kr}}{\partial a^{kr}}e^{ag}=e^{bg^k+ag}.\end{aligned}$$

The differential equation follows from

$$KL\mathscr{H}_p^{(k)}(a,b)=p\mathscr{H}_p^{(k)}(a,b),$$

which gives

$$\begin{aligned}&\left(a+kb\frac{\partial^{k-1}}{\partial a^{k-1}}\right)\left(\frac{\partial}{\partial a}\right)\mathscr{H}_p^{(k)}(a,b)\\ &= a\frac{\partial}{\partial a}\mathscr{H}_p^{(k)}(a,b)+kb\frac{\partial^k}{\partial a^k}\mathscr{H}_p^{(k)}(a,b)=p\mathscr{H}_p^{(k)}(a,b).\end{aligned}$$

Note that $\mathscr{H}_p^{(k)}(a,b)$ satisfies the differential relations

$$L\mathscr{H}_p^{(k)}(a,b)=\frac{\partial}{\partial a}\mathscr{H}_p^{(k)}(a,b)=p\mathscr{H}_{p-1}^{(k)}(a,b),$$

and

$$\frac{\partial}{\partial b}\mathscr{H}_p^{(k)}(a,b) = \frac{p!}{(p-k)!}\mathscr{H}_{p-k}^{(k)}(a,b)$$

from which we obtain

$$\begin{aligned}\frac{\partial^k}{\partial a^k}\mathscr{H}_p^{(k)}(a,b) &= p(p-1)\dots(p-k+1)\mathscr{H}_{p-k}^{(k)}(a) \\ &= \frac{p!}{(p-k)!}\mathscr{H}_{p-k}^{(k)}(a,b) = \frac{\partial}{\partial b}\mathscr{H}_p^{(k)}(a,b).\end{aligned}$$

Finally, using the equalities

$$\begin{aligned}\mathscr{H}_{p+1}^{(k)}(a,b) &= K\mathscr{H}_p^{(k)}(a,b) = \left(a + kb\frac{\partial^{k-1}}{\partial a^{k-1}}\right)\mathscr{H}_p^{(k)}(a,b) \\ &= a\mathscr{H}_p^{(k)}(a,b) + kb\frac{\partial^{k-1}}{\partial a^{k-1}}\mathscr{H}_p^{(k)}(a,b) \\ &= a\mathscr{H}_p^{(k)}(a,b) + kbL^{k-1}\mathscr{H}_p^{(k)}(a,b)\end{aligned}$$

and

$$\begin{aligned}L^{k-1}\mathscr{H}_p^{(k)}(a,b) &= p(p-1)\dots(p-k+2)\mathscr{H}_{p-k+1}^{(k)}(a,b) \\ &= \frac{p!}{(p-k+1)!}\mathscr{H}_{p-k+1}^{(k)},\end{aligned}$$

we get

$$H_{p+1}^{(k)}(a,b) = aH_p^{(k)}(a,b) + kb\frac{p!}{(p-k+1)!}H_{p-k+1}^{(k)}(a,b).$$

5.3 Laguerre Polynomials in Two Variables

A polynomial set, for instance, is obtained by using the isomorphism given in Remark 4.1.5. Consider the below defined Laguerre polynomials in two variables:

$$L_m(a,b) = m!\sum_{j=0}^{m}\frac{(-1)^j b^{m-j}a^j}{(m-j)!(j!)^2}, \quad L_m(a,0) = \frac{(-a)^m}{m!}, \tag{5.3.1}$$

which have a relationship with the ordinary Laguerre polynomials $\mathscr{L}_m(a)$ by

$$\check{L}_m(a,1) = \check{L}_m(a), \quad \check{L}_m(a,b) = b^m\mathscr{L}_m\left(\frac{a}{b}\right). \tag{5.3.2}$$

For simplicity, in the sequel we consider the polynomials

$$\mathscr{L}_m(a,b) = \check{L}_m(-a,b) = m! \sum_{j=0}^{m} \frac{a^{m-j} b^j}{(m-j)!\,(j!)^2}. \tag{5.3.3}$$

We call these polynomials as Laguerre polynomials in two-variables.

Theorem 5.3.1 *The Laguerre polynomials $\mathscr{L}_m(a,b)$ are quasi-monomials with respect to the operators*

$$A_* = \frac{\partial}{\partial a} a \frac{\partial}{\partial a}, \ \ B_* = b + \mathscr{D}_a^{-1}, \tag{5.3.4}$$

where

$$\mathscr{D}_a^{-1} \mathscr{F}(a) = \int_0^a \mathscr{F}(\tau)\, d\tau.$$

Proof. In fact $\mathscr{L}_m(a,b)$ satisfy the partial differential equation

$$\frac{\partial}{\partial b} \mathscr{L}_m(a,b) = \frac{\partial}{\partial a} a \frac{\partial}{\partial a} \mathscr{L}_m(a,b),$$

since

$$\begin{aligned}
\frac{\partial}{\partial b} \mathscr{L}_m(a,b) &= \frac{\partial}{\partial b} \left[m! \sum_{p=0}^{m} \frac{b^{m-p} a^p}{(m-p)!\,(p!)^2} \right] \\
&= m! \sum_{p=0}^{m-1} \frac{(m-p)\, b^{m-p-1} a^p}{(m-p)(m-p-1)!(p!)^2} \\
&= m(m-1)! \sum_{p=0}^{m-1} \frac{b^{m-p-1} a^p}{(m-p-1)!(p!)^2} \\
&= m \mathscr{L}_{m-1}(a,b) = A_* \mathscr{L}_m(a,b).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{\partial}{\partial a} a \frac{\partial}{\partial a} \mathscr{L}_m(a,b) &= \frac{\partial}{\partial a} a \left[m! \sum_{p=0}^{m-1} \frac{b^{m-p}\, p a^{p-1}}{(m-p)! p\,(p-1)! p!} \right] \\
&= \frac{\partial}{\partial a} \left[m! \sum_{p=1}^{m-1} \frac{b^{m-p} a^p}{(m-p)!(p-1)! p!} \right] \\
&= m! \sum_{p=1}^{m-1} \frac{b^{m-p}\, p a^{p-1}}{(m-p)! p\,(p-1)!^2} \\
&= m(m-1)! \sum_{p=0}^{m-1} \frac{b^{m-p-1} a^p}{(m-p-1)!(p!)^2}
\end{aligned}$$

$$= m\mathscr{L}_{m-1}(a,b) = A_*\mathscr{L}_m(a,b).$$

Then, considering the corresponding initial condition $\mathscr{L}_m(a,0) = \frac{a^m}{m!}$ in (5.3.1), we have

$$\mathscr{L}_m(a,b) = e^{b\frac{\partial}{\partial a}a\frac{\partial}{\partial a}}\mathscr{L}_m(a,0) = e^{b\frac{\partial}{\partial a}a\frac{\partial}{\partial a}}\left(\frac{a^m}{m!}\right). \tag{5.3.5}$$

In fact, $A_* = \Phi = \frac{\partial}{\partial a}a\frac{\partial}{\partial a}$ obviously commutes with $e^{b\frac{\partial}{\partial a}a\frac{\partial}{\partial a}}$. Furthermore, using Definition (5.3.3), recalling (5.3.4) and using the commutator between b and $\mathscr{D}_a^{-1}$, we can write

$$\begin{aligned}
\mathscr{L}_m(a,b) &= e^{b\frac{\partial}{\partial a}a\frac{\partial}{\partial a}}\left(\frac{a^m}{m!}\right) = \sum_{k=0}^{\infty}\frac{b^k}{k!m!}\left(\frac{\partial}{\partial a}a\frac{\partial}{\partial a}\right)^k a^m \\
&= \sum_{k=1}^{m}\frac{b^k}{k!m!}\frac{(m!)^2}{((m-k)!)^2}a^{m-k} \\
&= \sum_{k=1}^{m}\binom{m}{k}b^k\frac{a^{m-k}}{(m-k)!} = (b+\mathscr{D}_a^{-1})^m(1).
\end{aligned}$$

Hence,

$$\mathscr{L}_m(a,b) = e^{b\frac{\partial}{\partial a}a\frac{\partial}{\partial a}}\left(\frac{a^m}{m!}\right) = (b+\mathscr{D}_a^{-1})^m(1) = B_*^m(1).$$

On the other hand, clearly

$$\begin{aligned}
[A_*,B_*]f(a,b) &= \left[\frac{\partial}{\partial a}a\frac{\partial}{\partial a}\left(b+\mathscr{D}_a^{-1}\right) - \left(b+\mathscr{D}_a^{-1}\right)\frac{\partial}{\partial a}a\frac{\partial}{\partial a}\right]f(a,b) \\
&= \frac{\partial}{\partial a}a\frac{\partial}{\partial a}bf + \frac{\partial}{\partial a}a\frac{\partial}{\partial a}\mathscr{D}_a^{-1}f - b\frac{\partial}{\partial a}a\frac{\partial}{\partial a}f - \mathscr{D}_a^{-1}\frac{\partial}{\partial a}a\frac{\partial}{\partial a}f \\
&= \frac{\partial}{\partial a}abf_a + \frac{\partial}{\partial a}af - b\frac{\partial}{\partial a}af_a - \mathscr{D}_a^{-1}\frac{\partial}{\partial a}af_a \\
&= bf_a + abf_{aa} + f + af_a - bf_a - abf_{aa} - af_a \\
&= f(a,b),
\end{aligned}$$

which gives

$$[A_*,B_*] = 1.$$

This completes the proof. ■

Remark 5.3.2 *The differential isomorphism $T \equiv T_a$ has been introduced in Remark (4.1.5). Also, the Laguerre polynomials and the Hermite polynomials and their relations have been discussed in [3] . One can realize that under the action T_a, the Laguerre polynomials $\mathscr{L}_m(a,b)$ correspond to the Gould-Hopper polynomials $\mathscr{H}_m^{(1)}(a,b) = (a+b)^m$ i.e.,*

$$\mathscr{H}_m^{(1)}(a,b) = (a+b)^m = \sum_{k=0}^{m} \binom{m}{k} b^{m-k} a^k.$$

$$\begin{aligned} T_a \mathscr{H}_m^{(1)}(a,b) &= \sum_{k=0}^{m} \binom{m}{k} b^{m-k} T_a\left(a^k\right) \\ &= \sum_{k=0}^{m} \binom{m}{k} b^{m-k} \frac{a^k}{k!} = m! \sum_{k=0}^{m} \frac{a^k b^{m-k}}{(m-k)!\,(k!)^2} \\ &= \mathscr{L}_m(a,b). \end{aligned}$$

5.3.1 Differential Equation

From

$$\left(b + \mathscr{D}_a^{-1}\right)\left(\frac{\partial}{\partial a} a \frac{\partial}{\partial a}\right) \mathscr{L}_m(a,b) = m \mathscr{L}_m(a,b),$$

one can find that

$$\begin{aligned} & \left(b + \mathscr{D}_a^{-1}\right)\left(\frac{\partial}{\partial a} a \frac{\partial}{\partial a}\right) \mathscr{L}_m(a,b) \\ = & \left(b + \mathscr{D}_a^{-1}\right)\left(\frac{\partial}{\partial a} a m \mathscr{L}_{m-1}(a,b)\right) \\ = & \left(b + \mathscr{D}_a^{-1}\right)\left(m \mathscr{L}_{m-1}(a,b) + m(m-1)\mathscr{L}_{m-2}(a,b)\,a\right) \\ = & m \mathscr{L}_{m-1}(a,b)\,b + m(m-1)\mathscr{L}_{m-2}(a,b)\,ab \\ & + \mathscr{D}_a^{-1} m \mathscr{L}_{m-1}(a,b) + m(m-1)\mathscr{D}_a^{-1}\mathscr{L}_{m-2}(a,b)\,a \\ = & m \mathscr{L}_{m-1}(a,b)\,b + abm(m-1)\mathscr{L}_{m-2}(a,b) + \mathscr{L}_m(a,b) + \\ & + m(m-1)\frac{a \mathscr{L}_{m-1}(a,b)}{m-1} - \frac{m(m-1)}{m(m-1)} \mathscr{L}_m(a,b) \\ = & b \frac{\partial}{\partial a} \mathscr{L}_m(a,b) + ab \frac{\partial^2}{\partial a^2} \mathscr{L}_m(a,b) + a \frac{\partial}{\partial a} \mathscr{L}_m(a,b) \end{aligned}$$

$$= \left[(a+b)\frac{\partial}{\partial a} + ab\frac{\partial^2}{\partial a^2} \right] \mathscr{L}_m(a,b).$$

Hence, $\mathscr{L}_m(a,b)$ satisfy the differential equation

$$ba\frac{\partial^2}{\partial a^2}\mathscr{L}_m(a,b) + (b+a)\frac{\partial}{\partial a}\mathscr{L}_m(a,b) = m\mathscr{L}_m(a,b). \tag{5.3.6}$$

Note that, from the homogenity property, $\mathscr{L}_m(a,b)$ also satisfy the Euler equation

$$a\frac{\partial}{\partial a}\mathscr{L}_m(a,b) + b\frac{\partial}{\partial b}\mathscr{L}_m(a,b) = m\mathscr{L}_m(a,b).$$

5.3.2 Ordinary Generating Function

In Section 4.1, (iii) implies that

$$\mathscr{L}_m(a,b) = B_*^m(1). \tag{5.3.7}$$

We get

$$\begin{aligned} \sum_{m=0}^{\infty} g^m \mathscr{L}_m(a,b) &= \sum_{m=0}^{\infty} g^m B_*^m(1) = \sum_{m=0}^{\infty} g^m \left(b+\mathscr{D}_a^{-1}\right)^m(1) \\ &= \frac{1}{1-g\left(b+\mathscr{D}_a^{-1}\right)}(1). \end{aligned} \tag{5.3.8}$$

Furthermore,

$$\begin{aligned} \frac{1}{1-g\left(b+\mathscr{D}_a^{-1}\right)} &= \frac{1}{1-gb-g\mathscr{D}_a^{-1}} = \frac{1}{(1-gb)\left(1-\frac{g}{1-gb}\mathscr{D}_a^{-1}\right)} \\ &= \frac{1}{1-gb}\sum_{s=0}^{\infty}\left[\frac{g}{1-gb}\right]^s \mathscr{D}_a^{-s}. \end{aligned}$$

Recalling the operator $\mathscr{D}_a^{-s}$ in Remark 4.2.5, we obtain

$$\frac{1}{1-gb}\sum_{s=0}^{\infty}\left[\frac{g}{1-gb}\right]^s \frac{a^s}{s!} = \frac{1}{1-gb}e^{\frac{ga}{1-gb}}$$

and

$$\sum_{m=0}^{\infty} g^m \mathscr{L}_m(a,b) = \frac{1}{1-gb}e^{\frac{ga}{1-gb}}, \quad |gb|<1.$$

5.3.3 Exponential Generating Function

Assume that $\mathscr{L}_0(a,b) = 1$. Then we can write the following form as

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{g^m}{m!}\mathscr{L}_m(a,b) &= \sum_{m=0}^{\infty}\frac{g^m}{m!}(B_*(1))^m(1)\\ &= e^{gB_*(1)} = e^{g\left(b+\mathscr{D}_a^{-1}\right)}(1). \end{aligned} \tag{5.3.9}$$

Now we will use the Weyl identity. Considering the operators

$$\mathscr{A} = bg, \quad \mathscr{B} = g\mathscr{D}_a^{-1},$$

we see that

$$\begin{aligned}[\mathscr{A},\mathscr{B}]f(a,b) &= \left[bg^2\mathscr{D}_a^{-1} - g\mathscr{D}_a^{-1}bg\right]f(a,b) = bg^2\mathscr{D}_a^{-1}f - g^2b\mathscr{D}_a^{-1}f = 0\\ \left[bg, g\mathscr{D}_a^{-1}\right] &= 0.\end{aligned}$$

Thus, from the Weyl identity,

$$e^{gb+g\mathscr{D}_a^{-1}} = e^0 e^{gb} e^{g\mathscr{D}_b^{-1}}(1) = e^{gb}\sum_{r=0}^{\infty}\frac{\left(g\mathscr{D}_a^{-1}\right)^r}{r!}(1) = e^{gb}\sum_{r=0}^{\infty}\frac{(ga)^r}{(r!)^2}$$

and,

$$\sum_{m=0}^{\infty}\frac{g^m}{m!}\mathscr{L}_m(a,b) = e^{gb}\mathscr{C}_0(-ga),$$

where $\mathscr{C}_0$ is introduced as the 0-order Tricomi function. In general,

$$\mathscr{C}_r(a) = \sum_{p=0}^{\infty}\frac{(-1)^p a^p}{p!(r+p)!}. \tag{5.3.10}$$

for every integer r.

Remark 5.3.3 *The image of the exponential function under the isomorphism T results the Tricomi function $\mathscr{C}_0(-a)$ since*

$$Te^a = \sum_{p=0}^{\infty}\frac{T(a^p)}{p!} = \sum_{p=0}^{\infty}\frac{a^p}{(p!)^2}.$$

Clearly,

$$\begin{aligned}-\frac{\partial}{\partial a}a\frac{\partial}{\partial a}\mathscr{C}_0(ga) &= -\frac{\partial}{\partial a}a\frac{\partial}{\partial a}\sum_{p=0}^{\infty}\frac{(-ga)^p}{(p!)^2} = -\frac{\partial}{\partial a}a\sum_{p=0}^{\infty}-\frac{g^p pa^{p-1}}{k(k-1)!k!}\\ &= -\frac{\partial}{\partial a}\sum_{p=1}^{\infty}-\frac{g^p a^p}{(p-1)!p!} = -\sum_{p=1}^{\infty}-\frac{g^p pa^{p-1}}{(p-1)!p(p-1)!}\end{aligned}$$

$$= \sum_{p=0}^{\infty} \frac{g^{p+1} a^p}{(p!)^2} = g\mathscr{C}_0 (ag). \tag{5.3.11}$$

5.3.4 Recurrence Relation

Now, we will follow some steps to derive the recurrence relation of the classical Laguerre polynomials. Let us consider the relation

$$\sum_{m=0}^{\infty} \mathscr{L}_m (a) t^m = \frac{1}{1-t} e^{-\frac{at}{1-t}}$$

we get by taking derivative with respect to t on both sides and making series manipulations, we arrive to the following recurrence relation:

$$(m+1) \mathscr{L}_{m+1} (a) = (2m+1-a) \mathscr{L}_m (a) - m\mathscr{L}_{m-1} (a).$$

Replacing a by $\frac{a}{b}$, and multiplying both sides by b^{m+1}, we get the following recurrence formula:

$$(m+1) \mathscr{L}_{m+1} \left(-\frac{a}{b}\right) b^{m+1} = b^{m+1} \left[\left(2m+1+\frac{a}{b}\right) \mathscr{L}_m \left(-\frac{a}{b}\right) - m\mathscr{L}_{m-1} \left(-\frac{a}{b}\right)\right]$$

$$\Rightarrow (m+1) \mathscr{L}_{m+1} (-a,b) = [(2m+1) b + a] \mathscr{L}_m (a,b) - mb^2 \mathscr{L}_{m-1} (a,b). \tag{5.3.12}$$

5.3.5 Laguerre-Type Exponentials

For every positive integer r, the $r^{th} K-$exponential function is defined in the following way:

$$\begin{aligned}
e_1 (a) &= \mathscr{T}_a (e^a) = \sum_{p=0}^{\infty} \frac{\mathscr{T}_a (a^p)}{p!} = \sum_{p=0}^{\infty} \frac{a^p}{(p!)^2} \\
e_2 (a) &= \mathscr{T}_a^2 (e^a) = \mathscr{T} \left[\sum_{p=0}^{\infty} \frac{\mathscr{T}_a (a^p)}{p!}\right] = \mathscr{T} \left[\sum_{p=0}^{\infty} \frac{a^p}{(p!)^2}\right] = \sum_{p=0}^{\infty} \frac{\mathscr{T} (a^p)}{(p!)^2} = \sum_{p=0}^{\infty} \frac{a^p}{(p!)^3} \\
&\vdots \\
e_r (a) &= \mathscr{T}_a^r (e^a) = \sum_{p=0}^{\infty} \frac{a^p}{(p!)^{r+1}}.
\end{aligned}$$

For $r = 0$, we have $e_0 (a) = e^a$. Consider the operator (containing $r+1$ derivatives):

$$\mathscr{D}_{r\mathscr{L}} = \mathscr{D}a\ldots\mathscr{D}a\mathscr{D}a\mathscr{D} = S(r+1,1)\,\mathscr{D} + S(r+1,2)\,a\mathscr{D}^2 + \cdots + S(r+1,r+1)\,a^r\mathscr{D}^{r+1},$$

where, $S(r,k)$ denotes the Stirling numbers of the second kind.

Theorem 5.3.4 *[18] The r^{th} Laguerre-type exponential $e_r(ka)$ is an eigenfunction of the operator $\mathscr{D}_r\mathscr{L}$, for any $k \in \mathbb{C}$. In other words:*

$$\mathscr{D}_r\mathscr{L}e_r(ka) = ke_r(ka).$$

One can easily see that $\mathscr{D}_0\mathscr{L} = \mathscr{D}$ and we have:

$$\mathscr{D}e^{ka} = ke^{ka}.$$

Proof. Direct calculation yield that

$$\begin{aligned}
\mathscr{D}_1\mathscr{L}e_1(ka) &= (\mathscr{D}a\mathscr{D})\sum_{m=0}^{\infty} k^m\frac{a^m}{(m!)^2} = \mathscr{D}a\sum_{m=1}^{\infty}\frac{k^m m a^{m-1}}{m(m-1)!m!}\\
&= \mathscr{D}\sum_{m=1}^{\infty}\frac{k^m a^m}{(m-1)!m!} = \sum_{m=1}^{\infty}\frac{k^m m a^{m-1}}{m(m-1)!(m-1)!} = \sum_{m=0}^{\infty}k^{m+1}\frac{a^m}{(m!)^2}\\
&= ke_1(ka)\\
\mathscr{D}_2\mathscr{L}e_2(ka) &= (\mathscr{D}a\mathscr{D}a\mathscr{D})\sum_{m=0}^{\infty} k^m\frac{a^m}{(m!)^3} = \mathscr{D}a\mathscr{D}a\sum_{m=1}^{\infty}\frac{k^m m a^{m-1}}{m(m-1)!(m!)^2}\\
&= \mathscr{D}a\mathscr{D}\sum_{m=1}^{\infty}\frac{k^m a^m}{(m-1)!(m!)^2} = \mathscr{D}a\sum_{m=1}^{\infty}\frac{k^m m a^{m-1}}{(m-1)!m(m-1)!m!}\\
&= \mathscr{D}\sum_{m=1}^{\infty}\frac{k^m a^m}{(m-1)!^2 m!} = \sum_{m=1}^{\infty}\frac{k^m m a^{m-1}}{(m-1)!^2 m(m-1)!}\\
&= \sum_{m=1}^{\infty}\frac{k^m a^{m-1}}{(m-1)!^3} = \sum_{m=0}^{\infty}\frac{k^{m+1}a^m}{(m!)^3} = ke_2(ka)\\
&\vdots\\
\mathscr{D}_r\mathscr{L}e_r(ka) &= ke_r(ka).
\end{aligned}$$

■

Clearly, the $r^{th}K$-exponential function satisfies $e_r(0) = 1$ for all r, and for $a \geqslant 0$ is an increasing convex function. Moreover,

$$e^a = e_0(a) > e_1(a) > e_2(a) > \cdots > e_r(a) > \cdots \quad \forall a > 0.$$

According to [19], for each $t = 1, 2, 3, \ldots$, we have

$$(\mathscr{D}a\mathscr{D})^t = \mathscr{D}^t a^t \mathscr{D}^t, \ \ (\mathscr{D}a\mathscr{D}a\mathscr{D})^t = \mathscr{D}^t a^t \mathscr{D}^t a^t \mathscr{D}^t.$$

5.4 The Isomorphism T_a

In Remark (4.1.5), consider the space of analytic functions of the variable a, as $A = A_a$ and a differential isomorphism acting on this space as $T = T_a$; i.e.

$$\mathscr{D} = \frac{d}{da} \to \mathscr{D}_{\mathscr{L}} = \mathscr{D}a\mathscr{D}; \ \ a \to \mathscr{D}_a^{-1},$$

where

$$\mathscr{D}_a^{-1}\mathscr{F}(a) = \int_0^a \mathscr{F}(\varphi)\,d\varphi, \quad \mathscr{D}_a^{-t}\mathscr{F}(a) = \frac{1}{(t-1)!}\int_0^a (a-\varphi)^{t-1}\mathscr{F}(\varphi)\,d\varphi,$$

so that

$$T_a\left(a^t\right) = \mathscr{D}_a^{-t}(1) = \frac{1}{(t-1)!}\int_0^a (a-\varphi)^{t-1}\,d\varphi = \frac{a^t}{t!}. \tag{5.4.1}$$

Note that

$$\begin{aligned} T_a(e^a) &= \sum_{r=0}^{\infty}\frac{T_a(a^r)}{r!} = \sum_{r=0}^{\infty}\frac{a^r}{(r!)^2} = e_1(a) \\ T_a^2(e^a) &= \sum_{r=0}^{\infty}\frac{T_a(a^r)}{(r!)^2} = \sum_{r=0}^{\infty}\frac{a^r}{(r!)^3} = e_2(a). \end{aligned}$$

5.4.1 Iterations of The Isomorphism T_a

Using the isomorphism $T = T_a$,a demonstration for a set of generalized Laguerre derivatives can be as below:

$$\begin{aligned} T_a\mathscr{D}_1\mathscr{L} &= T_a(\mathscr{D}a\mathscr{D})f(a) = \left(\mathscr{D}a\mathscr{D}\mathscr{D}^{-1}\mathscr{D}a\mathscr{D}\right)f(a) = \mathscr{D}a\mathscr{D}\mathscr{D}^{-1}\mathscr{D}af_a \\ &= \mathscr{D}a\mathscr{D}\mathscr{D}^{-1}[f_a + af_{aa}] = \mathscr{D}a\mathscr{D}[f + af_a - f] = \mathscr{D}a\mathscr{D}[af_a] \\ &= \mathscr{D}a[f_a + af_{aa}] = \mathscr{D}\left[af_a + a^2 f_{aa}\right] = f_a + af_{aa} + 2af_{aa} + a^2 f_{aaa} \\ &= \mathscr{D} + 3a\mathscr{D}^2 + a^2\mathscr{D}^3 = \mathscr{D}_2 L, \end{aligned}$$

$$
\begin{aligned}
T_a\mathscr{D}_2\mathscr{L} &= T_a(\mathscr{D}a\mathscr{D}a\mathscr{D})f(a) = \left(\mathscr{D}a\mathscr{D}a\mathscr{D}\mathscr{D}^{-1}\mathscr{D}a\mathscr{D}\right)f(a) = \mathscr{D}a\mathscr{D}a\mathscr{D}\mathscr{D}^{-1}\mathscr{D}af_a \\
&= \mathscr{D}a\mathscr{D}a\mathscr{D}\mathscr{D}^{-1}[f_a+af_{aa}] = \mathscr{D}a\mathscr{D}a\mathscr{D}(f+af_a-f) = \mathscr{D}a\mathscr{D}a\mathscr{D}(af_a)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathscr{D}a\mathscr{D}a(f_a+af_{aa}) = \mathscr{D}a\mathscr{D}\left(af_a+a^2f_{aa}\right) \\
&= \mathscr{D}a\left(f_a+af_{aa}+2af_{aa}+a^2f_{aaa}\right) \\
&= \mathscr{D}\left(af_a+a^2f_{aa}+2a^2f_{aa}+a^3f_{aaa}\right) = f_a+7af_{aa}+6a^2f_{aaa}+a^3f_{aaaa} \\
&= \mathscr{D}+7a\mathscr{D}^2+6a^2\mathscr{D}^3+a^3\mathscr{D}^4 = \mathscr{D}_3L
\end{aligned}
$$

and in general by induction

$$
T_a^{k-1}\mathscr{D}_1\mathscr{L} = T_a^{k-1}(\mathscr{D}a\mathscr{D}) = \mathscr{D}a\mathscr{D}a\mathscr{D}\cdots a\mathscr{D} = \mathscr{D}_k\mathscr{L} \tag{5.4.2}
$$

where the last operator contains $k+1$ ordinary derivatives. The above relation provides a useful demonstration for the the generalized Laguerre derivatives using the iterations of the isomorphism T_a. Also, the actions of T_a on all functions belonging to $A = A_a$ can be observed. Considering the above mentioned definition, the following relation can be derived.

$$
\mathscr{D}_a^{-r}(1) = \frac{a^r}{r!},\; T_a\mathscr{D}_a^{-1}(1) = \mathscr{D}_{T_a}^{-1}(1) \Rightarrow \mathscr{D}_{T_a}^{-r}(1) = \frac{a^r}{(r!)^2}
$$

and, by induction,

$$
T_a^{r-1}\mathscr{D}_a^{-1}(1) = \mathscr{D}_{T_a^{k-1}}^{-1}(1) \Rightarrow \mathscr{D}_{T_a^{k-1}}^{-r}(1) = \frac{a^n}{(r!)^k}.
$$

5.5 Hermite-Based Appell Polynomials

The 3-variable Hermite polynomials (3VHP) $\mathscr{H}_m(a,b,c)$ are introduced in [6, p. 114 (22)] by

$$
\mathscr{H}_m(a,b,c) = m!\sum_{l=0}^{\left[\frac{m}{3}\right]} \frac{c^l\,\mathscr{H}_{m-3l}(a,b)}{l!\,(m-3l)!}, \tag{5.5.1}
$$

which are quasi-monomials under the action of the operators

$$\begin{aligned} L &= a+2b\frac{\partial}{\partial a}+3c\frac{\partial^2}{\partial a^2}, \\ K &= \frac{\partial}{\partial a}. \end{aligned} \tag{5.5.2}$$

The following properties holds true:

$$\begin{aligned} m\mathscr{H}_m(a,b,c) &= LK(\mathscr{H}_m(a,b,c)) = L\frac{\partial}{\partial a}(\mathscr{H}_m(a,b,c)) \\ m\mathscr{H}_m(a,b,c) &= L(m\mathscr{H}_{m-1}(a,b,c)) \\ m\mathscr{H}_m(a,b,c) &= \left(a+2b\frac{\partial}{\partial a}+3c\frac{\partial^2}{\partial a^2}\right)(m\mathscr{H}_{m-1}(a,b,c)) \end{aligned}$$

$$\left(a\frac{\partial}{\partial a}+2b\frac{\partial^2}{\partial a^2}+3c\frac{\partial^3}{\partial a^3}-m\right)\mathscr{H}_m(a,b,c)=0. \tag{5.5.3}$$

The generating function,

$$\begin{aligned} \sum_{m=0}^{\infty}\mathscr{H}_m(a,b,c)\frac{g^m}{m!} &= \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{c^p\mathscr{H}_{m-3p}(a,b)g^m}{p!(m-3p)!} \\ &= \sum_{p=0}^{\infty}\frac{(cg^3)^p}{p!}\sum_{m=0}^{\infty}\mathscr{H}_m(a,b)\frac{g^m}{m!} \\ &= e^{cg^3}\sum_{m=0}^{\infty}\mathscr{H}_m(a,b)\frac{g^m}{m!} = e^{cg^3}e^{ag+bg^2} \\ &= e^{ag+bg^2+cg^3}. \end{aligned} \tag{5.5.4}$$

Also, the polynomials $\mathscr{H}_m(a,b,c)$ satisfy the following relations

$$\begin{aligned} \frac{\partial}{\partial a}\mathscr{H}_m(a,b,c) &= m\mathscr{H}_{m-1}(a,b,c), \\ \frac{\partial}{\partial b}\mathscr{H}_m(a,b,c) &= m(m-1)\mathscr{H}_{m-2}(a,b,c), \\ \frac{\partial}{\partial c}\mathscr{H}_m(a,b,c) &= m(m-1)(m-2)\mathscr{H}_{m-3}(a,b,c), \end{aligned}$$

$$\frac{\partial}{\partial b}\mathscr{H}_m(a,b,c)=\frac{\partial^2}{\partial a^2}\mathscr{H}_m(a,b,c),\ \frac{\partial}{\partial c}\mathscr{H}_m(a,b,c)=\frac{\partial^3}{\partial a^3}\mathscr{H}_m(a,b,c)$$

which in view of the initial condition

$$\mathscr{H}_m(a,0,0) = a^m \tag{5.5.5}$$

gives the following operational definition for $\mathscr{H}_m(a,b,c)$,

$$\begin{aligned}
\exp\left(b\frac{\partial^2}{\partial a^2} + c\frac{\partial^3}{\partial a^3}\right)(a^m) &= \sum_{r=0}^{\infty} \frac{\left(b\frac{\partial^2}{\partial a^2} + c\frac{\partial^3}{\partial a^3}\right)^r}{r!}(a^m) \\
&= \sum_{r=0}^{\infty}\sum_{k=0}^{r} \frac{\binom{r}{k}\left(b\frac{\partial^2}{\partial a^2}\right)^k\left(c\frac{\partial^3}{\partial a^3}\right)^{r-k}}{r!}(a^m)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{r=0}^{\infty}\sum_{k=0}^{r} \frac{r!}{(r-k)!k!r!} b^k \frac{\partial^{2k}}{\partial a^{2k}} c^{r-k} \frac{\partial^{3r-3k}}{\partial a^{3r-3k}} a^m \\
&= \sum_{r=0}^{\infty}\sum_{k=0}^{\left[\frac{m-3r}{2}\right]} \frac{b^k}{(r-k)!k!} \frac{m!}{m-3r+3k!} c^{r-k} \frac{\partial^{2k}}{\partial a^{2k}} a^{m-3r+3k}
\end{aligned}$$

$$\begin{aligned}
&= m!\sum_{r=0}^{\left[\frac{m}{3}\right]}\sum_{k=0}^{\left[\frac{m-3r}{2}\right]} \frac{b^k c^{r-k} a^{m-3r+k}(m-3r+3k)!}{(r-k)!k!(m-3r+3k)!(m-3r+k)!} \\
&= m!\sum_{r=0}^{\left[\frac{m}{3}\right]}\sum_{k=0}^{\left[\frac{m-3r}{2}\right]} \frac{c^r}{r!} \frac{b^k a^{m-3r-2k}}{k!(m-3r-2k)!} = H_m(a,b,c)
\end{aligned}$$

$$H_m(a,b,c) = \exp\left(b\frac{\partial^2}{\partial a^2} + c\frac{\partial^3}{\partial a^3}\right)(a^m). \tag{5.5.6}$$

The polynomial set $\{\mathscr{A}_m(x)\}$ $(m = 0,1,2,\ldots)$ is an Appell set ($\mathscr{A}_m$ being of degree exactly m) if either

(i) $\frac{d}{dx}\mathscr{A}_m(x) = m\mathscr{A}_{m-1}(x)$, $m = 0,1,2,\ldots$, or

(ii) there exists a formal power series $\mathscr{A}(p) = \sum\limits_{m=0}^{\infty} a_m \frac{p^m}{m!}$, $a_0 \neq 0$ such that

$$\mathscr{A}(p)\exp(xp) = \sum_{m=0}^{\infty} \mathscr{A}_m(x)\frac{p^m}{m!}. \tag{5.5.7}$$

It is clear from the above definition that $\mathscr{A}_m(x) = \sum\limits_{k=0}^{m} \binom{m}{k} a_{m-k} x^k$.

We recall some of the members of Appell family:

(i) If $\mathscr{A}(p) = \frac{p}{(e^p-1)}$, then $\mathscr{A}_m(x) = \mathscr{B}_m(x)$, the Bernoulli polynomials [17].

(ii) If $\mathscr{A}(p) = \frac{2}{(e^p+1)}$, then $\mathscr{A}_m(x) = E_m(x)$, the Euler polynomials [17].

(iii) If $\mathscr{A}(p) = \frac{p^\gamma}{(e^p-1)^\gamma}$, then $\mathscr{A}_m(x) = \mathscr{B}_m^{(\gamma)}(x)$, the generalized Bernoulli polynomials [13].

(iv)If $\mathscr{A}(p) = \frac{2^\gamma}{(e^p+1)^\gamma}$, then $\mathscr{A}_m(x) = E_m^{(\gamma)}(x)$, the generalized Euler polynomials [13].

(v)If $\mathscr{A}(p) = \gamma_1\gamma_2\ldots\gamma_k p^k\left[(e^{\gamma_1 p}-1)(e^{\gamma_2 p}-1)\ldots(e^{\gamma_k p}-1)\right]^{-1}$, then $\mathscr{A}_m(x)$ is the Bernoulli polynomials of order k [14].

(vi) If $\mathscr{A}(p) = \frac{p^k}{e^p - \sum\limits_{s=0}^{k-1}\left(\frac{p^s}{s!}\right)}$, then $\mathscr{A}_m(x) = \mathscr{B}_m^{[k-1]}(x)$, $k \geq 1$, the new generalized Bernoulli polynomials [4].

(vii) If $\mathscr{A}(p) = 2^k\left[(e^{\gamma_1 p}+1)(e^{\gamma_2 p}+1)\ldots(e^{\gamma_k p}+1)\right]^{-1}$, then $\mathscr{A}_m(x)$ is the Euler polynomials of order k [14].

$(viii)$ If $\mathscr{A}(p) = \exp\left(\varepsilon_0 + \varepsilon_1 p + \varepsilon_2 p^2 + \cdots + \varepsilon_{n+1}p^{n+1}\right)$, $\varepsilon_{n+1} \neq 0$, then $\mathscr{A}_m(x)$ is the generalized Gould-Hopper polynomials [12], including the Hermite polynomials when $n = 1$ and classical 2-orthogonal polynomials when $n = 2$.

(ix) If $\mathscr{A}(p) = \frac{1}{(1-p)^{k+1}}$, then $\mathscr{A}_m(x) = m!\mathscr{G}_m^{(k)}(x)$, the Miller-Lee polynomials [1],[8], including the truncated exponential polynomials $e_m(x)$, when $k = 0$ and modified Laguerre polynomials $f_m^{(\gamma)}(x)$ [16], when $m = \alpha - 1$.

(x)If $\mathscr{A}(p) = \frac{2p}{(e^p+1)}$, then $\mathscr{A}_m(x) = \mathscr{G}_m(x)$, the Genocchi polynomials [9].

To generate Hermite-based Appell polynomials associated with 3VHP $\mathscr{H}_m(a,b,c)$, we introduce the generating function

$$\begin{aligned}\mathscr{G}(a,b,c;p) &= \mathscr{A}(p)\exp(Lp)\\ &= \mathscr{A}(p)\exp\left(\left(a+2b\frac{\partial}{\partial a}+3c\frac{\partial^2}{\partial a^2}\right)p\right).\end{aligned}$$

Now, decoupling the exponential operator appearing in (5.3.3), by using the Berry decoupling identity [11]

$$e^{\mathscr{C}+\mathscr{D}} = e^{\frac{t^2}{12}}e^{\left(\left(-\frac{t}{2}\right)\mathscr{C}^{\frac{1}{2}}+\mathscr{C}\right)}e^{\mathscr{D}}, \qquad [\mathscr{C},\mathscr{D}] = t\mathscr{C}^{\frac{1}{2}}, \tag{5.5.8}$$

we get the generating function for Hermite-Based Appell polynomials ${}_{\mathscr{H}}\mathscr{A}_m(x,y,z)$ in the form

$$\begin{aligned}\mathscr{G}(x,y,z;p) &= \mathscr{A}(p)\exp\left(xp+yp^2+zp^3\right) = \mathscr{A}(p)\sum_{m=0}^{\infty}\mathscr{H}_m(x,y,z)\frac{p^m}{m!}\\ &= \sum_{k=0}^{\infty}a_k\frac{p^k}{k!}\sum_{m=0}^{\infty}\mathscr{H}_m(x,y,z)\frac{p^m}{m!} = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}a_k\mathscr{H}_m(x,y,z)\frac{p^{m+k}}{m!k!}\\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{m}a_k\mathscr{H}_{m-k}(x,y,z)\frac{p^m}{(m-k)!k!}\end{aligned}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{m}a_k\mathscr{H}_{m-k}(x,y,z)\binom{m}{k}\frac{p^m}{m!} = \sum_{m=0}^{\infty}{}_{\mathscr{H}}\mathscr{A}_m(x,y,z)\frac{p^m}{m!}. \tag{5.5.9}$$

Differentiating (5.5.9) partially with respect to x, y and z, we get the following differential recurrence relations satisfied by the Hermite-Appell polynomials ${}_{\mathscr{H}}\mathscr{A}_m(x,y,z)$:

$$\begin{aligned}\frac{\partial}{\partial x}\,{}_{\mathscr{H}}\mathscr{A}_m(x,y,z) &= m\,{}_{\mathscr{H}}\mathscr{A}_{m-1}(x,y,z),\\ \frac{\partial}{\partial y}\,{}_{\mathscr{H}}\mathscr{A}_m(x,y,z) &= m(m-1)\,{}_{\mathscr{H}}\mathscr{A}_{m-2}(x,y,z),\\ \frac{\partial}{\partial z}\,{}_{\mathscr{H}}\mathscr{A}_m(x,y,z) &= m(m-1)(m-2)\,{}_{\mathscr{H}}\mathscr{A}_{m-3}(x,y,z).\end{aligned} \tag{5.5.10}$$

From relations (5.5.10), we observe that ${}_{\mathscr{H}}\mathscr{A}_m(x,y,z)$ are solutions of the equations

$$\begin{aligned}\frac{\partial}{\partial y}\,{}_{\mathscr{H}}\mathscr{A}_m(x,y,z) &= \frac{\partial^2}{\partial x^2}\,{}_{\mathscr{H}}\mathscr{A}_m(x,y,z),\\ \frac{\partial}{\partial z}\,{}_{\mathscr{H}}\mathscr{A}_m(x,y,z) &= \frac{\partial^3}{\partial x^3}\,{}_{\mathscr{H}}\mathscr{A}_m(x,y,z),\end{aligned} \tag{5.5.11}$$

under the following initial condition

$$ {}_{\mathscr{H}}\mathscr{A}_m(x,0,0) = \sum_{k=0}^{m} a_k \binom{m}{k} \mathscr{H}_{m-k}(x,0,0) = \mathscr{A}_m(x) \tag{5.5.12} $$

Thus from (5.5.11) and (5.5.12), it follows that:

$$
\begin{aligned}
& \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3} \right) \{\mathscr{A}_m(x)\} \\
= & \sum_{t=0}^{m} \binom{m}{t} a_t x^{m-t} \sum_{l=0}^{\infty} \frac{\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3} \right)^l}{l!} \\
= & \sum_{t=0}^{m} \binom{m}{t} a_t \sum_{l=0}^{\infty} \sum_{p=0}^{l} \frac{\binom{l}{p} \left(y\frac{\partial^2}{\partial x^2} \right)^p \left(z\frac{\partial^3}{\partial x^3} \right)^{l-p}}{l!} x^{m-t} \\
= & \sum_{t=0}^{m} \binom{m}{t} a_t \sum_{l=0}^{\infty} \sum_{p=0}^{l} \frac{l!}{(l-p)!p!l!} y^p \frac{\partial^{2p}}{\partial x^{2p}} z^{l-p} \frac{\partial^{3l-3p}}{\partial x^{3l-3p}} x^{m-t}
\end{aligned}
$$

then taking derivative on both sides $3l-3p$ times,

$$
\begin{aligned}
= & \sum_{t=0}^{m} \binom{m}{t} a_t \sum_{l=0}^{\infty} \sum_{p=0}^{l} \frac{1}{(p-l)!p!} y^p \frac{\partial^{2p}}{\partial x^{2p}} z^{l-p} \\
& \frac{(m-t)!}{(m-t-3l+3p)!} x^{m-t-3l+3p}
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{t=0}^{m} \binom{m}{t} a_t \sum_{l=0}^{\infty} \sum_{p=0}^{\left[\frac{m-t-3l}{2}\right]} \frac{(m-t)!}{(l-p)!p!(m-t-3l+3p)!} y^p z^{l-p} \\
& \frac{\partial^{2p}}{\partial x^{2p}} x^{m-t-3l+3p}
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{t=0}^{m} \binom{m}{t} a_t \sum_{l=0}^{\infty} \sum_{p=0}^{\left[\frac{m-t-3l}{2}\right]} \frac{(m-t)! y^p z^{l-p}}{(l-p)!p!(m-t-3l+3p)!} \\
& \frac{(m-t-3l+3p)!}{(m-t-3l+p)!} x^{m-t-3l+p}
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{t=0}^{m} \binom{m}{t} a_t \sum_{l=0}^{\left[\frac{m-t}{3}\right]} \sum_{p=0}^{\left[\frac{m-t-3l}{2}\right]} \frac{(m-t)! z^l y^p x^{m-t-3l-2p}}{l!p!(m-t-3l-2p)!} \\
= & \sum_{t=0}^{m} \binom{m}{t} a_t (m-t)! \sum_{l=0}^{\left[\frac{m-t}{3}\right]} \frac{z^l}{l!} \sum_{p=0}^{\left[\frac{m-t-3l}{2}\right]} \frac{y^p x^{m-t-3l-2p}}{(m-t-3l-2p)!p!} \\
= & \sum_{t=0}^{m} \binom{m}{t} a_t (m-t)! \sum_{l=0}^{\left[\frac{m-t}{3}\right]} \frac{z^l}{l!} \mathscr{H}_{m-t-3l}(x,y) \\
= & \sum_{t=0}^{m} \binom{m}{t} a_t \mathscr{H}_{m-t}(x,y,z) = {}_{\mathscr{H}}\mathscr{A}_m(x,y,z).
\end{aligned}
$$

Therefore, we get

$$ {}_{\mathscr{H}}\mathscr{A}_m(x,y,z) = \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\{\mathscr{A}_m(x)\}. \tag{5.5.13} $$

For example, the Hermite-Bernoulli ${}_{\mathscr{H}}\mathscr{B}_m(x,y,z)$ and Hermite-Euler polynomials ${}_{\mathscr{H}}E_m(x,y,z)$ are defined by means of the operational definitions

$$ {}_{\mathscr{H}}B_m(x,y,z) = \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\{B_m(x)\}, \tag{5.5.14} $$

and

$$ {}_{\mathscr{H}}E_m(x,y,z) = \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\{E_m(x)\}. \tag{5.5.15} $$

For $\mathscr{A}(t) = \frac{t}{(e^t-1)}$, i.e. corresponding to the generating function for Bernoulli polynomials $\mathscr{B}_m(x)$ [17]

$$ \frac{t}{(e^t-1)}\exp(xt) = \sum_{m=0}^{\infty} \mathscr{B}_m(x)\frac{t^m}{m!}, \quad |t| < 2\pi, \tag{5.5.16} $$

we get the following generating function for Hermite-Bernoulli polynomials ${}_{\mathscr{H}}\mathscr{B}_m(x,y,z)$:

$$ \frac{t}{(e^t-1)}\exp\left(xt + yt^2 + zt^3\right) = \sum_{m=0}^{\infty} {}_{\mathscr{H}}\mathscr{B}_m(x,y,z)\frac{t^m}{m!}. \tag{5.5.17} $$

Next, for $\mathscr{A}(t) = \frac{2}{(e^t+1)}$, i.e. corresponding to the generating function for Euler polynomials $E_m(x)$ [17]

$$ \frac{2}{(e^t+1)}\exp(xt) = \sum_{m=0}^{\infty} E_m(x)\frac{t^m}{m!}, \quad |t| < \pi, \tag{5.5.18} $$

we get the following generating function for Hermite-Euler polynomials ${}_{\mathscr{H}}E_m(x,y,z)$:

$$ \frac{2}{(e^t+1)}\exp\left(xt + yt^2 + zt^3\right) = \sum_{m=0}^{\infty} {}_{\mathscr{H}}E_m(x,y,z)\frac{t^m}{m!}. \tag{5.5.19} $$

Again, for $\mathscr{A}(t) = \frac{1}{(1-t)^{p+1}}$, i.e. corresponding to the generating function for Miller-Lee polynomials $\mathscr{G}_m^{(p)}(x)$ [8, p. 21, (1.11)]

$$ \frac{1}{(1-t)^{p+1}}\exp(xt) = \sum_{m=0}^{\infty} \mathscr{G}_m^{(p)}(x)\, t^m, \quad |t| < 1, \tag{5.5.20} $$

we get the following generating function for Hermite-Miller-Lee polynomials $_{\mathscr{H}}\mathscr{G}_m^{(p)}(x,y,z)$:

$$\frac{1}{(1-t)^{p+1}}\exp\left(xt+yt^2+zt^3\right)=\sum_{m=0}^{\infty}{}_HG_m^{(p)}(x,y,z)\,t^m, \tag{5.5.21}$$

which for $p=0$, gives the generating function for Hermite-truncated exponential polynomials $_{\mathscr{H}}e_m(x,y,z)$:

$$\frac{1}{(1-t)}\exp\left(xt+yt^2+zt^3\right)=\sum_{m=0}^{\infty}{}_{\mathscr{H}}e_m(x,y,z)\,t^m \tag{5.5.22}$$

and for $p=\beta-1$, gives the generating function for Hermite-modified Laguerre polynomials $_{\mathscr{H}}f_m^{(\beta)}(x,y,z)$:

$$\frac{1}{(1-t)^{\beta}}\exp\left(xt+yt^2+zt^3\right)=\sum_{m=0}^{\infty}{}_{\mathscr{H}}f_m^{(\beta)}(x,y,z)\,t^m. \tag{5.5.23}$$

Further, we recall that the Bernoulli polynomials $\mathscr{B}_m(x)$ are defined by means of the following series:

$$\mathscr{B}_m(x)=\sum_{k=0}^{m}\binom{m}{k}\mathscr{B}_px^{m-k},\ \ m\geqslant 0, \tag{5.5.24}$$

where $\mathscr{B}_m=\mathscr{B}_m(0)$ are the Bernoulli numbers defined by the generating function

$$\frac{t}{(e^t-1)}=\sum_{m=0}^{\infty}\mathscr{B}_m\frac{t^m}{m!}. \tag{5.5.25}$$

Now, operating $\exp\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}\right)$ on both sides of (5.5.24), we find

$$\exp\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}\right)\{\mathscr{B}_m(x)\}=\sum_{k=0}^{m}\binom{m}{k}\mathscr{B}_p\exp\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}\right)\left\{x^{m-k}\right\}, \tag{5.5.26}$$

which on using the operational definitions (5.5.15) and (5.5.7) on the L.H.S. and R.H.S. respectively, yields the series defining the Hermite-Bernoulli polynomials $_{\mathscr{H}}\mathscr{B}_m(x,y,z)$ in terms of 3VHP $\mathscr{H}_m(x,y,z)$ as

$${}_{\mathscr{H}}\mathscr{B}_m(x,y,z)=\sum_{k=0}^{m}\binom{m}{k}\mathscr{B}_p\mathscr{H}_{m-k}(x,y,z). \tag{5.5.27}$$

Similarly, from the series defining the Euler polynomials $E_m(x)$:

$$E_m(x) = \sum_{k=0}^{m} 2^{-k} \binom{m}{k} E_k \left(x - \frac{1}{2} \right)^{m-k}, \tag{5.5.28}$$

where $E_m = 2^m E_m\left(\frac{1}{2}\right)$ are Euler numbers defined by the generating function

$$\frac{2e^t}{(e^{2t}+1)} = \sum_{m=0}^{\infty} E_m \frac{t^m}{m!}, \tag{5.5.29}$$

we get the series definition for Hermite-Euler polynomials ${}_{\mathscr{H}}E_m(x,y,z)$ in terms of 3VHP $\mathscr{H}_m(x,y,z)$ as

$${}_{\mathscr{H}}E_m(x,y,z) = \sum_{k=0}^{m} 2^{-k} \binom{m}{k} E_k \mathscr{H}_{m-k}\left(x - \frac{1}{2}, y, z \right). \tag{5.5.30}$$

Thus, we conclude that the series definition for Hermite-Appell polynomials ${}_{\mathscr{H}}\mathscr{A}_m(x,y,z)$ can be obtained from the series defining the corresponding Appell polynomials on replacing the monomial x^m by the 3VHP $\mathscr{H}_m(x,y,z)$.

5.5.1 Applications

Several identities involving Appell polynomials are known. The formalism developed in the previous section can be used to obtain the corresponding identities involving Hermite-Appell polynomials by operating $\exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)$ on both sides of a given relation.

First, we recall the following functional equations involving Bernoulli polynomials $\mathscr{B}_m(x)$ [15, p. 26]:

$$\begin{aligned}
\mathscr{B}_m(x+1) - \mathscr{B}_m(x) &= mx^{m-1}, \ m = 0,1,2,\ldots, \\
\sum_{k=0}^{p-1} \binom{p}{k} \mathscr{B}_k(x) &= px^{p-1}, \ p = 2,3,4,\ldots, \\
\mathscr{B}_m(kx) &= k^{m-1} \sum_{l=0}^{k-1} \mathscr{B}_m\left(x + \frac{l}{k}\right), \ m = 0,1,2,\ldots; k = 1,2,3,\ldots.
\end{aligned}$$

Now, performing the operation $\exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)$ on the above equations and using the operational definitions (5.5.6) and (5.5.14) on the resultant equations we get the following identities involving Hermite-Bernoulli polynomials ${}_{\mathscr{H}}\mathscr{B}_m(x,y,z)$:

$$\begin{aligned}
&{}_{\mathscr{H}}\mathscr{B}_m(x+1,y,z)-{}_{\mathscr{H}}\mathscr{B}_m(x,y,z)\\
=\ &\exp\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}\right)\mathscr{B}_m(x+1)-\exp\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}\right)\mathscr{B}_m(x)\\
=\ &\exp\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}\right)[\mathscr{B}_m(x+1)-\mathscr{B}_m(x)]=\exp\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}\right)mx^{m-1}\\
=\ &m\mathscr{H}_{m-1}(x,y,z),\quad m=0,1,2,\ldots
\end{aligned}\tag{5.5.31}$$

$$\begin{aligned}
&\sum_{m=0}^{n-1}\binom{n}{m}\,{}_{\mathscr{H}}\mathscr{B}_m(x,y,z)\\
=\ &\sum_{m=0}^{n-1}\binom{n}{m}\exp\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}\right)\mathscr{B}_n(x)\\
=\ &\exp\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}\right)\sum_{m=0}^{n-1}\binom{n}{m}\mathscr{B}_n(x)=n\exp\left(y\frac{\partial^2}{\partial x^2}+z\frac{\partial^3}{\partial x^3}\right)x^{n-1}\\
=\ &n\mathscr{H}_{n-1}(x,y,z),\ (n=2,3,4,\ldots)
\end{aligned}\tag{5.5.32}$$

and

$$\begin{aligned}
&{}_{\mathscr{H}}\mathscr{B}_n\left(mx,m^2y,m^3z\right)\\
=\ &\exp\left(m^2y\frac{\partial^2}{\partial x^2}+m^3z\frac{\partial^3}{\partial x^3}\right)\{\mathscr{B}_n(mx)\}\\
=\ &\exp\left(m^2y\frac{\partial^2}{\partial x^2}+m^3z\frac{\partial^3}{\partial x^3}\right)m^{n-1}\sum_{l=0}^{m-1}\mathscr{B}_n\left(x+\frac{l}{m}\right)\\
=\ &m^{n-1}\sum_{l=0}^{m-1}\exp\left(m^2y\frac{\partial^2}{\partial x^2}+m^3z\frac{\partial^3}{\partial x^3}\right)\mathscr{B}_n\left(x+\frac{l}{m}\right)\\
=\ &m^{n-1}\sum_{l=0}^{m-1}{}_{\mathscr{H}}\mathscr{B}_n\left(x+\frac{l}{m},y,z\right),\ (n=0,1,2,\ldots,m=1,2,3,\ldots).
\end{aligned}\tag{5.5.33}$$

Similarly, corresponding to the functional equations involving Euler polynomials $E_n(x)$ [15, p. 30]:

$$E_m(x+1)+E_m(x)=2x^m,$$

$$E_m(kx) = k^m \sum_{l=0}^{k-1} (-1)^l E_m\left(x + \frac{l}{k}\right), \;\; m = 0,1,2,...;k\ odd,$$

we find the following identities involving Hermite-Euler polynomials ${}_{\mathscr{H}}E_n(x,y,z)$:

$${}_{\mathscr{H}}E_n(x+1,y,z) + {}_{\mathscr{H}}E_n(x,y,z) = 2\mathscr{H}_n(x,y,z). \tag{5.5.34}$$

$${}_{\mathscr{H}}E_n\left(mx, m^2y, m^3z\right) \;\; = \;\; m^n \sum_{l=0}^{m-1} (-1)^l \; {}_{\mathscr{H}}E_n\left(x + \frac{l}{m}, y, z\right) \quad (n = 0,1,2,\ldots, m\ odd) \tag{5.5.35}$$

Further, we recall the following relations between Bernoulli and Euler polynomials [15, pp. 29-30]

$$\mathscr{B}_m(x) = 2^{-k} \sum_{l=0}^{k} \binom{k}{l} \mathscr{B}_{k-l} E_k(x), \;\; (m = 0,1,2,\ldots),$$

$$E_m(x) = \frac{2^{m+1}}{(m+1)}\left[\mathscr{B}_{m+1}\left(\frac{x+1}{2}\right) - \mathscr{B}_{m+1}\left(\frac{x}{2}\right)\right], \;\; (n = 0,1,2,\ldots),$$

$$E_m(kx) = -\frac{2k^m}{(m+1)} \sum_{l=0}^{k-1} (-1)^l \mathscr{B}_{m+1}\left(x + \frac{l}{k}\right), \;\; (m = 0,1,2,\ldots;k\ even).$$

Using the operational definitions (5.5.14) and (5.5.15), and performing the operation $\exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)$ yield the following relations between Hermite-Bernoulli and Hermite-Euler polynomials:

$$\begin{aligned}
& {}_{\mathscr{H}}\mathscr{B}_n(x,y,z) \\
= \;\; & \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\{\mathscr{B}_n(x)\} \\
= \;\; & \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right) 2^{-n} \sum_{m=0}^{n} \binom{n}{m} \mathscr{B}_{n-m} E_m(2x) \\
= \;\; & \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right) 2^{-n} \sum_{m=0}^{n} \binom{n}{m} \mathscr{B}_{n-m} 2^n \sum_{l=0}^{m} (-1)^l E_n\left(2x + \frac{l}{m}\right) \\
= \;\; & 2^{-n} \sum_{m=0}^{n} \binom{n}{m} \mathscr{B}_{n-m} 2^n \sum_{l=0}^{m} (-1)^l \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right) E_n\left(2x + \frac{l}{m}\right) \\
= \;\; & 2^{-n} \sum_{m=0}^{n} \binom{n}{m} \mathscr{B}_{n-m} 2^n \sum_{l=0}^{m} (-1)^l \; {}_{\mathscr{H}}E_n\left(2x + \frac{l}{m}, y, z\right)
\end{aligned}$$

$$= 2^{-n} \sum_{m=0}^{n} \binom{n}{m} \mathscr{B}_{n-m} \ {}_{\mathscr{H}}E_n(2x,4y,8z), \ n = 0,1,2,\ldots \tag{5.5.36}$$

$$\begin{aligned} &{}_{\mathscr{H}}E_n(x,y,z) \\ = \ & \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\{E_n(x)\} \end{aligned}$$

$$\begin{aligned} = \ & \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\frac{2^{n+1}}{n+1}\left[\mathscr{B}_{n+1}\left(\frac{x+1}{2}\right) - \mathscr{B}_{n+1}\left(\frac{x}{2}\right)\right] \\ = \ & \frac{2^{n+1}}{n+1}\left[\exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\mathscr{B}_{n+1}\left(\frac{x+1}{2}\right) - \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\mathscr{B}_{n+1}\left(\frac{x}{2}\right)\right] \\ = \ & \frac{2^{n+1}}{n+1}\left[{}_{\mathscr{H}}\mathscr{B}_{n+1}\left(\frac{x+1}{2},\frac{y}{4},\frac{z}{8}\right) - {}_{\mathscr{H}}\mathscr{B}_{n+1}\left(\frac{x}{2},\frac{y}{4},\frac{z}{8}\right)\right] \end{aligned} \tag{5.5.37}$$

$$\begin{aligned} &{}_{\mathscr{H}}E_n\left(mx,m^2y,m^3z\right) \\ = \ & \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)E_n(mx) \\ = \ & \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)\left[-\frac{2m^n}{(n+1)}\sum_{l=0}^{m-1}(-1)^l B_{n+1}\left(x+\frac{l}{m}\right)\right] \\ = \ & -\frac{2m^n}{(n+1)}\sum_{l=0}^{m-1}(-1)^l \exp\left(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\right)B_{n+1}\left(x+\frac{l}{m}\right) \\ = \ & -\frac{2m^n}{(n+1)}\sum_{l=0}^{m-1}(-1)^l \ {}_{\mathscr{H}}B_{n+1}\left(x+\frac{l}{m},y,z\right), \ n = 0,1,2,\ldots; m \ even. \end{aligned} \tag{5.5.38}$$

We consider the following recently derived recurrence relation involving Genocchi polynomials $\mathscr{G}_n(x)$ [9, p. 1038, (43)]

$$2mx^{m-1} = \mathscr{G}_{m+1}(x) + \mathscr{G}_m(x),$$

which yields the following recurrence relation involving 3VHP $\mathscr{H}_n(x,y,z)$ and Hermite-Genocchi polynomials ${}_{\mathscr{H}}\mathscr{G}_n(x,y,z)$:

$${}_{\mathscr{H}}\mathscr{G}_{n+1}(x) + \ {}_{\mathscr{H}}\mathscr{G}_n(x) = 2n\mathscr{H}_{n-1}(x,y,z). \tag{5.5.39}$$

Also, corresponding to the summation formula involving Genocchi polynomials $\mathscr{G}_n(x)$

[9, p. 1038, (43)]

$$\sum_{p=1}^{l}(-1)^{p}(x+p)^{n}=\frac{1}{2(n+1)}\left[(-1)^{l}\mathscr{G}_{n+1}(x+l+1)-\mathscr{G}_{n+1}(x)\right],$$

we find the following summation formula involving 3VHP $\mathscr{H}_n(x,y,z)$ and Hermite-Genocchi polynomials ${}_{\mathscr{H}}\mathscr{G}_n(x,y,z)$:

$$\begin{aligned}&\sum_{k=1}^{n}(-1)^{k}\mathscr{H}_n(x+k,y,z)\\ =\ &\frac{1}{2(n+1)}\left[(-1)^{m}{}_{\mathscr{H}}\mathscr{G}_{n+1}(x+m+1,y,z)-{}_{\mathscr{H}}\mathscr{G}_{n+1}(x,y,z)\right]. \qquad (5.5.40)\end{aligned}$$

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